An Optimal-Order Error Estimate
for the Discontinuous Galerkin Method

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Abstract. In this paper a new approach is developed for analyzing the discontinuous Galerkin method for hyperbolic equations. For a model problem in R^2, the method is shown to converge at a rate $O(h^{n+1})$ when applied with nth degree polynomial approximations over a semiuniform triangulation, assuming sufficient regularity in the solution.

1. Introduction. We shall analyze the discontinuous Galerkin method in the context of a model hyperbolic equation:

$$(1) \qquad \alpha_1 \frac{\partial u}{\partial x} + \alpha_2 \frac{\partial u}{\partial y} = f(x,y), \qquad (x,y) \in \Omega \subset R^2,$$

$$u(x,y) \text{ given for } (x,y) \in \Gamma_{\text{in}}(\Omega).$$

Here $\Gamma_{\text{in}}(\Omega)$ is the "inflow" portion of the boundary Γ of Ω, defined by

$$\Gamma_{\text{in}}(\Omega) = \{(x,y) \in \Gamma \mid \alpha \cdot n < 0\},$$

where n is the unit outer normal to Ω, and α is assumed to be a constant vector, with $\alpha_1^2 + \alpha_2^2 = 1$ and $\alpha_2 > 0$.

Assuming Ω is a polygon and that it has been divided into triangles, it is always possible to order the triangles $\{T_1, T_2, \dots\}$ such that for each k the domain of dependence of T_k consists of some subset of $\Gamma_{\text{in}}(\Omega)$ and $\{T_1, \dots, T_{k-1}\}$ [4]. With such an ordering, one can develop a finite element approximation in an explicit fashion, triangle by triangle. To date, the most important application of this type of finite element approach has been in solving the neutron transport equation. See, e.g., [3], [5]. Two such finite element methods have been analyzed theoretically—the discontinuous Galerkin method [2], [4], and a continuous method [1].

Our concern here is the discontinuous method, which we describe as follows: For an arbitrary domain D, we denote by $\mathbf{P}_n(D)$ the space of polynomials of degree $\leq n$ over D. We seek an approximate solution u_h such that for each triangle T, $u_h|_T \in \mathbf{P}_n(T)$ and

$$(2) \qquad ((u_h)_\alpha, v_h) - \int_{\Gamma_{\text{in}}(T)} (u_h^+ - u_h^-) v_h \, \alpha \cdot n = (f, v_h), \quad \text{all } v_h \in \mathbf{P}_n(T),$$

where $(\ ,\ )$ denotes the $L^2(T)$-inner product. Here we have used the more compact notation $(u_h)_\alpha$ for the directional derivative of u_h with respect to α, and, for a point $P \in \Gamma_{\text{in}}(T)$,

$$u_h^\pm(P) \equiv \lim_{\varepsilon \to 0+} u_h(P \pm \varepsilon \alpha).$$

The development of u_h starts from an interpolant of the given initial condition along $\Gamma_{\mathrm{in}}(\Omega)$. Note that there are two types of triangles: Those with one inflow side and those with two, hereafter referred to as type I and type II triangles. Provision in (2) for a discontinuity across the inflow, $\Gamma_{\mathrm{in}}(T)$, allows the same space of test and trial functions, $\mathbf{P}_n(T)$, to be used for a triangle of either type.

The discontinuous Galerkin method was first analyzed by Lesaint and Raviart [4], who established an L^2 convergence rate of $O(h^n)$ for an arbitrary triangulation, where h is the mesh size. Later, Johnson and Pitkäranta [2], in an analysis using variational and Fourier techniques, obtained improved error estimates in $L^p(\Omega)$, $p > 1$, including

$$(3) \qquad \|u_h - u\|_{L^2(\Omega)} \leq C h^{n+1/2} \|u\|_{H^{n+1}(\Omega)}.$$

In this paper we develop a new method of analysis, utilizing exact representations for u_h on triangle boundaries. We use our approach to obtain an error estimate of the form

$$(4) \qquad \|u_h - u\|_{L^2(\Omega)} \leq C h^{n+1} \|u\|_{H^{n+2}(\Omega)},$$

assuming some uniformity in the triangulation. The symbol C represents a generic constant, independent of u and h. The latter estimate indicates a higher (optimal) order of convergence while requiring an additional derivative. It arises because the error in the finite element solution is oscillatory, and is damped in type II triangles.

In Section 2 we derive relevant results applicable to single triangles of both types. In Section 3 we assemble our results into the estimate (4) for the case of a semiuniform triangulation of a periodic domain. In Section 4 we present several corroborating computational examples. We also note that the $O(h^{n+1})$ convergence rate appears to be quite robust and can occur for irregular triangulations. In an appendix we provide a brief analysis of a modified version of the discontinuous Galerkin method in which continuity is enforced across the inflow side of type I triangles. The resulting method is slightly less costly to apply. We obtain an $O(h^n)$ error estimate for general meshes and show that the higher-order estimate (4) remains valid under the same uniformity conditions as for the discontinuous Galerkin method.

2. Local Properties of the Approximate Solution. In this section we establish some basic properties of the approximate solution over a single triangle. To do this, we need additional notation.

For a generic triangle T we use as independent variables s (parallel to α) and t (perpendicular to α), as indicated in Figure 2.1. Triangle T is described by

$$T = \{(s,t) \mid s \in [s_{\mathrm{in}}(t), s_{\mathrm{out}}(t)], \ t \in [t_0, t_1]\},$$

and both $\Gamma_{\mathrm{in}}(T)$ and $\Gamma_{\mathrm{out}}(T)$ can be parameterized by $t \in [t_0, t_1]$. We set $h \equiv t_1 - t_0$ and assume that T satisfies a minimum angle condition independent of h. We shall use several L^2 projections: An interior projection P_n with range $\mathbf{P}_n(T)$, and boundary projections P_{in} and P_{out}. If $\Gamma_{\mathrm{in}}(T)$ consists of side(s) Γ_j, the corresponding interval $[t_0, t_1]$ consists of subinterval(s) Δt_j, and $P_{\mathrm{in}} : L^2(\Gamma_{\mathrm{in}}(T)) \to \{v(t) \mid v|_{\Delta t_j} \in \mathbf{P}_n(\Delta t_j)\}$. We define P_{out} in the same way as P_{in}. Note that the range of P_{in} (P_{out}) is continuous only for a type I (type II) triangle. We also use the notation

$$u_{\mathrm{in}}(t) = u|_{\Gamma_{\mathrm{in}}(T)}, \quad u_{h,\mathrm{in}}^{\pm}(t) = u_h^{\pm}|_{\Gamma_{\mathrm{in}}(T)}, \quad e_{\mathrm{in}}^{-}(t) = P_{\mathrm{in}}u_{\mathrm{in}}(t) - u_{h,\mathrm{in}}^{-}(t),$$
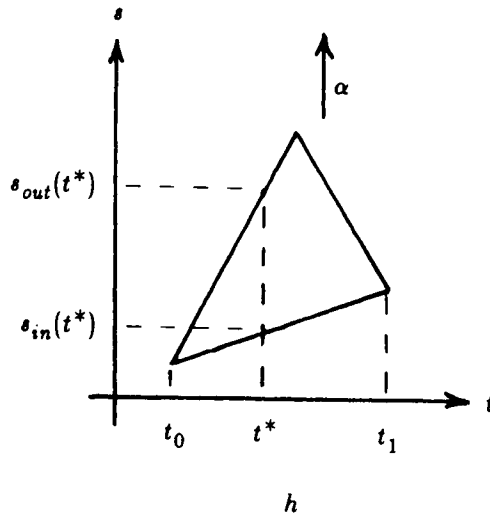
FIGURE 2.1

with an analogous set of quantities defined on $\Gamma_{\text{out}}(T)$. Finally, we denote by $\|\ \|_k$ the norm in the Sobolev space $H^k(T)$, with k omitted as a superscript when it has value zero, and by $|\ |$ an L^2-norm taken with respect to $t \in [t_0, t_1]$ over the inflow or outflow boundary of T.

In the lemmas that follow, we establish some basic properties of u_h over T.

LEMMA 2.1 (JOHNSON & PITKÄRANTA [2]). u_h *is well defined in T and has the local stability property*

$$(5) \qquad \|u_h\| + h^{1/2}|u_h^-|_{\Gamma_{\text{out}}(T)} \le C\{h^{1/2}|u_h^-|_{\Gamma_{\text{in}}(T)} + h\|f\|\}.$$

LEMMA 2.2. *For a type* II *triangle,*

$$(6) \qquad \text{(i)} \quad u_{h,\text{out}}^-(t) = P_{\text{out}}\left[u_{h,\text{in}}^-(t) + \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} f(s,t)\,ds\right],$$

$$(7) \qquad \text{(ii)} \quad e_{\text{out}}^-(t) = P_{\text{out}}e_{\text{in}}^-(t).$$

Proof. (i) We rewrite (2) as

$$(8) \quad ((u_h)_\alpha, v_h) + \int_{t_0}^{t_1} [u_{h,\text{in}}^+(t) - u_{h,\text{in}}^-(t)]v_{h,\text{in}}(t)\,dt = (f, v_h), \qquad v_h \in \mathbf{P}_n(T),$$

where $v_{h,\text{in}} = v_h|_{\Gamma_{\text{in}}(T)}$. Integration by parts yields

$$(9) \qquad -(u_h, (v_h)_\alpha) + \int_{t_0}^{t_1} [u_{h,\text{out}}^-(t)v_{h,\text{out}}(t) - u_{h,\text{in}}^-(t)v_{h,\text{in}}(t)]\,dt = (f, v_h).$$

If we take $v_h(s,t) = w(t)$, a polynomial of degree n in t, then (9) becomes

$$(10) \qquad \int_{t_0}^{t_1} [u_{h,\text{out}}^-(t) - u_{h,\text{in}}^-(t)]w(t)\,dt = \int_{t_0}^{t_1}\left(\int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} f(s,t)\,ds\right)w(t)\,dt.$$

But for a type II triangle, $w(t)$ is an arbitrary element in the range of P_{out}. Thus

$$P_{\text{out}}[u_{h,\text{out}}^-(t) - u_{h,\text{in}}^-(t)] = P_{\text{out}}\int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} f(s,t)\,ds,$$

which is equivalent to the desired result.

(ii) The exact solution to (1) satisfies

$$(11) \qquad u_{\text{out}}(t) = u_{\text{in}}(t) + \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} f(s,t)\,ds.$$

We obtain part (ii) of the lemma by applying P_{out} to (11), replacing $P_{\text{out}}u_{\text{in}}$ by $P_{\text{out}}P_{\text{in}}u_{\text{in}}$ (valid because range(P_{out}) $\subset$ range(P_{in}) for a type II triangle), then subtracting (6). $\square$

LEMMA 2.3. *For a type* I *triangle,*

$$(12) \qquad e_{\text{out}}^-(t) = e_{\text{in}}^-(t) + v(t),$$

where

$$(13) \qquad\qquad \text{(i)} \quad P_{\text{in}}v(t) = 0 \quad (\textit{thus } v(t) \perp e_{\text{in}}^-(t)),$$

$$(14) \qquad\qquad \text{(ii)} \quad |v| \le Ch^{n+1/2}\|u\|_{n+1}.$$

Proof. (i) (10) remains valid for a type I triangle, but now $w(t)$ is an arbitrary element in the range of P_{in}. Thus

$$(15) \qquad P_{\text{in}}u_{h,\text{out}}^-(t) = u_{h,\text{in}}^-(t) + P_{\text{in}}\int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} f(s,t)\,ds.$$

To (11) we apply P_{in}, then write $P_{\text{in}}u_{\text{out}} = P_{\text{in}}P_{\text{out}}u_{\text{out}}$, and subtract (15). The result is (i).

(ii) Consider the effect of replacing f in (8) by $P_{n-1}f$. [For the degenerate case $n = 0$, we define $P_{-1} \equiv 0$.] The solution $u_h^*(s,t)$ to (8) would become

$$(16) \qquad u_h^*(s,t) = u_{h,\text{in}}^-(t) + \int_{s_{\text{in}}(t)}^{s} P_{n-1}f\,ds,$$

which has no discontinuity across the inflow boundary. This can be shown by direct substitution into (8). Moreover, the stability result (5) applied to $u_h^* - u_h$ yields

$$(17) \qquad \begin{aligned} |u_{h,\text{out}}^* - u_{h,\text{out}}| &\le Ch^{1/2}\|(I - P_{n-1})f\| \\ &\le Ch^{n+1/2}\|f\|_n \le Ch^{n+1/2}\|u\|_{n+1}. \end{aligned}$$

Thus,

$$(18) \qquad u_{h,\text{out}}^-(t) = u_{h,\text{in}}^-(t) + \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} P_{n-1}f\,ds + \varepsilon, \qquad |\varepsilon| \le Ch^{n+1/2}\|u\|_{n+1}.$$

The exact solution can be written in an analogous form:

$$(19) \qquad u_{\text{out}}(t) = u_{\text{in}}(t) + \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} P_{n-1}f\,ds + \varepsilon', \qquad \varepsilon' = \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (I - P_{n-1})f\,ds.$$

To (19) we apply P_{out},

$$(20) \qquad P_{\text{out}}u_{\text{out}} = P_{\text{in}}u_{\text{in}} + (P_{\text{out}} - P_{\text{in}})u_{\text{in}} + \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} P_{n-1}f\,ds + P_{\text{out}}\varepsilon',$$

where we have used the fact that $\int_{s_{in}(t)}^{s_{out}(t)} P_{n-1}f\,ds$ is in the range of P_{out} for a type I triangle. We then bound $(P_{out} - P_{in})u_{in}$ and $P_{out}\varepsilon'$ as follows:

$$
\begin{aligned}
|P_{out}\varepsilon'| &= \max_{v_h(t)\in\text{range}(P_{out})} \frac{((I - P_{n-1})f, v_h(t))}{|v_h(t)|} \\
&\leq \max_{v_h(t)\in\text{range}(P_{out})} \frac{\|(I - P_{n-1})f\|\,\|v_h(t)\|}{|v_h(t)|} \\
&\leq Ch^{1/2}\|(I - P_{n-1})f\| \leq Ch^{n+1/2}\|u\|_{n+1},
\end{aligned}
$$

$$
\begin{aligned}
|(P_{out} - P_{in})u_{in}| &\leq |(I - P_{out})u_{in}| + |(I - P_{in})u_{in}| \\
&\leq 2|(I - P_{in})u_{in}| \quad (\text{since range}(P_{in}) \subset \text{range}(P_{out}) \\
&\hspace{6cm} \text{for a type I triangle)} \\
&\leq 2|u - u_I|_{\Gamma_{in}(T)}, \qquad u_I \equiv standard \text{ interpolant for } u \text{ in } \mathbf{P}_n(T) \\
&\leq Ch^{n+1/2}\|u\|_{n+1}.
\end{aligned}
$$

Subtracting (18) from (20) and applying the above bounds, we obtain (ii).   □

The preceding lemmas lead to an $O(h^{n+1/2})$ error estimate for u_h over an arbitrary triangulation of Ω. For, on a type I triangle, via (12)–(14),

$$
(21) \qquad |e_{out}^-|^2 = |e_{in}^-|^2 + |v|^2 \leq |e_{in}^-|^2 + Ch^{2n+1}\|u\|_{n+1}^2,
$$

and, on a type II triangle, via (7),

$$
(22) \qquad |e_{out}^-|^2 \leq |e_{in}^-|^2.
$$

Assuming that, at a certain point, u_h has been developed in $\Omega_j \subseteq \Omega$, we sum (21) and (22) over all the triangles of Ω_j to obtain

$$
(23) \qquad |e_{out}^-|_{\Gamma_{out}(\Omega_j)}^2 \leq |e_{in}^-|_{\Gamma_{in}(\Omega_j)}^2 + Ch^{2n+1}\|u\|_{n+1,\Omega_j}^2.
$$

Thus, if $u_h^-|_{\Gamma_{in}(\Omega)}$ is chosen to be a standard interpolant of the given initial data,

$$
|e_{h,out}^-|_{\Gamma_{out}(\Omega_j)} \leq Ch^{n+1/2}\|u\|_{n+1,\Omega_j},
$$

which implies

$$
|u - u_h^-|_{\Gamma_{out}(\Omega_j)} \leq Ch^{n+1/2}\|u\|_{n+1,\Omega_j}.
$$

An analogous interior estimate,

$$
(24) \qquad \|u - u_h\|_{\Omega_j} \leq Ch^{n+1/2}\|u\|_{n+1,\Omega_j},
$$

can then be obtained by a suitable application of Lemma 2.1. This result has already been obtained by Johnson and Pitkäranta [2] using variational and Fourier analysis techniques. We shall use our approach to show that u_h converges at a rate $O(h^{n+1})$ under certain conditions. The key observation is that (22) fails to exploit any potential damping of the error as it is projected across a type II triangle via (7). We proceed with the development of a mechanism which accounts for this.

Consider the situation where there are two adjoining type I triangles T and a *translate* $T_1 = \{(s + k, t + h) \mid (s, t) \in T\}$, as shown in Figure 2.2. For a function w defined on $T \cup T_1$, we define

$$
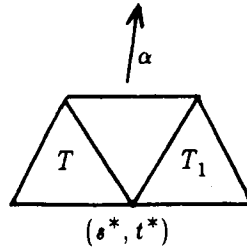\Delta w(s,t) \equiv w(s + k, t + h) - w(s, t), \qquad (s, t) \in T.
$$

FIGURE 2.2

Via superposition, we may regard Δu on T as the solution of

$$(\Delta u)_\alpha = \Delta f,$$

$$\Delta u = \Delta u_{\text{in}} \quad \text{on } \Gamma_{\text{in}}(T),$$

and Δu_h as its finite element approximation. Thus from Lemma 2.3, we infer that

$$\Delta e_{\text{out}}^-(t) = \Delta e_{\text{in}}^-(t) + \Delta v(t),$$

where

$$|\Delta v| \le Ch^{n+1/2}\|\Delta u\|_{n+1}.$$

Let us denote by (s^*, t^*) the coordinates of the common vertex of T and T_1. Then

$$\begin{aligned}
\|\Delta u\|_{n+1}^2 &= \sum_{0 \le |k| \le n+1} \|(D^k u(s+k, t+h) - D^k u(s^*, t^*)) \\
&\qquad\qquad - (D^k u(s,t) - D^k u(s^*, t^*))\|^2 \\
&\le 2 \sum_{0 \le |k| \le n+1} \{\|D^k u(s,t) - D^k u(s^*, t^*)\|_{T \cup T_1}\}^2 \\
&\le Ch^2 \sum_{1 \le |k| \le n+2} \|D^k u\|_{T \cup T_1}^2 \\
&\le Ch^2 \|u\|_{n+2, T \cup T_1}^2.
\end{aligned}$$

We summarize as follows:

LEMMA 2.4. *For a type* I *triangle T with an adjoining translate T_1 as in Figure 2.2,*

$$\Delta e_{\text{out}}^-(t) = \Delta e_{\text{in}}^-(t) + \Delta v(t),$$

where

$$|\Delta v(t)| \le Ch^{n+3/2}\|u\|_{n+2, T \cup T_1}.$$

3. Error Estimate for a Semiuniform Triangulation. We now extend the results of Section 2 to an entire triangulation, constructed as follows. We take $\Omega \equiv [0, 2\pi] \times [0, 1]$ as the domain of (1) and assume periodicity with respect to the first variable, so that only an initial condition $u(x, 0)$ is needed. We partition Ω into layers

$$(25) \qquad S_j = \{(x, y) \in [0, 2\pi] \times [y_j, y_{j+1}]\}, \qquad j = 0, \ldots, m-1,$$

where $0 = y_0 < y_1 < y_2 < \cdots < y_m = 1$. We then divide each layer into "half-layers" of congruent type I and type II triangles, as shown in Figure 3.1. The finite element solution can be thought of as developing a half-layer at a time, and in
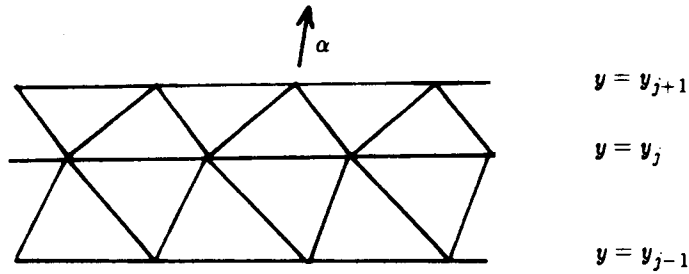
FIGURE 3.1

parallel over the triangles comprising each half-layer. Note that all the triangles in this triangulation of Ω have the same horizontal side length. We shall consider a family of such triangulations satisfying two additional conditions:

H1. The angles of all triangles of Ω are uniformly bounded away from zero.

H2. $|\alpha \cdot n|$ is uniformly bounded away from zero.

We also need some new notation. We denote by P_j the L^2-projection into (in general, discontinuous) piecewise nth degree polynomials in t with respect to the grid points along $y = y_j$, and by Q_j the analogous projection into piecewise constants at level j. We then define

$$u_j(t) = u|_{y=y_j}, \quad u_{h,j}^-(t) = u_h^-|_{y=y_j}, \quad e_j^-(t) = P_j u_j(t) - u_{h,j}^-(t),$$

and, for an arbitrary function $z_j(t)$ defined along $y = y_j$, $\Delta z_j(t) \equiv z_j(t + h) - z_j(t)$. In general, a subscript j, as in $z_j(t)$, will denote a piecewise nth degree polynomial with respect to the grid points along level y_j. Similarly, $z_{j+1/2}$ will signify a piecewise nth degree polynomial with respect to the outflow boundary of the type I portion of S_j. Finally, we denote by $|z_j|$ the L^2-norm of $z_j(t)$ over level j.

By applying Lemmas 2.2–2.4 to the jth layer we obtain

LEMMA 3.1. *There holds*

$$(26) \qquad e_{j+1}^-(t) = P_{j+1}(e_j^-(t) + v_{j+1/2}(t)),$$

where

$$(27) \qquad \text{(i)} \quad |v_{j+1/2}| \leq C h^{n+1/2} \|u\|_{n+1,S_j},$$

$$(28) \qquad \text{(ii)} \quad Q_j v_{j+1/2} = 0,$$

$$(29) \qquad \text{(iii)} \quad |\Delta v_{j+1/2}| \leq C h^{n+3/2} \|u\|_{n+2,S_j}.$$

Our strategy for establishing an $O(h^{n+1})$ convergence rate will involve showing that the influence of $v_{j+1/2}$ declines exponentially as the solution advances beyond the jth layer. At a given level, the cumulative effect of all previous $v_{j+1/2}$'s will then be that of a geometrically decaying sum. Observe that if $z_j(t)$ is a nonvanishing piecewise polynomial at level j satisfying $Q_j z_j = 0$ and $\Delta z_j = 0$ (implying that $z_j(t)$ has period h), then $z_{j+1} \equiv P_{j+1} z_j$ satisfies $|z_{j+1}| < |z_j|$. Further, $Q_{j+1} z_{j+1} = 0$ and $\Delta z_{j+1} = 0$, so subsequent projection onto a new level will produce additional damping. Our situation is similar, although complicated by the fact that $|\Delta v_{j+1/2}|$, while small in relation to $|v_{j+1/2}|$, is in general nonzero. In the following lemma,
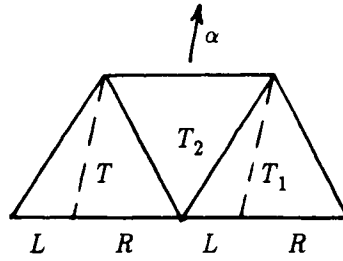
FIGURE 3.2

we maintain $e_j^-(t)$ inductively in the form $e_j^-(t) = w_j(t) + r_j(t)$, where $Q_j w_j = 0$, and monitor the growth of w_j, Δw_j and r_j.

LEMMA 3.2. *Suppose* $e_j^-(t)$ *has a representation*

$$e_j^-(t) = w_j(t) + r_j(t),$$

where $Q_j w_j = 0$. *Then, by* (26) *and* (28), $e_{j+1}^-(t)$ *can be represented as*

$$e_{j+1}^-(t) = w_{j+1}(t) + r_{j+1}(t),$$

where

$$\text{(i)} \quad Q_{j+1} w_{j+1} = 0,$$

(30) $\quad$ (ii) $\quad |w_{j+1}| \le \lambda |w_j| + |v_{j+1/2}|,$

(31) $\quad$ (iii) $\quad |\Delta w_{j+1}| \le \lambda |\Delta w_j| + |\Delta v_{j+1/2}|,$

(32) $\quad$ (iv) $\quad |r_{j+1}| \le |r_j| + |\Delta v_{j+1/2}| + |\Delta w_j|,$

where $\lambda < 1$ *is a constant, independent of* h.

Proof. By projecting the grid points along $y = y_{j+1}$ backward to $y = y_j$ along characteristics, we partition the subintervals of $y = y_j$ into left and right sections, as indicated in Figure 3.2 by L and R. We define

(33)
$$\bar{w}_{j+1/2}(t) = \begin{cases} w_j(t), & t \in L, \\ w_j(t+h), & t \in R, \end{cases}$$
$$\bar{v}_{j+1/2}(t) = \begin{cases} v_{j+1/2}(t), & t \in L, \\ v_{j+1/2}(t+h), & t \in R. \end{cases}$$

Note that

(34) $\qquad |\bar{w}_{j+1/2} - w_j| \le |\Delta w_j|, \qquad |\bar{v}_{j+1/2} - v_{j+1/2}| \le |\Delta v_{j+1/2}|.$

We set

(35) $\qquad\qquad\qquad w_{j+1} \equiv P_{j+1}(\bar{w}_{j+1/2} + \bar{v}_{j+1/2}),$

(36) $\qquad\qquad r_{j+1} \equiv P_{j+1}(r_j + w_j - \bar{w}_{j+1/2} + v_{j+1/2} - \bar{v}_{j+1/2}).$

Using (26) and the assumed representation for e_j^-, we obtain

$$e_{j+1}^- = P_{j+1}(w_j + r_j + v_{j+1/2}) = w_{j+1} + r_{j+1}.$$

We now prove (i)–(iv).

(i) We regard $\bar{w}_{j+1/2}(t)$ as being defined on the inflow to the type II portion of S_j (its discontinuities lie at the grid points of this "half-level"). Note that in Figure 3.2, $\bar{w}_{j+1/2}|_{\Gamma_{in}(T_2)}$ arises from $w_j|_{\Gamma_{in}(T_1)}$ by "breaking off" the right section of the latter and "appending" it on the left. Thus

$$\int_{\Gamma_{in}(T_2)} \bar{w}_{j+1/2}(t)\, dt = \int_{\Gamma_{in}(T_1)} w_j(t)\, dt,$$

and $Q_j w_j = 0$ implies that $Q_{j+1}\bar{w}_{j+1/2} = 0$. Analogously, one may show that $Q_{j+1}\bar{v}_{j+1/2} = 0$. Thus $Q_{j+1}w_{j+1} = Q_{j+1}(\bar{w}_{j+1/2} + \bar{v}_{j+1/2}) = 0$.

(ii) If $\bar{w}_{j+1/2}(t)|_{\Gamma_{in}(T_2)}$ is a polynomial (as opposed to a piecewise polynomial), then it must vanish identically. For, if $\bar{w}_{j+1/2}(t)|_{\Gamma_{in}(T_2)}$ is a polynomial, then

$$\xi_j(t) \equiv \begin{cases} w_j(t), & t \in \Gamma_{in}(T_1), \\ w_j(t+h), & t \in \Gamma_{in}(T) \end{cases}$$

must be a polynomial on $\Gamma_{in}(T) \cup \Gamma_{in}(T_1)$, with $\xi_j|_{\Gamma_{in}(T)}$ and $\xi_j|_{\Gamma_{in}(T_1)}$ periodic images of each other. This is impossible unless $\xi_j \equiv$ constant. But $Q_j w_j = 0$, so $\xi_j \equiv 0$, which implies that $\bar{w}_{j+1/2}(t)|_{\Gamma_{in}(T_2)} \equiv 0$. Hence $\bar{w}_{j+1/2}(t)|_{\Gamma_{in}(T_2)}$, if nonzero, is not a polynomial and will therefore undergo a decrease in norm when projected onto $\Gamma_{out}(T_2)$. Thus,

(37) $\qquad |P_{j+1}\bar{w}_{j+1/2}|_{\Gamma_{out}(T_2)} \le \lambda|\bar{w}_{j+1/2}|_{\Gamma_{in}(T_2)} = \lambda|w_j|_{\Gamma_{in}(T_1)}, \quad \lambda < 1.$

Moreover, the constant λ may be taken to be uniformly less than one and independent of h, in view of Assumption H2. Application of (37) over the entire layer then yields

$$|P_{j+1}\bar{w}_{j+1/2}| \le \lambda|w_j|,$$

which, with (35) and the fact that

$$|P_{j+1}\bar{v}_{j+1/2}| \le |\bar{v}_{j+1/2}| = |v_{j+1/2}|,$$

proves (ii).

(iii) This is analogous to (ii) and follows in the same way via superposition.

(iv) This follows from (36) and (34). □

The solutions of (30)–(32) are

$$|w_j| \le \lambda^j |w_0| + \sum_{k=0}^{j-1} \lambda^k |v_{j-k-1/2}|,$$

$$|\Delta w_j| \le \lambda^j |\Delta w_0| + \sum_{k=0}^{j-1} \lambda^k |\Delta v_{j-k-1/2}|,$$

$$|r_j| \le |r_0| + \sum_{k=0}^{j-1} (|\Delta v_{k+1/2}| + |\Delta w_k|).$$

Thus,

(38) $$|w_j| \le \lambda^j |w_0| + \frac{1}{1-\lambda} \max_{0 \le k \le j-1} |v_{k+1/2}|,$$

$$|\Delta w_j| \le \lambda^j |\Delta w_0| + \frac{1}{1-\lambda} \max_{0 \le k \le j-1} |\Delta v_{k+1/2}|,$$

(39) $$|r_j| \le |r_0| + \frac{1}{1-\lambda} |\Delta w_0| + \frac{2-\lambda}{1-\lambda} j \max_{0 \le k \le j-1} |\Delta v_{k+1/2}|.$$

Summation of (38) and (39) yields

$$
|e_j^-| \le \lambda^j |w_0| + \frac{1}{1-\lambda}|\Delta w_0| + |r_0|
$$

(40)

$$
+ \frac{1}{1-\lambda}\left\{\max_{0\le k \le j-1}|v_{k+1/2}| + (2-\lambda)j\max_{0\le k\le j-1}|\Delta v_{k+1/2}|\right\}.
$$

We now set

$$
w_0 = (I - Q_0)e_0^-, \qquad r_0 = Q_0 e_0^-
$$

so that, at level 0, the premise of Lemma 3.2 is satisfied. Then

$$
|w_0| \le |e_0^-|, \quad |r_0| \le |e_0^-|, \quad |\Delta w_0| \le 2|e_0^-|.
$$

Substituting these bounds into (40) and applying (27) and (29), we obtain

LEMMA 3.3. *The error at level j satisfies*

$$
|e_j^-| \le \frac{1}{1-\lambda}\left\{4|e_0^-| + Ch^{n+1/2}\left(\max_{0\le k\le j-1}\|u\|_{n+1,S_k}\right.\right.
$$

(41)

$$
\left.\left. + jh\max_{0\le k\le j-1}\|u\|_{n+2,S_k}\right)\right\}.
$$

We now make the regularity assumption

H3. $\|u\|_{n+2,S_j} \le C\sqrt{h}\|u\|_{n+2,\Omega}$,

implying, in a sense, that $\|u\|_{n+2,\Omega}$ is distributed uniformly over the layers of Ω. We also assume that the finite element solution at level 0 is a standard interpolant of the exact initial condition, so that

$$
|e_0^-| \le Ch^{n+1/2}\|u\|_{n+1,S_0}.
$$

Using the fact that there are $O(h^{-1})$ layers in all (implied by Assumption H1), we conclude that

$$
\max_{0\le j\le m}|e_j^-| \le Ch^{n+1}\|u\|_{n+2,\Omega},
$$

and, as a result,

(42)
$$
\max_{0\le j\le m}|u_h^- - u|_{y=y_j} \le Ch^{n+1}\|u\|_{n+2,\Omega}.
$$

To obtain an interior estimate of the error in u_h, we write (2) in the form

$$
((u_h)_\alpha, v_h) - \int_{\Gamma_{\text{in}}(T)}(u_h^+ - u_h^-)v_h\alpha\cdot n = (u_\alpha, v_h),
$$

and note that for the standard *continuous* nth degree piecewise polynomial interpolant $u_I \approx u$,

$$
((u_I)_\alpha, v_h) - \int_{\Gamma_{\text{in}}(T)}(u_I^+ - u_I^-)v_h\alpha\cdot n = ((u_I)_\alpha, v_h).
$$

Subtracting, we obtain

$$
((u_h - u_I)_\alpha, v_h) - \int_{\Gamma_{\text{in}}(T)}[(u^h - u_I)^+ - (u_h - u_I)^-]v_h\alpha\cdot n = ((u - u_I)_\alpha, v_h).
$$

Application of the local stability result (5) over the triangles in S_j then yields

$$
\|u_h - u_I\|_{S_j} \le C\{\sqrt{h}|u_h^- - u_I^-|_{y=y_j} + h\|(u - u_I)_\alpha\|_{S_j}\}
$$
$$
\le C\{h^{n+3/2}\|u\|_{n+2,\Omega} + h^{n+1}\|u\|_{n+1,S_j}\},
$$

using (42) and relevant approximation properties of u_I. Thus,

$$\|u_h - u_I\|_\Omega^2 \leq C\{mh^{2n+3}\|u\|_{n+2,\Omega}^2 + h^{2n+2}\|u\|_{n+1,\Omega}^2\}$$
$$\leq Ch^{2n+2}\|u\|_{n+2,\Omega}^2$$

and

$$\|u_h - u\|_\Omega \leq \|u_h - u_I\|_\Omega + \|u_I - u\|_\Omega \leq Ch^{n+1}\|u\|_{n+2,\Omega}.$$

We summarize as follows:

THEOREM 3.1. *Under Assumptions* H1–H3, *the discontinuous Galerkin approximation u_h satisfies*

$$(43) \qquad \|u_h - u\|_\Omega + \max_{0 \leq j \leq m} |u_h^- - u|_{y=y_j} \leq Ch^{n+1}\|u\|_{n+2,\Omega}.$$

We conclude this section with several remarks. We note first that the $O(h^{n+1})$ error estimate is a result of the damping which accompanies projection of the error from the grid at one level to the interlacing grid at the next level. Specifically, the fact that $Q_j w_j = 0$ in the representation $e_j^-(t) = w_j(t) + r_j(t)$ implies that w_j is highly oscillatory and hence damped by P_{j+1}. Our computational experimentation indicates that the $O(h^{n+1})$ convergence rate is not confined to the assumptions made in our analysis, and that it can occur even for irregular triangulations. We believe that the basic mechanism involved is the damping of an oscillatory error via (7). Some of our assumptions were motivated by ease of exposition rather than necessity, e.g., that of a periodic domain. Also, H2 is not essential, for in the limiting case where $\alpha \cdot n = 0$ in a layer S_j, $P_{j+1} = P_j$ and (26) becomes $e_{j+1}^-(t) = e_j^-(t)$ (because Lemmas 2.2–2.4 imply that $P_j v_{j+1/2} = 0$). Thus, there is no growth in the error over such a layer.

4. Computational Results. We use as a test problem

$$(44) \qquad \frac{1}{2}\frac{\partial u}{\partial x} + \frac{\sqrt{3}}{2}\frac{\partial u}{\partial y} = 0, \qquad x \in (-\infty, \infty),\ t > 0,$$

with $u(x,0)$ a cubic B-spline with knots at $x = -1, -.5, 0, .5, 1.0$ and maximum value 1 at $x = 0$. The exact solution is thus the same cubic B-spline, propagated along characteristics at an angle of $60°$ with the x-axis. We triangulate Ω by means

TABLE 4.1

L^2 errors; piecewise constant approximation

Δx	error at $y = 1$	ratio	error at $y = 2$	ratio
1	.5338	* * * *	.5803	* * * *
.5	.3383	1.58	.4166	1.39
.25	.2156	1.57	.3009	1.38
.125	.1267	1.70	.1969	1.53
.0625	.6989(−1)	1.81	.1175	1.68
.03125	.3693(−1)	1.89	.6524(−1)	1.80
.015625	.1902(−1)	1.94	.3458(−1)	1.89

of a uniform set of right isosceles triangles with their hypotenuses parallel to the
x-axis, producing a configuration similar to that depicted in Figure 3.1. Table 4.1
shows the L^2 error in the piecewise constant discontinuous Galerkin approximation
at $y = 1$ and $y = 2$, as well as ratios of successive values of the error as the subin-
terval size Δx is repeatedly halved. The exact solution has sufficient smoothness to
make the estimate (43) applicable, and the results are consistent with the predicted
$O(h)$ rate of convergence.

Table 4.2 shows the analogous computational results for the case of piecewise
linear approximation. Again, the optimal order of convergence is occurring, as
predicted.

TABLE 4.2

L^2 *errors; piecewise linear approximation*

Δx	error at $y = 1$	ratio	error at $y = 2$	ratio
1	.2597	* * * *	.2650	* * * *
.5	.7253(−1)	3.58	.1061	2.50
.25	.2109(−1)	3.44	.2598(−1)	4.08
.125	.5164(−2)	4.08	.5784(−2)	4.49
.0625	.1279(−2)	4.04	.1351(−2)	4.28
.03125	.3173(−3)	4.03	.3248(−3)	4.16
.015625	.7982(−4)	4.02	.7977(−4)	4.07

To illustrate that the optimal order of convergence may occur even for nonuni-
form triangulations, we randomly perturbed each triangle vertex, the x coordinate
by as much as $.15\Delta x$, and a proportionate amount for the y coordinate, and re-
peated the experiment. The L^2 errors in the piecewise constant and piecewise lin-
ear discontinuous Galerkin approximations are shown in Tables 4.3 and 4.4. They
again indicate an optimal order of convergence. We have not conducted a thorough
study of the conditions under which the optimal order of convergence occurs, but
this phenomenon appears to be quite robust.

TABLE 4.3

L^2 *errors; piecewise constant approximation; randomly perturbed grid points*

Δx	error at $y = 1$	ratio	error at $y = 2$	ratio
1	.5382	* * * *	.5867	* * * *
.5	.3402	1.58	.4153	1.43
.25	.2212	1.54	.2964	1.40
.125	.1238	1.79	.1946	1.52
.0625	.6906(−1)	1.79	.1152	1.69
.03125	.3631(−1)	1.90	.6367(−1)	1.81
.015625	.1866(−1)	1.95	.3365(−1)	1.89

TABLE 4.4

L^2 errors; piecewise linear approximations; randomly perturbed grid points

Δx	error at $y = 1$	ratio	error at $y = 2$	ratio
1	.2412	* * * *	.2497	* * * *
.5	.7649(−1)	3.15	.1040	2.40
.25	.2074(−1)	3.69	.3198(−1)	3.25
.125	.6473(−2)	3.20	.6124(−1)	5.22
.0625	.1410(−2)	4.59	.1498(−2)	4.09
.03125	.3615(−3)	3.90	.3663(−3)	4.09
.015625	.8889(−4)	4.07	.9655(−4)	3.79

Appendix. We briefly consider a variant of the discontinuous Galerkin method in which u_h is defined by (2) in type II triangles, and by

$$(45) \qquad \begin{aligned} ((u_h)_\alpha, v_h) &= (f, v_h), \quad \text{all } v_h \in \mathbf{P}_{n-1}(T), \\ u_h^+ &= u_h^- \quad \text{on } \Gamma_{\text{in}}(T) \end{aligned}$$

in type I triangles. We assume $n \geq 1$ so that the inner product conditions in (45) are nonvacuous. Since the approximate solution u_h is now *continuous* along the inflow to type I triangles, it has fewer degrees of freedom; thus type I triangles yield smaller linear algebraic systems. This could produce a possibly significant saving in computational expense for small values of n. For example, with $n = 2$, (45) reduces to a linear system of order 3 vs. 6 for the discontinuous Galerkin method.

To obtain a closed form representation for u_h in a type I triangle T, observe that the inner product conditions in (45) are equivalent to

$$(u_h)_\alpha = P_{n-1}f.$$

Integration along characteristics then yields the unique solution

$$(46) \qquad u_h(s, t) = u_{h,\text{in}}^-(t) + \int_{s_{\text{in}}(t)}^{s} P_{n-1}f\, ds, \qquad (s, t) \in T.$$

As noted in the proof of Lemma 2.3, the discontinuous Galerkin method will produce the *same* approximate solution as the above (cf. (16)) if $(I - P_{n-1})f = 0$ in T. In particular, for the homogeneous version of (1), $f \equiv 0$, there is no difference between the two solutions.

To analyze the error in the modified scheme, we evaluate (46) at $s_{\text{out}}(t)$,

$$(47) \qquad u_{h,\text{out}}^-(t) = u_{h,\text{in}}^-(t) + \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} P_{n-1}f\, ds,$$

then apply P_{out} to (11):

$$(48) \qquad P_{\text{out}}u_{\text{out}}(t) = P_{\text{in}}u_{\text{in}}(t) + (P_{\text{out}} - P_{\text{in}})u_{\text{in}}(t) + P_{\text{out}}\int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} f\, ds.$$

Subtracting (48) from (47), we obtain

$$e_{\text{out}}^-(t) = e_{\text{in}}^-(t) + v(t),$$

where

$$v(t) = (P_{\text{in}} - P_{\text{out}})u_{\text{in}}(t) + P_{\text{out}} \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (P_{n-1} - I)f \, ds.$$

As in Lemma 2.3,

$$|v| \leq Ch^{n+1/2}\|u\|_{n+1,T}.$$

However, in place of (13), we have a weaker orthogonality condition,

$$(49) \qquad\qquad\qquad\qquad P_{\text{in}}^* v(t) = 0,$$

where P_{in}^* is defined to be the L^2 projection into polynomials of degree $\leq n-1$ on $\Gamma_{\text{in}}(T)$. [In obtaining the above relation, we have used the fact that, for a polynomial $w(t)$ of degree $\leq n-1$, $(w(t), \int_{s_{\text{in}}(t)}^{s_{\text{out}}(t)} (I - P_{n-1})f \, ds) = \int_T w(I - P_{n-1})f = 0.$] As a result of the reduced orthogonality, we have only

$$(50) \qquad\qquad |e_{\text{out}}^-|^2 \leq (1 + O(h))|e_{\text{in}}^-|^2 + Ch^{2n}\|u\|_{n+1,T}^2$$

(using the arithmetic-geometric mean inequality) instead of (21). When combined with (22), inequality (50) leads to an $O(h^n)$ estimate for the L^2 error in u_h.

However, the $O(h^{n+1})$ estimate (43) remains applicable to the modified method under the same assumptions as for the fully discontinuous method. This follows from the observation that Lemma 2.4 is still valid, and the fact that the orthogonality condition (49) is sufficient to make $Q_j v_{j+1/2} = 0$ (cf. (28)) for the semiuniform triangulation of Section 3. Thus, all the analysis in Section 3, and in particular Theorem 3.1, apply to the modified method.

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