An Optimal Lognormal Approximation to Lognormal Sum Distributions

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***Abstract*—Sums of lognormal random variables occur in many problems in wireless communications because signal shadowing is well modeled by the lognormal distribution. The lognormal sum distribution is not known in the closed form and is difficult to compute numerically. Several approximations to the distribution have been proposed and employed in applications. Some widely used approximations are based on the assumption that a lognormal sum is well approximated by a lognormal random variable. Here, a new paradigm for approximating lognormal sum distributions is presented. A linearizing transform is used with a linear minimax approximation to determine an optimal lognormal approximation to a lognormal sum distribution. The accuracies of the new method are quantitatively compared to the accuracies of some well-known approximations. In some practical cases, the optimal lognormal approximation is several orders of magnitude more accurate than previous approximations. Efficient numerical computation of the lognormal characteristic function is also considered.**

***Index Terms*—Approximation methods, cochannel interference, distribution functions, Fourier transforms, lognormal distributions.**

I. Introduction

A SUM of lognormal random variables (RVs) occurs in many important communications problems. Two examples are the analysis of cochannel interference in cellular mobile systems and the computation of outage probabilities. The lognormal distribution is used to model the attenuation caused by signal shadowing in both of these cases. Determining the probability distribution of a sum of lognormal RVs is a longstanding problem in wireless communications [1]–[12]. Several approximate solutions to the probability distribution of a sum of independent lognormal RVs have been reported, including Wilkinson's [1], Schwartz-Yeh's [1], and Farley's [1] methods. Some past works have focused on comparing the strengths and weaknesses of these methods [1]–[5]. However, none are clearly better than the others and well accepted. The asymptotic character of lognormal sum distributions was given in [6] and further clarified in [3]. Recently, upper and lower bounds to the sum distributions were provided in [7].

The well-known approach for obtaining the probability distribution of a sum of independent RVs uses the characteristic function (CF). This approach is totally general because the probability density function (pdf) of a sum of independent RVs has a CF equal to the product of the CFs of the summands [13]. However, the CF of a lognormal RV is not known. Moreover, numerical integration is difficult due to the oscillatory integrand and the slow decay rate of the tail of the lognormal density function [3], [14].

In this paper, a new paradigm for constructing approximations to lognormal sum distributions is presented. We find an optimal lognormal approximation to a lognormal sum distribution by using a transformation that linearizes a lognormal distribution and then deriving the minimax linear approximation in the transformed domain. Applying the minimax error criterion in the transformed domain effectively weights the relative error of the approximation in the tails of the distribution. This minimax approximation is, thus, optimal in this sense, although other measures of optimality are possible. In the course of developing the optimal minimax approximation, we examine the goodness of the assumption that a lognormal sum distribution can be approximated by a lognormal distribution. In support of these investigations, we examine some numerical integration methods to select a method for computing the lognormal CF numerically. The accuracies of some well-known lognormal sum distribution approximations are tested over a wide useful range of probabilities, wider than those undertaken in previous comparisons.

II. Sums of Lognormal RVs

A lognormal RV has the property that its logarithm has a normal (Gaussian) distribution. Let $X = \ln Y$. If the RV X is normally distributed and its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-m_X)^2}{2\sigma_X^2}\right] \tag{1}$$

with mean m_X and standard deviation σ_X, the RV Y is said to be lognormally distributed and the pdf of Y is given by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma_X y} \exp\left[-\frac{(\ln y - m_X)^2}{2\sigma_X^2}\right], & y > 0, \\ 0, & y \le 0. \end{cases} \tag{2}$$

In wireless engineering applications, it is more convenient to express the signal power in decibel units. Define the Gaussian RV $V = 10\log_{10} Y$ with mean m_V and standard deviation σ_V, both of which are in decibel units. In a mobile radio environment, the parameter σ_V is sometimes called the dB spread, having typical values between 6 and 12 dB for practical channels depending on

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the severity of shadowing [2], [8]. The RVs X and V are related as

$$X = \lambda V \tag{3a}$$
$$m_X = \lambda m_V \tag{3b}$$
$$\sigma_X = \lambda \sigma_V \tag{3c}$$

where $\lambda = \ln(10)/10 = 0.230\,26$. Thus, the pdf of the RV Y can equivalently be represented by

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\lambda\sigma_V y} \exp\left[-\frac{(10\log_{10} y - m_V)^2}{2\sigma_V^2}\right], & y > 0, \\ 0, & y \le 0. \end{cases} \tag{4}$$

Observe that a lognormal RV is specified by two parameters, m and σ. If using decibel units, m and σ refer to m_V and σ_V, respectively; otherwise, m and σ refer to m_X and σ_X, respectively.

Sums of lognormal RVs occur in a number of practical problems [8]–[10]. For example, in wireless systems the total cochannel interference signal in a shadowed propagation environment is often modeled as a sum of N lognormally distributed signals. In general, the distribution of a sum of lognormal RVs is difficult to determine, which has led to the development of several approximation methods [1]–[9]. A widely used assumption is that the sum distribution is well approximated by another lognormal RV, i.e.,

$$W = \sum_{i=1}^{N} Y_i = e^{X_1} + e^{X_2} + \cdots + e^{X_N} \approx e^Z \tag{5}$$

where the RV Z possesses a normal distribution. Both Wilkinson's method and the method of Schwartz and Yeh are based on this assumption [1]. Wilkinson's method estimates the mean and standard deviation of Z by matching the first and second moments of both sides of (5). The method of Schwartz and Yeh provides exact expressions for the first two moments of a sum with two independent summands; a recursive procedure is used to estimate the first two moments for sums with more than two summands. Farley's method [1] is also widely used [3], [10], [12] to approximate lognormal sum distributions. It is not based on the assumption that the sum distribution is approximately lognormal. In [2], Farley's approximation was proved to be a strict lower bound to the complementary cumulative distribution function (cdf) of a lognormal sum distribution.

III. Characteristic Function and Numerical Evaluation

A standard method for computing the pdf of a sum of independent RVs is to use CFs. The CF of an RV X is defined as [13]

$$\Phi_X(w) = E[e^{jwX}] = \int_{-\infty}^{\infty} f_X(x) e^{jwx} dx \tag{6}$$

TABLE I
Numerical Approaches to the Computation of Lognormal CF

Method	Comments
Modified Hermite Polynomial	Can not use for practical values of σ.
Trapezoidal Rule	Accurate but low efficiency.
Simpson's Rule	Accurate but low efficiency.
Fast Fourier Transform (FFT)	· Choice of sampling frequency is complicated due to limited knowledge of the transform of the lognormal CF. · Impractical for large dB spreads ($\sigma \ge 10$).
Modified Clenshaw-Curtis	· Accurate. · High efficiency.

where $E[\cdot]$ denotes the expectation operation and $j = \sqrt{-1}$. The pdf is determined by the inverse transformation

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_X(w) e^{-jwx} dw \tag{7}$$

and the cdf $F_X(x)$ can be expressed as

$$F_X(x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi_X(w)}{jw} e^{-jwx} dw \tag{8}$$

$$= \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \left[\frac{\mathrm{Im}\{\Phi_X(w)\}\cos(wx)}{w} - \frac{\mathrm{Re}\{\Phi_X(w)\}\sin(wx)}{w} \right] dw \tag{9}$$

where $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ denote the real and imaginary parts of x, respectively. Equation (8) uses the integration property of Fourier transforms [15, 3.3.7] and (9) is obtained from (8) by recognizing that the imaginary part of the integrand in (8) must be integrated to zero because $F_X(x)$ is real. Considering $f_X(x) = 0 (x \le 0)$ for a lognormal RV, (9) is simplified to [16]

$$F_X(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\mathrm{Re}\{\Phi_X(w)\}\sin(wx)}{w} dw \tag{10}$$

$$= 1 - \frac{2}{\pi} \int_{0}^{\infty} \frac{\mathrm{Im}\{\Phi_X(w)\}\cos(wx)}{w}. \tag{11}$$

As is well known, the CF of a sum of independent RVs is the product of the CFs of the summands [13]. Eqs. (7) and (9) then represent, in principle, a way to determine the pdf and cdf, respectively, of a sum of RVs. However, the CF of a lognormal RV is not known in the closed form. Furthermore, numerical computation of the CF is difficult because of the slow rate of decay of the lognormal pdf; each of the real and imaginary parts of the integrand is oscillatory and the slowly decaying envelope of the integrand results in the addition of a large number of areas of nearly equal magnitude and alternating sign. To the best of the authors' knowledge, the only graph of the CF of a lognormal RV

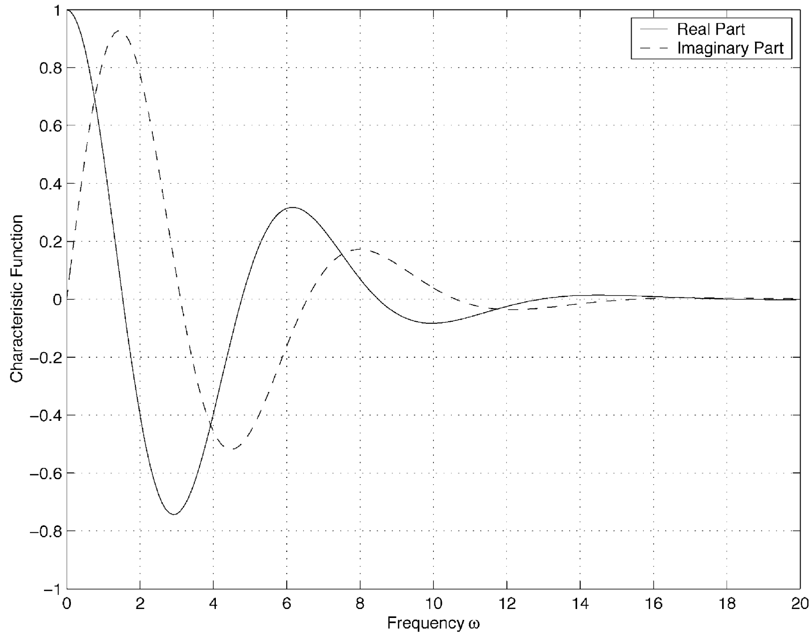

Fig. 1. Real and imaginary parts of a lognormal CF with $m = 0$ and $\sigma = 0.25$ (after [14, Fig. 1]).

in the literature was published in [14], in which a modified Hermite polynomial series was derived for the CF of a lognormal RV. Using this series, the real and imaginary parts of the lognormal CF were plotted for $\sigma = 0.25$. Note that $\sigma = 0.25$ or $\sigma^2 = 0.0625$ corresponds to 1.09-dB spread [see (3c) and the related discussion]. Practical problems in wireless communications involve lognormal RVs with values of decibels spread ranging from 6 to 12 dB. Our empirical tests indicate that the modified Hermite polynomial series in [14] can only be used for some small values of σ and small values of w. For example, for $\sigma = 1.0$ and $w = 5$, or for $\sigma = 0.25$ and $w = 30$, the series did not converge. These tests were done using MATLAB on a Linux system running on an IBM IntelliStation M Pro with Intel Xeon processor.

In order to assess the accuracies of previous approximations and our new minimax approximation over a wide range of probabilities, we investigated the numerical computation of the lognormal CF [17]. The goal was to find a numerical integration method that gives reliable results over a wide range of sum probabilities, a wider range than can be tested using computer simulation. Table I summarizes the methods and our empirical assessment of the suitability of each for computing the CF for values of σ and w of interest. In our tests, the modified Clenshaw–Curtis method proved to be the best algorithm to use. It gave accurate results and was much more efficient than the well-known Trapezoidal and Simpson's rules. (Note that efficiency becomes increasingly important as w increases.) The fast Fourier transform (FFT) is efficient; however, it is not possible to bound the aliasing error, as the CF is not known *a priori*, complicating the determination of an appropriate sampling frequency. Moreover, we found empirically that the FFT method was impractical for large values of dB spread ($\sigma > 10$ dB) because very large sampling frequencies and correspondingly excessively large memory storage in a computer system were needed to achieve reasonable accuracies in the desired CFs. The modified Clenshaw–Curtis method, based on the algorithm in [18], is the most efficient among these methods for computing the CF. While Table I summarizes the usefulness of the different methods for computing a lognormal CF, more needs to be said. Once having computed the CF, we must form a product of appropriate CFs to obtain the CF of a lognormal sum and then numerically integrate the sum CF to obtain the pdf or cdf. This corresponds to executing a double numerical integration, which requires the square of the numerical cost of just computing the CF. At this point, the efficiencies of the methods becomes pivotal. As a test of both the efficiency and the accuracy of a method, we executed this procedure for a single lognormal RV. That is, we compute the CF numerically and then invert this CF numerically to obtain the original pdf or cdf. We were not able to do this using either the Trapezoidal or Simpson's rule due to the excessive computation times needed. The modified Clenshaw–Curtis method functioned well, requiring seconds to perform this double transform and return to the original function. In the remainder of this paper, the modified Clenshaw–Curtis method is used for the numerical computation of lognormal CFs and lognormal sum cdfs.

Fig. 1 shows the real and imaginary components of the CF of a lognormal RV with $m = 0$ and $\sigma = 0.25$ corresponding to 1.09-dB spread. This figure replicates [14, Fig. 1] and the results are numerically in agreement. As stated previously, practical values of dB spread for wireless systems range from 6 to 12 dB. Figs. 2 and 3 show the real and imaginary parts of the lognormal CF for $\sigma = 6\ \mathrm{dB}$ and $\sigma = 12\ \mathrm{dB}$, respectively, both with $m = 0$. The curves are similar in shape for $\sigma = 6\ \mathrm{dB}$ and $\sigma = 12\ \mathrm{dB}$, but do not resemble the curves for the small value of σ illustrated in Fig. 1.

In all our examples, we set $m = 0$: There is no loss in generality in doing so. Consider $Y = e^X$ for $m \neq 0$. One has $Y = e^{X-m+m} = e^m e^{X-m}$ and we see that a lognormal RV

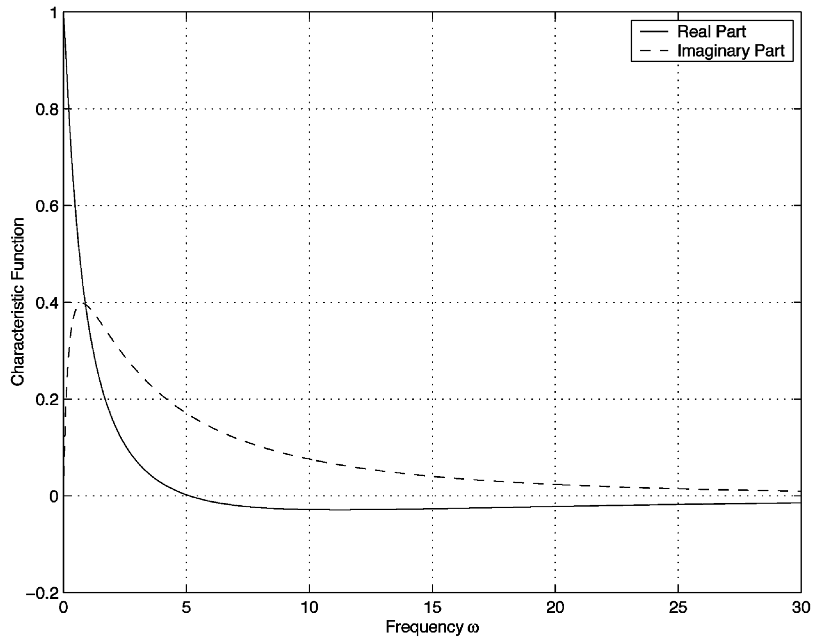

Fig. 2. Real and imaginary parts of a lognormal CF with $m = 0$ and $\sigma = 6$ dB.

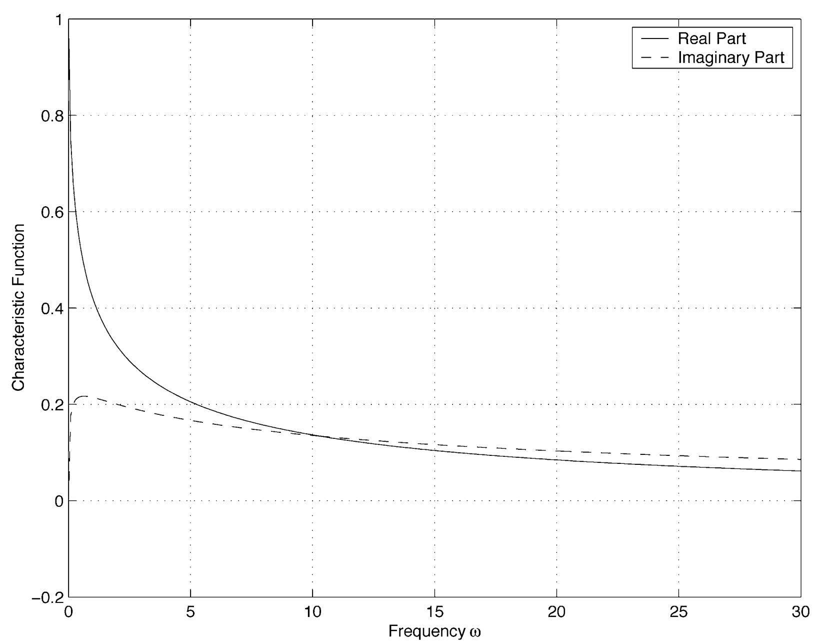

Fig. 3. Real and imaginary parts of a lognormal CF with $m = 0$ and $\sigma = 12$ dB.

with $m \neq 0$ is identical to a lognormal RV with $m = 0$ scaled by e^m. Similarly,

$$\begin{aligned}\Phi_Y(w) &= E[e^{jwY}] = E\left[\exp\{jwe^{X-m+m}\}\right] \\ &= E\left[\exp\left\{j(we^m)e^{X-m}\right\}\right]\end{aligned}$$

and the CF of a lognormal RV with $m \neq 0$ is obtained by scaling the frequencies of the CF of the corresponding lognormal RV having $m = 0$ by e^m. Further, note that while m is equivalently a scale factor, the parameter σ defines different distributions. Thus, the lognormal distribution with parameters m and σ is essentially a family of distributions characterized by a single parameter σ. Each value of σ represents a different distribution, but a different value of m represents a scaling of any other distribution having the same value of σ. This is in contrast to the corresponding Gaussian distributions with parameters m and σ. The parameter m translates the Gaussian distribution while the parameter σ corresponds to a scaling. Hence, the Gaussian distributions with parameters m and σ are not a family of different distributions in this sense. This fact is relevant to the problem of approximating the "lognormal distribution." If an approximation is to be accurate for arbitrary values of σ, one is actually approximating an infinite number of distributions that all belong to the "lognormal distribution family."

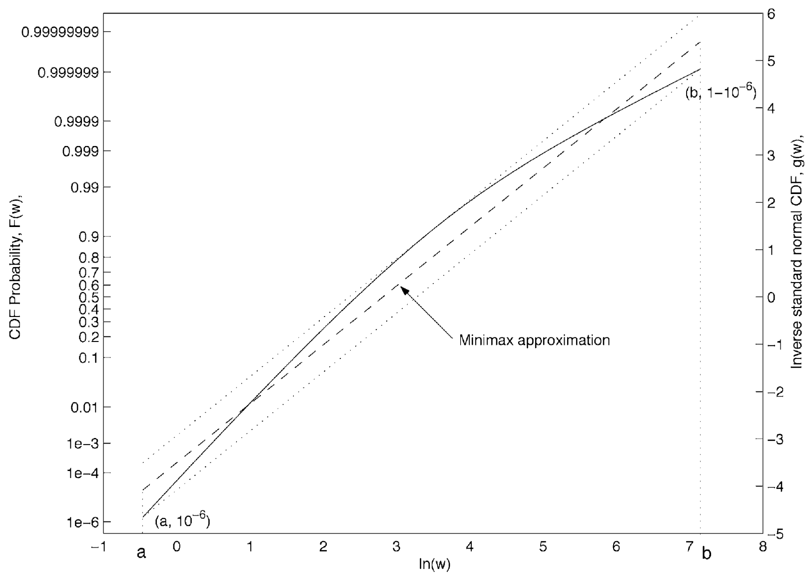

Fig. 4. Minimax approximation to a lognormal sum plotted on lognormal probability paper.

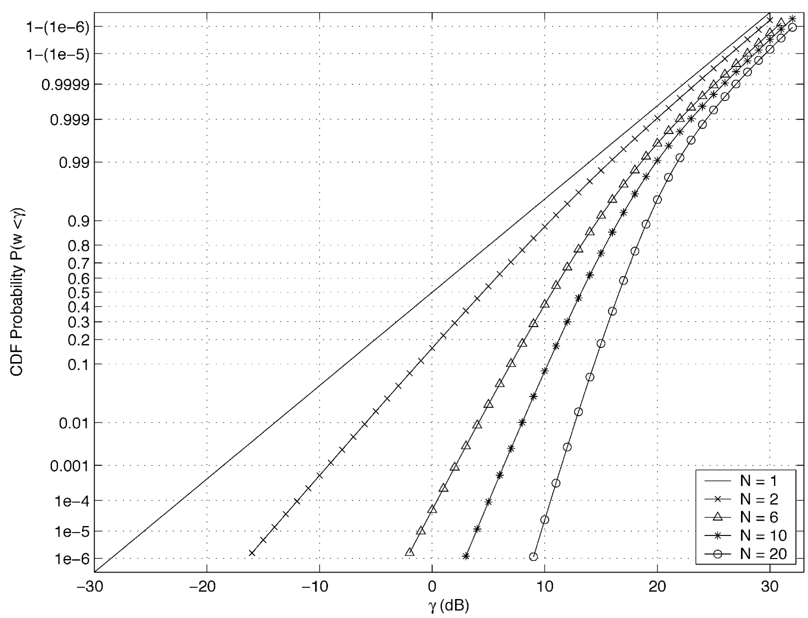

Fig. 5. The cdf of a sum of N i.i.d. lognormal RVs with $m = 0$ dB and $\sigma = 6$ dB.

Having the CF of a lognormal RV at hand, by computing (6), the cdf of a sum of independent lognormal RVs is computed by numerically integrating (10) or (11) with

$$\Phi_W(w) = \prod_{i=1}^{N} \Phi_{Y_i}(w).$$

The modified Clenshaw–Curtis method [18] is again used. Note that

$$\begin{aligned}\Phi_W(w) &= \prod_{i=1}^{N} \left\{ \hat{\Phi}_{Y_i}(w) + \varepsilon_i \right\} \\ &= \prod_{i=1}^{N} \hat{\Phi}_{Y_i}(w) + \sum_{j=1}^{N} \varepsilon_j \prod_{\substack{i=1 \\ i\neq j}}^{N} \hat{\Phi}_{Y_i}(w) \\ &\quad + \text{higher order terms in } \{\varepsilon_1, \cdots, \varepsilon_N\}\end{aligned}$$

where $\hat{\Phi}(w)$ denotes the numerical approximation to $\Phi(w)$, Δ denotes the error in $\hat{\Phi}_W(w)$, and ε_i denotes the error in $\hat{\Phi}_{Y_i}(w)$. One has

$$\hat{\Phi}_W(w) = \prod_{i=1}^{N} \hat{\Phi}_{Y_i}(w)$$

and

$$\begin{aligned}\Delta &= \Phi_W(w) - \hat{\Phi}_W(w) \\ &= \sum_{j=1}^{N} \varepsilon_j \prod_{\substack{i=1 \\ i\neq j}}^{N} \hat{\Phi}_{Y_i}(w) \\ &\quad + \text{higher order terms in } \{\varepsilon_1, \cdots, \varepsilon_N\}. \qquad (12)\end{aligned}$$

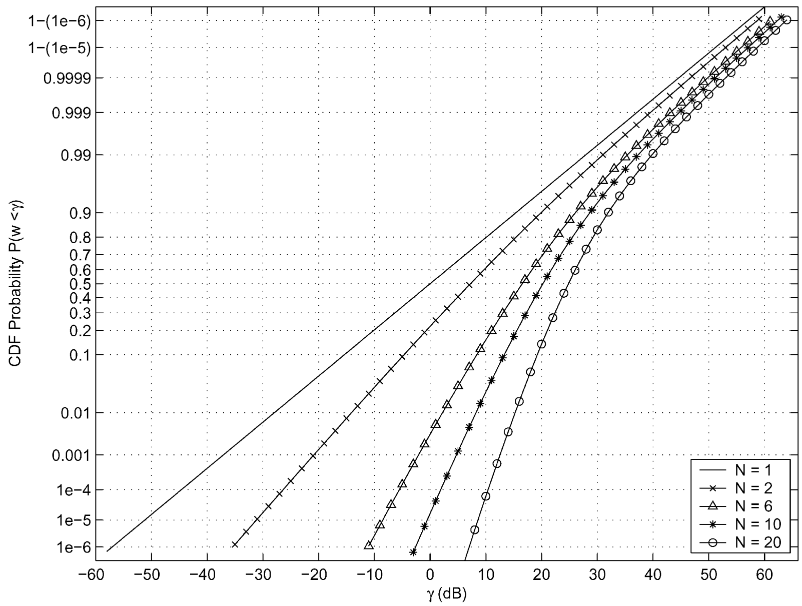

Fig. 6. The cdf of a sum of N i.i.d. lognormal RVs with $m = 0$ dB and $\sigma = 12$ dB.

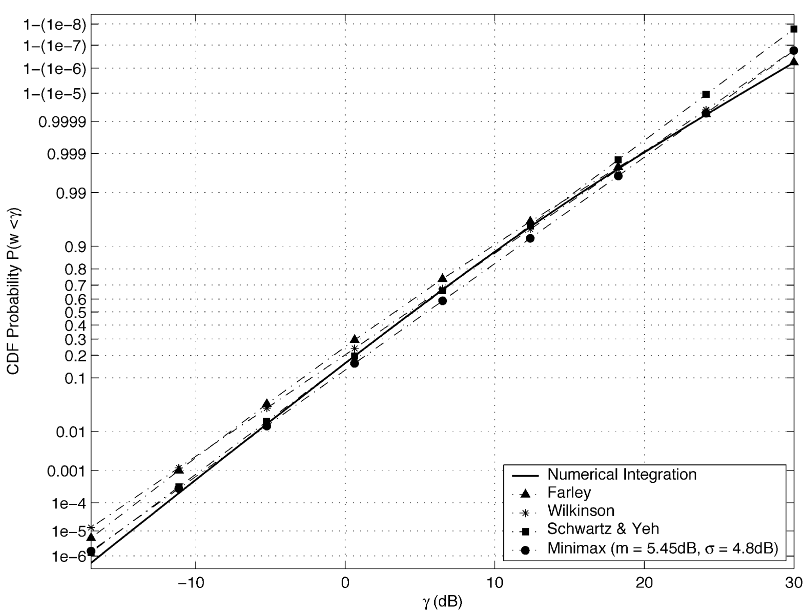

Fig. 7. The cdf of a sum of two i.i.d. lognormal RVs with $m = 0$ dB and $\sigma = 6$ dB.

Using the triangle inequality gives

$$|\Delta| \leq \left| \sum_{j=1}^{N} \varepsilon_j \prod_{\substack{i=1 \\ i\neq j}}^{N} \hat{\Phi}_{Y_i}(w) \right| + |\text{higher order terms in } \{\varepsilon_1, \cdots, \varepsilon_N\}| . \tag{13}$$

Now, $|\Phi(w)| \leq 1$ for any CF $\Phi(w)$ [13]. Thus, $|\hat{\Phi}_{Y_i}(w)|$, if accurately computed, will be less than 1, or very nearly so. Then, under the assumption that all the numerical approximations are computed to a good level of accuracy, one has

$$|\Delta| \leq \sum_{i=1}^{N} |\varepsilon_i| + |\text{higher -order terms in } \{\varepsilon_1, \cdots, \varepsilon_N\}| . \tag{14}$$

Equation (14) indicates that decreasing the error tolerance in the numerical computation of each summand CF in proportion to the number of summands N should ensure that a given desired error tolerance in $\Phi_W(w)$ is achieved, when the error tolerances are small.

In this paper, all approximations are tested for values of the cdf in the range of probabilities from 10^{-6} to $(1-10^{-6})$. Note that accuracy in both tails of the cdf are of interest; small outage probabilities correspond to small values of the cdf while small values of the complementary cdf $(1 - \text{cdf})$ are of interest in cochannel interference problems. This range is greater than the ranges examined in past work [1], [3], [7]. Our range is intended to represent probabilities of interest in practical system design for present wireless systems and for

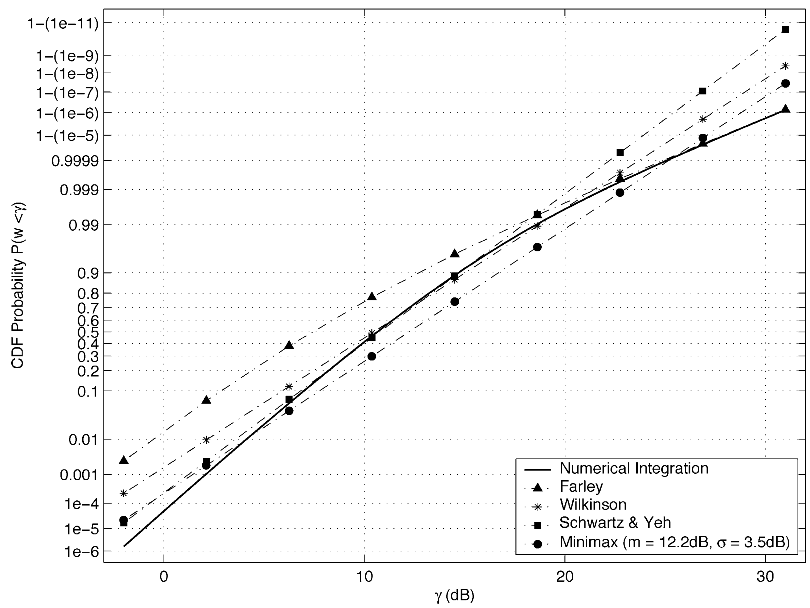

Fig. 8. The cdf of a sum of six i.i.d. lognormal RVs with $m = 0$ dB and $\sigma = 6$ dB.

future wireless systems in which performance demands are becoming increasingly stringent. As a device to readily see how much an approximation deviates from a lognormal distribution, we use "lognormal probability scales" for our graphs. Recall that a Gaussian cdf plots as a straight line on "probability paper" [19]. Using "probability paper" but transforming the abscissa into log(abscissa) yields graph scales on which a lognormal cdf plots as a straight line. Deviations from "lognormality" then are easily seen and appreciated. An explanation of how to design lognormal probability paper follows.

On lognormal probability paper, the cdf $F(x)$ of a distribution is transformed according to

$$g(x) = F_N^{-1}[F(x)] \tag{15}$$

where $F_N^{-1}(x)$ is the inverse function of the standard normal cdf $F_N(x)$ having zero mean and unit variance. Note that if $F(x)$ is the zero mean, unit variance normal cdf, then $g(x) = x$. Also, if $F(x)$ is a normal cdf with mean m and variance σ^2, then $F(x) = F_N((x-m)/\sigma)$ and, in this case, $g(x) = (x/\sigma) - (m/\sigma)$. Similarly, if $F(x)$ is a lognormal cdf with the parameters m and σ, then $F(x) = F_N((\ln x - m)/\sigma)$ and, in this case

$$g(x) = \frac{\ln x}{\sigma} - \frac{m}{\sigma} \tag{16}$$

in which $g(x)$ is a linear function of $\ln x$. The cdf of a lognormal sum (determined numerically) is also transformed using (15), but $g(x)$ is no longer a linear function of $\ln x$. We plot the data pairs $\{\ln x, g(x)\}$ on a two-dimensional (2-D) coordinate system and label the corresponding probability values on the vertical axis using the one-to-one transformation $F(x) = F_N[g(x)]$. Such a lognormal probability plot is demonstrated in Fig. 4 for a sum of six independent identically distributed (i.i.d.) lognormal RVs with $m = 0$ dB and $\sigma = 6$ dB. Note that the horizontal axis is a log-scale and that the right vertical axis is uniformly spaced in terms of values of the inverse standard normal cdf. The left vertical axis is graduated according to the corresponding probabilities.

Figs. 5 and 6 show the cdfs of lognormal sums for $\sigma = 6$ dB and $\sigma = 12$ dB, respectively, both with $m = 0$ dB. It is seen that straight lines (lognormal approximations) can only be accurate for a limited subset of the range of probabilities considered, even for $N = 2$. Furthermore, as expected, the lognormal approximation (5) becomes increasingly poor as the number of summands increases both for $\sigma = 6$ dB and $\sigma = 12$ dB. Since the lognormal approximation is a simple and useful way to approximate the lognormal sum distribution, it is clearly of interest to optimize, in some way, the lognormal approximation to minimize, in some way, the error of the approximation.

IV. Minimax Approximation

When considering a lognormal approximation to a lognormal sum distribution on the lognormal probability paper, we need to find a linear function (i.e., a straight line) to best fit the cdf curve of the sum. The minimax approximation is the best in the sense that it minimizes the maximum absolute distance between the approximate function and the true function over a specified interval. Let a true function $s(x)$ be continuous on $[a, b]$ and let $p_n(x)$ be any polynomial of degree n. The minimax polynomial approximation of degree n to $s(x)$ is determined by minimizing

$$\max_{x\in[a,b]} |s(x) - p_n(x)|. \tag{17}$$

The existence and uniqueness of such a best (minimax) approximate function is proved in [20] for any continuous function $s(t)$ defined on a finite interval $[a, b]$. In addition, the alternation theorem [21] provides a rigorous necessary and sufficient condition for determining the minimax approximation. Consider the function $E(x) = s(x) - p_n(x)$ (then $E(x)$ is the algebraic error of the approximation at the argument x) and let E_m denote the maximum value of $|E(x)|$ on $[a, b]$. A necessary and sufficient condition that a polynomial of degree n, $p_n(x)$, is the unique minimax approximation to $s(x)$ is that there are at least $n+2$

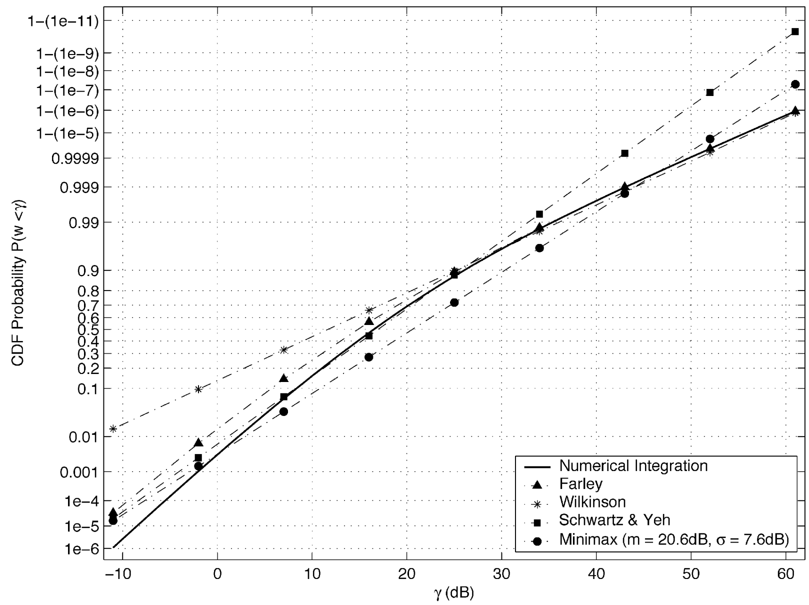

Fig. 9. The cdf of a sum of six i.i.d. lognormal RVs with $m = 0$ dB and $\sigma = 12$ dB.

points on $[a, b]$ where $E(x)$ achieves E_m with alternating signs. Further details and proof of this useful theorem are given in [21]. Returning to the problem of determining a lognormal approximation to a lognormal sum distribution, we conclude that there are at least three maximum deviations with alternating signs between the optimal lognormal approximation curve (the straight line) and the sum-distribution curve.

Now, let $t = \ln x$ and $G(t) = g(e^t)$. A linear function on the lognormal probability paper $p(t) = c_0 + c_1 t$ is to be found with constants c_0 and c_1, determined according to

$$\min_{c_0, c_1} \max_{t \in [a,b]} |G(t) - p(t)| \tag{18}$$

where $F(e^a) = 10^{-6}$ and $F(e^b) = 1 - 10^{-6}$. Inspection of the cdf curves in Figs. 5 and 6, as well as other examples in [17], implies an assumption[1] that the cdf of a lognormal sum distribution is a concave function with $G''(t) < 0$ on lognormal probability paper. Therefore, $G'(t)$ is a monotonically decreasing function. Since $E'(t) = G'(t) - p'(t) = G'(t) - c_1$, $E'(t)$ is also a monotonically decreasing function that indicates that $E(t)$ is a concave function. Considering that there exist at least three distinct points $a \le t_1 < t_0 < t_2 \le b$ that have the maximum error with alternating signs, the maximum error magnitude E_m must occur at the end points of the interval $[a, b]$. Hence, $t_1 = a$ and $t_2 = b$. The third point of the maximum error magnitude t_0 is determined by setting $E'(t_0) = G'(t_0) - c_1 = 0$, giving

$$c_1 = G'(t_0). \tag{19}$$

Equation (19) shows that a line tangent to the lognormal cdf at the point t_0 is parallel to the minimax approximation. Further, this line is also parallel to the line through the end points (a, 10^{-6}), and (b, $1 - 10^{-6}$); this property follows from the convexity of $G(t)$ and the fact that $E(a) = E(b) = -E_m$. Thus,

$$c_1 = \frac{G(b) - G(a)}{b - a}. \tag{20}$$

[1]It is difficult to theoretically prove the concavity of the cdf of a lognormal sum distribution in the transformed domain since the knowledge of the cdf and the pdf of a lognormal sum is lacking.

These relationships are illustrated in Fig. 4. The constant c_0 can be determined using the fact that the point $(t_0, [G(t_0) + G(a) + c_1(t_0 - a)]/2)$ is on the minimax line. One has

$$c_0 = \frac{1}{2}\left[G(a) + G(t_0)\right] - c_1 \frac{a + t_0}{2} \tag{21}$$

where t_0 is given by (19). Consequently, the mean m^* and the standard deviation σ^* of the optimal lognormal approximation are given by (16)

$$m^* = \frac{-c_0}{c_1} \tag{22}$$

$$\sigma^* = \frac{1}{c_1}. \tag{23}$$

An optimal lognormal approximation to a lognormal sum distribution was presented in this section. The approximation is optimal in the sense that it minimizes the maximum absolute error in the transformed domain. Minimizing the maximum error in the transformed domain gives significant weighting to the relative error of the approximation in the tails. Note that even large relative errors in the tails are negligible as compared with small or moderate relative errors near the mode of the distribution in the original (untransformed) domain, although it is the tails of the distribution that are of interest in outage and interference problems. In the proposed method, the approximation preserves the lognormal form. This is highly desirable due to its simplicity and analytical tractability. This approach is very simple and novel. Furthermore, the paradigm proposed in this section can be used to construct lognormal approximations that are accurate in any desired region using the minimax criterion. More generally, the paradigm can be generalized to use other criteria.

V. Examples and Comparisons

In the following, several examples of sums of independent lognormal RVs are considered. Sums with i.i.d. summands and sums with summands that are not i.i.d. are examined for different numbers N of summands. The accuracy of the minimax approximation is compared to the accuracies of previous approximations.

A. Sums of i.i.d. Lognormal RVs

Figs. 7–9 show the cdfs of sums of i.i.d. lognormal RVs and their approximations derived from Wilkinson's, Schwartz and Yeh's, and Farley's methods, as well as the new minimax approximation. One can see in Fig. 7 that all of the methods give relatively good approximations when $N = 2$ because the sum distribution of two variates is very close to a lognormal distribution. One can see in all of Figs. 7–9 that Schwartz and Yeh's method gives excellent approximations for sum distributions in the range [0.1, 0.9], but has significant error in the tail values of the complementary cdf. In Fig. 8 when $\sigma = 6$ dB and $N = 6$, the maximum discrepancy between Schwartz and Yeh's approximation and the distribution is almost five orders of magnitude. In general, Schwartz and Yeh's approximation severely underestimates the values of the tails of complementary cdfs [17]. Its accuracy for values of cdf tails is better than that for values of complementary cdf tails. In contrast, the simple Wilkinson's method provides better approximation to values of the complementary cdf tail than Schwartz and Yeh's method. Its accuracy improves as the value of the dB spread increases. For 12-dB spread, the complementary cdf tail values approximated by Wilkinson's method are very accurate. On the other hand, Wilkinson's estimates of the tail values of the cdfs are worse than those of Schwartz and Yeh's. The maximum deviation exceeds four orders of magnitude for the case of 12-dB spread and $N = 6$ illustrated in Fig. 9. As stated in [4], Farley's approach is an upper bound to the cdf and a strict lower bound to the complementary cdf tails, but the discrepancy becomes worse as the dB spread becomes smaller or as the number of summands N becomes greater. In Fig. 8 when $\sigma = 6$ dB and $N = 6$, the maximum deviation is about three orders of magnitude. On the other hand, Farley's approximation follows the shape of the true distribution curve for large values of γ and is quite accurate in this region. The simple lognormal form, however, is lost.

If a lognormal approximation to a lognormal sum distribution is employed, it is equivalent to a straight line on the lognormal probability paper; many lines could be drawn, tangent to or intersecting different points of the lognormal sum-distribution curve. Neither Schwartz and Yeh's nor Wilkinson's approximations are good for the range of probabilities in Figs. 7–9. The minimax approximation provides good balance over the whole range, giving more accurate approximations than Schwartz and Yeh's and Wilkinson's methods. The maximum error of the former is two to three orders of magnitude less than the maximum error of the latter two in some cases in our examples.

Table II gives the mean m^* and dB spread σ^* of the minimax approximations to lognormal sum distributions for values of $N = 2, 6,$ and 10 and for dB spreads of 6–12 dB, which will be useful for reference.

TABLE II
MINIMAX APPROXIMATION OF THE SUMS OF i.i.d. LOGNORMAL RVs ($m = 0$ dB)

dB spreads of e^{X_i}	Number N	Minimax m^* (dB)	Minimax σ^* (dB)
6	2	5.45	4.8
	6	12.2	3.5
	10	15.0	3.0
7	2	6.08	5.7
	6	13.6	4.2
	10	16.5	3.7
8	2	6.62	6.5
	6	14.9	4.9
	10	18.1	4.3
9	2	7.38	7.4
	6	16.4	5.6
	10	19.7	4.9
10	2	8.0	8.2
	6	17.7	6.2
	10	21.4	5.5
11	2	8.76	9.1
	6	19.2	6.9
	10	23.2	6.2
12	2	9.44	9.9
	6	20.6	7.6
	10	24.9	6.8

B. Sums of Lognormal RVs With Different Power Means

Fig. 10 shows the cdf of a sum of six independent summands with the same dB spread (12 dB), but different values of m (different power means). The power means are equally distributed in $[-25\text{ dB},\ 25\text{ dB}]$. We observe that methods by Wilkinson, Schwartz and Yeh, and Farley yield approximations with similar qualities to those in the i.i.d. cases. Schwartz and Yeh's method again provides good approximations to the cdf in the range [0.1, 0.9], but it has great deviations in the two tails. Wilkinson's method is better than Schwartz and Yeh's method for values of the tail of the complementary cdf. Compared with the sum distribution in Fig. 9, which has i.i.d. summands ($m = 0$ dB, $\sigma = 12$ dB), the sum distribution in Fig. 10 is closer to a lognormal distribution. Thus, the approximations of previous methods are better than those for sums with i.i.d. summands. For example, the maximum deviation of Schwartz and Yeh's method decreases from four orders of magnitude in Fig. 9 to two orders of magnitude in Fig. 10; this occurs because changing the power mean of a lognormal RV is equivalent to scaling the RV. When the power means (scalings) are substantially different, the largest of the scaled RVs dominates the sum and the CF tends toward the CF of the single, dominant RV, which is a lognormal RV.

C. Sums of Lognormal RVs With Different dB Spreads

Fig. 11 shows the cdf of a sum that has summands with the same power mean ($m_i = 0$), but different dB spreads. The summands have dB spreads spaced in the range [6 dB, 12 dB]. We plot the cdf of the summand with the greatest dB spread as well. It is seen that, for large values of argument, the cdf of the sum tends to the cdf of the summand with the greatest dB spread. This example clearly demonstrates this asymptotic character of the sum distribution detailed in [3] and [6]. Comparing the sum distribution with the sum distribution for the case of i.i.d. summands, the sum distribution deviates from a lognormal distribution more greatly. Thus, the approximations given by Wilkinson's and Schwartz and Yeh's methods degrade. An example is the following. In Fig. 8 for the sum of six i.i.d. summands with 6-dB spread, the maximum error given by Schwartz and Yeh's method is about five orders of magnitude, whereas

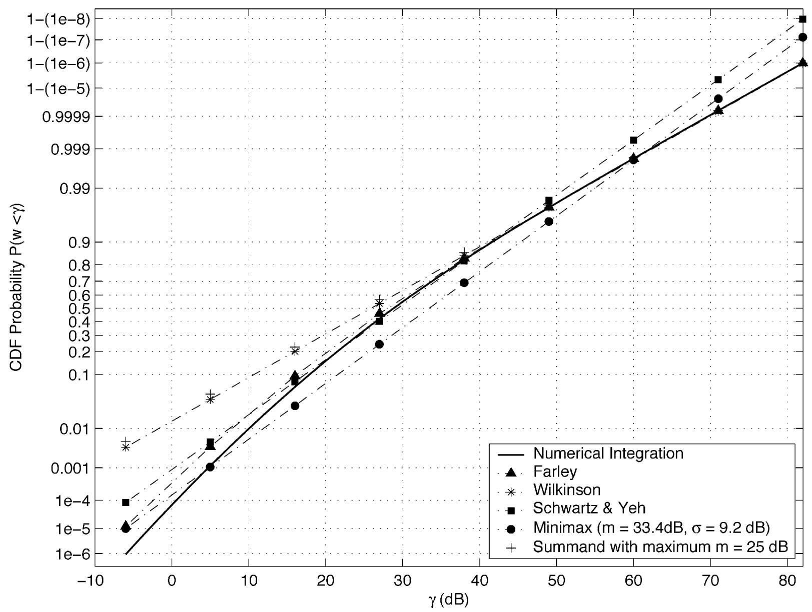

Fig. 10. The cdf of a sum of six lognormal RVs with the same dB spreads ($\sigma = 12$ dB) and different power means ($m_1 = -25$ dB, $m_2 = -15$ dB, $m_3 = -5$ dB, $m_4 = 5$ dB, $m_5 = 15$ dB, and $m_6 = 25$ dB).

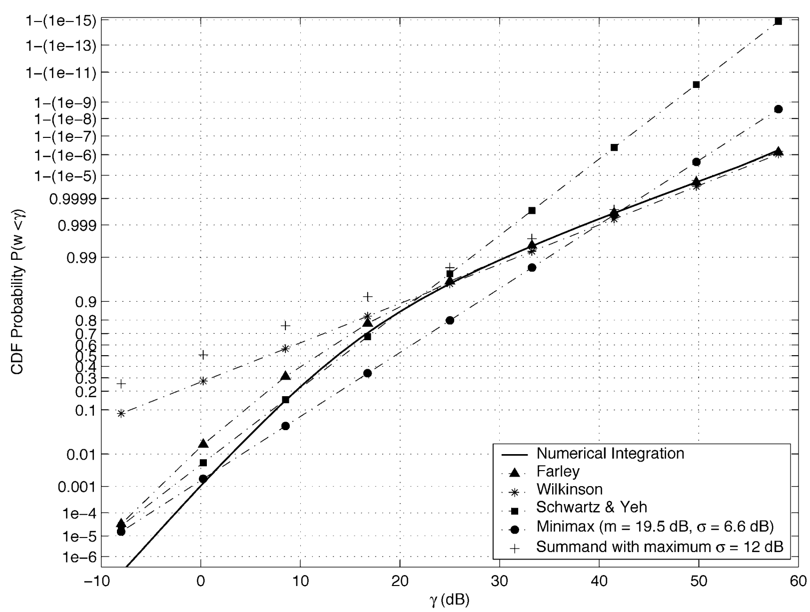

Fig. 11. CDF of a sum of six lognormal RVs with the same m values ($m = 0$ dB) and different dB spreads ($\sigma_1 = 6$ dB, $\sigma_2 = 8$ dB, $\sigma_3 = 9$ dB, $\sigma_4 = 10$ dB, $\sigma_5 = 11$ dB, and $\sigma_6 = 12$ dB).

the maximum error in Fig. 11 for summands with different dB spreads is over nine orders of magnitude. Similarly, Wilkinson's method is also poor. However, the maximum errors decrease when the differences among the dB spreads of the summands decrease, as expected.

VI. Conclusion

We have examined the goodness of the well-accepted assumption that a sum of independent lognormal RVs is also lognormally distributed. It was found that this assumption is good for sums of $N = 2$ i.i.d. summands, but is poor when the number of summands increases or the difference among the dB spreads of the summands increases. Three previous approximate approaches—methods by Schwartz and Yeh, Wilkinson, and Farley—have been compared with the results obtained by numerical computation. It was seen that none are valid over a wide range of parameters. The approximations obtained by Schwartz and Yeh's method deviate significantly in the tails of the complementary cdfs, whereas the approximations given by Wilkinson's method are not good in the tails of the cdfs, especially for large dB spreads (12 dB). Farley's approximation is, generally, good for large values of argument, but worse than other methods for small values of argument when the dB spread is small or the number of summands is large.

A new paradigm for approximating the distributions of lognormal sums was given in this paper; it is simple and, in some sense, optimal. This novel approach was developed from the linearization of the lognormal cdf curve. Then, the minimax approximation was developed to determine a best lognormal approximation to a lognormal sum distribution in the transformed distribution domain. This approach reduces the relative error in the tails of the approximating distribution. Our work shows that this approximation can be better than other methods in some applications, sometimes reducing the approximation error by several orders of magnitude.

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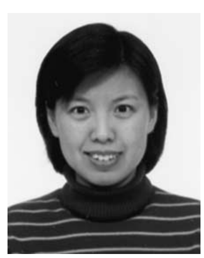

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