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An Operator-theoretic Approach to Invariant Integrals on Quantum Homogeneous $\mathbf{SU_{n,1}}$-spaces

By

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Abstract

An operator-theoretic approach to invariant integrals on non-compact quantum spaces is introduced on the examples of quantum ball algebras. In order to describe an invariant integral, operator algebras are associated to the quantum space which allow an interpretation as "rapidly decreasing" functions and as functions with compact support. If an operator representation of a first order differential calculus over the quantum space is known, then it can be extended to the operator algebras of integrable functions. The important feature of the approach is that these operator algebras are topological spaces in a natural way. For suitable representations and with respect to the bounded and weak operator topologies, it is shown that the algebra of functions with compact support is dense in the algebra of closeable operators used to define these algebras of functions and that the infinitesimal action of the quantum symmetry group is continuous.

§1. Introduction

The development of quantum mechanics at the beginning of the past century resulted in the discovery that nuclear physics is governed by non-commutative quantities. Recently, there have been made various suggestions that spacetime may be described by non-commutative structures at Planck scale. Within this approach, quantum groups might play a fundamental role.

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They can be viewed as q-deformations of a classical Lie group or Lie algebra and allow thus an interpretation as generalized symmetries. At the present stage, the theory is still in the beginning. Before constructing physical models, one has to establish the mathematical foundations—most important, the machineries of differential and integral calculus.

In this paper, we deal with integral calculus on non-compact quantum spaces. The integration theory on compact quantum groups is well established and was mainly developed by S. L. Woronowicz [22]. He proved the existence of a unique normalized invariant functional (Haar functional) on compact quantum groups. If one turns to the study of non-compact quantum groups or non-compact quantum spaces, one faces new difficulties which do not occur in the compact case. For instance, we do not expect that there exists a normalized invariant functional on the polynomial algebra of the quantum space. The situation is analogous to the classical theory of locally compact spaces, where one can only integrate functions which vanish sufficiently rapidly at infinity.

Our aim is to define appropriate classes of quantized integrable functions for non-compact q-deformed manifolds. The ideas are similar to those in [15] and [17], where a space of finite functions was associated to the the quantum disc and to the quantum matrix ball, respectively (see also the review article [20]). However, our treatment will make this construction more general and will allow us to consider a wider class of integrable functions. Furthermore, the invariant integral resembles the well-known quantum trace—an observation that provides us with a rather natural proof of its invariance. Admittedly, we do not elaborate harmonic analysis on quantum homogeneous $\mathrm{SU}_{n,1}$-spaces. For this, one needs additional properties, for instance the self-adjointness of Casimir operators.

Starting point of our approach will be what we call an operator expansion of the action. Suppose we are given a Hopf *-algebra $\mathcal{U}$ and a $\mathcal{U}$-module *-algebra $\mathcal{X}$ with action $\triangleright$. Let $\pi : \mathcal{X} \to \mathcal{L}^+(D)$ be a *-representation. (Precise definitions will be given below.) If for any $Z \in \mathcal{U}$ there exists a finite number of operators $L_i, R_i \in \mathcal{L}^+(D)$ such that

$$\pi(Z \triangleright x) = \sum_i L_i \pi(x) R_i, \qquad x \in \mathcal{X}, \tag{1}$$

then we say that we have an *operator expansion of the action*. Obviously, it is sufficient to know the operators L_i, R_i for the generators of $\mathcal{U}$. The operators L_i, R_i are not unique as it can be seen by replacing L_i and R_i by $(-L_i)$ and $(-R_i)$.

Let us briefly outline our method of dealing with invariant integrals on non-compact quantum spaces. Assume that $\mathfrak{g}$ is a finite-dimensional complex semi-simple Lie algebra. Let $\mathcal{U}_q(\mathfrak{g})$ denote the corresponding quantized universal enveloping algebra. With the adjoint action $\mathrm{ad}_q(X)(Y) := X_{(1)}YS(X_{(2)})$, $\mathcal{U}_q(\mathfrak{g})$ becomes a $\mathcal{U}_q(\mathfrak{g})$-module *-algebra. It is a well-known fact that, for finite dimensional representations ρ of $\mathcal{U}_q(\mathfrak{g})$, the quantum trace formula $\mathrm{Tr}_{\,q}(X) := \mathrm{Tr}\,\rho(XK_{2\omega}^{-1})$, $X \in \mathcal{U}_q(\mathfrak{g})$, defines an ad_q-invariant linear functional on $\mathcal{U}_q(\mathfrak{g})$. Here, the element $K_{2\omega} \in \mathcal{U}_q(\mathfrak{g})$ is taken such that $K_{2\omega}^{-1}XK_{2\omega} = S^2(X)$.

Now consider a $\mathcal{U}_q(\mathfrak{g})$-module *-algebra $\mathcal{X}$ and a *-representation $\pi : \mathcal{X} \to \mathcal{L}^+(D)$. In our examples, the operator expansion (1) of the $\mathcal{U}_q(\mathfrak{g})$-action on $\mathcal{X}$ will resemble the adjoint action. Furthermore, it can be extended to the *-algebra $\mathcal{L}^+(D)$ turning $\mathcal{L}^+(D)$ into a $\mathcal{U}_q(\mathfrak{g})$-module *-algebra. The quantum trace formula suggests that we can try to define an invariant integral by replacing $K_{2\omega}$ by the operator that realizes the operator expansion of $K_{2\omega}$ and taking the trace on the Hilbert space completion of D. Since we deal with unbounded operators, this can only be done for an appropriate class of operators, say $\mathbb{B}$.

First of all, the invariant functional should be well defined. Next, we wish that $\mathbb{B}$ is a $\mathcal{U}_q(\mathfrak{g})$-module *-algebra. This means that $\mathbb{B}$ should be stable under the action defined by the operator expansion. If we choose $\mathbb{B}$ such that the closures of its elements are of trace class and that multiplying the elements of $\mathbb{B}$ by any operator appearing in the operator expansion yields an element of $\mathbb{B}$, then $\mathbb{B}$ is certainly stable under the action of $\mathcal{U}_q(\mathfrak{g})$ on $\mathcal{L}^+(D)$ and the invariant functional is well defined on $\mathbb{B}$. Our intention is to interpret $\mathbb{B}$ as the rapidly decreasing functions on a q-deformed manifold. For this reason, we suppose additionally that $\mathbb{B}$ is stable under multiplication by elements of $\mathcal{X}$.

Clearly, the assumptions on $\mathbb{B}$ are satisfied by the *-algebra $\mathbb{F}$ of finite rank operators in $\mathcal{L}^+(D)$. The elements of $\mathbb{F}$ are considered as functions with finite support on the q-deformed manifold. If we think of $\mathcal{U}_q(\mathfrak{g})$ as generalized differential operators, then we can think of $\mathbb{B}$ and $\mathbb{F}$ as infinitely differentiable functions since both algebras are stable under the action of $\mathcal{U}_q(\mathfrak{g})$.

The algebras $\mathbb{B}$ and $\mathbb{F}$ were mainly introduced in order to treat invariant integration theory on q-deformed manifolds. Nevertheless, our approach also allows to include differential calculi. By means of an operator representation of a first order differential calculus over $\mathcal{X}$, one can build a differential calculus over the operator algebras $\mathbb{B}$ and $\mathbb{F}$. In this case, we view the differential calculus over $\mathbb{B}$ and $\mathbb{F}$ as an extension of the differential calculus over $\mathcal{X}$.

Our approach has the following advantage. The algebras $\mathcal{X}$ (more exactly, $\pi(\mathcal{X})$), $\mathbb{B}$, and $\mathbb{F}$ are subalgebras of $\mathcal{L}^+(D)$. In particular, they are subspaces of the topological space $\mathcal{L}(D, D^+)$. Therefore we can view this algebras as topological spaces in a rather natural way. As a consequence, it makes sense to discuss topological concepts such as continuity, density, etc.

In this paper, we study the quantum ball algebra $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ as a $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-algebra [16]. Since our approach to invariant integrals is based on Hilbert space representations, we shall specify *-representations of $\mathcal{O}_q(\mathrm{Mat}_{n,1})$. We do not require that they are irreducible. It is another notable fact that our approach works also for non-irreducible representations.

When $n = 1$, $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ is referred to as quantum disc algebra $\mathcal{O}_q(\mathrm{U})$ [15]. As the algebraic relations and the *-representations of $\mathcal{O}_q(\mathrm{U})$ are comparatively simple, it will serve as a guiding example in order to motivate and illustrate our ideas and, therefore, we shall discuss it in a greater detail.

§2. Preliminaries

§2.1. Algebraic preliminaries

Throughout this paper, q stands for a real number such that $0 < q < 1$, and we abbreviate $\lambda = q - q^{-1}$.

Let $\mathcal{U}$ be a Hopf algebra. The comultiplication, the counit, and the antipode of a Hopf algebra are denoted by Δ, ε, and S, respectively. For the comultiplication Δ, we employ the Sweedler notation: $\Delta(x) = x_{(1)} \otimes x_{(2)}$. The main objects of our investigation are $\mathcal{U}$-module algebras. An algebra $\mathcal{X}$ is called a *left $\mathcal{U}$-module algebra* if $\mathcal{X}$ is a left $\mathcal{U}$-module with action $\triangleright$ satisfying

$$f\triangleright(xy) = (f_{(1)}\triangleright x)(f_{(2)}\triangleright y), \quad x, y \in \mathcal{X},\ f \in \mathcal{U}. \tag{2}$$

For an algebra $\mathcal{X}$ with unit 1, we additionally require

$$f\triangleright 1 = \varepsilon(f)1, \quad f \in \mathcal{U}. \tag{3}$$

Let $\mathcal{X}$ be a *-algebra and $\mathcal{U}$ a Hopf *-algebra. Then $\mathcal{X}$ is said to be a *left $\mathcal{U}$-module *-algebra* if $\mathcal{X}$ is a left $\mathcal{U}$-module algebra such that the following compatibility condition holds

$$(f\triangleright x)^* = S(f)^*\triangleright x^*, \quad x \in X,\ f \in \mathcal{U}. \tag{4}$$

By an *invariant integral* we mean a linear functional h on $\mathcal{X}$ such that

$$h(f\triangleright x) = \varepsilon(f)h(x), \quad x \in \mathcal{X},\ f \in \mathcal{U}. \tag{5}$$

Synonymously, we refer to it as *$\mathcal{U}$-invariant.*

A *first order differential calculus* (abbreviated as FODC) over an algebra $\mathcal{X}$ is a pair (Γ, d), where Γ is an $\mathcal{X}$-bimodule and $\mathrm{d}: \mathcal{X} \to \Gamma$ a linear mapping, such that

$$\mathrm{d}(xy) = x \cdot \mathrm{d}y + \mathrm{d}x \cdot y, \quad x, y \in \mathcal{X}, \qquad \Gamma = \mathrm{Lin}\{\, x{\cdot}\mathrm{d}y{\cdot}z\,;\, x, y, z \in \mathcal{X}\,\}.$$

(Γ, d) is called a *first order differential *-calculus* over a *-algebra $\mathcal{X}$ if the complex vector space Γ carries an involution * such that

$$(x \cdot \mathrm{d}y \cdot z)^* = z^* \cdot \mathrm{d}(y^*) \cdot x^*, \quad x, y, z \in \mathcal{X}.$$

Let $(a_{ij})_{i,j=1}^n$ be the Cartan matrix of $\mathrm{sl}(n+1, \mathbb{C})$, that is, $a_{jj} = 2$ for $j = 1, \ldots, n$, $a_{j,j+1} = a_{j+1,j} = -1$ for $j = 1, \ldots, n-1$ and $a_{ij} = 0$ otherwise. The Hopf algebra $\mathcal{U}_q(\mathrm{sl}_{n+1})$ is generated by K_j, K_j^{-1}, E_j, F_j, $j = 1, \ldots, n$, subjected to the relations

(6)
$$K_iK_j = K_jK_i,\;\; K_j^{-1}K_j = K_jK_j^{-1} = 1,\;\; K_iE_j = q^{a_{ij}}E_jK_i,\;\; K_iF_j = q^{-a_{ij}}F_jK_i,$$

$$E_iE_j - E_jE_i = 0,\;\; i \neq j \pm 1,\;\; E_j^2E_{j\pm1} - (q+q^{-1})E_jE_{j\pm1}E_j + E_{j\pm1}E_j^2 = 0, \tag{7}$$

$$F_iF_j - F_jF_i = 0,\;\; i \neq j \pm 1,\;\; F_j^2F_{j\pm1} - (q+q^{-1})F_jF_{j\pm1}F_j + F_{j\pm1}F_j^2 = 0, \tag{8}$$

$$E_iF_j - E_jF_i = 0,\;\; i \neq j,\;\; E_jF_j - F_jE_j = \lambda^{-1}(K_j - K_j^{-1}),\;\; j = 1, \ldots, n. \tag{9}$$

The comultiplication Δ, counit ε, and antipode S are given by

$$\Delta(E_j) = E_j \otimes 1 + K_j \otimes E_j,\;\; \Delta(F_j) = F_j \otimes K_j^{-1} + 1 \otimes F_j,\;\; \Delta(K_j) = K_j \otimes K_j,$$
$$\varepsilon(K_j) = \varepsilon(K_j^{-1}) = 1, \quad \varepsilon(E_j) = \varepsilon(F_j) = 0,$$
$$S(K_j) = K_j^{-1}, \quad S(E_j) = -K_j^{-1}E_j, \quad S(F_j) = -F_jK_j.$$

Consider the involution on $\mathcal{U}_q(\mathrm{sl}_{n+1})$ determined by
(10)
$$K_i^* = K_i,\; E_j^* = K_jF_j,\; F_j^* = E_jK_j^{-1},\; j \neq n,\; E_n^* = -K_nF_n,\; F_n^* = -E_nK_n^{-1}.$$

The corresponding Hopf *-algebra is denoted by $\mathcal{U}_q(\mathrm{su}_{n,1})$.

If $n = 1$, we write K, K^{-1}, E, F rather than K_1, K_1^{-1}, E_1, F_1. Then the algebraic relations read

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{11}$$

$$EF - FE = \lambda^{-1}(K - K^{-1}), \tag{12}$$

$$K^* = K, \quad E^* = -KF, \quad F^* = -EK^{-1}. \tag{13}$$

For $n > 1$, the generators K_j, K_j^{-1}, E_j, F_j, $j = 1, \ldots, n-1$ with relations (6)–(10) generate the Hopf *-algebra $\mathcal{U}_q(\mathrm{su}_n)$.

§2.2. Operator-theoretic preliminaries

We shall use the letters $\mathcal{H}$ and $\mathcal{K}$ to denote complex Hilbert spaces. If I is an index set and $\mathcal{H} = \oplus_{i\in I}\mathcal{H}_i$, where $\mathcal{H}_i = \mathcal{K}$ for all $i \in I$, we denote by η_i the vector of $\mathcal{H}$ which has the element $\eta \in \mathcal{K}$ as its i-th component and zero otherwise. It is understood that $\eta_i = 0$ whenever $i \notin I$.

If T is a closable densely defined operator on $\mathcal{H}$, we denote by $D(T)$, $\sigma(T)$, $\bar{T}$, and T^* the domain, the spectrum, the closure, and the adjoint of T, respectively. A self-adjoint operator A is called strictly positive if $A \geq 0$ and $\ker A = \{0\}$. We write $\sigma(A) \sqsubseteq (a, b]$ if $\sigma(A)\subseteq[a, b]$ and a is not an eigenvalue of A. By definition, two self-adjoint operators strongly commute if their spectral projections mutually commute.

Let D be a dense linear subspace of $\mathcal{H}$. Then the vector space

$$\mathcal{L}^+(D) := \{\, x \in \mathrm{End}(D)\,;\, D \subset D(x^*),\ x^*D \subset D\,\}$$

is a unital *-algebra of closeable operators with the involution $x \mapsto x^+ := x^*\lceil D$ and the operator product as its multiplication. Since it should cause no confusion, we shall continue to write x^* in place of x^+. Unital *-subalgebras of $\mathcal{L}^+(D)$ are called *O*-algebras*.

Two *-subalgebras of $\mathcal{L}^+(D)$ which are not O*-algebras will be of particular interest: The *-algebra of all finite rank operators

(14)
$$\mathbb{F}(D) := \{\, x \in \mathcal{L}^+(D)\,;\, \bar{x} \text{ is bounded},\ \dim(\bar{x}\mathcal{H}) < \infty,\ \bar{x}\mathcal{H} \subset D,\ \bar{x}^*\mathcal{H} \subset D\,\}$$

and, given an O*-algebra $\mathfrak{A}$,

(15)
$$\mathbb{B}_1(\mathfrak{A}) := \{\, t\in\mathcal{L}^+(D)\,;\, \bar{t}\mathcal{H} \subset D,\ \bar{t}^*\mathcal{H} \subset D,\ \overline{atb} \text{ is of trace class for all } a, b\in\mathfrak{A}\,\}.$$

It follows from [11, Lemma 5.1.4] that $\mathbb{B}_1(\mathfrak{A})$ is a *-subalgebra of $\mathcal{L}^+(D)$. Obviously, we have $\mathbb{F}(D) \subset \mathbb{B}_1(\mathfrak{A})$ and $1 \notin \mathbb{B}_1(\mathfrak{A})$ if $\dim(\mathcal{H}) = \infty$. An operator $A \in \mathbb{F}(D)$ can be written as $A = \sum_{i=1}^n \alpha_i e_i \otimes f_i$, where $n \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $f_i, e_i \in D$, and $(e_i \otimes f_i)(x) := \langle f_i, x\rangle e_i$ for $x \in D$.

Assume that $\mathfrak{A}$ is an O*-algebra on a dense domain $D_{\mathfrak{A}}$. A natural choice for a topology on $D_{\mathfrak{A}}$ is the *graph topology* $t_{\mathfrak{A}}$ generated by the family of seminorms

$$\{\, ||\cdot||_a \,\}_{a\in\mathfrak{A}}, \quad ||\varphi||_a := ||a\varphi||, \quad \varphi \in D_{\mathfrak{A}}. \tag{16}$$

$\mathfrak{A}$ is called *closed* if the locally convex space $D_{\mathfrak{A}}$ is complete. The *closure* $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ is defined by

$$D_{\bar{\mathfrak{A}}} := \cap_{a\in\mathfrak{A}} D(\bar{a}), \quad \bar{\mathfrak{A}} := \{\, \bar{a}\lceil D_{\bar{\mathfrak{A}}}\,;\, a \in \mathfrak{A}\,\}. \tag{17}$$

By [11, Lemma 2.2.9], $D_{\bar{\mathfrak{A}}}$ is complete.

Let $D'_{\mathfrak{A}}$ denote the strong dual of the locally convex space $D_{\mathfrak{A}}$. Then the conjugate space $D^+_{\mathfrak{A}}$ is the topological space $D'_{\mathfrak{A}}$ with the addition defined as before and the multiplication replaced by $\alpha \cdot f := \bar{\alpha} f$, $\alpha \in \mathbb{C}$, $f \in D'_{\mathfrak{A}}$. For $f \in D^+_{\mathfrak{A}}$ and $\varphi \in D_{\mathfrak{A}}$, we shall write $\langle f, \varphi\rangle$ rather than $f(\varphi)$. The vector space of all continuous linear operators mapping $D_{\mathfrak{A}}$ into $D^+_{\mathfrak{A}}$ is denoted by $\mathcal{L}(D_{\mathfrak{A}}, D^+_{\mathfrak{A}})$. In the case $\mathfrak{A} = \mathcal{L}^+(D)$, we write t instead of $t_{\mathfrak{A}}$ and $\mathcal{L}(D, D^+)$ instead of $\mathcal{L}(D_{\mathfrak{A}}, D^+_{\mathfrak{A}})$. We assign to $\mathcal{L}(D_{\mathfrak{A}}, D^+_{\mathfrak{A}})$ the *bounded topology* τ_b generated by the system of semi-norms

$$\{\, p_M \,;\, M \subset D_{\mathfrak{A}},\ \text{bounded}\,\},\quad p_M(A) := \sup\nolimits_{\varphi,\psi\in M} |\langle A\varphi, \psi\rangle|,\quad A \in \mathcal{L}(D_{\mathfrak{A}}, D^+_{\mathfrak{A}})$$

and the *weak operator topology* τ_{ow} generated by the system of semi-norms

$$\{\, p_M \,;\, M \subset D_{\mathfrak{A}},\ \text{finite}\,\},\quad p_M(A) := \sup\nolimits_{\varphi,\psi\in M} |\langle A\varphi, \psi\rangle|,\quad A \in \mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}).$$

Note that $\mathfrak{A} \subset \mathcal{L}(D_{\mathfrak{A}}, D^+_{\mathfrak{A}})$ for any O*-algebra $\mathfrak{A}$. Furthermore, it is known that $\mathcal{L}^+(D_{\mathfrak{A}}) \subset \mathcal{L}(D_{\mathfrak{A}}, D^+_{\mathfrak{A}})$ if $D_{\mathfrak{A}}$ is a Fréchet space.

We say that $\mathfrak{A}$ is a *commutatively dominated O*-algebra on the Fréchet domain* $D_{\mathfrak{A}}$ if it satisfies the following assumptions (which are consequences from the definitions given in [8, 11]). There exist a self-adjoint operator A on $\mathcal{H}$ and a sequence of Borel measurable real-valued functions r_n, $n \in \mathbb{N}$, such that $1 \le r_1(t)$, $r_n(t)^2 \le r_{n+1}(t)$, $r_n(A)\lceil D_{\mathfrak{A}} \in \mathfrak{A}$, and $D_{\mathfrak{A}} = \cap_{n\in\mathbb{N}} D(r_n(A))$.

By a *-representation π of a *-algebra $\mathfrak{A}$ on a domain D we mean a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{L}^+(D)$. For notational simplicity, we usually suppress the representation and write x instead of $\pi(x)$ when no confusion can arise. If each decomposition $\pi = \pi_1 \oplus \pi_2$ of π as direct sum of *-representations π_1 and π_2 implies that $\pi_1 = 0$ or $\pi_2 = 0$, then π is said to be *irreducible.*

Given a *-representation π, it follows from [11, Proposition 8.1.12] that the mapping

$$\bar{\pi} : \mathfrak{A} \to \mathcal{L}^+(D(\bar{\pi})),\quad \bar{\pi}(a) := \overline{\pi(a)}\lceil D(\bar{\pi}),$$

defines a *-representation on $D(\bar{\pi}) := \cap_{a\in\mathfrak{A}} D(\overline{\pi(a)})$. $\bar{\pi}$ is called the *closure* of π and π is said to be *closed* if $\bar{\pi} = \pi$.

If we consider *-representations of *-algebras, we shall restrict ourself to representations which are in a certain sense “well behaved”. This means that we shall impose some regularity conditions on the (in general) unbounded operators under consideration. The requirements will strongly depend on the situation. For further discussion on “well behaved” representations, see [13, 2, 1].

Suppose that $\mathcal{X}$ is a *-algebra and $\pi : \mathcal{X} \to \mathcal{L}^+(D)$ a *-representation. Each symmetric operator $C \in \mathcal{L}^+(D)$ gives rise to a first order differential

*-calculus $(\Gamma_{\pi,C}, \mathrm{d}_{\pi,C})$ over $\mathcal{X}$ defined by

$$\Gamma_{\pi,C} := \mathrm{Lin}\{\, \pi(x)(C\pi(y) - \pi(y)C)\pi(z)\,;\, x, y, z \in \mathcal{X}\,\} \text{ and} \tag{18}$$

$$\mathrm{d}_{\pi,C} : \mathcal{X} \to \Gamma_{\pi,C}, \quad \mathrm{d}_{\pi,C}(x) := \mathrm{i}(C\pi(x) - \pi(x)C), \quad x \in \mathcal{X}, \tag{19}$$

where i denotes the imaginary unit (see [12]). Let (Γ, d) be a first order differential *-calculus over $\mathcal{X}$. Then $(\Gamma_{\pi,C}, \mathrm{d}_{\pi,C})$ is called a *commutator representation of* (Γ, d), if there exits a linear mapping $\rho : \Gamma \to \Gamma_{\pi,C}$ such that $\rho(x{\cdot}\mathrm{d}y{\cdot}z) = \pi(x)\mathrm{d}_{\pi,C}(y)\pi(z)$ and $\rho(\gamma^*) = \rho(\gamma)^*$ for all $x, y, z \in \mathcal{X}$, $\gamma \in \Gamma$.

We close this subsection by stating three auxiliary lemmas.

Lemma 2.1. *Let A be a self-adjoint operator and let w be an unitary operator on a Hilbert space $\mathcal{H}$ such that*

$$qwA \subseteq Aw. \tag{20}$$

i. *Then the spectral projections of A corresponding to $(-\infty, 0)$, $\{0\}$, and $(0, \infty)$ commute with w.*

ii. *Suppose additionally that A is strictly positive. Then there exists a self-adjoint operator A_0 on a Hilbert space $\mathcal{H}_0$ with $\sigma(A_0) \sqsubseteq (q, 1]$ such that, up to unitary equivalence, $\mathcal{H} = \oplus_{n=-\infty}^{\infty} \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$, and*

$$A\eta_n = q^n A_0 \eta_n, \quad w\eta_n = \eta_{n+1},$$

where $\eta \in \mathcal{H}_0$ and $n \in \mathbb{Z}$.

Proof. (i): Let $e(\mu)$ denote the spectral projections of A. Since w is unitary, (20) implies that $A = qwAw^*$ and hence $e(q\mu) = we(\mu)w^*$. This proves (i).

(ii): Let $\mathcal{H}_n := e((q^{n+1}, q^n])\mathcal{H}$ and $A_n := A\lceil\mathcal{H}_n$, $n \in \mathbb{Z}$. Since A is strictly positive, $\mathcal{H} = \oplus_{n=-\infty}^{\infty} \mathcal{H}_n$. Now $e((q^{n+1}, q^n]) = we((q^n, q^{n-1}])w^*$ yields $w\mathcal{H}_n = \mathcal{H}_{n+1}$. Up to unitary equivalence, we can assume that $\mathcal{H}_n = \mathcal{H}_0$ and $w\eta_n = \eta_{n+1}$ for $\eta \in \mathcal{H}_0$. Moreover, $A\eta_n = q^n w^n A w^{n*} \eta_n = q^n w^n A_0 \eta_0 = q^n A_0 \eta_n$. □

Lemma 2.2. *Let A be a self-adjoint operator and let w be a linear isometry on a Hilbert space $\mathcal{H}$ such that*

$$swA \subseteq Aw \tag{21}$$

for some fixed positive real number $s \neq 1$. Suppose that A has an eigenvalue λ such that the eigenspace $\mathcal{H}_0 := \ker(A - \lambda)$ coincides with $\ker w^*$. Then the eigenspace $\mathcal{H}_n := \ker(A - s^n\,\lambda)$ coincides with $w^n\mathcal{H}_0$ for each $n \in \mathbb{N}$.

Proof. Taking adjoints in (21) gives $s^{-1}w^*A \subseteq Aw^*$. Let $n \in \mathbb{N}_0$, $\varphi \in \mathcal{H}_n$, and $\psi \in \mathcal{H}_{n+1}$. Then $Aw\varphi = swA\varphi = s^{n+1}\lambda w\varphi$ and $Aw^*\psi = s^{-1}w^*A\psi = s^n\lambda w^*\psi$. Hence $w\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and $w^*\mathcal{H}_{n+1} \subset \mathcal{H}_n$. Since $\mathcal{H}_{n+1} \perp \mathcal{H}_0$, we have $ww^*\psi = \psi$. This together with $w^*w = 1$ implies that $w\lceil\mathcal{H}_n$ is a bijective mapping from $\mathcal{H}_n$ onto $\mathcal{H}_{n+1}$ with inverse $w^*\lceil\mathcal{H}_{n+1}$. □

Lemma 2.3. *Let $\epsilon \in \{\pm 1\}$. Assume that x is a closed densely defined operator on a Hilbert space. Then the coincidence of domains $\mathcal{D}(xx^*) = \mathcal{D}(x^*x)$ and the relation*

$$xx^* - q^2x^*x = \epsilon(1-q^2) \tag{22}$$

hold if and only if x is unitarily equivalent to an orthogonal direct sum of operators determined as follows:

$\epsilon = 1$:

- *(I) $x\eta_n = (1-q^{2n})^{1/2}\eta_{n-1}$ on the Hilbert space $\oplus_{n=0}^{\infty}\mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$.*
- *$(II)_A$ x is the minimal closed operator on $\oplus_{n=-\infty}^{\infty}\mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$, with $x\eta_n = (1+q^{2n}A)^{1/2}\eta_{n-1}$, where A is a self-adjoint operator on $\mathcal{H}_0$ such that $\sigma(A) \sqsubseteq (q^2, 1]$.*
- *$(III)_u$ $x = u$, where u is a unitary operator.*

$\epsilon = -1$:

x is the minimal closed operator on the Hilbert space $\oplus_{n=1}^{\infty}\mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_1$, with $x\eta_n = (q^{-2n}-1)^{1/2}\eta_{n+1}$.

Proof. Direct calculations show that the operators described in Lemma 2.3 satisfy (22). Suppose now we are given an operator x satisfying the assumptions of the lemma. Recall that x^*x is self-adjoint for every closed densely defined operator x. Let $e(\mu)$ denote the spectral projections of the self-adjoint operator $Q = \epsilon - x^*x$. For $\varphi \in D(Q^2) = D((x^*x)^2)$, it follows from (22) that

$$Qx^*\varphi = x^*(\epsilon - xx^*)\varphi = x^*(\epsilon - q^2x^*x - \epsilon(1-q^2))\varphi = q^2x^*Q\varphi, \tag{23}$$

$$xQ\varphi = (\epsilon - xx^*)x\varphi = (\epsilon - q^2x^*x - \epsilon(1-q^2))x\varphi = q^2Qx\varphi. \tag{24}$$

The cases $\epsilon = 1$ and $\epsilon = -1$ will be analyzed separately.

$\epsilon = 1$: Let $x^* = ua$ be the polar decomposition of x^*. Note that

$$a^2 = xx^* = 1 - q^2 + q^2x^*x = 1 - q^2Q \geq 1 - q^2, \tag{25}$$

which implies, in particular, that $\ker a = \ker u = 0$, so u is an isometry. Inserting $\varphi = a^{-1}\psi$ in (23), where $\psi \in D(Q^2)$, one obtains $Qu\psi = q^2uaQa^{-1}\psi = q^2uQ\psi$. Since $D(Q^2)$ is a core for Q, it follows that $q^2uQ \subseteq Qu$. By taking adjoints, one also gets $u^*Q \subseteq q^2Qu^*$. Furthermore, $\varphi \in \ker x = \ker x^*x = \ker u^*$ if and only if $(Q-1)\varphi = 0$. If $\ker u^* \neq \{0\}$, Lemma 2.2 implies that $\mathcal{K} := \oplus_{n=0}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n = \ker(Q - q^{2n})$, is a reducing subspace for u and Q. Moreover, $x\lceil\mathcal{K} = (1-q^2Q)^{1/2}u^*\lceil\mathcal{K}$ is unitarily equivalent to an operator of the form (I).

It suffices now to prove the assertion under the additional assumption that $\ker u^* = \{0\}$. By Lemma 2.1(i), we can treat the cases where Q is strictly positive, zero, or strictly negative separately.

If Q were strictly positive, then it would be unbounded by Lemma 2.1(ii), which contradicts (25). Hence we can discard this case. If $Q = 0$, then $x = u^*$ is unitarily equivalent to an operator of the form $(III)_u$. When Q is strictly negative, Lemma 2.1(ii) applied to the relation $q^2u(-Q) \subseteq (-Q)u$ shows that $x = (1-q^2Q)^{1/2}u^*$ is unitarily equivalent to an operator of the form $(II)_A$.

$\epsilon = -1$: In this case, we use the polar decomposition $x = vb$ of x. From

$$b^2 = x^*x = -1 - Q = q^{-2}(xx^* + 1 - q^2) \geq q^{-2} - 1, \tag{26}$$

it follows that $\ker b = \ker v = \{0\}$ so that v is an isometry. Using (24) and arguing as above, one obtains $q^{-2}vQ \subseteq Qv$ and $q^2v^*Q \subseteq Qv^*$. Note that, in the present case, $Q \leq -q^{-2}$ by (26). Therefore $\ker v^* \neq \{0\}$ since otherwise Lemma 2.1 would imply that 0 belongs to the spectrum of Q. Now $\varphi \in \ker v^* = \ker x^* = \ker xx^*$ if and only if $Q\varphi = (-1 - x^*x)\varphi = (-1 - q^{-2}(1-q^2))\varphi = -q^{-2}\varphi$. From Lemma 2.2, it follows that $\mathcal{K} := \oplus_{n=1}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n = \ker(Q + q^{-2n})$, is a reducing subspace for v and Q. In particular, $x\lceil\mathcal{K} = v(-1-Q)^{1/2}\lceil\mathcal{K}$ is unitarily equivalent to an operator of the form stated in the lemma. Finally, we conclude that $\mathcal{H} = \mathcal{K}$ since the restriction of v^* to a non-zero orthogonal complement of $\mathcal{K}$ would be injective, which is impossible as noted before. □

Remark. For $\epsilon = 1$, a characterization of irreducible representations of (22) can be found in [10] as a special case of the results therein. For $\epsilon = -1$, the irreducible representations of (22) were obtained in [3] by assuming in the proof that x^*x has eigenvectors.

§3. Quantum Disc Algebra

§3.1. Invariant integrals associated with the quantum disc algebra

The quantum disc algebra $\mathcal{O}_q(\mathrm{U})$ is defined as the *-algebra generated by z and z^* with relation [5, 9]

$$z^* z - q^2 z z^* = 1 - q^2. \tag{27}$$

By (27), it is obvious that $\mathcal{O}_q(\mathrm{U}) = \mathrm{Lin}\{z^n z^{*m}\,;\, n, m \geq 0\}$. Set

$$y := 1 - zz^*. \tag{28}$$

Then $y = y^*$ and

$$yz = q^2 zy, \quad yz^* = q^{-2} z^* y. \tag{29}$$

From $zz^* = 1 - y$, $z^* z = 1 - q^2 y$, and (29), we deduce

$$z^n z^{*n} = (y; q^{-2})_n, \quad z^{*n} z^n = (q^2 y; q^2)_n, \tag{30}$$

where $(t;q)_0 := 1$ and $(t;q)_n := \prod_{k=0}^{n-1}(1 - q^k t)$, $n \in \mathbb{N}$. In particular, each element $f \in \mathcal{O}_q(\mathrm{U})$ can be written as

$$f = \sum_{n=0}^{N} z^n p_n(y) + \sum_{n=1}^{M} p_{-n}(y) z^{*n}, \quad N, M \in \mathbb{N}, \tag{31}$$

with polynomials p_n in y.

The left action $\triangleright$ which turns $\mathcal{O}_q(\mathrm{U})$ into a $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebra can be found in [15, 19] or [5]. On generators, it takes the form

$$K^{\pm 1} \triangleright z = q^{\pm 2} z, \qquad E \triangleright z = -q^{1/2} z^2, \qquad F \triangleright z = q^{1/2}, \tag{32}$$

$$K^{\pm 1} \triangleright z^* = q^{\mp 2} z^*, \qquad E \triangleright z^* = q^{-3/2}, \qquad F \triangleright z^* = -q^{5/2} z^{*2}. \tag{33}$$

Recall our notational conventions regarding representations. For instance, if $\pi : \mathcal{O}_q(\mathrm{U}) \to \mathcal{L}^+(D)$ is a representation, we write f instead of $\pi(f)$ and $X \triangleright f$ in instead of $\pi(X \triangleright f)$, where $f \in \mathcal{O}_q(\mathrm{U})$, $X \in \mathcal{U}_q(\mathrm{su}_{1,1})$. The key observation of this subsection is the following simple operator expansion.

Lemma 3.1. *Let $\pi : \mathcal{O}_q(\mathrm{U}) \to \mathcal{L}^+(D)$ be a *-representation of $\mathcal{O}_q(\mathrm{U})$ such that y^{-1} belongs to $\mathcal{L}^+(D)$. Set $A := q^{-1/2}\lambda^{-1} z$ and $B := -y^{-1} A^*$. Then the formulas*

$$K \triangleright f = y f y^{-1}, \quad K^{-1} \triangleright f = y^{-1} f y, \tag{34}$$

$$E \triangleright f = Af - y f y^{-1} A, \tag{35}$$

$$F \triangleright f = B f y - q^2 f y B \tag{36}$$

define an operator expansion of the action $\triangleright$ *for* $f \in \mathcal{O}_q(\mathrm{U})$*. The same formulas applied to* $f \in \mathcal{L}^+(D)$ *turn the* O^**-algebra* $\mathcal{L}^+(D)$ *into a* $\mathcal{U}_q(\mathrm{su}_{1,1})$*-module* *-*algebra.*

Proof. We take Equations (34)–(36) as definition and show that the action $\triangleright$ defined in this way turns $\mathcal{L}^+(D)$ into a $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebra. To verify that $\triangleright$ is well defined, we use the commutation relations

$$yA = q^2Ay, \quad yB = q^{-2}By, \quad AB - BA = -\lambda^{-1}y^{-1} \tag{37}$$

which are easily obtained by applying (27) and (29). For $f \in \mathcal{L}^+(D)$, we have

$$\begin{aligned}(EF - FE){\triangleright}f &= ABfy + yfBA - BAfy - yfAB\\ &= (AB - BA)fy - yf(AB - BA)\\ &= \lambda^{-1}(yfy^{-1} - y^{-1}fy) \;=\; \lambda^{-1}(K - K^{-1}){\triangleright}f.\end{aligned}$$

The relations (11) are handled in the same way, so we conclude that the action is well defined.

We continue by verifying (2)–(4). Since the action is associative, it is sufficient to prove (2)–(4) for generators. Let $f, g \in \mathcal{L}^+(D)$. Then

$$\begin{aligned}(E{\triangleright}f)g + (K{\triangleright}f)(E{\triangleright}g) &= (Af - yfy^{-1}A)g + yfy^{-1}(Ag - ygy^{-1}A)\\ &= Afg - yfgy^{-1}A \;=\; E{\triangleright}(fg),\end{aligned}$$

$$(E{\triangleright}f)^* = f^*A^* - A^*y^{-1}f^*y = -f^*yB + q^{-2}Bf^*y = q^{-2}F{\triangleright}f^* = S(E)^*{\triangleright}f^*,$$

and $E{\triangleright}1 = A - yy^{-1}A = 0 = \varepsilon(E)1$. The generators F, K and K^{-1} are treated analogously. Summarizing, we have shown that the action $\triangleright$ defined by (34)–(36) equips $\mathcal{L}^+(D)$ with the structure of a $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebra.

It remains to prove that (34)–(36) define an operator expansion of the action $\triangleright$ given by (32) and (33). Since $\pi(\mathcal{O}_q(\mathrm{U}))$ is a *-subalgebra of the $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebra $\mathcal{L}^+(D)$, it is sufficient to verify (34)–(36) for the generators of $\mathcal{U}_q(\mathrm{su}_{1,1})$ and $\mathcal{O}_q(\mathrm{U})$ (see Equation (2)). From the definition of A and y, it follows by using (27) and (29) that

$$E{\triangleright}z = Az - yzy^{-1}A = q^{-1/2}\lambda^{-1}(z^2 - q^2z^2) = -q^{1/2}z^2, \tag{38}$$

$$E{\triangleright}z^* = Az^* - yz^*y^{-1}A = q^{-5/2}\lambda^{-1}(q^2zz^* - z^*z) = q^{-3/2}. \tag{39}$$

The other relations of (32) and (33) are proved similarly. This completes the proof. □

Recall that the left adjoint action $\mathrm{ad}_L(a)(b) := a_{(1)}bS(a_{(2)})$, $a, b \in \mathcal{U}_q(\mathrm{su}_{1,1})$, turns $\mathcal{U}_q(\mathrm{su}_{1,1})$ into a $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebra. For the generators E, F, and K, we obtain $\mathrm{ad}_L(E)(b) = Eb - KbK^{-1}E$, $\mathrm{ad}_L(F)(b) = FbK - q^2bKF$, and $\mathrm{ad}_L(K)(b) = KbK^{-1}$. There is an obvious formal coincidence of this formulas with (34)–(36) but A, B, and y do not satisfy the relations of E, F, and K because the last equation of (37) differs from (12).

We mentioned that for a finite dimensional representation ρ of $\mathcal{U}_q(\mathrm{su}_{1,1})$ the quantum trace

$$\mathrm{Tr}_q\, a := \mathrm{Tr}\, \rho(aK^{-1})$$

defines an invariant integral on $\mathcal{U}_q(\mathrm{su}_{1,1})$ (see [6, Proposition 7.1.14]). The proof does not involve the whole set of relations of $\mathcal{U}_q(\mathrm{su}_{1,1})$ but the trace property and the relation $K^{-1}fK = S^2(f)$ for all $f \in \mathcal{U}_q(\mathrm{su}_{1,1})$. The last relation reads on generators as $K^{-1}KK = K$, $K^{-1}EK = q^{-2}E$, $K^{-1}FK = q^2F$ and these equations are also satisfied if we replace K by y, E by A, and F by B.

The main result of this section, achieved in Proposition 3.1 below, is a trace formula for an invariant integral on the operator algebras $\mathbb{B}_1(\mathfrak{A})$ and $\mathbb{F}(D)$ from Subsection 2.2 by using above observations. Note that we cannot have a normalized invariant integral on $\mathcal{O}_q(\mathrm{U})$; if there were an invariant integral h on $\mathcal{O}_q(\mathrm{U})$ satisfying $h(1) = 1$, then we would obtain

$$1 = h(1) = q^{-1/2}h(F \triangleright z) = q^{-1/2}\varepsilon(F)h(z), \tag{40}$$

a contradiction since $\varepsilon(F) = 0$.

Proposition 3.1. *Suppose that $\pi : \mathcal{O}_q(\mathrm{U}) \to \mathcal{L}^+(D)$ is a *-representation of $\mathcal{O}_q(\mathrm{U})$ such that $y^{-1} \in \mathcal{L}^+(D)$. Let $\mathfrak{A}$ be the O*-algebra generated by the operators z, z^*, and y^{-1}. Then the *-algebras $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$ defined in* (14) *and* (15), *respectively, are $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebras, where the action is given by* (34)–(36). *The linear functional*

$$h(g) := c\, \mathrm{Tr}\, \overline{gy^{-1}}, \quad c \in \mathbb{R}, \tag{41}$$

defines an invariant integral on both $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$.

Proof. Obviously, by the definition of $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$, we have $afb \in \mathbb{F}(D)$ and $agb \in \mathbb{B}_1(\mathfrak{A})$ for all $f \in \mathbb{F}(D)$, $g \in \mathbb{B}_1(\mathfrak{A})$, $a, b \in \mathfrak{A}$, so both algebras are stable under the action of $\mathcal{U}_q(\mathrm{su}_{1,1})$. By Lemma (3.1), this action turns $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$ into $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebras.

The proof of the invariance of h uses the trace property $\mathrm{Tr}\, \overline{agb} = \mathrm{Tr}\, \overline{gba} = \mathrm{Tr}\, \overline{bag}$ which holds for all $g \in \mathbb{B}_1(\mathfrak{A})$ and all $a, b \in \mathfrak{A}$ (see [11]). Since the action is associative and ε a homomorphism, we only have to prove the invariance of h for generators. Let $g \in \mathbb{B}_1(\mathfrak{A})$. Then

$$h(E \triangleright g) = \mathrm{Tr}\,(\overline{Agy^{-1}} - \overline{ygy^{-1}Ay^{-1}}) = \mathrm{Tr}\,\overline{Agy^{-1}} - \mathrm{Tr}\,\overline{Agy^{-1}} = 0 = \varepsilon(E)h(g).$$

Similarly, $h(F \triangleright g) = 0 = \varepsilon(F)h(g)$ and $h(K^{\pm 1} \triangleright g) = h(g) = \varepsilon(K^{\pm 1})h(g)$. Hence h defines an invariant integral on $\mathbb{B}_1(\mathfrak{A})$. It is obvious that the restriction of h to $\mathbb{F}(D)$ gives an invariant integral on $\mathbb{F}(D)$. □

Commonly, the algebra $\mathcal{O}_q(\mathrm{U})$ is considered as the polynomial functions on the quantum disc. Observe that $agb \in \mathbb{B}_1(\mathfrak{A})$ for all $g \in \mathbb{B}_1(\mathfrak{A})$ and all polynomial functions $a, b \in \mathcal{O}_q(\mathrm{U})$. Note, furthermore, that the action of E and F satisfies a "twisted" Leibniz rule. If we think of $\mathcal{U}_q(\mathrm{su}_{1,1})$ as an algebra of "generalized differential operators", then we can think of $\mathbb{B}_1(\mathfrak{A})$ as the algebra of infinitely differentiable functions which vanish sufficiently rapidly at "infinity" and of $\mathbb{F}(D)$ as the infinitely differentiable functions with compact support.

§3.2. Topological aspects of *-representations

This subsection is concerned with some topological aspects of representations of $\mathcal{O}_q(\mathrm{U})$. The "well behaved" operators satisfying the defining relation of $\mathcal{O}_q(\mathrm{U})$ are described in Lemma 2.3. Here we restate Lemma 2.3 by considering only irreducible *-representations and specifying the domain on which the algebra acts. As we require that y^{-1} exists, we exclude the case $(III)_u$ in which $y = 0$. Let $\{\eta_j\}_{j\in J}$ denote the canonical basis in the Hilbert space $\mathcal{H} = l_2(J)$, where $J = \mathbb{N}_0$ or $J = \mathbb{Z}$.

(I) The operators z, z^*, and y act on $D := \mathrm{Lin}\{\, \eta_n\, ;\, n \in \mathbb{N}_0 \,\}$ by

$$z\eta_n = \lambda_{n+1}\eta_{n+1}, \quad z^*\eta_n = \lambda_n\eta_{n-1}, \quad y\eta_n = q^{2n}\eta_n.$$

$(II)_\alpha$ Let $\alpha \in [0, 1)$. The actions of z, z^*, and y on $D := \mathrm{Lin}\{\, \eta_n\, ;\, n \in \mathbb{Z} \,\}$ are given by

$$z\eta_n = \lambda_{\alpha,n+1}\eta_{n+1}, \quad z^*\eta_n = \lambda_{\alpha,n}\eta_{n-1}, \quad y\eta_n = -q^{2(\alpha+n)}\eta_n.$$

Here, $\lambda_n = (1 - q^{2n})^{1/2}$ and $\lambda_{\alpha,n} = (1 + q^{2(\alpha+n)})^{1/2}$. Obviously, $y^{-1} \in \mathcal{L}^+(D)$ in both cases.

Let $\mathfrak{A}_0$ be the O*-algebra on D generated by z, z^*, and $y^{-1} = (1 - zz^*)^{-1}$. If we equip D with the graph topology $t_{\mathfrak{A}_0}$, D is not complete. The situation becomes better if we pass to the closure $\mathfrak{A}$ of $\mathfrak{A}_0$. By (17), $\mathfrak{A}$ is an O*-algebra on $D_{\mathfrak{A}} := \cap_{a\in\mathfrak{A}_0} D(\bar{a})$. Some topological facts concerning $\mathfrak{A}$ and $\mathcal{L}^+(D_{\mathfrak{A}})$ are collected in the following lemma and the next proposition.

Lemma 3.2. *Suppose we are given an irreducible *-representation of type* (I) *or* $(II)_\alpha$*. Let* $\mathfrak{A}$ *be the* O^**-algebra defined in the preceding paragraph.*

i. $\mathfrak{A}$ *is a commutatively dominated* O^**-algebra on a Fréchet domain.*

ii. $D_{\mathfrak{A}}$ *is nuclear, in particular,* $D_{\mathfrak{A}}$ *is a Fréchet–Montel space.*

Proof. (i): The operator y is essentially self-adjoint on $D_{\mathfrak{A}}$ and so is

$$T := 1 + y^2 + y^{-2}. \tag{42}$$

Let $\varphi \in D_{\mathfrak{A}}$. A standard argument shows that, for each polynomial $p(y, y^{-1})$, there exist $k \in \mathbb{N}$ such that $||p(y, y^{-1})\varphi|| \leq ||T^k\varphi||$. By using (30), we get the estimates

$$||z^n p(y,y^{-1})\varphi|| \leq (||\bar{p}(y,y^{-1})(q^2y;q^2)_n\, p(y,y^{-1})\varphi||\,||\varphi||)^{1/2} \leq ||T^l\varphi||,$$
$$||z^{*n} p(y,y^{-1})\varphi|| \leq (||\bar{p}(y,y^{-1})(y;q^{-2})_n\, p(y,y^{-1})\varphi||\,||\varphi||)^{1/2} \leq ||T^{l'}\varphi||$$

for some $l, l' \in \mathbb{N}$. Since $T \geq 2$ and $T^k \leq T^m$ for $k \leq m$, we can find for each finite sequence $k_1, \ldots, k_N \in \mathbb{N}$ a $k_0 \in \mathbb{N}$ such that $\sum_{j=1}^N ||T^{k_j}\varphi|| \leq ||T^{k_0}\varphi||$. By (31), (29), and the definition of $\mathfrak{A}$, it follows that each $f \in \mathfrak{A}$ can be written as $f = \sum_{n=0}^N z^n p_n(y, y^{-1}) + \sum_{n=1}^M z^{*n} p_{-n}(y, y^{-1})$. From the foregoing, we conclude that there exist $m \in \mathbb{N}$ such that $||f\varphi|| \leq ||T^m\varphi||$, consequently $||\cdot||_f \leq ||\cdot||_{T^m}$. This shows that the family $\{||\cdot||_{T^{2^k}}\}_{k\in\mathbb{N}}$ generates the graph topology and $D_{\mathfrak{A}} = \cap_{k\in\mathbb{N}} D(\bar{T}^{2^k})$, which proves (i).

(ii): By (i), the graph topology is metrizable. It follows from [11, Proposition 2.2.9 and Corollary 2.3.2.(ii)] that $D_{\mathfrak{A}}$ is a reflexive Fréchet space, in particular, $D_{\mathfrak{A}}$ is barreled. To see that $D_{\mathfrak{A}}$ is nuclear, consider $E_n := (\overline{D_{\mathfrak{A}}}, ||\cdot||_{T^n})$, where the closure of $D_{\mathfrak{A}}$ is taken in the norm $||\cdot||_{T^n}$, and the embeddings $\iota_{n+1} : E_{n+1} \to E_n$, where ι_{n+1} denotes the identity on E_{n+1}, $n \in \mathbb{N}$. It is easy to see that the operator $\bar{T}^{-1} : \mathcal{H} \to \mathcal{H}$ is a Hilbert–Schmidt operator and that the canonical basis $\{e_j\}_{j\in J}$, where $J = \mathbb{N}_0$ in case (I) and $J = \mathbb{Z}$ in case (II), is a complete set of eigenvectors. The set $\{f_j^n\}_{j\in J}$, $f_j^n = ||T^n e_j||^{-1} e_j$ constitutes an orthonormal basis in E_n, and we have

$$\sum_{j\in J} ||\iota_{n+1}(f_j^{n+1})||_{T^n}^2 = \sum_{j\in J} ||T^n f_j^{n+1}||^2 = \sum_{j\in J} ||T^n(||T^{n+1}e_j||^{-1}e_j)||^2$$
$$= \sum_{j\in J} ||T^{-1}e_j||^2 < \infty$$

which shows that ι_{n+1} is a Hilbert–Schmidt operator. From this, we conclude that $D_{\mathfrak{A}}$ is a nuclear space since the family $\{||\cdot||_{T^n}\}_{n\in\mathbb{N}}$ of Hilbert semi-norms generates the topology on $D_{\mathfrak{A}}$. As each nuclear space is a Schwartz space and as each barreled Schwartz space is a Montel space, $D_{\mathfrak{A}}$ is a Montel space. $\square$

Proposition 3.2. *Suppose we are given an irreducible *-representation of type* (I) *or* $(II)_\alpha$. *Assume that* $\mathfrak{A}$ *is the closed* O^**-algebra defined above.*

i. $\mathbb{F}(D_{\mathfrak{A}})$ *is dense in* $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$ *with respect to the bounded topology* τ_b.

ii. *The* $\mathcal{U}_q(\mathrm{su}_{1,1})$*-action on* $\mathcal{L}^+(D_{\mathfrak{A}})$ *is continuous with respect to* τ_b.

Proof. (i) follows immediately from Lemma 3.2(ii) and [11, Theorem 3.4.5].

(ii): Let $x \in \mathcal{L}^+(D_{\mathfrak{A}})$ and $a, b \in \mathfrak{A}$. According to [11, Proposition 3.3.4(ii)], the multiplication $x \mapsto axb$ is continuous. By Lemma 3.1, the action of $\mathcal{U}_q(\mathrm{su}_{1,1})$ is given by a finite linear combination of such expressions, hence it is continuous. $\square$

The algebra $\mathbb{F}(D)$ is the linear span of operators $\eta_m \otimes \eta_n$, where $n, m \in \mathbb{N}_0$ for the type (I) representation and $n, m \in \mathbb{Z}$ for type (II) representations. Since $D \subset D_{\mathfrak{A}}$, we can consider $\mathbb{F}(D)$ as a subalgebra of $\mathbb{F}(D_{\mathfrak{A}})$ and, moreover, as a $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebra. The interest in $\mathbb{F}(D)$ stems from the fact that the operators $\eta_n \otimes \eta_m$ are more suitable for calculations. With a little extra effort, we can deduce from Proposition 3.2 that the linear span of this operators is dense in $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$.

Corollary 3.1. $\mathbb{F}(D)$ *is dense in* $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$ *with respect to the bounded topology* τ_b.

Proof. In view of Proposition 3.2(i), it is sufficient to show that $\mathbb{F}(D_{\mathfrak{A}})$ lies in the closure of $\mathbb{F}(D)$. With T defined in (42), consider the set of Borel measurable functions

$$\mathcal{R} := \{\, r : \sigma(\bar{T}) \to [0, \infty)\,;\ \sup_{t \in \sigma(\bar{T})} r(t) t^{2^n} < \infty \,\}.$$

It follows from Lemma 3.2(i) and [8, Proposition 3.4] that the family of seminorms

$$\{||\cdot||_r\}_{r \in \mathcal{R}}, \quad ||a||_r := ||r(\bar{T}) a\, r(\bar{T})||, \quad a \in \mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+),$$

(the norm $||\cdot||$ being the operator norm in $\mathcal{L}(\mathcal{H})$) generates the topology τ_b.

Let $\varphi, \psi \in D_{\mathfrak{A}}$. Note that $||r(\bar{T})(\varphi \otimes \psi) r(\bar{T})|| \leq ||r(\bar{T})||^2 ||\varphi||\, ||\psi||$. With $\alpha_n, \beta_n \in \mathbb{C}$, write $\varphi = \sum_{n \in J} \alpha_n \eta_n$, $\psi = \sum_{n \in J} \beta_n \eta_n$, where $J = \mathbb{N}_0$ or $J = \mathbb{Z}$

according to the type of representation considered. For $k \in \mathbb{N}$, set $\varphi_k := \sum_{|n|\le k} \alpha_n \eta_n$ and $\psi_k := \sum_{|n|\le k} \beta_n \eta_n$. Clearly, $\varphi_k, \psi_k \in \mathbb{F}(D)$. Now

$$\begin{aligned} ||\varphi \otimes \psi - \varphi_k \otimes \psi_k||_r &= ||r(\bar{T})(\varphi \otimes \psi - \varphi_k \otimes \psi_k) r(\bar{T})|| \\ &\le ||r(\bar{T})||^2 ||\varphi - \varphi_k||\, ||\psi|| + ||r(\bar{T})||^2 ||\varphi_k||\, ||\psi - \psi_k|| \to 0 \end{aligned}$$

as $k \to \infty$ for all $r \in \mathcal{R}$, hence $\varphi \otimes \psi$ lies in the closure of $\mathbb{F}(D)$. Since $\mathbb{F}(D_{\mathfrak{A}})$ is the linear span of operators $\varphi \otimes \psi$, the assertion follows. □

Proposition 3.2 and Corollary 3.1 show how $\mathbb{F}(D)$ and $\mathbb{F}(D_{\mathfrak{A}})$ are related to the image of $\mathcal{O}_q(\mathrm{U})$ in $\mathcal{L}^+(D_{\mathfrak{A}})$: By density and continuity, $\mathbb{F}(D)$ and $\mathbb{F}(D_{\mathfrak{A}})$ carry the whole information about the action of $\mathcal{U}_q(\mathrm{su}_{1,1})$ on $\mathcal{L}^+(D_{\mathfrak{A}}) \subset \mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$ and, in particular, on $\mathcal{O}_q(\mathrm{U}) \subset \mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$.

It would be desirable to have also the converse statement, that is, to obtain the action on $\mathbb{F}(D)$ (or $\mathbb{F}(D_{\mathfrak{A}})$) by taking the closure of $\mathcal{O}_q(\mathrm{U})$ in $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$. Unfortunately, this is not possible in the above setting. From [11, Theorem 4.5.4], it follows that, for unbounded representations of type (II), the bounded topology τ_b coincides with the finest locally convex topology on $\mathfrak{A}$. Since $\mathfrak{A}$ is complete with respect to the finest locally convex topology, it is closed with respect to τ_b.

For $\mathbb{F}(D)$ to be in the closure of $\mathcal{O}_q(\mathrm{U})$, we can consider a different locally convex topology on D. Let D' be the vector space of all formal series $\sum_{j\in J} \alpha_j \eta_j$, where $J = \mathbb{N}$ or $J = \mathbb{Z}$. There exists a dual pairing $\langle \cdot, \cdot \rangle$ of D' and D given by

$$\left\langle \sum_{j\in J} \alpha_j \eta_j, \sum_{|n|\le n_0} \beta_n \eta_n \right\rangle = \sum_{|n|\le n_0} \alpha_n \beta_n.$$

We equip D and D' with the weak topologies arising from this dual pairing. To $\mathcal{L}(D, D')$, the vector space of all continuous linear mappings from D into D', we assign the weak operator topology τ_{ow}, that is, the topology generated by the family of semi-norms

$$\{p_{\varphi,\psi}\}_{\varphi,\psi\in D}, \quad p_{\varphi,\psi}(a) := |\langle a\varphi, \psi \rangle|, \quad a \in \mathcal{L}(D, D').$$

Then $\mathcal{O}_q(\mathrm{U})$ is dense in $\mathcal{L}(D, D')$ with respect to τ_{ow} and the action of $\mathcal{U}_q(\mathrm{su}_{1,1})$ on $\mathcal{L}^+(D)$ defined by (34)–(36) is continuous. This is essentially the method for constructing the space $D(U_q)'(= \mathcal{L}(D, D'))$ of distributions on the quantum disc as performed in [15]. The topological space $D(U_q)$ of finite functions on the quantum disc defined in [15] is homeomorphic to $\mathbb{F}(D)$ with the weak operator topology τ_{ow}.

We now give another description of $\mathbb{F}(D)$.

Lemma 3.3. *Let $\mathcal{F}(\sigma(\bar{y}))$ be the set of (Borel measurable) functions on $\sigma(\bar{y})$ with finite support, that is,*

$$\mathcal{F}(\sigma(\bar{y})) = \{\, \psi : \sigma(\bar{y}) \to \mathbb{C}\,;\, \#\{t \in \sigma(\bar{y}); \psi(t) \neq 0\} < \infty \,\}.$$

Each $f \in \mathbb{F}(D)$ can be written as

$$f = \sum_{n=0}^{N} z^n \psi_n(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y}) z^{*n}, \quad N, M \in \mathbb{N},$$

where $\psi_k \in \mathcal{F}(\sigma(\bar{y}))$, $k = -M, \ldots, N$.

Conversely, if $\psi_k \in \mathcal{F}(\sigma(\bar{y}))$, then $\sum_{n=0}^{N} z^n \psi_n(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y}) z^{*n} \in \mathbb{F}(D)$.

Proof. To see this, consider the functions

$$\delta_k(t) \;:=\; \begin{cases} 1 & : \;\text{for}\;\; t = q^{2k} \\ 0 & : \;\text{for}\;\; t \neq q^{2k} \end{cases}$$

if we are given a type (I) representation, and

$$\delta_k(t) \;:=\; \begin{cases} 1 & : \;\text{for}\;\; t = -q^{2\alpha+2k} \\ 0 & : \;\text{for}\;\; t \neq -q^{2\alpha+2k} \end{cases}$$

if we are given a representation of type $(II)_\alpha$. Note that $\delta_k(\bar{y})$ is the projection on $\mathcal{H}$ with range $\mathbb{C}\eta_k$, that is, $\delta_k(\bar{y}) = \eta_k \otimes \eta_k$.

Each $\psi_n \in \mathcal{F}(\sigma(\bar{y}))$ can be written as a finite sum $\sum_k \psi_{n,k}\delta_k(t)$, where $\psi_{n,k} = \psi_n(q^{2k})$ for the type (I) representation and $\psi_{n,k} = \psi_n(-q^{2\alpha+2k})$ for type $(II)_\alpha$ representations. Furthermore, we have

$$z^n \delta_k(\bar{y}) = z^n(\eta_k \otimes \eta_k) = (z^n \eta_k) \otimes \eta_k \in \mathbb{F}(D),$$
$$\delta_k(\bar{y}) z^{*n} = (\eta_k \otimes \eta_k) z^{*n} = \eta_k \otimes (z^n \eta_k) \in \mathbb{F}(D),$$

hence $\sum_{n=0}^{N} z^n \psi_n(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y}) z^{*n} \in \mathbb{F}(D)$ whenever $\psi_n \in \mathcal{F}(\sigma(\bar{y}))$ for all $n = -M, \ldots, N$.

On the other hand, for $k \leq n$, we can write

$$\eta_n \otimes \eta_k = \gamma_{n,k}(z^{n-k}\eta_k) \otimes \eta_k = \gamma_{n,k} z^{n-k} \delta_k(\bar{y}),$$
$$\eta_k \otimes \eta_n = \gamma_{n,k} \eta_k \otimes (z^{n-k}\eta_k) = \gamma_{n,k} \delta_k(\bar{y}) z^{*n-k},$$

where $\gamma_{n,k} = (q^{2(k+1)}; q^2)_{n-k}^{-1/2}$ and $\gamma_{n,k} = (-q^{2(\alpha+k+1)}; q^2)_{n-k}^{-1/2}$ for the representations of type (I) and type $(II)_\alpha$, respectively. Hence any linear combination of $\eta_m \otimes \eta_l$ is equivalent to a linear combination of $z^n \delta_k(\bar{y})$ and $\delta_k(\bar{y}) z^{*n}$.

Summing over equal powers of z and z^* yields coefficients of z^n and z^{*n} of the form $\sum_k \psi_{n,k}\delta_k(\bar{y})$, $\psi_{n,k} \in \mathbb{C}$, and the functions $\sum_k \psi_{n,k}\delta_k(t)$ belong to $\mathcal{F}(\sigma(\bar{y}))$ since all sums are finite. □

A similar result can be obtained by considering the following set of (Borel measurable) functions

$$\mathcal{S}(\sigma(\bar{y})) := \{\, \psi : \sigma(\bar{y}) \to \mathbb{C}\,;\ \sup_{t\in\sigma(\bar{y})} |t^k\psi(t)| < \infty \ \text{ for all } k \in \mathbb{Z}\,\}.$$

Lemma 3.4. *The element* $f = \sum_{n=0}^{N} z^n\psi_n(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y})z^{*n}$, $\psi_n \in \mathcal{S}(\sigma(\bar{y}))$, $N, M \in \mathbb{N}$, *belongs to* $\mathbb{B}_1(\mathfrak{A})$. *The operators* $\psi(\bar{y})$, $\psi \in \mathcal{S}(\sigma(\bar{y}))$, *satisfy on* $D_{\mathfrak{A}}$ *the commutation rules*

$$z\psi(\bar{y}) = \psi(q^2\bar{y})z, \quad z^*\psi(\bar{y}) = \psi(q^{-2}\bar{y})z^*. \tag{43}$$

The linear space

$$\mathcal{S}(D) := \left\{ \sum_{n=0}^{N} z^n\psi_n(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y})z^{*n}\,;\ \psi_k \in \mathcal{S}(\sigma(\bar{y})),\ -M \le k \le N \right\}$$

forms a $\mathcal{U}_q(\mathrm{su}_{1,1})$*-module *-subalgebra of* $\mathcal{L}^+(D_{\mathfrak{A}})$.

Proof. By definition of $\mathbb{B}_1(\mathfrak{A})$, $a\psi b \in \mathbb{B}_1(\mathfrak{A})$ for all $a, b \in \mathfrak{A}$ whenever $\psi \in \mathbb{B}_1(\mathfrak{A})$. Fix $a \in \mathfrak{A}$. From the proof of Lemma 3.2(i), we know that $\{||\cdot||_{\bar{T}^n}\}_{n\in\mathbb{N}}$, $T = 1 + y^2 + y^{-2}$, generates the graph topology on $D_{\mathfrak{A}}$, so there exist $n_a \in \mathbb{N}$ such that $||a\varphi|| \le ||T^{n_a}\varphi||$ for all $\varphi \in D_{\mathfrak{A}}$. Consequently, $||aT^{-n_a}\varphi|| \le ||\varphi||$, hence $\overline{aT^{-n_a}}$ and $\overline{T^{-n_a}a^*}$ are bounded. The operators $\overline{\psi_n(\bar{y})T^m}$, $\psi_n \in \mathcal{S}(\sigma(\bar{y}))$, $m \in \mathbb{N}$, are bounded by the definition of $\mathcal{S}(\sigma(\bar{y}))$, and $\bar{T}^{-1}$ is of trace class. From this facts, we conclude that

$$\overline{a\psi_n(\bar{y})b} = \overline{aT^{-n_a}}\,\overline{\psi_n(\bar{y})T^{n_a+n_b+1}}\bar{T}^{-1}\overline{T^{-n_b}b}$$

is of trace class. This shows that the operator f from Lemma 3.4 belongs to $\mathbb{B}_1(\mathfrak{A})$.

The commutation relations (43) are satisfied if we restrict the operators to $D \subset D_{\mathfrak{A}}$. Consider the O*-algebra generated by the elements $\psi(\bar{y})\lceil D$, $\psi \in \mathcal{S}(\sigma(\bar{y}))$, and $a\lceil D$, $a \in \mathfrak{A}$. Since the operators $\psi(\bar{y})$ are bounded, the closure of this algebra is contained in $\mathcal{L}^+(D_{\mathfrak{A}})$. Taking the closure of an O*-algebra does not change the commutation relations, hence Equation (43) holds.

Recall that $\mathfrak{A}$ is the linear span of operators of the form $z^n p_n(y, y^{-1})$ and $p_{-n}(y, y^{-1})z^{*n}$, where $p_n(y, y^{-1})$ and $p_{-n}(y, y^{-1})$ are polynomials in y and y^{-1}.

Note, furthermore, that $p(t,t^{-1})\psi(t) \in \mathcal{S}(\sigma(\bar{y}))$ for all $\psi(t) \in \mathcal{S}(\sigma(\bar{y}))$ and all polynomials $p(t,t^{-1})$. Now it follows from (29), (30), (43), and the definition of $\mathcal{S}(D)$ that $\mathcal{S}(D)$ is stable under the $\mathcal{U}_q(\mathrm{su}_{1,1})$-action defined in Lemma 3.1. Similarly, using (30), (43), and the definition of $\mathcal{S}(D)$, it is easy to check that $\mathcal{S}(D)$ forms a *-algebra. Therefore, by Lemma 3.1, $\mathcal{S}(D)$ is a $\mathcal{U}_q(\mathrm{su}_{1,1})$-module *-algebra. □

The description of $\mathbb{F}(D)$ and $\mathcal{S}(D)$ by functions $\psi : \sigma(\bar{y}) \to \mathbb{C}$ suggests that we can consider the elements of $\mathbb{F}(D)$ and $\mathcal{S}(D)$ as infinitely differentiable functions with compact support and which are rapidly decreasing, respectively. Note that $\mathbb{F}(D) \neq \mathbb{F}(D_{\mathfrak{A}})$ (e.g., $\eta \otimes \eta \notin \mathbb{F}(D)$ for $\eta = \sum_{n=0}^{\infty} \alpha_n \eta_n \in \mathcal{D}_{\mathfrak{A}}$ if an infinite number of α_n are non-zero), and $\mathcal{S}(D) \neq \mathbb{B}_1(\mathfrak{A})$ (e.g., $f = \sum_{k=0}^{\infty} \exp(-\bar{y}^{2^k})\delta_k(\bar{y})z^{*k} \in \mathbb{B}_1(\mathfrak{A})$, $f \notin \mathcal{S}(D)$).

Clearly, $\mathbb{F}(D) \subset \mathcal{S}(D)$. On $\mathcal{S}(D)$, the invariant integral can be expressed nicely in terms of the Jackson integral. The Jackson integral is defined by

$$\int_0^1 \varphi(t)d_qt = (1-q)\sum_{k=0}^{\infty}\varphi(q^k)q^k \quad \text{and} \quad \int_0^{\infty} \varphi(t)d_qt = (1-q)\sum_{k=-\infty}^{\infty}\varphi(q^k)q^k.$$

Proposition 3.3. *Suppose that*

$$\psi = \sum_{n=1}^{N} z^n\psi_n(\bar{y}) + \psi_0(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y})z^{*n} \in \mathcal{S}(D).$$

Let h denote the invariant integral defined in Proposition 3.1*. For irreducible type (I) representations, we have*

$$h(\psi) = c(1-q^2)^{-1}\int_0^1 \psi_0(t)t^{-2}d_{q^2}t,$$

and, for irreducible type $(II)_\alpha$ representations, we have

$$h(\psi) = cq^{-2\alpha}(1-q^2)^{-1}\int_0^{\infty} \psi_0(-q^{2\alpha}t)t^{-2}d_{q^2}t.$$

Proof. Since $\langle \eta_k, z^n\psi_n(\bar{y})y^{-1}\eta_k\rangle = \langle \eta_k, \psi_{-n}(\bar{y})z^{*n}y^{-1}\eta_k\rangle = 0$ for all $n \neq 0$, we obtain for type $(II)_\alpha$ representations

$$\begin{aligned} h(\psi) &= c\,\mathrm{Tr}\,\overline{\psi y^{-1}} = c\sum_{k=-\infty}^{\infty}\langle \eta_k, \psi_0(\bar{y})y^{-1}\eta_k\rangle = c\sum_{k=-\infty}^{\infty}\psi_0(-q^{2\alpha}q^{2k})q^{-2(\alpha+k)} \\ &= cq^{-2\alpha}(1-q^2)^{-1}\int_0^{\infty}\psi_0(-q^{2\alpha}t)t^{-2}d_{q^2}t. \end{aligned}$$

The proof for type (I) representations is similar. □

§3.3. Application: differential calculus

The bimodule structure of a first order differential *-calculus (Γ, d) over $\mathcal{O}_q(\mathrm{U})$ has been described in [15] and [12]. The commutation relations are given by

$$\mathrm{d}z\, z = q^2 z\, \mathrm{d}z, \quad \mathrm{d}z\, z^* = q^{-2} z^*\, \mathrm{d}z, \quad \mathrm{d}z^*\, z = q^2 z\, \mathrm{d}z^*, \quad \mathrm{d}z^*\, z^* = q^{-2} z^*\, \mathrm{d}z^*.$$

Our aim is to extend this FODC to the classes of integrable functions defined in Subsection 3.2. To this end, we use a commutator representation of the FODC. A faithful commutator representation of the above differential calculus can be found in [12] and is obtained as follows. Given a *-representation π of $\mathcal{O}_q(\mathrm{U})$ from Subsection 3.2, consider the direct sum $\rho := \pi \oplus \pi$ on $D \oplus D \subset \mathcal{H} \oplus \mathcal{H}$ and set

$$C := (1-q^2)^{-1} \begin{pmatrix} 0 & \pi(z) \\ \pi(z^*) & 0 \end{pmatrix}.$$

Then the differential mapping $\mathrm{d}_{\rho,C}$ defined in (19) is given by

$$\mathrm{d}_{\rho,C}(f) = \mathrm{i}[C, \rho(f)] = (1-q^2)^{-1}\, \mathrm{i} \begin{pmatrix} 0 & \pi(zf - fz) \\ \pi(z^* f - f z^*) & 0 \end{pmatrix}, \; f \in \mathcal{O}_q(\mathrm{U}).$$

Clearly, $C \in \mathcal{L}^+(D \oplus D)$, so we can extend $\mathrm{d}_{\rho,C}$ to $\mathcal{L}^+(D \oplus D)$, that is,

$$\mathrm{d}_{\rho,C}(x) := \mathrm{i}[C, x], \quad x \in \mathcal{L}^+(D \oplus D).$$

The same formula applies to any *-subalgebra of $\mathcal{L}^+(D \oplus D)$. Note that we can consider $\mathcal{L}^+(D)$ as a *-subalgebra of $\mathcal{L}^+(D \oplus D)$ by identifying $A \in \mathcal{L}^+(D)$ with the operator $A \oplus A$ acting on $D \oplus D$. In particular, the algebras $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$ from Proposition 3.1 become *-subalgebras of $\mathcal{L}^+(D \oplus D)$. In this way, we obtain a FODC over these algebras.

For z and z^*, we have

$$\mathrm{d}_{\rho,C}(z) = \mathrm{i} \begin{pmatrix} 0 & 0 \\ \pi(y) & 0 \end{pmatrix}, \quad \mathrm{d}_{\rho,C}(z^*) = \mathrm{i} \begin{pmatrix} 0 & -\pi(y) \\ 0 & 0 \end{pmatrix}.$$

For functions $\psi(\bar{y})$, the differential mapping $\mathrm{d}_{\rho,C}$ can be expressed in terms of the q-differential operator D_q defined by $\mathrm{D}_q f(x) = (x - qx)^{-1}(f(x) - f(qx))$. It follows from

$$(1-q^2)^{-1}(z\psi(\bar{y}) - \psi(\bar{y})z) = zy(y - q^2 y)^{-1}(\psi(\bar{y}) - \psi(q^2\bar{y})) = z\mathrm{D}_{q^2}\psi(\bar{y})y,$$

$$(1-q^2)^{-1}(z^*\psi(\bar{y}) - \psi(\bar{y})z^*) = y(y - q^2 y)^{-1}(\psi(q^2\bar{y}) - \psi(\bar{y}))z^* = -q^{-2}\mathrm{D}_{q^2}\psi(\bar{y})z^* y$$

that

$$\mathrm{d}_{\rho,C}(\psi(\bar{y})) = -\,\mathrm{i}\,\rho(z)\mathrm{D}_{q^2}\psi(\bar{y})\mathrm{d}_{\rho,C}(z) - \mathrm{i}\,q^{-2}\mathrm{D}_{q^2}\psi(\bar{y})\rho(z^*)\mathrm{d}_{\rho,C}(z^*).$$

In particular, the "δ-distributions" $\delta_k(\bar{y})$ are differentiable.

§4. Quantum Ball Algebras

§4.1. Algebraic relations

Let $n \in \mathbb{N}$ and $q \in (0,1)$. We denote by $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ the *-algebra generated by $z_1, \ldots, z_n, z_1^*, \ldots, z_n^*$ obeying the relations

$$z_k z_l = q z_l z_k, \quad k < l, \tag{44}$$

$$z_l^* z_k = q z_k z_l^*, \quad k \neq l, \tag{45}$$

$$z_k^* z_k = q^2 z_k z_k^* - (1-q^2) \sum_{j=k+1}^{n} z_j z_j^* + (1-q^2), \quad k < n, \tag{46}$$

$$z_n^* z_n = q^2 z_n z_n^* + (1-q^2). \tag{47}$$

Equations (44)–(47) are called *twisted canonical commutation relations* [10] and $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ is also known as *q-Weyl algebra* [6]. Here we consider it as a special case of the quantum balls introduced in [16] because the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ defined below is taken from the latter. A profound study of quantum balls including results on invariant integrals can be found in [17, 18].

The following hermitian elements Q_k will play a crucial role throughout this section. Set

$$Q_k := 1 - \sum_{j=k}^{n} z_j z_j^*, \;\; k \leq n, \quad Q_{n+1} := 1. \tag{48}$$

Equations (46), (47), and (48) imply immediately

$$z_k^* z_k - q^2 z_k z_k^* = (1-q^2) Q_{k+1}, \quad z_k^* z_k - z_k z_k^* = (1-q^2) Q_k, \tag{49}$$

$$z_k z_k^* = Q_{k+1} - Q_k, \quad z_k^* z_k = Q_{k+1} - q^2 Q_k. \tag{50}$$

Furthermore, one easily shows by using Equations (44)–(48) that

$$Q_k z_j = z_j Q_k, \qquad j < k, \qquad Q_k z_j = q^2 z_j Q_k, \qquad j \geq k, \tag{51}$$

$$Q_k z_j^* = z_j^* Q_k, \qquad j < k, \qquad Q_k z_j^* = q^{-2} z_j^* Q_k, \qquad j \geq k. \tag{52}$$

As a consequence,

$$Q_k Q_l = Q_l Q_k, \quad \text{for all } k, l \leq n+1. \tag{53}$$

For $I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$, $J = (j_1, \ldots, j_n) \in \mathbb{N}_0^n$, set $z^I := z_1^{i_1} \cdots z_n^{i_n}$, $z^{*J} := z_1^{*j_1} \cdots z_n^{*j_n}$ and define $I \cdot J = (i_1 j_1, \ldots, i_n j_n) \in \mathbb{N}_0^n$. We write 0 instead of $(0, \ldots, 0)$. It follows from (51)–(53) together with the defining relations (44)–(47) that each $f \in \mathcal{O}_q(\mathrm{Mat}_{n,1})$ can be expressed as a finite sum

$$f = \sum_{I \cdot J = 0} z^I p_{IJ}(Q_1, \ldots, Q_n) z^{*J} \tag{54}$$

with polynomials $p_{IJ}(Q_1, \ldots, Q_n)$ in $Q_1, \ldots, Q_n$.

The $\mathcal{U}_q(\mathrm{su}_{n,1})$-action $\triangleright$ which turns $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ into a $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-algebra is given by the following formulas [16].

$$\begin{array}{lrcllrcll}
j \neq n: & E_j \triangleright z_{j+1} & = & q^{-1/2} z_j, & & E_j \triangleright z_k & = & 0, & k \neq j+1,\\
& E_j \triangleright z_j^* & = & -q^{-3/2} z_{j+1}^*, & & E_j \triangleright z_k^* & = & 0, & k \neq j,\\
& F_j \triangleright z_j & = & q^{1/2} z_{j+1}, & & F_j \triangleright z_k & = & 0, & k \neq j,\\
& F_j \triangleright z_{j+1}^* & = & -q^{3/2} z_j^*, & & F_j \triangleright z_k^* & = & 0, & k \neq j+1,
\end{array}$$

$$\begin{array}{rclrclrcl}
K_j \triangleright z_j & = & q z_j, & K_j \triangleright z_{j+1} & = & q^{-1} z_{j+1}, & K_j \triangleright z_k & = & z_k,\ k \neq j, j+1,\\
K_j \triangleright z_j^* & = & q^{-1} z_j^*, & K_j \triangleright z_{j+1}^* & = & q z_{j+1}^*, & K_j \triangleright z_k^* & = & z_k^*,\ k \neq j, j+1,
\end{array}$$

$$\begin{array}{lrcllrcl}
j = n: & E_n \triangleright z_n & = & -q^{1/2} z_n^2, & k < n: & E_n \triangleright z_k & = & -q^{1/2} z_n z_k,\\
& E_n \triangleright z_n^* & = & q^{-3/2}, & & E_n \triangleright z_k^* & = & 0,\\
& F_n \triangleright z_n & = & q^{1/2}, & & F_n \triangleright z_k & = & 0,\\
& F_n \triangleright z_n^* & = & -q^{5/2} z_n^{*2} & & F_n \triangleright z_k^* & = & -q^{5/2} z_k^* z_n^*,\\
& K_n \triangleright z_n & = & q^2 z_n, & & K_n \triangleright z_k & = & q z_k,\\
& K_n \triangleright z_n^* & = & q^{-2} z_n^*, & & K_n \triangleright z_k^* & = & q^{-1} z_k^*.
\end{array}$$

If $n = 1$, we recover the relations of the quantum disc algebra. For $n > 1$, we obtain by omitting the elements K_n, K_n^{-1}, E_n, and F_n a $\mathcal{U}_q(\mathrm{su}_n)$-action on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ such that $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ becomes a $\mathcal{U}_q(\mathrm{su}_n)$-module *-algebra. Note that, by Equation (2), it is sufficient to describe the action on generators.

§4.2. Representations of the *-algebra $\mathcal{O}_q(\mathrm{Mat}_{n,1})$

Irreducible *-representations of the twisted canonical commutation relations have been classified in [10] under the condition that $1 - Q_1$ is essentially self-adjoint. In this subsection, we describe *-representations of the twisted canonical commutation relations without requiring the representation to be irreducible. The result was obtained by applying repeatedly Lemmas 2.1–2.3

and using similar methods as in [14]. For $n = 1$, the outcome of Proposition 4.1 is basically the series $\epsilon = 1$ of Lemma 2.3.

Recall our notational conventions from Subsection 2.2 regarding direct sums of a Hilbert space $\mathcal{K}$. Let A be a self-adjoint operator on $\mathcal{K}$ such that $\sigma(A) \sqsubseteq (q^2, 1]$. Then the expression $\mu_j(A)$, $j \in \mathbb{Z}$, stands for the operator $\mu_j(A) = (1 + q^{-2j}A)^{1/2}$. We shall also abbreviate $\lambda_j = (1 - q^{2j})^{1/2}$ and $\beta_j = (q^{-2j} - 1)^{1/2}$ for $j \in \mathbb{N}_0$.

Proposition 4.1. *Assume that $m, k, l \in \mathbb{N}_0$ satisfy $m + l + k = n$. Let $\mathcal{K}$ denote a Hilbert space. Set*

$$\mathcal{H} := \oplus_{i_n,\dots,i_{n-m+1}=0}^{\infty} \oplus_{i_k=-\infty}^{\infty} \oplus_{i_{k-1},\dots,i_1=1}^{\infty} \mathcal{H}_{i_n\dots i_1},$$

where $\mathcal{H}_{i_n\dots i_1} = \mathcal{K}$, and

$$D := \mathrm{Lin}\{\eta_{i_n\dots i_1}\,;\,\eta \in \mathcal{K},\ i_n,\dots,i_{n-m+1} \in \mathbb{N}_0,\ i_k \in \mathbb{Z},\ i_{k-1},\dots,i_1 \in \mathbb{N}\}.$$

(For $l > 0$, we retain the notation $\eta_{i_n\dots i_1}$ and do not write $\eta_{i_n\dots i_{n-m+1}, i_k\dots i_1}$.) Consider the operators $z_1, \dots, z_n$ acting on D by

$$\begin{aligned}
&(m, 0, k): \\
&\quad z_j\eta_{i_n\dots i_1} = q^{i_{j+1}+\dots+i_n}\lambda_{i_j+1}\eta_{i_n\dots i_j+1\dots i_1}, \quad \text{if } k < j \le n, \\
&\quad z_k\eta_{i_n\dots i_1} = q^{i_{k+1}+\dots+i_n}\mu_{i_k-1}(A^2)\eta_{i_n\dots i_k-1\dots i_1}, \\
&\quad z_j\eta_{i_n\dots i_1} = q^{-(i_{j+1}+\dots+i_k)+(i_{k+1}+\dots+i_n)}\beta_{i_j-1}A\eta_{i_n\dots i_j-1\dots i_1}, \text{ if } 1 \le j < k,
\end{aligned}$$

and, for $l > 0$,

$$\begin{aligned}
&(m, l, k): \\
&\quad z_j\eta_{i_n\dots i_1} = q^{i_{j+1}+\dots+i_n}\lambda_{i_j+1}\eta_{i_n\dots i_j+1\dots i_1}, \quad \text{if } n - m < j \le n, \\
&\quad z_{n-m}\eta_{i_n\dots i_1} = q^{i_{n-m+1}+\dots+i_n}v\eta_{i_n\dots i_k-1\dots i_1}, \\
&\quad z_j \equiv 0, \quad \text{if } k < j < n - m, \\
&\quad z_k\eta_{i_n\dots i_1} = q^{-i_k+i_{n-m+1}+\dots+i_n}A\eta_{i_n\dots i_k-1\dots i_1}, \\
&\quad z_j\eta_{i_n\dots i_1} = q^{-(i_{j+1}+\dots+i_k)+(i_{n-m+1}+\dots i_n)}\beta_{i_j-1}A\eta_{i_n\dots i_j-1\dots i_1}, \text{ if } 1 \le j < k.
\end{aligned}$$

(If $k = 0$, then the indices $i_1, \dots, i_k$ are omitted; similarly, if $m = 0$, then the indices $i_{n-m+1}, \dots, i_n$ are omitted.) In both series, A denotes a self-adjoint operator acting on the Hilbert space $\mathcal{K}$ such that $\sigma(A) \sqsubseteq (q, 1]$. In the series (m, l, k), $l > 0$, v is a unitary operator on $\mathcal{K}$ such that $Av = vA$.

*Then the operators $z_1, \dots, z_n$ define a *-representation of $\mathcal{O}_q(\mathrm{Mat}_{n,1})$, where the action of z_j^*, $j = 1, \dots, n$ is obtained by restricting the adjoint of*

z_j to D. Representations belonging to different series (m,k,l) or to different operators A and v are not unitarily equivalent. A representation of this series is irreducible if and only if $\mathcal{K}=\mathbb{C}$. In this case, v is a complex number of modulus one and $A\in(q,1]$. Only the representations $(m,0,k)$ are faithful.

Proof. Direct calculations show that the formulas given in Proposition 4.1 define a *-representation of $\mathcal{O}_q(\mathrm{Mat}_{n,1})$. Clearly, if a *-representation of these series is irreducible, then A and v must be complex numbers and $\mathcal{K}=\mathbb{C}$. The converse statement was shown in [10]. That the representations $(m,0,k)$ are faithful is proved by showing that for each $x\in\mathcal{O}_q(\mathrm{Mat}_{n,1})$, $x\neq 0$, there exist $\eta_{i_n\dots i_1},\eta_{j_n\dots j_1}\in\mathcal{H}$ such that the matrix element $\langle\eta_{i_n\dots i_1},x\eta_{j_n\dots j_1}\rangle$ is non-zero. The vectors can easily be found by writing x in the standard form (54) and observing that z_j, z_j^* act as shift operators. We omit the details. The other assertions of the proposition are obvious. □

Remark. The operators Q_j are given by

$$Q_j\eta_{i_n\dots i_1}=q^{2(i_j+\dots+i_n)}\eta_{i_n\dots i_1},\quad \text{if } n-m<j\le n, \tag{55}$$

$$Q_j\equiv 0,\quad \text{if } k<j\le n-m, \tag{56}$$

$$Q_j\eta_{i_n\dots i_1}=-q^{-2(i_j+\dots+i_k)+2(i_{n-m+1}+\dots+i_n)}A^2\eta_{i_n\dots i_1},\quad \text{if } 1\le j\le k. \tag{57}$$

The numbers $m,l,k\in\mathbb{N}_0$ correspond to the signs of the operators Q_j, that is, we have $Q_n\ge\dots\ge Q_{n-m+1}>0$ if $m>0$, $Q_{n-m}=\dots=Q_{k+1}=0$ if $l>0$, and $0>Q_k\ge\dots\ge Q_1$ if $k>0$. The only bounded representations are the series $(m,l,0)$.

Classically, we can view $\mathbb{CP}^n$ as homogeneous $\mathrm{SU}_{n,1}$-spaces. The $\mathrm{SU}_{n,1}$-action on $\mathbb{CP}^n$ has two open orbits: the unit ball and complex hyperbolic space, the complement of the unit ball under standard embedding into the projective space. The bounded representations $(n,0,0)$ in Proposition 4.1 correspond to the unit balls, thus the name "quantum ball" for the algebra $\mathcal{O}_q(\mathrm{Mat}_{n,1})$. The representations $(m,0,k)$ with $k\neq 0$ correspond to the complements of balls and are related to open leaves of the complex hyperbolic space. Note that, in this case, $\sum_{j=1}^n z_jz_j^*>1$ since $Q_1<0$ for $k\neq 0$.

§4.3. Invariant integrals associated with the quantum ball algebra

Let D be a dense subspace of a Hilbert space $\mathcal{H}$ and assume that we are given a *-representation $\pi:\mathcal{O}_q(\mathrm{Mat}_{n,1})\to\mathcal{L}^+(D)$. Throughout this subsection, we shall impose the following conditions on the representation:

Firstly, the closures $\overline{Q}_j$ of the operators Q_j, $j=1,\dots,n$, are self-adjoint and strongly commute. Secondly, the operators $\overline{Q}_j$ are injective and D is an invariant linear subspace of $D(|\overline{Q}_j|^{-1/2})$ so that $|\overline{Q}_j|^{-1/2}\lceil D \in \mathcal{L}^+(D)$. Thirdly, $|\overline{Q}_l|^{1/2}z_j \subset z_j|\overline{Q}_l|^{1/2}$ for $j<l$ and $|\overline{Q}_l|^{1/2}z_j \subset qz_j|\overline{Q}_l|^{1/2}$ for $j\geq l$. Fourthly, each of the operators $\overline{Q}_j$ is either positive or negative definite, i.e., $\overline{Q}_j=\epsilon_j|\overline{Q}_j|$, where $\epsilon_j=\pm 1$.

Note that these conditions imply that the operators $|Q_j|^{p/2} := |\overline{Q}_j|^{p/2}\lceil D$, $p\in\mathbb{Z}$, are elements of $\mathcal{L}^+(D)$ since $|Q_j| := |\overline{Q}_j|\lceil D = |Q_j|^{-1}Q^2 \in \mathcal{L}^+(D)$. Note also that the representations of type $(n-k,0,k)$, $k=0,\dots,n$, from Proposition 4.1 satisfy these conditions. In this case, $\epsilon_j=1$ for $j>k$ and $\epsilon_j=-1$ for $j\leq k$. The same is true for extensions of such representations obtained by taking the closure of the O^*-algebra generated by z_j, $|Q_j|^{-1/2}$, $j=1,\dots,n$.

To describe an invariant integral associated with the quantum ball algebra, we proceed as in Subsection 3.1. The crucial step is to find an operator expansion of the action. To begin, we prove some useful operator relations.

Lemma 4.1. *Define*

$$\rho_l = |Q_l|^{1/2}|Q_{l+1}|^{-1}|Q_{l+2}|^{1/2},\quad l<n,\qquad \rho_n=|Q_1|^{1/2}|Q_n|^{1/2}, \tag{58}$$

$$A_l = -q^{-5/2}\lambda^{-1}Q_{l+1}^{-1}z_{l+1}^{*}z_l,\quad l<n,\qquad A_n=q^{-1/2}\lambda^{-1}z_n, \tag{59}$$

$$B_l=\rho_l^{-1}A_l^{*},\quad l<n,\qquad B_n=-\rho_n^{-1}A_n^{*}. \tag{60}$$

The operators ρ_l, A_l, and B_l satisfy the following commutation relations:

$$\rho_i\rho_j=\rho_j\rho_i,\ \ \rho_j^{-1}\rho_j=\rho_j\rho_j^{-1}=1,\ \ \rho_iA_j=q^{a_{ij}}A_j\rho_i,\ \ \rho_iB_j=q^{-a_{ij}}B_j\rho_i, \tag{61}$$

$$A_iA_j-A_jA_i=0,\ \ i\neq j\pm1,\quad A_j^2A_{j\pm1}-(q+q^{-1})A_jA_{j\pm1}A_j+A_{j\pm1}A_j^2=0, \tag{62}$$

$$B_iB_j-B_jB_i=0,\ \ i\neq j\pm1,\quad B_j^2B_{j\pm1}-(q+q^{-1})B_jB_{j\pm1}B_j+B_{j\pm1}B_j^2=0, \tag{63}$$

$$A_iB_j-A_jB_i=0,\ \ i\neq j,\qquad A_jB_j-B_jA_j=\lambda^{-1}(\epsilon_{j+2}\epsilon_j\rho_j-\rho_j^{-1}),\ \ j<n, \tag{64}$$

$$A_nB_n-B_nA_n = -\lambda^{-1}\rho_n^{-1}, \tag{65}$$

where $(a_{ij})_{i,j=1}^n$ denotes the Cartan matrix of $\mathrm{sl}(n+1,\mathbb{C})$.

Proof. Our assumptions on the representation imply that

$$|Q|_l^{1/2}z_j=z_j|Q|_l^{1/2},\qquad |Q|_l^{1/2}z_j^{*}=z_j^{*}|Q|_l^{1/2},\qquad j<l, \tag{66}$$

$$|Q|_l^{1/2}z_j=qz_j|Q|_l^{1/2},\qquad |Q|_l^{1/2}z_j^{*}=q^{-1}z_j^{*}|Q|_l^{1/2},\qquad j\geq l. \tag{67}$$

Now Equations (61)–(65) are easily shown by repeated application of the commutation rules from Subsection 4.1 and Equations (66) and (67). For instance, we have

$$\begin{aligned}
A_lB_l - B_lA_l &= q^{-5}\lambda^{-2}\rho_l^{-1}(q^2Q_{l+1}^{-1}z_{l+1}^*z_lz_l^*z_{l+1}Q_{l+1}^{-1} - z_l^*z_{l+1}Q_{l+1}^{-2}z_{l+1}^*z_l)\\
&= q^{-1}\lambda^{-2}\rho_l^{-1}Q_{l+1}^{-2}\big((Q_{l+2}-q^2Q_{l+1})(Q_{l+1}-Q_l)-(Q_{l+2}-Q_{l+1})(Q_{l+1}-q^2Q_l)\big)\\
&= \lambda^{-1}\rho_l^{-1}(Q_{l+2}Q_{l+1}^{-2}Q_l - 1) = \lambda^{-1}(\epsilon_{l+2}\epsilon_l\rho_l - \rho_l^{-1})
\end{aligned}$$

for $l < n$. □

Remark. By (65), the operators A_l, B_l, and ρ_l do not satisfy the defining relations of $\mathcal{U}_q(\mathrm{su}_{n,1})$. If $n > 1$, then we get only for the series $(n, 0, 0)$ a *-representation of $\mathcal{U}_q(\mathrm{su}_n)$ by assigning K_j to ρ_j, E_j to A_j, and F_j to B_j, $j < n$.

To see this, observe that we must have $\epsilon_{j+2} = \epsilon_j$ by (64). But $\epsilon_{n-1} = 1$ since $Q_{n+1} = 1$, and $\epsilon_n = 1$ since $\epsilon_n|Q_n| = Q_{n-1} + z_{n-1}z_{n-1}^* > 0$ by (50) (cf. the remarks after Proposition 4.1), so $\epsilon_n = \cdots = \epsilon_1 = 1$.

Although Equations (61)–(65) do not yield a representation of $\mathcal{U}_q(\mathrm{su}_{n,1})$, the analogy to (6)–(9) is obvious, so it is natural to try to define an operator expansion of the action by imitating the adjoint action. That this can be done is the assertion of the next lemma. Again, we write f instead of $\pi(f)$ and $X{\triangleright}f$ instead of $\pi(X{\triangleright}f)$ for $f \in \mathcal{O}_q(\mathrm{Mat}_{n,1})$, $X \in \mathcal{U}_q(\mathrm{su}_n)$.

Lemma 4.2. *With the operators ρ_l, A_l, and B_l defined in Lemma 4.1, set*

$$K_j \triangleright f = \rho_j f \rho_j^{-1}, \qquad K_j^{-1} \triangleright f = \rho_j^{-1} f \rho_j, \tag{68}$$

$$E_j \triangleright f = A_j f - \rho_j f \rho_j^{-1} A_j, \tag{69}$$

$$F_j \triangleright f = B_j f \rho_j - q^2 f \rho_j B_j \tag{70}$$

*for $j = 1, \ldots, n$. Then Equations (68)–(70) applied to $f \in \mathcal{O}_q(\mathrm{Mat}_{n,1})$ define an operator expansion of the action $\triangleright$ on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$. The same formulas applied to $f \in \mathcal{L}^+(D)$ turn the O^*-algebra $\mathcal{L}^+(D)$ into a $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-algebra.*

Proof. The lemma is proved by direct verifications. We start by showing that $\mathcal{L}^+(D)$ with the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action defined by (68)–(70) becomes a $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-algebra. That the action satisfies (2)–(4) is readily seen if we replace in the proof of Lemma 3.1 $y^{\pm 1}$ by $\rho_j^{\pm 1}$, A by A_j, and B by B_j. By using Lemma 4.1, it is easy to check that the action is consistent with (6)–(9). For example, (61) implies

$$(K_iE_j)\triangleright f = \rho_i(A_jf - \rho_j f\rho_j^{-1}A_j)\rho_i^{-1} = q^{a_{ij}}(A_j\rho_i f\rho_i^{-1} - \rho_j\rho_i f\rho_i^{-1}\rho_j^{-1}A_j)$$
$$= (q^{a_{ij}}E_jK_i)\triangleright f$$

for all $f \in \mathcal{L}^+(D)$ and $i, j = 1, \ldots, n$.

It remains to prove that (68)–(70) define an operator expansion of the action. That (68) yields the action of $K_l^{\pm 1}$ on z_j and z_j^* is easily verified by using (66) and (67). Let $l < n$. Applying (44), (45), (66), and (67), we get $\rho_l z_j \rho_l^{-1} A_l = A_l z_j$ and $\rho_l z_j^* \rho_l^{-1} A_l = A_l z_j^*$ whenever $j \notin \{l, l+1\}$, hence $E_l\triangleright z_j = E_l\triangleright z_j^* = 0$. Similarly, $E_l\triangleright z_l = E_l\triangleright z_{l+1}^* = 0$. Equation (69) applied to z_{l+1} and z_l^* gives

$$E_l\triangleright z_{l+1} = A_l z_{l+1} - q^{-1}z_{l+1}A_l = -q^{-3/2}\lambda^{-1}Q_{l+1}^{-1}(z_{l+1}^*z_{l+1} - z_{l+1}z_{l+1}^*)z_l$$
$$= q^{-1/2}z_l,$$

$$E_l\triangleright z_l^* = A_l z_l^* - q^{-1}z_l^*A_l = -q^{-5/2}\lambda^{-1}z_{l+1}^*(q^2 z_l z_l^* - z_l^* z_l)Q_{l+1}^{-1} = -q^{3/2}z_{l+1}^*,$$

where we used (49). The action of E_n on z_j and z_j^*, $j = 1, \ldots, n$, is calculated analogously. The corresponding relations for the generators F_j follow by using $F_j\triangleright f = -(-1)^{\delta_{nj}}q^2(E_j\triangleright f^*)^*$. □

Let $\omega_1, \ldots, \omega_n$ be the simple roots of the Lie algebra sl_{n+1}. For $\gamma = \sum_{j=1}^n p_j\omega_j$, we write $K_\gamma = K_1^{p_1}\cdots K_n^{p_n}$. Recall that, for a finite dimensional representation σ of $\mathcal{U}_q(\mathrm{su}_{n,1})$, the quantum trace

$$\mathrm{Tr}_{q,L}\, a := \mathrm{Tr}\, \sigma(aK_{2\omega}^{-1})$$

defines an invariant integral on $\mathcal{U}_q(\mathrm{su}_{n,1})$, where ω denotes the half-sum of all positive roots (see [6, Proposition 7.14]). $K_{2\omega}$ is chosen such that $XK_{2\omega} = K_{2\omega}S^2(X)$ for all $X \in \mathcal{U}_q(\mathrm{su}_{n,1})$. In Subsection 3.1, we replaced $K\,(= K_{2\omega})$ by y and proved the existence of invariant integrals on appropriate classes of functions. Our aim is to generalize this result to $\mathcal{O}_q(\mathrm{Mat}_{n,1})$.

The half-sum of positive roots is given by $\omega = \frac{1}{2}\sum_{l=1}^n l(n-l+1)\omega_l$. Consider $\Gamma := \prod_{l=1}^n \rho_l^{-l(n-l+1)}$. Inserting the definition of ρ_l gives

$$\Gamma = |Q_1|^{-n}|Q_2|\cdots|Q_n|,\ \ n>1, \quad \Gamma = |Q_1|^{-1},\ \ n=1, \tag{71}$$

since $-\frac{1}{2}(l-1)(n-l+2) + l(n-l+1) - \frac{1}{2}(l+1)(n-l) = 1$ for $1 < l \le n$. The operator $|Q_1|$ appears in the definition of Γ twice, in ρ_1^{-n} and ρ_n^{-n}, in each factor to the power $-n/2$. For $n = 1$, Equation (71) is trivial (cf. Equation

(58)). The following proposition shows that Γ enables us to define an invariant functional resembling the quantum trace.

Note that z_n, z_n^*, $K_n^{\pm 1}$, E_n, and F_n satisfy the relations of the quantum disc algebra, in particular, Equation (40) applies. Therefore we cannot have a normalized invariant integral on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$.

Proposition 4.2. *Let $\mathfrak{A}$ be the O^*-algebra generated by the operators z_j, z_j^*, and $|Q_j|^{-1/2}$, $j = 1, \ldots, n$. Then the *-algebras $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$ defined in* (14) *and* (15), *respectively, are $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-algebras, where the action is given by* (68)–(70). *The linear functional*

$$h(f) := c \,\mathrm{Tr}\, \overline{f\Gamma}, \quad c \in \mathbb{R}, \tag{72}$$

defines an invariant integral on both $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$.

Proof. From the definition of $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$, it is obvious that both algebras are stable under the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action defined by (68)–(70), in particular, by Lemma 4.2, they are $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-algebras.

We proceed as in the proof of Proposition 3.1 and show the invariance of h for generators by using the trace property $\mathrm{Tr}\,\overline{agb} = \mathrm{Tr}\,\overline{gba} = \mathrm{Tr}\,\overline{bag}$ for all $g \in \mathbb{B}_1(\mathfrak{A})$, $a, b \in \mathfrak{A}$. Let $g \in \mathbb{B}_1(\mathfrak{A})$. Clearly, ρ_l commutes with Γ, hence

$$h(K_l^{\pm 1} \triangleright g) = \mathrm{Tr}\, \overline{\rho_l^{\pm 1} g \rho_l^{\mp 1} \Gamma} = \mathrm{Tr}\, \overline{g\Gamma} = \varepsilon(K_l^{\pm 1}) h(g).$$

It follows from the definition of Γ and from (61) that $A_l\Gamma = q^2 \Gamma A_l$ for all l since $-(l-1)(n-l+2) + 2l(n-l+1) - (l+1)(n-l) = 2$. Hence $\rho_l^{-1} A_l \Gamma = \Gamma A_l \rho_l^{-1}$ and therefore

$$h(E_l \triangleright g) = \mathrm{Tr}\,(\overline{A_l g\Gamma} - \overline{\rho_l g \rho_l^{-1} A_l \Gamma}) = \mathrm{Tr}\, \overline{A_l g \Gamma} - \mathrm{Tr}\, \overline{A_l g \Gamma} = 0 = \varepsilon(E_l) h(g).$$

A similar reasoning shows $h(F_l \triangleright g) = 0 = \varepsilon(F_l) h(g)$. □

Remark. As in Subsection 3.1, we consider $\mathbb{B}_1(\mathfrak{A})$ as the algebra of infinitely differentiable functions which vanish sufficiently rapidly at “infinity” and $\mathbb{F}(D)$ as the infinitely differentiable functions with compact support.

§4.4. Topological aspects of *-representations

Our first aim is to find a suitable topology on $\mathcal{L}(D, D^+)$ such that the algebras of integrable functions as well as the quantum ball algebra are dense in $\mathcal{L}^+(D) \subset \mathcal{L}(D, D^+)$ and that the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action on $\mathcal{L}^+(D)$ defined in Lemma 4.2 is continuous. Then, by extending the action continuously from a dense set to its closure, the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action on the algebras of integrable functions is completely determined by the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action on the quantum ball algebra and vice versa. In the case of the bounded topology τ_b, we have seen in Subsection 3.2 that there are obstructions. However, the next proposition shows that the answer to this problem is affirmative in the case of the weak operator topology τ_{ow}.

Proposition 4.3. *Let (m, l, k), D, and $z_1, \ldots, z_n \in \mathcal{L}^+(D)$ be as in Proposition 4.1. Suppose that $\mathfrak{B}$ is the von Neumann algebra on $\mathcal{K}$ generated by A and v in the case $l > 0$ and $k > 0$, by A in the case $l = 0$ and $k > 0$, by v in the case $l > 0$ and $k = 0$, and by the identity on $\mathcal{K}$ in the case $l = k = 0$. Let $P_{i_n \ldots i_1} : \mathcal{K} \to D$ denote the embedding $P_{i_n \ldots i_1}\eta = \eta_{i_n \ldots i_1}$ and $P'_{i_n \ldots i_1} : D^+ \to \mathcal{K}$ its adjoint.*

i. *$\mathbb{F}(D)$ is dense in $\mathcal{L}(D, D^+)$ with respect to the weak operator topology τ_{ow}.*

ii. *The τ_{ow}-closure of the O^*-algebra generated by $z_1, \ldots, z_n$ coincides with*

$$\{\, C \in \mathcal{L}(D, D^+)\,;\, P'_{i_n \ldots i_1} C P_{j_n \ldots j_1} \in \mathfrak{B} \text{ for all possible } i_n, \ldots, i_1,\, j_n, \ldots, j_1 \,\}.$$

In particular, for irreducible representations, the O^*-algebra generated by $z_1, \ldots, z_n$ is dense in $\mathcal{L}(D, D^+)$.

iii. *For $l = 0$, the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action on $\mathcal{L}^+(D)$ defined in Lemma 4.2 is continuous with respect to τ_{ow}.*

Proof. (i): By the Double Commutant Theorem (see, e.g., [4]), $\mathbb{F}(D)$ is τ_{ow}-dense in the set of all norm-bounded operators, and by [7, Corollary 4.2], there exists a τ_b-dense space of norm-bounded operators belonging to $\mathcal{L}^+(D)$. From this, (i) follows.

(ii): Let $k, l > 0$. Note that finite orthogonal sums of spaces $\mathcal{H}_{i_n \ldots i_1}$ are invariant under the operators Q_j and that the topology τ_{ow} of operators on such finite orthogonal sums coincides with the weak operator topology defined in the theory of bounded operators. Therefore it follows from the Double Commutant Theorem and Equations (55)–(57) that operators C of the following form are in the τ_{ow}-closure of the O*-algebra generated by $Q_1, \ldots, Q_n$:

$$C\eta_{i_n \ldots i_1} = B\eta_{i_n \ldots i_1}, \quad C\eta_{j_n \ldots j_1} = 0 \ \text{ for } \ (j_n, \ldots, j_1) \neq (i_n, \ldots, i_1),$$

where B belongs to the von Neumann algebra generated by A. Recall from the formulas of Proposition 4.1 that $z^*_{n-m} z_k \eta_{i_n \ldots i_1} = q^{-i_k + 2(i_{n-m+1} + \cdots + i_n)} Av\eta_{i_n \ldots i_1}$

and $z_k^* z_{n-m}\eta_{i_n\dots i_1} = q^{-i_k+2(i_{n-m+1}+\dots+i_n)}Av^*\eta_{i_n\dots i_1}$. So another application of the Double Commutant Theorem shows that all operators C of the above form with $B \in \mathfrak{B}$ are in the τ_{ow}-closure of the O*-algebra generated by $z_1, \dots, z_n$. Observe that the non-zero operators z_j, z_j^* act as shift operators (multiplied on each of the spaces $\mathcal{H}_{i_n\dots i_1}$ by an invertible element of $\mathfrak{B}$). By taking appropriate products, we conclude that operators C of the form $C\eta_{i_n\dots i_1} = B\eta_{r_n\dots r_1}$, $B \in \mathfrak{B}$, $C\eta_{j_n\dots j_1} = 0$ for $(j_n, \dots, j_1) \neq (i_n, \dots, i_1)$ belong to this closure. But the linear span of such operators is τ_{ow}-dense in the space defined in (ii) and this space is τ_{ow}-closed in $\mathcal{L}(D, D^+)$. This proves (ii) in the case $k, l > 0$. The other cases are similar but easier.

(iii) follows immediately from the definition of τ_{ow} and the operator expansion of the action in Lemma 4.2. □

Remark. It was essential in the proof of Proposition 4.3(ii) that we considered the O*-algebra generated by $z_1, \dots, z_n$ on D and not its closure nor the closed O*-algebra $\mathfrak{A}$ defined below.

In the remaining part of this section, we shall restrict ourselves to irreducible *-representations of the series $(m, 0, k)$. As before, let D denote the linear space defined in Proposition 4.1. Then the operators $|Q_j|^{-1/2} = |\overline{Q}_j|^{-1/2}\lceil D$ belong to $\mathcal{L}^+(D)$. Let $\mathfrak{A}_0$ denote the O*-algebra on D generated by z_j, $|Q_j|^{-1/2}$, $j = 1, \dots, n$ and let $\mathfrak{A}$ be the closure of $\mathfrak{A}_0$ so that $D_{\mathfrak{A}}$ is the Fréchet space $\cap_{a\in\mathfrak{A}_0} D(\bar{a})$. It turns out that the topological properties of the closed O*-algebra $\mathfrak{A}$ are very similar to that of Subsection 3.2.

Lemma 4.3.

i. *$\mathfrak{A}$ is a commutatively dominated O*-algebra on a Fréchet domain.*

ii. *$D_{\mathfrak{A}}$ is nuclear, in particular, $D_{\mathfrak{A}}$ is a Fréchet–Montel space.*

Proof. The operator

$$T := 1 + Q_1^2 + \dots + Q_n^2 + Q_1^{-2} + \dots + Q_n^{-2} \tag{73}$$

is essentially self-adjoint on $D_{\mathfrak{A}}$, and $T > 2$. Let $\varphi \in D_{\mathfrak{A}}$. As in the proof of Lemma 3.2, we conclude from a standard argument that, for each polynomial $p = p(|Q_1|^{1/2}, \dots, |Q_n|^{1/2}, |Q_1|^{-1/2}, \dots, |Q_n|^{-1/2})$, there exist $k \in \mathbb{N}$ such that $||p\varphi|| \leq ||T^k\varphi||$. Furthermore, for each finite sequence $k_1, \dots, k_N \in \mathbb{N}$ and real numbers $\gamma_1, \dots, \gamma_N \in (0, \infty)$, we find $k_0 \in \mathbb{N}$ such that $\sum_{j=1}^N \gamma_j ||T^{k_j}\varphi|| \leq$

$||T^{k_0}\varphi||$. Let p be as above and let $I, J \in \mathbb{N}^n$ such that $I \cdot J = 0$. By (44)–(47) and (50)–(53), $(z^I p z^{*J})^*(z^I p z^{*J})$ is a polynomial in $|Q_j|^{1/2}$, $|Q_j|^{-1/2}$, $j = 1, \ldots, n$, say $\tilde{p}$. Thus there exist $k \in \mathbb{N}$ such that

$$||z^I p z^{*J}\varphi|| = \langle \tilde{p}\varphi, \varphi\rangle^{1/2} \leq (||\tilde{p}\varphi||\,||\varphi||)^{1/2} \leq ||T^k\varphi||.$$

From the definition of $\mathfrak{A}$, (54), (66), and (67), it follows that each $f \in \mathfrak{A}$ can be written as $f = \sum_{I\cdot J=0} z^I p_{IJ} z^{*J}$, where p_{IJ} are polynomials in $|Q_j|^{1/2}$, $|Q_j|^{-1/2}$, $j = 1, \ldots, n$. From the preceding arguments, we conclude that there exist $m \in \mathbb{N}$ such that $||f\varphi|| \leq ||T^m\varphi||$ for all $\varphi \in D_{\mathfrak{A}}$, therefore $||\cdot||_f \leq ||\cdot||_{T^m}$. This implies that the family $\{||\cdot||_{T^{2^k}}\}_{k\in\mathbb{N}}$ generates the graph topology on $D_{\mathfrak{A}}$ and $D_{\mathfrak{A}} = \cap_{k\in\mathbb{N}} D(\bar{T}^{2^k})$ which proves (i).

Note that the proof of Lemma 3.2(ii) is based on the observation that the operator $\bar{T}^{-1}$ is a Hilbert–Schmidt operator. One easily checks that this holds also for the operator T defined in (73). Now the rest of the proof runs completely analogous to that of Lemma 3.2. □

Proposition 4.4.

i. $\mathbb{F}(D_{\mathfrak{A}})$ *is dense in* $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$ *with respect to the bounded topology* τ_b.

ii. *The* $\mathcal{U}_q(\mathrm{su}_{n,1})$*-action on* $\mathcal{L}^+(D_{\mathfrak{A}})$ *is continuous with respect to* τ_b.

The proof of Proposition 4.4 is completely analogous to that of Proposition 3.2.

Corollary 4.1. *Let* $\mathbb{F}(D)$ *denote the* O^**-algebra of finite rank operators on* D *defined in* (14). *Then* $\mathbb{F}(D)$ *is a* $\mathcal{U}_q(\mathrm{su}_{n,1})$*-module *-subalgebra of* $\mathbb{F}(D_{\mathfrak{A}})$ *and* $\mathbb{F}(D)$ *is dense in* $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$ *with respect to* τ_b.

Proof. Since $D \subset D_{\mathfrak{A}}$, we can consider $\mathbb{F}(D)$ as a *-subalgebra of $\mathbb{F}(D_{\mathfrak{A}})$. It follows from Proposition 4.1 that $\mathbb{F}(D)$ is stable under the $\mathcal{U}_q(\mathrm{su}_{n,1})$-action defined in Lemma 4.2, in particular, it is a $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-algebra. The density of $\mathbb{F}(D)$ in $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$ can be proved in exactly the same way as in Corollary 3.1. □

Recall that the self-adjoint operators $\bar{Q}_j$, $j = 1, \ldots, n$, strongly commute. Set

$$\mathfrak{M} := \sigma(\bar{Q}_1) \times \cdots \times \sigma(\bar{Q}_n).$$

By the spectral theorem of self-adjoint operators, we can assign to each (Borel measurable) function $\psi : \mathfrak{M} \to \mathbb{C}$ an operator $\psi(\bar{Q}_1, \ldots, \bar{Q}_n)$ such that

$$\psi(\bar{Q}_1, \ldots, \bar{Q}_n)\eta_{i_n\ldots i_1} = \psi(t_{i_1}, \ldots, t_{i_n})\eta_{i_n\ldots i_1},$$

where $t_{i_j} = q^{2(i_j+\cdots+i_n+\alpha)}$ for $j > k$, $t_{i_j} = -q^{-2(i_j+\cdots+i_k)+2(i_{k+1}+\cdots+i_n+\alpha)}$ for $j \le k$, and $A = q^{2\alpha}$. (A denotes the operator appearing in the type $(m, 0, k)$ representations for $k > 0$. If $k = 0$, set $\alpha = 0$.) Define

$$\mathcal{S}(\mathfrak{M}) := \{\, \psi : \mathfrak{M} \to \mathbb{C}\, ; \sup_{(t_1,\ldots,t_n)\in\mathfrak{M}} |t_1^{s_1}\cdots t_n^{s_n}\psi(t_1,\ldots,t_n)| < \infty \text{ for all } s_1,\ldots,s_n \in \mathbb{Z}\},$$

$$\mathcal{S}(D) := \left\{ \sum_{I\cdot J=0} z^I \psi_{IJ}(\bar{Q}_1,\ldots,\bar{Q}_n) z^{*J}\, ;\, \psi_{IJ} \in \mathcal{S}(\mathfrak{M}),\ \#\{\psi_{IJ} \neq 0\} < \infty \right\}.$$

Lemma 4.4. *With the action defined in Lemma 4.2, $\mathcal{S}(D)$ becomes a $\mathcal{U}_q(\mathrm{su}_{n,1})$-module *-subalgebra of $\mathbb{B}_1(\mathfrak{A})$. The operators z_j, z_j^*, $j = 1,\ldots,n$, and $\psi(\bar{Q}_1,\ldots,\bar{Q}_n)$, $\psi \in \mathcal{S}(\mathfrak{M})$, satisfy the following commutation rules*

$$\psi(\bar{Q}_1,\ldots,\bar{Q}_j,\bar{Q}_{j+1},\ldots,\bar{Q}_n)z_j = z_j\psi(q^2\bar{Q}_1,\ldots,q^2\bar{Q}_j,\bar{Q}_{j+1},\ldots,\bar{Q}_n)$$

$$z_j^*\psi(\bar{Q}_1,\ldots,\bar{Q}_j,\bar{Q}_{j+1},\ldots,\bar{Q}_n) = \psi(q^2\bar{Q}_1,\ldots,q^2\bar{Q}_j,\bar{Q}_{j+1},\ldots,\bar{Q}_n)z_j^*.$$

The proof of Lemma 4.4 differs from that of Lemma 3.4 only in notation, the argumentation to establish the result remains the same.

Since $\mathbb{F}(D) \subset \mathbb{F}(D_{\mathfrak{A}})$ and $\mathcal{S}(D) \subset \mathbb{B}_1(\mathfrak{A})$, we can consider $\mathbb{F}(D)$ and $\mathbb{B}_1(\mathfrak{A})$ as algebras of infinite differentiable functions with compact support and which are rapidly decreasing, respectively. It is not difficult to see that $\mathbb{F}(D)$ is the set of all $\sum_{I\cdot J=0} z^I\psi_{IJ}(\bar{Q}_1,\ldots,\bar{Q}_n)z^{*J} \in \mathcal{S}(D)$, where the functions $\psi_{IJ} \in \mathcal{S}(\mathfrak{M})$ have finite support. On $\mathcal{S}(D)$, we have the following explicit formula of the invariant integral.

Proposition 4.5. *Set $\mathfrak{M}_0 := \sigma(\bar{Q}_1)\backslash\{0\} \times \cdots \times \sigma(\bar{Q}_n)\backslash\{0\}$. Assume that $f = \sum_{I\cdot J=0} z^I\psi_{IJ}(\bar{Q}_1,\ldots,\bar{Q}_n)z^{*J} \in \mathcal{S}(D)$. Then the invariant integral h defined in Proposition 4.2 is given by*

$$h(f) = c \sum_{(t_1,\ldots,t_n)\in\mathfrak{M}_0} \psi_{00}(t_1,\ldots,t_n)|t_1|^{-n}|t_2|\cdots|t_n|.$$

(If $n = 1$, then $t_2,\ldots,t_n$ are omitted.)

Proof. Recall that $h(f) = c\,\mathrm{Tr}\,\overline{f\Gamma}$, where Γ is given by (71). If $I \neq (0,\ldots,0)$ or $J \neq (0,\ldots,0)$, then $\langle \eta_{i_n\ldots i_1}, z^I\psi_{IJ}(\bar{Q}_1,\ldots,\bar{Q}_n)z^{*J}\Gamma\eta_{i_n\ldots i_1}\rangle = 0$ since, by Proposition 4.1, $\psi_{IJ}(\bar{Q}_1,\ldots,\bar{Q}_n)$ and Γ are diagonal and z^I and z^{*J} act as shift operator on $\mathcal{H}$. Hence only $\psi_{00}(\bar{Q}_1,\ldots,\bar{Q}_n)\Gamma$ contributes to the trace.

For each tuple $(t_1, \dots, t_n) \in \mathfrak{M}_0$, there exists exactly one tuple $(i_1, \dots, i_n)$ such that $\eta_{i_n \dots i_1} \in D$ and $Q_j \eta_{i_n \dots i_1} = t_j \eta_{i_n \dots i_1}$, $j = 1, \dots, n$. This can be seen inductively; Q_n determines i_n uniquely, and if $i_n, \dots, i_{n-k+1}$ are fixed, then Q_{n-k} determines uniquely i_{n-k} (see the remark after Proposition 4.1). Since the vectors $\eta_{i_n \dots i_1}$ constitute an orthonormal basis of eigenvectors of the Q_j's, and since Γ is given by $\Gamma = |Q_1|^n |Q_2|^{-1} \cdots |Q_n|^{-1}$ for $n > 1$, $\Gamma = |Q_1|$ for $n = 1$, the assertion follows. □

In the following, let $n > 1$. We noted in Subsection 4.1 that the action of the elements E_j, F_j, $K_j^{\pm 1}$, $j = 1, \dots, n-1$ on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ induces a $\mathcal{U}_q(\mathrm{su}_n)$-action which turns $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ into a $\mathcal{U}_q(\mathrm{su}_n)$-module *-algebra. $\mathcal{U}_q(\mathrm{su}_n)$ is regarded as a compact real form of $\mathcal{U}_q(\mathrm{sl}_n)$. Naturally, the compactness should manifest in the existence of a normalized invariant integral on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$. This is indeed the case. Consider a irreducible *-representation of type $(n, 0, 0)$. Then the operators Q_j, $j = 1, \dots, n$, are bounded, and Q_1 is of trace class. In Proposition 4.2, a $\mathcal{U}_q(\mathrm{su}_{n,1})$-invariant functional h was given by $h(f) := c \,\mathrm{Tr}\, \overline{f\Gamma}$, where $\Gamma = |Q_1|^{-n} |Q_2| \cdots |Q_n|$. Note that the proof of Proposition 4.2 uses only the commutation relations of Γ with A_i, B_i, and ρ_i, $i = 1, \dots, n$. The crucial observation is that Q_1 commutes with A_j, B_j, and ρ_j, $j = 1, \dots, n-1$. Therefore the commutation relations used in proving the invariance of h remain unchanged if we multiply Γ by Q_1^{n+1}. Furthermore, ΓQ_1^{n+1} is of trace class. This suggests that $h(f) := c \,\mathrm{Tr}\, f \Gamma Q_1^{n+1}$ defines a $\mathcal{U}_q(\mathrm{su}_n)$-invariant integral on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$. The only difficulty is that the definitions of A_j, B_j, and $\rho_j^{\pm 1}$ involve the unbounded operators Q_j^{-1}, therefore we cannot freely apply the trace property in proving the invariance of h. Nevertheless, a modified proof will establish the result.

Proposition 4.6. *Let $n > 1$ and set $c := \prod_{k=1}^{n} (1 - q^{2k})^{-1}$. Suppose we are given an irreducible *-representation of $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ of type $(n, 0, 0)$. Then the linear functional*

$$h(f) := c \,\mathrm{Tr}\, f \Gamma Q_1^{n+1} = c \,\mathrm{Tr}\, f Q_1 \cdots Q_n\,, \qquad f \in \mathcal{O}_q(\mathrm{Mat}_{n,1}), \tag{74}$$

defines a normalized $\mathcal{U}_q(\mathrm{su}_n)$-invariant integral on $\mathcal{O}_q(\mathrm{Mat}_{n,1})$.

Proof. First note that the vectors $\eta_{i_n \dots i_1}$, $i_1, \dots, i_n \in \mathbb{N}_0$, form a complete set of eigenvectors of the positive operator Q_1 with corresponding eigenvalues $q^{2(i_1 + \cdots + i_n)}$. As $\sum_{i_1, \cdots, i_n \in \mathbb{N}_0} q^{2(i_1 + \cdots + i_n)} < \infty$, Q_1 is of trace class. This implies that $f \Gamma Q_1^{n+1} = f Q_1 \cdots Q_n$ is of trace class for all $f \in \mathcal{O}_q(\mathrm{Mat}_{n,1})$ since the representations of the series $(n, 0, 0)$ are bounded. Therefore h is well defined. An easy calculation shows that $h(1) = 1$.

As in the proof of Proposition 3.1, it suffices to verify the invariance of h for the generators of $\mathcal{U}_q(\mathrm{su}_n)$. Recall that $\mathcal{O}_q(\mathrm{Mat}_{n,1})$ is the linear span of the elements $z^I p_{IJ} z^{*J}$, where $I, J \in \mathbb{N}_0^n$, $I \cdot J = 0$, and p_{IJ} is a polynomial in Q_i, $i = 1, \dots, n$. If $I \neq 0$ or $J \neq 0$, then the same arguments as in the proof of Proposition 4.5 show

(75)
$$0 = \langle \eta_{i_n \dots i_1}, \rho_j^{\pm 1} z^I p_{IJ} z^{*J} \rho_j^{\mp 1} \Gamma Q_1^{n+1} \eta_{i_n \dots i_1} \rangle = \langle \eta_{i_n \dots i_1}, z^I p_{IJ} z^{*J} \Gamma Q_1^{n+1} \eta_{i_n \dots i_1} \rangle.$$

Hence $h(K_j^{\pm 1} \triangleright (z^I p_{IJ} z^{*J})) = \varepsilon(K_j^{\pm 1}) h(z^I p_{IJ} z^{*J}) = 0$. If $I = J = 0$, then $K_j^{\pm 1} \triangleright p_{00} = \rho_j^{\pm 1} p_{00} \rho_j^{\mp 1} = p_{00}$, so $h(K_j^{\pm 1} \triangleright p_{00}) = h(p_{00}) = \varepsilon(K_j^{\pm 1}) h(p_{00})$.

Recall that $A_k = -q^{-5/2} \lambda^{-1} Q_{k+1}^{-1} z_{k+1}^* z_k$, $k < n$. If $I \neq (0, \dots, 1, \dots, 0)$ or $J \neq (0, \dots, 1, \dots, 0)$ with 1 in the $(k+1)$th and kth positions, respectively, we have similarly to Equation (75)

$$\begin{aligned} 0 &= \langle \eta_{i_n \dots i_1}, A_k z^I p_{IJ} z^{*J} \Gamma Q_1^{n+1} \eta_{i_n \dots i_1} \rangle \\ &= \langle \eta_{i_n \dots i_1}, \rho_k z^I p_{IJ} z^{*J} \rho_k^{-1} A_k \Gamma Q_1^{n+1} \eta_{i_n \dots i_1} \rangle. \end{aligned}$$

Thus $h(E_k \triangleright (z^I p_{IJ} z^{*J})) = \varepsilon(E_k) h(z^I p_{IJ} z^{*J}) = 0$.

Now let p denote an arbitrary polynomial in Q_i, $i = 1, \dots, n$. Then, by the definition of Γ and repeated application of the commutation rules of Q_i with z_j and z_j^*, we obtain

$$\begin{aligned} Q_{k+1}^{-1} z_{k+1}^* z_k z_{k+1} p z_k^* \Gamma Q_1^{n+1} &= z_{k+1}^* z_k z_{k+1} p z_k^* Q_1 \cdots Q_k Q_{k+2} \cdots Q_n, \\ \rho_k z_{k+1} p z_k^* \rho_k^{-1} Q_{k+1}^{-1} z_{k+1}^* z_k \Gamma Q_1^{n+1} &= z_{k+1} p z_k^* Q_1 \cdots Q_k Q_{k+2} \cdots Q_n z_{k+1}^* z_k. \end{aligned}$$

All operators on the right hand sides are bounded and Q_1 is of trace class, in particular, the trace property applies. Therefore, the difference of the traces of the right hand sides vanishes. Hence

$$\begin{aligned} h(E_k \triangleright (z_{k+1} p z_k^*)) &= c \,\mathrm{Tr}\, (A_k z_{k+1} p z_k^* - \rho_k z_{k+1} p z_k^* \rho_k^{-1} A_k) \Gamma Q_1^{n+1} = 0 \\ &= \varepsilon(E_k) h(z_{k+1} p z_k^*) \end{aligned}$$

which establishes the invariance of h with respect to E_k, $k = 1, \dots, n-1$.

To verify that h is invariant with respect to F_k, $k = 1, \dots, n-1$, note that $h(f^*) = \overline{h(f)}$ for all $f \in \mathcal{O}_q(\mathrm{Mat}_{n,1})$ since the operator ΓQ_1^{n+1} is self-adjoint. Thus, by (4) and the preceding,

$$h(F_k \triangleright f) = \overline{h(S(F_k)^* \triangleright f^*)} = -q^2 \overline{h(E_k \triangleright f^*)} = 0 = \varepsilon(F_k) h(f)$$

for all $f \in \mathcal{O}_q(\mathrm{Mat}_{n,1})$. □

Corollary 4.2. *Let* $f = \sum_{I \cdot J=0} z^I p_{IJ}(Q_1, \dots, Q_n) z^{*J} \in \mathcal{O}_q(\mathrm{Mat}_{n,1})$. *Then the invariant integral* h *defined in Proposition* 4.6 *is given by*

$$h(f) = c \sum_{j_1, \dots, j_n \in \mathbb{N}_0} p_{00}(q^{j_1}, \dots, q^{j_n}) q^{j_1} \cdots q^{j_n}.$$

Proof. Taking into account that $\mathfrak{M}_0 = \{(q^{j_1}, \dots, q^{j_n})\,;\, j_1, \dots, j_n \in \mathbb{N}_0\}$ for representations of the series $(n, 0, 0)$, Corollary 4.2 is verified by an obvious modification of the proof of Proposition 4.5. □

§5. Concluding remarks

In general, the definition of quantum spaces is completely algebraic. However, our definition of integrable functions involves operator algebras. This is not a disadvantage since operator algebras form a natural setting for the study of (non-compact) quantum spaces. For example, Hilbert space representations provide us with the powerful tool of spectral theory which allows to define functions of self-adjoint operators. We emphasize that different representations will lead to different algebras of integrable functions. If one accepts that representations carry information about the underlying quantum space (for instance, by considering the spectrum of self-adjoint operators), then representations can be used to distinguish between q-deformed manifolds which are isomorphic on purely algebraic level. It is another advantage of our method that it works in a unique way for different representations.

The crucial step of our approach was to find an operator expansion of the action. At first sight it seems a serious drawback that no direct method was given to obtain an operator expansion of the action. This problem can be removed by considering cross product algebras. Inside the cross product algebra, the action can be expressed by algebraic relations. Representations of cross product algebras lead therefore to an operator expansion of the action. Moreover, the operator expansion is given by the adjoint action so that our ideas concerning invariant integrals apply [21]. Hilbert space representations of some cross product algebras can be found in [14] and [21].

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