

AN OPTIMAL CONTROL PROBLEM INVOLVING A CLASS OF LINEAR TIME-LAG SYSTEMS

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Abstract

A class of convex optimal control problems involving linear hereditary systems with linear control constraints and nonlinear terminal constraints is considered. A result on the existence of an optimal control is proved and a necessary condition for optimality is given. An iterative algorithm is presented for solving the optimal control problem under consideration. The convergence property of the algorithm is also investigated. To test the algorithm, an example is solved.

1. Introduction

In [10], a feasible directions algorithm has been devised for solving a class of nonlinear time-lag optimal control problems with linear control constraints and nonlinear terminal inequality constraints. This feasible direction algorithm was first initiated by Polak and Mayne in [7] for an optimal control problem involving lumped-parameter system, but without time-delayed arguments. The convergence result of the algorithms reported in [10] and [7] is as follows:

If the sequence of feasible controls generated by the algorithm has an accumulation point in the strong topology of L_∞ , then it satisfies a necessary condition for optimality.

Clearly, this convergence result is not satisfactory in the sense that there is no guarantee of the existence of such an accumulation point.

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The main aim of this paper is to obtain a more natural convergence result than those of [10] and [7]. More precisely, we shall show that the sequence of feasible controls generated by the feasible directions algorithm has an accumulation point in the weak* topology of L_∞ . Furthermore, each of these accumulation points is shown to satisfy a necessary condition for optimality. However, this is possible only for a special case of [10]. More precisely, our result is only valid for the case in which the dynamic system is linear.

To demonstrate the efficiency of the algorithm, an example is solved.

2. The problem statement

Consider the following delay-differential equation defined on the fixed time interval $(0, T]$:

$$\dot{x}(t) = \sum_{j=0}^s \{ A_j(t)x(t - h_j) + B_j(t)u(t - h_j) \}, \tag{1a}$$

where

$$x \equiv [x_1, \dots, x_n]^T \in R^n, \quad u \equiv [u_1, \dots, u_r]^T \in R^r$$

are, respectively, the state and the control vectors; the superscript T denotes the transpose; A_j and B_j are, respectively, $n \times n$ and $n \times r$ matrix-valued functions defined on $[0, T + h_s]$. The h_j are time delays, ordered so that

$$0 = h_0 < h_1 \cdots < h_s < T; \quad s < \infty.$$

The initial function for the differential equation (1a) is

$$x(t) = \phi(t), \quad t \in [-h_s, 0]; \quad x(0) = \Gamma, \tag{1b}$$

where $\phi(t) \equiv [\phi_1, \dots, \phi_n]^T$ is a given piecewise continuous function from $[-h_s, 0)$ into R^n , and Γ is a given vector in R^n .

Let U be a compact and convex subset of R^r such that for all $u \in U$,

$$E_i^T u \leq b_i, \quad i = 1, \dots, p, \tag{2}$$

where E_i , $i = 1, \dots, p$, are r -vectors and b_i , $i = 1, \dots, p$, are real numbers. Let $\beta: [-h_s, 0) \rightarrow R^r$ be a given piecewise continuous function. Let \mathcal{U} be the class of all admissible controls defined by

$$\mathcal{U} \equiv \{ u : u \equiv [u_1, \dots, u_r]^T \text{ is a measurable function defined on } [-h_s, T] \text{ such that } u(t) \in U \text{ for all } t \in [0, T] \text{ and } u(t) = \beta(t) \text{ for all } t \in [-h_s, 0) \}.$$

Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote, respectively, the usual inner product and norm in a Euclidean space. For any

$$n \times r \text{ matrix } A \equiv \{A_{i,j}\}_{\substack{i=1,\dots,n, \\ j=1,\dots,r}}$$

let

$$\|A\| = \sum_{j=1}^r \sum_{i=1}^n |A_{i,j}|. \tag{3}$$

Let L_∞^q (resp. L^q) denote the Banach space $L_\infty([-h_s, T], R^q)$ (resp. $L_1([-h_s, T], R^q)$) of all essentially bounded measurable functions (resp. of all integrable functions) from $[-h_s, T]$ into R^q . The norms of L_∞^q and L^q are, respectively, defined by

$$\|u\|_\infty \equiv \text{ess sup}_{t \in [-h_s, T]} |u(t)|$$

and

$$\|u\|_1 \equiv \int_{-h_s}^T |u(t)| dt.$$

For each $u \in L_\infty^q$, Let $x(u)$ be the corresponding vector-valued function which is absolutely continuous on $(0, T]$ and satisfies the differential equation (1a) almost everywhere on $(0, T]$ and the initial condition (1b) everywhere on $[-h_s, 0]$. This function $x(u)$ is called the solution of the system (1) corresponding to $u \in L_\infty^q$. In view of Appendix (B1) of [4], pages 104–106, we observe that $x(u)$ can be expressed in terms of the Oguztörelı kernel matrix of the second kind, namely:

$$\begin{aligned} x(u)(t) = & N(0, t)\Gamma + \sum_{j=0}^s \int_{-h_j}^0 \{N(\tau + h_j, t)A_j(\tau + h_j)\phi(\tau)\} d\tau \\ & + \sum_{j=0}^s \int_0^t \{N(\tau, t)B_j(\tau)u(\tau - h_j)\} d\tau, \end{aligned} \tag{4}$$

where $N(\tau, t)$ (the Oguztörelı kernel matrix of the second kind), is the continuous $n \times n$ matrix defined on $[0, T] \times [0, T]$ which satisfies the system

$$\frac{\partial N(\tau, t)}{\partial t} = \sum_{j=0}^s A_j(t)N(\tau, t - h_j), \quad t \in [0, T], \quad \tau \in [0, T], \tag{5a}$$

$$N(\tau, \tau) = I \text{ (identity matrix),} \tag{5b}$$

$$N(\tau, t) = 0 \text{ for all } \tau > t. \tag{5c}$$

The terminal inequality constraints may now be stated as follows:

$$J^m(u) \equiv g^m(x(u)(T)) \leq 0, \quad m = 1, \dots, l, \tag{6}$$

where $g^m: R^n \rightarrow R$, $m = 1, \dots, l$, are real-valued functions defined on R^n .

Clearly, not all controls from \mathcal{U} will satisfy the constraints (6). Let \mathcal{F} denote the class of all those controls from \mathcal{U} such that the constraints (6) are satisfied. Elements from \mathcal{F} are called feasible controls and \mathcal{F} is called the class of feasible controls. In this paper, we assume that \mathcal{F} is not empty.

Our optimal control problem, denoted by P , may now be formally stated as follows:

Subject to the system (1), find a feasible control $u \in \mathcal{F}$ such that the cost functional

$$J^0(u) = g^0(x(u)(T)) + \int_0^T \{f(t, x(u)(t)) + u^T(t)M(t)u(t)\} dt \quad (7)$$

is minimized over \mathcal{F} .

We assume that the following conditions are satisfied.

(A1) $A_j, j = 0, \dots, s$, are real continuous $n \times n$ matrices defined on $[0, T + h_s]$; $B_j, j = 0, \dots, s$, are real continuous $n \times r$ matrices defined on $[0, T + h_s]$.

(A2) M is a $r \times r$ symmetric, positive semidefinite continuous matrix defined on $[0, T]$.

(A3) $x \rightarrow g^m(x), m = 0, \dots, l, (t, x) \rightarrow f(t, x)$ are continuously differentiable on R^n and $[0, T] \times R^n$, respectively.

3. Preparatory results

For each $u \in L^\infty$, let the functions $\lambda_m(u): [-h_s, T + h_s] \rightarrow R^n, m = 0, \dots, l$, be the solution of the adjoint system

$$\lambda_m(t) = - \sum_{j=0}^s A_j^T(t + h_j)\lambda_m(t + h_j), \quad m = 1, \dots, l, \quad t \in [0, T), \quad (8a)$$

$$\lambda_0(t) = - \sum_{j=0}^s A_j^T(t + h_j)\lambda_0(t + h_j) - \nabla_x f(t, x(u)(t)), \quad t \in [0, T), \quad (8b)$$

with the conditions

$$\lambda_m(T) = \nabla_x g^m(x(u)(T)), \quad m = 0, \dots, l, \quad (8c)$$

$$\lambda_m(t) = 0, \quad m = 0, \dots, l, \quad t \in [-h_s, 0), \quad (8d)$$

and

$$\lambda_m(t) = 0, \quad m = 0, \dots, l, \quad t \in [T, T + h_s). \quad (8e)$$

Following a similar approach as that used in the proof of Theorem 3.1 of [12], it can be shown that the Fréchet derivatives of the functionals J^0, J^1, \dots, J^l at

$u^0 \in L'_\infty$ are, respectively,

$$J_{u^0}^0(w) = - \int_{-h_s}^T \left\{ \sum_{j=0}^s \lambda_0^T(u^0)(t + h_j) B_j(t + h_j) + 2u^{0T}(t) M(t) \right\} w(t) dt, \quad (9a)$$

$$J_{u^m}^m(w) = - \int_{-h_s}^T \left\{ \sum_{j=0}^s \lambda_m^T(u^0)(t + h_j) B_j(t + h_j) \right\} w(t) dt, \quad m = 1, \dots, l, \quad (9b)$$

for all $w \in L'_\infty$.

REMARK 3.1. In view of the problem P , we see that the only non-standard feature is the inclusion of the terminal inequality constraints on the state variables. In this situation, one may include the constraints in the Pontryagin theory, by modifying boundary conditions for λ_0 as it is done in [3]. Alternatively, one may treat these constraints as those in mathematical programming problems. The latter approach is adopted in this paper. Thus, we need to calculate the “gradients” of these constraints. To do so, we take each constraint as a “cost functional”. Then, by applying the usual Pontryagin theory, we can write down the corresponding “Hamiltonian”. On this basis, we obtain readily the corresponding “adjoint system” (cf. 8(a)). The formula for the “gradient” of the constraint concerned is given in (9b). The reader may find more detail of this idea in Appendix V of [11], pages 291–292.

DEFINITION 3.1. Let $\{u^k\}$ be a sequence in \mathcal{U} and let \bar{u} be an element in \mathcal{U} . Then, we say that $u^k \xrightarrow{w^*} \bar{u}$ (or u^k converges to \bar{u} in the weak* topology) if

$$\int_{-h_s}^T u^k(t) v(t) dt \rightarrow \int_{-h_s}^T \bar{u}(t) v(t) dt$$

for all $v \in L'_1$.

DEFINITION 3.2. Let $\{u^k\}$ be a sequence in L'_∞ and let \bar{u} be an element in L'_∞ . Then we say that $u^k \xrightarrow{w^*} \bar{u}$ in L'_∞ (or u^k converges to \bar{u} in the weak* topology in L'_∞) if

$$\int_{-h_s}^T u^k(t) v(t) dt \rightarrow \int_{-h_s}^T \bar{u}(t) v(t) dt$$

for all $v \in L'_1$.

LEMMA 3.1. Consider the system (1). Let x be considered as a mapping from \mathcal{U} into L'_∞ , where the norm in the space L'_∞ is again denoted by $\|\cdot\|_\infty$. Then

(i) There exists a positive constant C , independent of $u \in \mathcal{U}$, such that

$$\|x(u)\|_\infty \leq C.$$

(ii) For each $t \in [0, T]$

$$|x(u^k)(t) - x(\bar{u})(t)| \rightarrow 0 \text{ whenever } u^k \xrightarrow{w^*} \bar{u}.$$

(iii) $\|x(u^k) - x(u)\|_\infty \rightarrow 0$ whenever $u^k \xrightarrow{w^*} \bar{u}$.

PROOF. The proof of (i) follows easily from (4), (A1) and the definition of \mathcal{U} . The proof of (ii) and (iii) requires (4) and (A1).

LEMMA 3.2. Consider the adjoint system (8). Let λ_m be considered as mappings from \mathcal{U} into L_∞^n . Then

(i) For each $m = 0, \dots, l$, there exists a positive constant C_m , independent of $u \in \mathcal{U}$, such that

$$\|\lambda_m(u)\|_\infty \leq C_m.$$

(ii) For each $m = 0, \dots, l$,

$$\|\lambda_m(u^k) - \lambda_m(\bar{u})\|_\infty \rightarrow 0 \text{ whenever } u^k \xrightarrow{w^*} \bar{u}.$$

PROOF. Using (8b) and (8c), we deduce from Cauchy’s inequality that for all $u \in \mathcal{U}$ and for all $t \in [0, T]$,

$$\begin{aligned} |\lambda_0(u)(t)| &\leq |\nabla_x g^0(x(u)(T))| + \int_t^T \sum_{j=0}^s \|A_j^T(\tau + h_j)\| |\lambda_0(u)(\tau + h_j)| d\tau \\ &\quad + \int_t^T |\nabla_x f(\tau, x(u)(\tau))| d\tau, \end{aligned} \tag{10}$$

where $\|\cdot\|$ is as defined in (3).

Since $x(u)(\cdot)$ is absolutely continuous on $[0, T]$ for each $u \in \mathcal{U}$, it follows from Lemma 3.1 and (A3) that there exists a constant $K_1 > 0$, independent of $u \in \mathcal{U}$, such that

$$|\nabla_x g^0(x(u)(T))| \leq K_1$$

and

$$|\nabla_x f(t, x(u)(t))| \leq K_1$$

for all $t \in [0, T]$.

Thus, we deduce from (10), (8e) and (A1) that

$$\|\lambda_0(u)\|_\infty \leq K_2 + K_3 N_1 (\|\lambda_0(u)\|_\infty), \tag{11}$$

where

$$K_2 \equiv K_1(T + 1), \tag{12}$$

$$K_3 \equiv \max_{\substack{j \in \{0, \dots, s\} \\ t \in [0, T]}} \|A_j(t)\| \tag{13}$$

and

$$N_1(w)(t) = \int_t^T w(\tau) d\tau. \tag{14}$$

Now, by using the same argument as that used to obtain the inequality (A4) of [12] (i.e. the inequality (4) in the Appendix of [12]) from the inequality (A2) of [12], we deduce from (11) that

$$\|\lambda_0(u)\|_\infty \leq \sum_{i=0}^\infty K_2 T^i K_3^i / i! \equiv C_0.$$

Similarly, for each $m = 1, \dots, l$, there exists a positive constant C_m such that

$$\|\lambda_m(u)\|_\infty \leq C_m$$

for all $u \in \mathcal{U}$.

It remains to establish the validity of the second part of the lemma. For this, let $\{u^k\}$ be a sequence of controls in \mathcal{U} which converge to \bar{u} in the weak* topology. Then, by using (8b) and (8c), we deduce from Cauchy's inequality that for any integer k ,

$$\begin{aligned} & |\lambda_0(u^k)(t) - \lambda_0(\bar{u})(t)| \\ & \leq |\nabla_x g^0(x(u^k)(T)) - \nabla_x g^0(x(\bar{u})(T))| \\ & \quad + \int_t^T \sum_{j=0}^s \|A_j^T(\tau + h_j)\| |\lambda_0(u^k)(\tau + h_j) - \lambda_0(\bar{u})(\tau + h_j)| d\tau \tag{15} \\ & \quad + \int_t^T |\nabla_x f(\tau, x(u^k)(\tau)) - \nabla_x f(\tau, x(\bar{u})(\tau))| d\tau. \end{aligned}$$

From Lemma 3.1 and (A3), we note that for any given $\epsilon > 0$, there exists an integer $\hat{k} > 0$ such that for all $k > \hat{k}$,

$$|\nabla_x g(x(u^k)(T)) - \nabla_x g(x(\bar{u})(T))| < \epsilon \tag{16}$$

and

$$|\nabla_x f(t, x(u^k)(t)) - \nabla_x f(t, x(\bar{u})(t))| < \epsilon \tag{17}$$

for all $t \in [0, T]$.

On the basis of (15), it follows from (A1), (16) and (17) that

$$\|\lambda_0(u^k) - \lambda_0(\bar{u})\|_\infty < K_2 \epsilon + K_3 N_1 (\|\lambda_0(u^k) - \lambda_0(\bar{u})\|_\infty), \tag{18}$$

where K_2 , K_3 and N_1 are defined in (12), (13) and (14), respectively.

Then, by using the same argument as that used to obtain the inequality (A4) of [12] from the inequality (A2) of [12], we deduce from (18) that

$$\|\lambda_0(u^k) - \lambda_0(\bar{u})\|_\infty \leq \sum_{i=0}^\infty K_1 T^i K_2^i \epsilon / i! \equiv K_3 \epsilon.$$

Therefore

$$\|\lambda_0(u^k) - \lambda_0(\bar{u})\|_\infty \rightarrow 0 \text{ whenever } u^k \xrightarrow{w^*} \bar{u}.$$

Similarly, for each $m = 1, \dots, l$, we have

$$\|\lambda_m(u^k) - \lambda_m(\bar{u})\|_\infty \rightarrow 0 \text{ whenever } u^k \xrightarrow{w^*} \bar{u}.$$

This completes the proof of the lemma.

LEMMA 3.3. *For each $m = 0, 1, \dots, l$, and for each $w \in L'_\infty$, $J_u^m(w)$ is continuous in u with respect to the weak* topology in the sense that*

$$J_u^m(w) \rightarrow J_{\bar{u}}^m(x) \text{ whenever } u^k \xrightarrow{w^*} \bar{u}.$$

PROOF. The proof follows easily from (9), (A1), (A2) and part (ii) of Lemma 3.2.

LEMMA 3.4. *For each $m = 0, 1, \dots, l$, and for each $u \in \mathcal{U}$, $J_u^m(w)$ is continuous in w with respect to the weak* topology in L'_∞ in the sense that*

$$J_u^m(w^k) \rightarrow J_u^m(\bar{w}) \text{ whenever } w^k \xrightarrow{w^*} \bar{w} \text{ in } L'_\infty.$$

PROOF. The proof follows easily from (9), (A1), (A2) and part (i) of Lemma 3.2.

4. Existence of optimal control

In this section, we shall show that the optimal control problem has a solution. More precisely, we have

THEOREM 4.1. *If \mathcal{F} is nonempty, then the problem P has a solution.*

PROOF. From part (i) of Lemma 3.1 and the continuity property of $x(u)(\cdot)$, we can find a compact set Z in R^n such that

$$x(u)(t) \in Z$$

for all $u \in \mathcal{U}$ and for all $t \in [0, T]$. Now, by using this result, together with the compactness of the set U in R^r and the assumptions (A2) and (A3), it can be shown (by method of [9]) that the cost functional $J^0(u)$ is bounded on \mathcal{U} . Thus,

$$\inf_{u \in \mathcal{F}} J^0(u) = \sigma > -\infty.$$

Let $\{u^k\} \subset \mathcal{F}$ be a sequence such that

$$\lim_{k \rightarrow \infty} J^0(u^k) = \sigma.$$

Since U is a compact and convex subset of R^l , \mathcal{U} is sequentially compact in the weak* topology (cf. Theorem I.6.2 of [11], page 31). Thus, there exists a function $u^* \in \mathcal{U}$ and a subsequence of the sequence $\{u^k\}$, again indexed by k , such that $u^k \rightarrow u^*$ in the weak* topology as $k \rightarrow \infty$.

It remains to show that u^* is an optimal control of the problem P . From Lemma 3.1, (A3) and (6), it follows that, for each $m = 1, \dots, l$,

$$g^m(x(u^*)(T)) = \lim_{k \rightarrow \infty} g^m(x(u^k)(T)) \leq 0.$$

This means that $u^* \in \mathcal{F}$. Again from Lemma 3.1 and (A3), we have

$$g^0(x(u^*)(T)) = \lim_{k \rightarrow \infty} g^0(x(u^k)(T)) \tag{19}$$

and

$$\int_0^T f(t, x(u^*)(t)) dt = \lim_{k \rightarrow \infty} \int_0^T f(t, x(u^k)(t)) dt. \tag{20}$$

From (A2), it is clear that the function $u^T(t)M(t)u(t)$ is convex with respect to u . Thus, from Theorem 1.2.14 of [1], page 26, we have

$$\liminf_{k \rightarrow \infty} \int_0^T (u^k)^T(t)M(t)u^k(t) dt \geq \int_0^T u^{*T}(t)M(t)u^*(t) dt. \tag{21}$$

Now, by virtue of (7), it follows from (19), (20) and (21) that

$$\sigma = \liminf_{k \rightarrow \infty} J^0(u^k) \geq J^0(u^*).$$

Recall that the sequence $\{u^k\} \subset \mathcal{F}$ is chosen such that

$$\lim_{k \rightarrow \infty} J^0(u^k) = \inf_{u \in \mathcal{F}} J^0(u) = \sigma.$$

Thus

$$\liminf_{k \rightarrow \infty} J^0(u^k) = \lim_{k \rightarrow \infty} J^0(u^k) = \inf_{u \in \mathcal{F}} J^0(u) = \sigma.$$

Hence

$$\inf_{u \in \mathcal{F}} J^0(u) \geq J^0(u^*).$$

But, we have already shown that

$$u^* \in \mathcal{F}.$$

Thus it is clear that

$$J^0(u^*) = \inf_{u \in \mathcal{F}} J^0(u).$$

Thus, in turn, implies that $u^* \in \mathcal{F}$ is an optimal control for the problem P , as there could not be any other *feasible* control which will make J^0 smaller than $\inf_{u \in \mathcal{F}} J^0(u)$, and hence $J(u^*)$. Thus, the proof is complete.

REMARK 4.1. By examining the arguments given for Theorem 4.1, it is easy to see that \mathcal{F} is sequentially compact in the weak* topology.

5. Subproblem

For the problem P , we want to minimize the cost functional $J^0(u)$, subject to constraints $J^m(u) \leq 0, m = 1, \dots, l$, which have been built as part of the adjoint system (i.e. (8a) and (8c)–(8e)). We shall do this by a “feasible direction” method similar to that of [10] (and hence [7]). Thus, at each iteration of the algorithm, it is required to calculate a search direction which decreases J^0 , while pointing into the region specified by (6). Each of these search directions corresponds to a subproblem which is itself an optimal problem and can be solved by a standard convex program (Meyer-Polak proximity algorithm, see [6], pages 234–238). In what follows, we shall formulate the subproblem and describe an algorithm to solve it.

For any $\epsilon > 0$ and $u \in \mathcal{F}$, let $I_\epsilon(u)$ denote the ϵ -active index set, i.e.

$$I_\epsilon(u) \equiv \{0\} \cup \{m \in \{1, \dots, l\} : J^m(u) \geq -\epsilon\}. \tag{22}$$

for any subset I of the set $\{0, 1, \dots, l\}$, let $\psi_I : \mathcal{F} \rightarrow R$ be defined by

$$\psi_I(u) = \min_{w \in \mathcal{U} - u} \max_{m \in I} \{J_u^m(w)\},$$

where

$$\begin{aligned} \mathcal{U} - u &\equiv \{v - u : v \in \mathcal{U}\} \\ &\equiv \{v : v \in \mathcal{U}\} - u. \end{aligned}$$

In view of Lemma 4.1 of [13], it is clear that for each $u \in \mathcal{F}$, there exists a $w_I^u \in \mathcal{U} - u$ such that

$$\psi_I(u) = \max_{m \in I} \{J_u^m(w_I^u)\}.$$

The subproblem can now be stated as:

Problem (Ps). For each $u \in \mathcal{F}$ and $\epsilon > 0$, find a search direction $w_\epsilon^u \in \mathcal{U} - u$ which minimizes

$$\max_{m \in I_\epsilon(u)} \{J_u^m(w)\},$$

i.e.,

$$\begin{aligned} \psi_{I_\epsilon(u)}(u) &= \min_{w \in \mathcal{U} - u} \max_{m \in I_\epsilon(u)} \{J_u^m(w)\} \\ &= \max_{m \in I_\epsilon(u)} \{J_u^m(w_\epsilon^u)\}. \end{aligned}$$

To present an algorithm for solving this problem, we need some preparation.

Let n_ϵ^u denote the cardinality of the set I_ϵ^u and let $I_\epsilon^u \equiv \{i(1), i(2), \dots, i(q)\}$, where $q \equiv n_\epsilon^u$ and $i(1), \dots, i(q)$ are arranged in the natural order with $i(1) = 0$.

Let

$$C_\epsilon^u = \left\{ \xi = [\xi_1, \dots, \xi_q]^T : \xi_k = J_u^{i(k)}(w), k = 1, \dots, q, w \in \mathcal{U} - u \right\}. \quad (23)$$

Since $\mathcal{U} - u$ is convex and compact in the weak* topology, it follows from Lemma 3.4 that C_ϵ^u is a compact and convex subset of R^q .

The Meyer-Polask proximity algorithm requires a subprocedure for finding a point $\xi_{\epsilon,c}^u \in C_\epsilon^u$ which minimizes $\langle c, \xi \rangle$ subject to $\xi \in C_\epsilon^u$ for each $c \equiv [c_1, \dots, c_q]^T \in R^q$. From (9) and (23), this is equivalent to finding a search direction $w_{\epsilon,c}^u \in \mathcal{U} - u$ which minimizes

$$\left\{ \sum_{j=0}^s \sum_{k=1}^q c_k \lambda_{i(k)}^T(u)(t + h_j) B_j(t + h_j) + 2c_1 u^T(t) M(t) \right\} w(t)$$

over all $w \in \mathcal{U} - u$ and for all $t \in [-h_s, T]$, where $\lambda_{i(k)}$, $k = 1, \dots, q$, is the solution of the corresponding system (8).

LEMMA 5.1. *Let c be a given constant vector in R^q , where $q \equiv n_\epsilon^u$. Then, for each $u \in \mathcal{F}$, there exists a $w_{\epsilon,c}^u \in \mathcal{U} - u$ such that*

$$\begin{aligned} & \min_{w \in \mathcal{U} - u} \left\{ \sum_{j=0}^s \sum_{k=1}^q c_k \lambda_{i(k)}^T(u)(t + h_j) B_j(t + h_j) + 2c_1 u^T(t) M(t) \right\} w(t) \\ \text{(i)} \quad & = \left\{ \sum_{j=0}^s \sum_{k=1}^q c_k \lambda_{i(k)}^T(u)(t + h_j) B_j(t + h_j) + 2c_1 u^T(t) M(t) \right\} w_{\epsilon,c}^u(t) \\ & \text{for all } t \in [-h_s, T]. \end{aligned}$$

$$\langle c, \xi_{\epsilon,c}^u \rangle \leq \langle c, \xi \rangle$$

$$\begin{aligned} \text{(ii)} \quad & \text{for all } \xi \in C_\epsilon^u, \text{ where } \xi_{\epsilon,c}^u \equiv [(\xi_{\epsilon,c}^u)_1, \dots, (\xi_{\epsilon,c}^u)_q]^T \text{ with} \\ & (\xi_{\epsilon,c}^u)_k \equiv J^{i(k)}(w_{\epsilon,c}^u), k = 1, \dots, q. \end{aligned}$$

PROOF. The proof follows as a special case of Lemma 4.1 of [10].

Now, let

$$D_{\epsilon,a}^u \equiv \left\{ \eta \in R^{n_\epsilon^u} : \eta_j \in [b_j(u, \epsilon), a] \right\},$$

where $b(u, \epsilon) \in R^q$ denotes the vector

$$b(u, \epsilon) \equiv \left[(\xi_{\epsilon,e_1}^u)_1, (\xi_{\epsilon,e_2}^u)_2, \dots, (\xi_{\epsilon,e_q}^u)_q \right]^T,$$

where $q = n_\epsilon^u$ and for each $j = 1, \dots, n_\epsilon^u$, e_j denotes the j th unit vector in $R^{n_\epsilon^u}$.

In what follows, ξ will denote a point in C_ε^u and η a point in $D_{\varepsilon,a}^u$. Then, the Meyer-Polak proximity algorithm can be used to determine the least value of a such that $C_\varepsilon^u \cap D_{\varepsilon,a}^u \neq \emptyset$ and to determine a point in the intersection. Let $\hat{\xi}_\varepsilon^u$ be such a point in the intersection. Then

$$\psi_{I_\varepsilon(u)}(u) = \max\{(\hat{\xi}_\varepsilon^u)_j : j = 1, \dots, n_\varepsilon^u\}.$$

We can now present the Meyer-Polak proximity algorithm for determining $\psi_{I_\varepsilon^u}(u)$.

ALGORITHM 5.1.

DATA: $u \in \mathcal{F}$, $\varepsilon > 0$, $I_\varepsilon(u)$.

Step 0. Set $\xi^0 = 0$. Set $w^0 = 0$.

 Compute $b = b(u, \varepsilon)$.

 Set $\eta^0 = b$.

 Set $a^0 = \max\{b_j : j = 1, 2, \dots, n_\varepsilon^u\}$.

 Set $i = 0$.

Step 1. If $|\xi^i - \eta^i| = 0$, stop.

 Else set $\sigma^i = \xi^i - \eta^i$.

Step 2. Compute $w_{\varepsilon,\sigma^i}^u$ and $\xi_{\varepsilon,\sigma^i}^u$.

 Set $\bar{w}^i = w_{\varepsilon,\sigma^i}^u$, $\bar{\xi}^i = \xi_{\varepsilon,\sigma^i}^u$.

Step 3. If $\langle \sigma^i, \bar{\xi}^i \rangle \leq \langle \sigma^i, \eta^i \rangle$, set $a^{i+1} = a^i$, $\bar{\eta}^i = \eta^i$.

 Else set $a^{i+1} = \min\{a : D_{\varepsilon,a}^u \cap H \neq \emptyset, a \geq a^i\}$,
 where $H = \{y \in R^{n_\varepsilon^u} : \langle y, \sigma^i \rangle = \langle \bar{\xi}^i, \sigma^i \rangle\}$.

 Compute any $\bar{\eta}^i \in D_{u, a^{i+1}}^\varepsilon \cap H$.

Step 4. Compute $\xi^{i+1} \in [\bar{\xi}^i, \xi^i]$, $\eta \in [\bar{\eta}^i, \bar{\eta}^i]$ such that

$|\xi^{i+1} - \eta^{i+1}| = \min\{|\xi - \eta| : \xi \in [\bar{\xi}^i, \xi^i], \eta \in [\bar{\eta}^i, \bar{\eta}^i]\}$.

 Set $w^{i+1} = w^i + (w^i - w^i)|\xi^i - \xi^{i+1}|/|\xi^i - \bar{\xi}^i|$.

 Set $i = i + 1$.

 Go to Step 1.

REMARK 5.1. For each i , ξ^i , which is a point in C_ε^u , is generated by $w^i \in \mathcal{U} - u$, i.e.

$$(\xi^i)_k = J_u^{i(k)}(w^i), \quad i(k) \in I_\varepsilon(u).$$

REMARK 5.2. From Remark 4.5 of [10], there exists an integer $i_0 > 0$ such that

$$\max_k \{(\xi^{i_0})_k\} - \max_k \{(\eta^{i_0})_k\} \leq \varepsilon/2.$$

Let $\bar{\psi}_\varepsilon(u)$, $\underline{\psi}_\varepsilon(u)$ and \bar{w}_ε^u denote $\max_k\{(\xi^{i_0})_k\}$, $\max_k\{(\eta^{i_0})_k\}$ and w^{i_0} , respectively. Then $\bar{\psi}_\varepsilon^u(u)$, $\underline{\psi}_\varepsilon^u(u)$ and \bar{w}_ε^u , which can be computed in a finite number of iterations, satisfy

$$\begin{aligned} \psi_{I_\varepsilon^u}(u) &\in [\underline{\psi}_\varepsilon(u), \bar{\psi}_\varepsilon(u)], \\ \bar{\psi}_\varepsilon(u) - \underline{\psi}_\varepsilon(u) &\leq \varepsilon/2, \end{aligned}$$

and

$$\bar{\psi}_\varepsilon(u) = \max_{j \in I_\varepsilon^u} \{J_u^j(\bar{w}_\varepsilon^u)\}.$$

6. An algorithm

In this section, we shall present a feasible direction algorithm for solving the problem P and study its convergence properties.

ALGORITHM 6.1.

DATA: $\alpha, \beta \in (0, 1)$ and $\varepsilon' > 0$.

Step 0. Select a $u^0 \in \mathcal{F}$; set $i = 0$.

Step 1. Set $\varepsilon_0 = \varepsilon'$; set $j = 0$.

Step 2. Compute, using Algorithm 5.1, a search direction $\bar{w}_{\varepsilon_j}^{u^i}$, together with $\bar{\psi}_{\varepsilon_j}(u^i)$.

Step 3. If $\bar{\psi}_{\varepsilon_j}(u^i) > -\varepsilon_j/2$, set $\varepsilon_{j+1} = \beta\varepsilon_j$, set $j = j + 1$ and go to Step 2.
If $\bar{\psi}_{\varepsilon_j}(u^i) \leq -\varepsilon_j/2$, proceed.

Step 4. Compute the smallest integer k such that

$$J^0(u^i + \beta^k \bar{w}_{\varepsilon_j}^{u^i}) - J^0(u^i) \leq \alpha\beta^k \bar{\psi}_{\varepsilon_j}(u^i), \tag{24}$$

$$J^m(u^i + \beta^k \bar{w}_{\varepsilon_j}^{u^i}) \leq 0, \quad m = 1, \dots, p. \tag{25}$$

Step 5. Set $u^{i+1} = u^i + \beta^k \bar{w}_{\varepsilon_j}^{u^i}$. Set $i = i + 1$ and go to Step 1.

In what follows, our aim is to study the convergent property of the Algorithm 6.1. To be more precise, we shall show that the sequence of feasible controls generated by the feasible directions Algorithm 6.1 has an accumulation point in

the weak* topology and that each of these accumulation points satisfies a necessary condition for optimality. Note that this weak* convergence development is similar to that given in [9] and [8]. However, there is no constraint on the state variables in [9], while the terminal inequality constraints on the state variables are *only* linear in [8]. In the present paper, we allow the terminal inequality constraints to be *nonlinear*. Furthermore, the technique used in this paper and that in [9] and [8] are completely different. To be more precise, a control parametrization technique is used in [9] and [8], while a feasible direction method is used here in the present paper.

Let

$$\mathcal{D} \equiv \{u \in \mathcal{F} : \psi_{I_0}(u) = 0\}.$$

For any feasible control $u \notin \mathcal{D}$, let

$$\bar{\varepsilon}(u) \equiv \max\{\varepsilon : \bar{\psi}_\varepsilon(u) \leq -\varepsilon/2, \quad \varepsilon = \beta^k \varepsilon', \quad k = 0, 1, \dots\}.$$

and let $k(u)$ be the smallest non-negative integer satisfying (24) and (25).

LEMMA 6.1. *For any subset I of the index set $\{0, 1, \dots, l\}$, $\psi_I(\cdot)$ is upper semicontinuous with respect to the weak* topology. (In other words, if the sequence $\{u^k\}$ converges to \bar{u} in the weak* topology, then, for any given $\varepsilon > 0$, there exists an integer $\hat{k} > 0$ such that for all $k > \hat{k}$, $\psi_I(u^k) \leq \psi_I(\bar{u}) + \varepsilon$.)*

PROOF. The proof is similar to that given for Lemma 5.1 of [10], except that Remark 3.2 and L_∞ topology of [10] are, respectively, replaced by Lemma 3.3 and the weak* topology in the proof of the present lemma.

DEFINITION 6.1. Let $\varepsilon > 0$ be given. Then, for any $m \in \{1, \dots, l\}$, the corresponding terminal inequality constraint of (6) is said to be ε -active at $\bar{u} \in \mathcal{U}$ if $m \in I_\varepsilon(\bar{u})$. (For the definition of $I_\varepsilon(\bar{u})$, see (22).) (That is, if $J^m(\bar{u}) \equiv g^m(x(\bar{u})(T)) \geq -\varepsilon$.)

LEMMA 6.2. *Let $\{u^i\}$ be an infinite sequence in \mathcal{F} converging to $\bar{u} \in \mathcal{D}$ in the weak* topology. Then, there exists an $\hat{\varepsilon} > 0$ and an integer i_0 such that*

$$\bar{\varepsilon}(u^i) \geq \hat{\varepsilon} > 0$$

for all $i \geq i_0$.

PROOF. Following a similar approach as that given for Lemma 5.2 of [13], we get for each $u \in \mathcal{F} \setminus \mathcal{D}$,

$$\bar{\varepsilon}(u) \geq \varepsilon(u), \quad (26)$$

where

$$\varepsilon(u) \equiv \max\{\varepsilon : \psi_{I_{\varepsilon}(u)}(u) \leq -\varepsilon, \varepsilon = \beta^k \varepsilon', k = 0, 1, 2, \dots\}. \quad (27)$$

It remains to show that there exists an $\hat{\varepsilon} > 0$ and an integer i_0 such that

$$\varepsilon(u^i) \geq \hat{\varepsilon} > 0 \quad (28)$$

for all $i \geq i_0$. For then by replacing u by u^i in (26), the conclusion of the lemma follows easily from (28) and (26).

To begin, we note that $\bar{u} \notin \mathcal{D}$. Thus, there exists a $\delta = \beta^{k_1} \varepsilon'$ such that $\psi_{I_0(\bar{u})}(\bar{u}) < -\delta$. Hence, there exists an $\varepsilon_1 = \beta^{k_2} \varepsilon'$ such that $\psi_{I_{\varepsilon_1}(\bar{u})}(\bar{u}) < -\delta$ for all $\varepsilon \in [0, \varepsilon_1]$. Let $\varepsilon_2 = \min[\varepsilon_1, \delta]$. Then, it is clear that

$$\psi_{I_{\varepsilon_2}(\bar{u})}(\bar{u}) \leq \psi_{I_{\varepsilon_1}(\bar{u})}(\bar{u}) < -\delta \leq -\varepsilon_2,$$

and hence

$$\varepsilon(\bar{u}) \geq \varepsilon_2 > 0. \quad (29)$$

From Lemma 3.1 and (A3), it follows that there exists a positive integer i_1 such that for all $i \geq i_1$, $x(u^i)(T)$ is sufficiently close to $x(\bar{u})(T)$ for no additional constraints to become $\varepsilon(\bar{u})$ -active. On this basis, we can deduce that, for all $i \geq i_1$,

$$I_{\varepsilon(\bar{u})}(u^i) \subset I_{\varepsilon(\bar{u})}(\bar{u}) \equiv \bar{I}.$$

Thus, for all $i \geq i_1$,

$$\psi_{I_{\varepsilon(\bar{u})}(u^i)}(u^i) \leq \psi_{\bar{I}}(u^i).$$

Since $\psi_{\bar{I}}(\cdot)$ is upper semicontinuous, there exists integer $i_0 \geq i_1$ such that

$$\psi_{I_{\varepsilon(\bar{u})}(u^i)}(u^i) \leq \psi_{\bar{I}}(u^i) \leq \beta \psi_{\bar{I}}(\bar{u}) \leq -\beta \varepsilon(\bar{u})$$

for all $i \geq i_0$. Therefore

$$\psi_{I_{\beta \varepsilon(\bar{u})}(u^i)}(u^i) \leq \psi_{I_{\varepsilon(\bar{u})}(u^i)}(u^i) \leq -\beta \varepsilon(\bar{u}), \quad (30)$$

for all $i \geq i_0$. Thus, from (27), (30) and (29), we get

$$\varepsilon(u^i) \geq \beta \varepsilon(\bar{u}) \geq \beta \varepsilon_2 \equiv \hat{\varepsilon} > 0,$$

for all $i \geq i_0$. Hence (28) holds and the proof of the lemma is complete.

LEMMA 6.3. *Let $\{u^i\}$ be an infinite sequence of feasible controls converging to $\bar{u} \notin \mathcal{D}$ in the weak* topology. Then, there exists a $\delta > 0$ and positive integers \hat{k} and i_0 such that for all $i \geq i_0$,*

$$\begin{aligned} k(u^i) &\leq \hat{k}, \\ J^0(\bar{u}^i) - J^0(u^i) &\leq -\delta, \\ J^m(\bar{u}^i) &\leq 0, \quad m = 1, 2, \dots, l, \end{aligned}$$

where

$$\bar{u}^i = u^i + \beta^{k(u^i)} w_{\epsilon(u^i)}^{u^i}.$$

PROOF. The proof is similar to that given for Proposition 3 of [7], except that Proposition 2 and ‘the uniform continuity of $\nabla g^0(\cdot)$ on any $L_\infty^m[0, 1]$ bounded set’ in the proof of Proposition 3 of [7] are being replaced, respectively, by Lemma 6.2 and Lemma 3.2 (ii) in the proof of the present lemma.

COROLLARY 6.1. *For any $u^i \in \mathcal{F} \setminus \mathcal{D}$, the computation of $\bar{\epsilon}(u^i)$ and $k(u^i)$ in Algorithm 6.1 requires only a finite number of iterations.*

THEOREM 6.1. *If $u^* \in \mathcal{F}$ is an optimal control of the problem P , then $u^* \in \mathcal{D}$.*

PROOF. The proof is similar to that given for Corollary 5.2 of [13], except that Lemma 5.2 of [13] is replaced by Lemma 6.3 in the proof of the present theorem.

THEOREM 6.2. *Let $\{u^i\}$ be a sequence of feasible controls generated by Algorithm 6.1. Then, either the sequence is finite, in which case the algorithm jams in the inner loops of Step 2 and Step 3, and the last element of the sequence is in \mathcal{D} , or the sequence is infinite and possesses at least one weak* accumulation point, where each of these weak* accumulation points is also in \mathcal{D} .*

PROOF. From Corollary 6.1, we note that the algorithm can jam only if the last u^i is in \mathcal{D} . Therefore, it remains to prove the second part of the theorem. Suppose that $\{u^i\}$ is infinite. Since \mathcal{F} is sequentially compact in the weak* topology (Remark 4.1), we conclude that the sequence possesses at least one weak* accumulation point in \mathcal{F} . Now, let $u^* \in \mathcal{F}$ be a w^* -accumulation point of $\{u^i\}$. Then, there exists a subsequence $\{u^{i(k)}\}$ of the sequence $\{u^i\}$ such that $u^{i(k)} \xrightarrow{w^*} u^*$ as $k \rightarrow \infty$. Since $\{J(u^i)\}$ is monotonically decreasing, it follows that

$$J^0(u^{i(k+1)}) - J^0(u^{i(k)}) \leq J^0(u^{i(k)+1}) - J^0(u^{i(k)}). \tag{31}$$

Suppose that u^* is not in \mathcal{D} . Then for sufficiently large k , we obtain from Lemma 6.3 and (31) that

$$J^0(u^{i(k+1)}) - J^0(u^{i(k)}) \leq -\delta, \tag{32}$$

where $\delta > 0$ is a constant independent of integers $i(k)$, $k = 1, 2, \dots$. However, $J^0(u^{i(k)})$ is a bounded and monotonically decreasing sequence of real numbers, and hence is convergent. Thus, by taking limit in (32), we obtain $0 \leq -\delta$. This is a contradiction and hence the proof is complete.

REMARK 6.1. From Remark 5.1 of [13], we recall that the convergence result of Theorem 6.2 is valid *only* if the initial control of Algorithm 6.1 is a feasible control.

REMARK 6.2. Since the problem considered in this paper is a special case of that considered in [10], an initial feasible control can be easily found by using the approach reported in Section 6 of [10]. More precisely, by using the control parametrization technique, this problem of obtaining an initial feasible control is reduced to the one amenable to the application of the algorithm in page 210 of [5].

7. An illustrative example

Consider the problem of minimizing

$$J(u) = \frac{3}{2} [x(u)(3)]^2 + \frac{1}{2} \int_0^3 [u(t)]^2 dt$$

subject to the delay-differential equation

$$\dot{x}(t) = x(t-1) + u(t), \quad t \in (0, 3],$$

with the initial condition

$$x(t) = 1, \quad t \in [-1, 0],$$

and the terminal inequality constraints

$$g^1(x(u)(3)) \equiv [x(u)(3)]^2 - 0.03 \leq 0, \quad (33a)$$

$$g^2(x(u)(3)) = 0.06 + [x(u)(3)]^2 \leq 0, \quad (33b)$$

where u is the control function with values in $[-3, 3]$.

Let the above problem be denoted by Q . This problem is adapted from that considered in Example 4.1 of [2], page 189. More precisely, there are no control constraints and terminal inequality constraints in the original problem. Let the original problem be denoted by \bar{Q} . From [2], we know that the optimal control of the unconstrained problem \bar{Q} is

$$u^*(t) = \begin{cases} \delta(-(t-2)^2/2 - 3/2), & t \in [0, 1] \\ \delta(t-3), & t \in [1, 2] \\ -\delta, & t \in [2, 3], \end{cases}$$

where δ is given (approximately) by

$$\delta = 0.56231.$$

The optimal cost is approximately 1.7338.

Note that u^* satisfies the control constraints and that both of the terminal inequality constraints are also satisfied when $u = u^*$. Thus, u^* is also the optimal control of the problem Q .

The problem Q is to be solved in two stages as described below. For both stages, the combination of the fourth order Runge-Kutta integration scheme together with the method of steps is used to integrate the system forward in time and the adjoint system backward in time, both over the interval $[0, 3]$, which is divided into 300 netpoints.

STAGE 1: To find a feasible control.

Let the interval $[0, 3]$ be divided into 5 subintervals such that the length of each interval is equal to 0.6. Using the initial control equal to zero throughout each of the subintervals (note that this control is *not* a feasible control because $g^2(x(0)(3)) = 37.9678 > 0$), a feasible control u^0 was found by using the procedure mentioned in Remark 6.2. It takes 70 iterations of the algorithm of [5], where the parameters a and b are chosen to be 0.01 and 0.5, respectively. This feasible control u^0 is

$$u^0(t) = \begin{cases} -1.6886, & t \in [0, 0.6) \\ -1.1728, & t \in [0.6, 1.2) \\ -0.8307, & t \in [1.2, 1.8) \\ -0.5723, & t \in [1.8, 2.4) \\ -0.5538, & t \in [2.4, 3.0). \end{cases} \quad (34)$$

STAGE 2: To find an optimal control.

Using u^0 as the initial feasible control and the parameters $(\alpha, \beta, \epsilon) = (0.2, 0.8, 0.01)$ (respectively $(\alpha, \beta, \epsilon) = (0.2, 0.8, 0.005)$), the 'optimal' cost obtained by the combination of Algorithm 5.1 and Algorithm 6.1 after 70 (respectively 100) iterations is 1.7339 (respectively 1.7341). These values are extremely close to the true optimal cost 1.7338.

For the case $\epsilon = 0.01$, the constraint (33b) was ϵ -active on 57 occasions throughout the 70 iterations. For the case $\epsilon = 0.005$, the constraint (33b) was ϵ -active on 27 occasions throughout the 100 iterations. Numerical results are listed in Tables 7.1 and 7.2. Graphs of $u^{70}(t)$ with $(\alpha, \beta, \epsilon) = (0.2, 0.8, 0.01)$ and the true optimal control are plotted in Figure 1 and Figure 2, respectively.

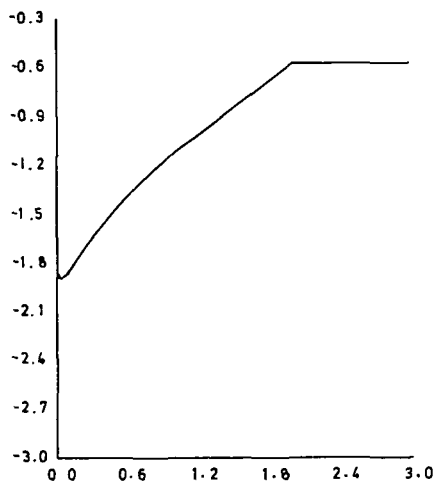


Figure 1. $u^{70}(t)$ $\alpha = 0.2, \beta = 0.8, \epsilon = 0.01$

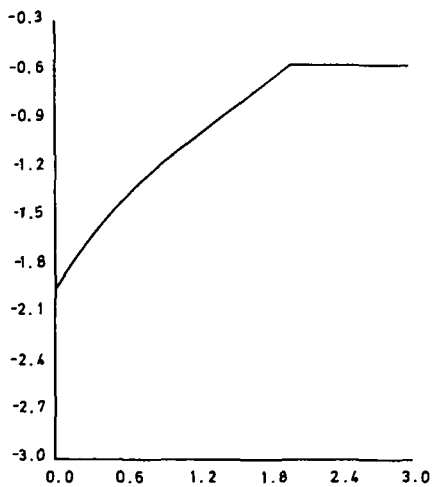


Figure 2. The true optimal control.

TABLE 7.1 Numerical Results
 $(\alpha = 0.2, \quad \beta = 0.8, \quad \epsilon = 0.01)$
 Initial Control is defined by (34)

i	$J(u')$
10	1.7404
20	1.7376
30	1.7350
40	1.7342
50	1.7341
60	1.7339
70	1.7339

TABLE 7.2 Numerical Results
 $(\alpha = 0.2, \quad \beta = 0.08 \quad \epsilon = 0.005)$
 Initial Control is defined by (34)

i	$J(u')$
10	1.7406
20	1.7380
30	1.7354
40	1.7348
50	1.7374
60	1.7346
70	1.7345
80	1.7342
90	1.7341
100	1.7341

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