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**AN OPTIMAL PARALLEL ALGORITHM FOR DETECTING  
WEAK VISIBILITY OF A SIMPLE POLYGON**

**Danny Z. Chen**

**CSD-TR-91-008  
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# An Optimal Parallel Algorithm for Detecting Weak Visibility of a Simple Polygon (Extended Abstract)

Danny Z. Chen\*

## Abstract

The problem of detecting the weak visibility of an  $n$ -vertex simple polygon  $P$  is that of finding whether or not  $P$  is weakly visible from one of its edges and (if it is) identifying every edge from which  $P$  is weakly visible. In this paper, we present an optimal parallel algorithm for solving this problem. Our algorithm runs in  $O(\log n)$  time using  $O(n/\log n)$  processors in the CREW-PRAM computational model, and is very different from the sequential algorithms for this problem. This algorithm also enables us to optimally solve, in parallel, several other problems on weakly visible polygons.

## 1 Introduction

Visibility is one of the most fundamental topics in computational geometry. Visibility problems arise in many application areas, such as computer graphics, vision, VLSI design, and robotics. Visibility problems also appear as subproblems in other geometric problems (like finding the shortest obstacle-avoiding paths and computing intersections between geometric figures). Therefore, a great deal of research has been devoted to finding efficient algorithms for solving various visibility problems, in both sequential and parallel computational models.

Weak visibility problems concern visibility with respect to "observers" in the shape of line segments. An important class of weak visibility problems deals with the case where the opaque object is the boundary of a simple polygon. For a point  $p$  in a polygon and a line segment  $s$ ,  $p$  is *weakly visible* from  $s$  if  $p$  is visible from some point on  $s$ . An example of such problems is that of computing the region inside a polygon that is weakly visible from a segment. For this problem, many sequential algorithms [4, 12, 14, 18, 23] and a parallel algorithm [11] have been discovered. For more examples of the weak visibility problems on simple polygons, see [2, 3, 6, 13, 15, 22].

This paper considers the problem of detecting the weak visibility of a simple polygon. An  $n$ -vertex simple polygon

$P$  is *weakly visible* if there exists an edge  $e$  of  $P$  such that any point of  $P$  is weakly visible from  $e$  ( $e$  is called a *weakly visible edge* of  $P$ ). The problem of detecting the weak visibility of  $P$  is that of finding whether or not  $P$  is weakly visible and (if it is) identifying *all* weakly visible edges of  $P$ . Note that this problem is a natural generalization of the well-known problem of computing the *kernel* of a simple polygon [19]. (Recall that a point is in the kernel of a polygon iff the whole polygon is visible from that point, and that a polygon with a nonempty kernel is called a *star-shaped* polygon [21].)

Avis and Toussaint [2] first consider the problem of detecting the weak visibility of a simple polygon. They present a sequential linear time algorithm for the following case: check whether or not a polygon  $P$  is weakly visible from a specified edge  $e$  of  $P$ . Another sequential linear time algorithm for this case was recently given in [13]. Using the algorithms in [2, 13], the problem of detecting the weak visibility of  $P$  can be trivially solved in  $O(n^2)$  time (by checking separately each edge of  $P$ ), but Sack and Suri [22] succeeded in finding a linear time algorithm for this problem. Our interest here is to solve this problem in parallel. The parallel computational model we use is the CREW-PRAM; this is the synchronous shared-memory model where multiple processors can simultaneously read from the same memory location but at most one processor is allowed to write to a memory location at each time unit.

Based on the observations of Sack and Suri [22], a sub-optimal parallel algorithm can be easily obtained by using the recent result of Goodrich *et al.* [11] on constructing a data structure that supports ray-shooting queries [4]. This algorithm first preprocesses  $P$  and builds the data structure in  $O(\log n)$  time using  $O(n)$  processors [11], and then does  $O(n)$  ray-shooting queries by using the data structure. The algorithm takes in total  $O(\log n)$  time and  $O(n \log n)$  work (the work complexity of a parallel algorithm is the total number of operations performed by the algorithm). Obviously, the work complexity of this algorithm is a factor of  $\log n$  away from optimality. The sequential algorithm in [22] manages to avoid doing the ray-shooting queries, but that method seems to be inherently sequential.

Our method for obtaining an optimal parallel algorithm is very different from the above approaches. We give geometric insights and parallel techniques which enable us to use the divide-and-conquer strategies and to avoid the difficulty of doing ray-shooting queries. Our algorithm runs in  $O(\log n)$  time using  $O(n/\log n)$  processors, and is obviously optimal. We also use this algorithm to optimally solve, in parallel, several other problems on weakly visible polygons (such as the shortest paths, triangulation, and one cruising guard [6] problems); these parallel solutions all take  $O(\log n)$  time using  $O(n/\log n)$  processors and avoid

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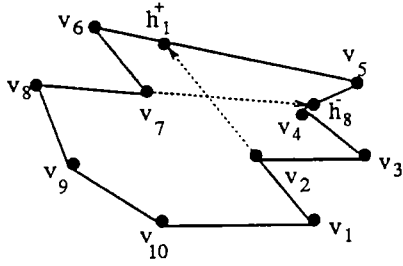


Figure 1: Illustrating  $HE_i^+$  and  $HE_i^-$ .

the preprocessing step of triangulating an arbitrary simple polygon. (The best known parallel algorithms for triangulating a simple polygon require  $O(n)$  processors in the CREW-PRAM [9, 24], or  $O(n/\log n)$  processors in the more powerful CRCW-PRAM [10], and  $O(\log n)$  time.) The geometric insights we present could be useful in solving other geometric problems.

There are two major subproblems solved by our weak visibility algorithm: (1) identifying all weakly visible edges for a star-shaped polygon whose kernel contains a convex vertex, and (2) checking whether or not a polygon is weakly visible from a specified edge (i.e., the case solved by [2, 13]). The solutions to these two subproblems could be interesting in their own right. The problem of detecting the weak visibility of a simple polygon is reduced to the two subproblems. The idea for the reduction is derived from the one used in [22], but our reduction procedure is very different from [22]. Owing to the limitation of space, we leave the reduction and the applications of the algorithm to the full paper.

The rest of this paper consists of 4 sections. Section 2 gives some notations and preliminary results on the weak visibility of a simple polygon. Section 3 discusses several geometric and computational observations needed by the algorithm. Sections 4 and 5 describe the algorithms for solving the two subproblems mentioned above, respectively.

## 2 Preliminaries

An  $n$ -vertex simple polygon  $P$  is given as a sequence  $(v_1, v_2, \dots, v_n)$  of its vertices, in the order in which they are visited by a *counterclockwise* walk along the boundary of  $P$ , starting from vertex  $v_1$ . An edge of  $P$  joining  $v_i$  and  $v_{i+1}$  is denoted by  $e_i = \overline{v_i v_{i+1}}$  ( $= \overline{v_{i+1} v_i}$ ), with the convention that  $v_{n+1} = v_1$ . The boundary of  $P$  is denoted by  $bd(P)$ .

Without loss of generality (WLOG), we assume that no edge of  $P$  is vertical and no three consecutive vertices of  $P$  are collinear.

Vertex  $v_i$  is *convex* if the interior angle of  $P$  at  $v_i$  is  $< \pi$ . Edge  $e_i$  is *convex* if both  $v_i$  and  $v_{i+1}$  are convex. For a vertex  $v_i$ , if  $v_{i+1}$  (resp.,  $v_{i-1}$ ) is nonconvex, then let  $r_i^+$  (resp.,  $r_i^-$ ) be a ray starting at  $v_i$  and containing  $e_i$  (resp.,  $e_{i-1}$ ). The set of all such  $r_i^+$ 's (resp.,  $r_i^-$ 's) is denoted by  $Ray^+(P)$  (resp.,  $Ray^-(P)$ ). Let  $h_i^+$  (resp.,  $h_i^-$ ) be the first point at which  $r_i^+$  (resp.,  $r_i^-$ ) hits  $bd(P) - e_i$  (resp.,  $bd(P) - e_{i-1}$ ) (i.e.,  $v_i$  is closer to  $h_i^+$  (resp.,  $h_i^-$ ) than to any other point in  $r_i^+ \cap (bd(P) - e_i)$  (resp.,  $r_i^- \cap (bd(P) - e_{i-1})$ )). Let  $h_i^+$  (resp.,  $h_i^-$ ) be on  $e_j - v_j$  (resp.,  $e_k - v_k$ ) for some  $j$  (resp.,  $k$ ). Then we call the consecutive edges that are on the portion of  $bd(P)$  from  $v_{i+2}$  (resp.,  $v_{i-2}$ ) counterclockwise (resp., clockwise) to  $v_j$  (resp.,  $v_k$ ) the *counterclockwise* (resp., *clockwise*) *hidden edges* of  $v_i$ . The set of counterclockwise (resp., clockwise) hidden edges of  $v_i$  is denoted by  $HE_i^+$  (resp.,  $HE_i^-$ ) (see

Fig. 1). For examples, in Fig. 1,  $HE_1^+ = \{e_3, e_4\}$  and  $HE_8^- = \{e_3\}$ . Note that  $HE_i^+$  (resp.,  $HE_i^-$ ) can be empty. The union of all  $HE_i^+$  (resp.,  $HE_i^-$ ) is denoted by  $HE^+(P)$  (resp.,  $HE^-(P)$ ). We henceforth call  $h_i^+$  (resp.,  $h_i^-$ ) the *first-hit* point of  $r_i^+$  (resp.,  $r_i^-$ ). The following lemma characterizes an important property of the weakly visible polygons.

**Lemma 1** (Sack and Suri [22]) *Suppose that polygon  $P$  is weakly visible from edge  $e_n$  and  $e_n$  is convex. Then for every  $i = 1, 2, \dots, n$ , the following holds: if  $h_i^+$  exists, then  $h_i^+$  is the first point on  $bd(P) - e_i$  collinear with  $e_i$  encountered in the counterclockwise walk along  $bd(P)$  starting from  $v_{i+1}$ , and if  $h_i^-$  exists, then  $h_i^-$  is the first point on  $bd(P) - e_{i-1}$  collinear with  $e_{i-1}$  encountered in the clockwise walk along  $bd(P)$  starting from  $v_{i-1}$ .*

**Proof.** See Lemma 3 of [22].  $\square$

It is shown in [22] that the set of weakly visible edges of  $P$ , denoted by  $WVE(P)$ , is equal to  $bd(P) - (HE^+(P) \cup HE^-(P))$  (see Theorem 1 of [22]). For convenience, we call the edges in  $HE^+(P) \cup HE^-(P)$  the *bad edges* of  $P$ . If  $HE^+(P)$  and  $HE^-(P)$  were available, then  $WVE(P)$  could be obtained in the desired complexity bounds. The main difficulty, therefore, is in computing  $HE^+(P)$  and  $HE^-(P)$ . WLOG, we will show the computation for  $HE^+(P)$  (the computation for  $HE^-(P)$  is similar).

A point  $p$  is represented by its  $x$ -coordinate and  $y$ -coordinate, denoted by  $x(p)$  and  $y(p)$ , respectively. For a line segment  $s$  (resp., a ray  $r$ ), the line containing  $s$  (resp.,  $r$ ) is denoted by  $l(s)$  (resp.,  $l(r)$ ). The slope of a line  $l$  (resp., a segment  $s$ , a ray  $r$ ) is denoted by  $slope(l)$  (resp.,  $slope(s)$ ,  $slope(r)$ ).

The chain on  $bd(P)$  from  $v_i$  counterclockwise to  $v_j$ ,  $i \neq j$ , is denoted by  $bd_{ij}$ . The size of a chain  $C$  is the number of line segments on  $C$ , denoted by  $|C|$ . For three non-collinear points  $p$ ,  $q$ , and  $r$ , we say that the directed chain from  $p$  to  $q$  to  $r$  makes a *left* (resp., *right*) turn if  $x(r)(y(p) - y(q)) + y(r)(x(q) - x(p)) + x(p)y(q) - x(q)y(p) > 0$  (resp.,  $< 0$ ). For a directed simple chain  $C = (p_1, p_2, \dots, p_k)$ ,  $k \geq 3$ ,  $C$  is said to only make left (resp., right) turns iff every subchain of the form  $(p_{i-1}, p_i, p_{i+1})$  makes a left (resp., right) turn,  $1 < i < k$ .

If polygon  $P$  is weakly visible from edge  $e_n$  and  $e_n$  is convex, then for  $1 \leq i \leq j \leq n$ , the (directed) shortest path from  $v_i$  to  $v_j$  inside  $P$  goes through only the vertices on  $bd_{ij}$ , and the shortest path only makes right turns (this fact is shown in [2, 13]). Hence, we call such a shortest path the *internal convex path* of  $bd_{ij}$ , and denote it by  $ICP(bd_{ij})$ .

Let  $l$  be a non-vertical line. We say a point  $p$  is *above* (resp., *below*)  $l$  iff the vertical line passing  $p$  intersects  $l$  at a point  $q$  such that  $y(q) < y(p)$  (resp.,  $y(q) > y(p)$ ). A segment  $s$  is said to be (properly) *above* (resp., *below*)  $l$  iff every point of  $s$  is above (resp., below)  $l$ . The upper (resp., lower) half-plane of  $l$  is the half-plane defined by  $l$  whose interior points are all above (resp., below)  $l$ . A left (resp., right) half-plane of a ray  $r$  is the half-plane whose boundary contains  $r$  and which is to the left (resp., right) of  $l(r)$ . For a set  $L$  of lines, we use  $UPCI(L)$  (resp.,  $LPCL(L)$ ) to denote the common intersection of the upper (resp., lower) half-planes for the lines in  $L$ .

Internal convex paths and common intersections of half-planes play important role in our algorithm. We will represent the internal convex paths and common intersections of half-planes by a data structure called the *hull tree* [8, 9] or *rank tree* [5]. This data structure supports efficient implementation for the parallel operations of search, concatenation, and split (see [8, 9, 5] for the details).

### 3 Some Useful Observations

This section gives some useful geometric observations and develops the computational machinery needed by the algorithm.

The following type of tests will be frequently done by the algorithm: given a set  $L$  of lines and a line segment  $s$ , find (i) whether there is a line  $l \in L$  such that  $s$  is *below*  $l$ , or (ii) whether  $s$  is *above all* the lines in  $L$ . We call such a test a *lines-vs-segment test* and denote it by  $Test(L, s)$ .

(Note: Another type of tests which is also needed by the algorithm is: (i') whether there is a line  $l \in L$  such that  $s$  is *above*  $l$ , or (ii') whether  $s$  is *below all* the lines in  $L$ . The tests for (i') and (ii') are handled similarly to the tests for (i) and (ii). Hence we omit the discussion for (i') and (ii').)

Doing  $Test(L, s)$  by a brute force method, which checks segment  $s$  against every line in  $L$ , is inefficient (it requires  $O(|L|)$  work and  $O(\log |L|)$  time). We would like to achieve  $O(\log |L|)$  time and  $O(|L|^\alpha)$  work for every  $Test(L, s)$  done by the algorithm, where  $\alpha$  is some constant,  $0 < \alpha < 1$ . Our method for doing the tests makes use of the common intersection of upper half-planes. It is clear that  $s$  is above all the lines in  $L$  iff  $s$  is properly contained in  $UPCI(L)$ . However, it is not necessarily true that if  $s$  does not intersect  $UPCI(L)$ , then  $s$  is below a line in  $L$ . Our solution to the tests is based on the following observation.

**Lemma 2** *Given a non-vertical segment  $s$  and a set  $L$  of lines, if  $L$  is partitioned into two subsets  $L'$  and  $L''$  such that  $L'$  (resp.,  $L''$ ) contains the lines of  $L$  whose slopes are all  $\geq$  slope( $s$ ) (resp.,  $<$  slope( $s$ )), then the following is true: (i)  $s$  is below a line in  $L$  iff  $s$  does not intersect either  $UPCI(L')$  or  $UPCI(L'')$ , and (ii)  $s$  is above all the lines in  $L$  iff  $s$  is properly contained in both  $UPCI(L')$  and  $UPCI(L'')$ .*

**Proof.** Omitted. See the full paper.  $\square$

The computational lemma below follows from Lemma 2.

**Lemma 3** *Given a non-vertical segment  $s$  and a set  $L$  of  $m$  lines, suppose that the slopes of all the lines in  $L$  are  $\geq$  (resp.,  $\leq$ ) slope( $s$ ) and that  $UPCI(L)$  is available. Then using  $k$  processors,  $Test(L, s)$  can be done in  $O(\log m / \log(k+1))$  time if  $UPCI(L)$  is stored in an array, and in  $O((\log m)^2 / \log(k+1))$  time if  $UPCI(L)$  is stored in a rank tree.*

**Proof.** Omitted. See the full paper.  $\square$

Note that in Lemma 3, if  $k = O(m^\alpha)$  for any constant  $\alpha$ ,  $0 < \alpha < 1$ , then the time complexities become  $O(1)$  (when  $UPCI(L)$  is stored in an array) and  $O(\log m)$  (when  $UPCI(L)$  is stored in a rank tree).

Based on Lemmas 2 and 3, the next lemma describes the basic operation done by the procedure for performing the lines-vs-segment tests.

**Lemma 4** *Given a non-vertical segment  $s$  and a set  $L$  of  $m$  lines, suppose that  $L$  is partitioned into  $m^{1/c}$  subsets  $L_1, L_2, \dots, L_{m^{1/c}}$  of equal size such that the slope of every line in  $L_{i+1}$  is  $\geq$  the slopes of all the lines in  $L_i$ , and that  $UPCI(L_1), \overline{UPCI}(L_2), \dots, \overline{UPCI}(L_{m^{1/c}})$  are available (each stored in a rank tree), where  $c > 1$  is a constant. Then, in  $O(\log m)$  time using  $O(m^c)$  processors, either the result of  $Test(L, s)$  is found, or the test range is restricted to a unique  $L_j$  (i.e.,  $Test(L, s)$  is completed by doing  $Test(L_j, s)$ ), where  $c'$  is a constant and  $1/c < c' < 1$ .*

**Proof.** There are two possible cases: (1) there is a unique subset  $L_j$  which contains two lines  $l'$  and  $l''$  such that

slope( $l'$ )  $<$  slope( $s$ )  $<$  slope( $l''$ ), or (2) there is no such  $L_j$ . In case (2), we apply Lemma 3 and do  $Test(L_i, s)$ , in parallel, for each  $i = 1, 2, \dots, m^{1/c}$ . Each  $Test(L_i, s)$  takes  $O(\log m)$  time using  $O(m^\alpha)$  processors (by Lemma 3), where  $\alpha = c' - (1/c)$  is a constant and  $0 < \alpha < 1$ . The answer to  $Test(L, s)$  can be easily obtained from the answers to the  $Test(L_i, s)$ 's (based on Lemma 2), in the desired complexity bounds. In case (1), suppose  $L_j$  is the unique subset which gives rise to this case. We first do  $Test(L_i, s)$  for each  $i \neq j$  (in  $O(\log m)$  time using  $O(m^\alpha)$  processors). If the answer to  $Test(L, s)$  can be derived from the answers to the  $m^{1/c} - 1$   $Test(L_i, s)$ 's (e.g., there is a line in  $L - L_j$  that is above  $s$ ), then we are done. Otherwise, the answers to the  $m^{1/c} - 1$   $Test(L_i, s)$ 's must be combined with the answer to  $Test(L_j, s)$  in order to obtain the result for  $Test(L, s)$ ; hence  $Test(L, s)$  will be completed by performing  $Test(L_j, s)$ .  $\square$

Note that if Lemma 4 can be recursively applied to  $Test(L_j, s)$ , then we only need to repeat the use of Lemma 4 a constant number of times in order to reduce the size of the test range to  $O(m^c)$  (at that point the brute force method can take over). In this way,  $Test(L, s)$  is processed in totally  $O(\log m)$  time using  $O(m^c)$  processors.

Lemmas 3 and 4 require that the common intersections of the upper half-planes be available *before* the tests are performed. The computation for the common intersection of  $m$  half-planes, in general, requires  $O(\log m)$  time and  $\Omega(m \log m)$  total work. In our situation, there can be as many as  $O(n)$  rays (and thus  $O(n)$  half-planes) to consider. It would be impossible to compute the common intersection of  $O(n)$  half-planes in  $O(\log n)$  time using  $O(n/\log n)$  processors if the  $O(n)$  rays were arbitrary. Next, we show that if polygon  $P$  is weakly visible from a convex edge, then it is possible to obtain a subset of  $Ray^+(P)$  (resp.,  $Ray^-(P)$ ), denoted by  $DR^+(P)$  (resp.,  $DR^-(P)$ ), with the following properties: (i)  $HE^+(P)$  (resp.,  $HE^-(P)$ ) can be computed by using only  $DR^+(P)$  (resp.,  $DR^-(P)$ ), and (ii)  $DR^+(P)$  (resp.,  $DR^-(P)$ ) can be easily partitioned into two subsets, each containing rays sorted by slopes. The rays in  $DR^+(P)$  (resp.,  $DR^-(P)$ ) are called the *dominating rays* of  $Ray^+(P)$  (resp.,  $Ray^-(P)$ ). We just discuss the case for  $DR^+(P)$  (the case for  $DR^-(P)$  is similar).

WLOG, we assume that  $P$  is weakly visible from convex edge  $e_n$ , that  $e_n$  is horizontal, and that  $l(e_n)$  is below  $P - e_n$ . We define the polar angle of a ray  $r_i^+ \in Ray^+(P)$ , denoted by  $\alpha(r_i^+)$ , as follows: let the starting vertex  $v_i$  of  $r_i^+$  be at the origin; then  $\alpha(r_i^+)$  is the angle from the positive  $x$ -axis counterclockwise to  $r_i^+$ . Note that  $0 \leq \alpha(r_i^+) < 2\pi$ . For rays  $r_i^+$  and  $r_j^+$  in  $Ray^+(P)$ ,  $i < j$ , we say  $r_i^+$  *dominates*  $r_j^+$  if  $\alpha(r_i^+) \geq \alpha(r_j^+)$ . Let  $DR^+(P)$  consist of the rays in  $Ray^+(P)$  that are not dominated by any ray in  $Ray^+(P)$ . The following lemma characterizes  $DR^+(P)$ .

**Lemma 5** *For rays  $r_i^+$  and  $r_j^+$  in  $Ray^+(P)$ ,  $i < j$ , if  $r_i^+$  dominates  $r_j^+$ , then  $HE_j^+ \subseteq HE_k^+$  for some  $k$ ,  $i \leq k < j$ .*

**Proof.** Let  $Q$  be the polygon formed by segment  $\overline{v_{i+1}h_i^+}$  and the subchain of  $bd(P)$  from  $v_{i+1}$  counterclockwise to  $h_i^+$  ( $h_i^+$  is the first-hit point of  $r_i^+$ ). WLOG, assume that  $h_i^+ \neq v_j$ . There are two possible cases: (a)  $v_j$  is in  $Q$  (see Fig. 2 (a)), and (b)  $v_j$  is not in  $Q$  (see Fig. 2 (b)). We first show that in case (a),  $HE_j^+ \subseteq HE_i^+$ . If  $HE_j^+$  were not a subset of  $HE_i^+$ , then  $h_j^+$  would have to be outside  $Q$ . For this to happen,  $r_j^+$  must intersect  $\overline{v_{i+1}h_i^+}$  before hitting  $h_j^+$  (since  $v_j$  is in  $Q$ ); furthermore,  $r_j^+$  must start in the

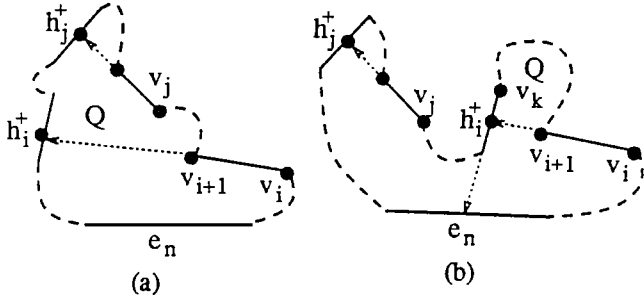


Figure 2: Illustrating the proof of Lemma 5.

right half-plane of  $r_i^+$  and hit  $h_j^+$  in the left half-plane of  $r_i^+$ . When  $\alpha(r_i^+) - \alpha(r_j^+) \leq \pi$ , such an intersection between  $r_j^+$  and  $v_{i+1}h_i^+$  is impossible because  $v_j$  is in the right half-plane of  $r_i^+$  and  $\alpha(r_i^+) \geq \alpha(r_j^+)$  (e.g., Fig. 2 (a)). When  $\alpha(r_i^+) - \alpha(r_j^+) > \pi$ , such an intersection between  $r_j^+$  and  $v_{i+1}h_i^+$  is also impossible for the following reason. That  $\alpha(r_i^+) - \alpha(r_j^+) > \pi$  implies that  $\alpha(r_i^+) > \pi$  and  $\alpha(r_j^+) < \pi$ . If  $r_j^+$  did not intersect  $bd(P) - e_j$  before crossing  $v_{i+1}h_i^+$ , then  $v_j$  would not be weakly visible from  $e_n$ , a contradiction. Hence  $HE_j^+ \subseteq HE_i^+$  in case (a).

In case (b), chain  $bd_{(i+1),j}$  first intersects the right half-plane of  $r_i^+$  and must later go to the left half-plane of  $r_i^+$  by crossing  $r_i^+$  at  $h_i^+$ . Let  $h_l^+$  be on  $e_l$ ,  $l > i + 1$ . Since  $P$  is weakly visible from  $e_n$ , the internal convex path  $ICP(bd_{(i+1),j})$  makes right turns only. It is not hard to see that there exists a vertex  $v_k$ ,  $l \leq k < j$ , such that  $v_{k+1}$  is a vertex on  $ICP(bd_{(i+1),j})$  and  $HE_j^+ \subseteq HE_k^+$  (Fig. 2 (b)).  $\square$

Based on Lemma 5, it is easy to compute  $DR^+(P)$  in  $O(\log n)$  time using  $O(n/\log n)$  processors (by doing parallel prefix [16, 17]). Note that the rays in  $DR^+(P)$  are sorted by polar angles. We further partition  $DR^+(P)$  into two subsets  $DR_1^+(P)$  and  $DR_2^+(P)$  such that the rays in each subset are sorted by slopes. This partition is done by splitting  $DR^+(P)$  using a ray whose polar angle is  $\pi$ .  $DR_1^+(P)$  (resp.,  $DR_2^+(P)$ ) contains the rays of  $DR^+(P)$  whose polar angles are all  $\leq$  (resp.,  $>$ )  $\pi$ . From now on, we assume that  $DR_1^+(P)$  and  $DR_2^+(P)$  are already available.

## 4 Detecting the Weak Visibility of a Star-Shaped Polygon

This section deals with the following problem: given that  $P$  is star-shaped and its kernel contains a convex vertex (say  $v_1$ ), compute the bad edges in  $HE^+(P)$  using  $DR^+(P)$ . Clearly,  $P$  is weakly visible from  $e_1$  (since  $P$  is visible from  $v_1$ ) and  $e_1$  is a convex edge. The algorithm for this problem has two phases. **Phase 1.A** computes the internal convex paths and the data structure storing the common intersections of the half-planes for the rays of  $DR^+(P)$ . This phase also identifies some bad edges. **Phase 1.B** completes the identification of all bad edges; it makes use of the internal convex paths and the data structure constructed in **Phase 1.A**. The computation consists of two separate parts: one using  $DR_1^+(P)$  and the other using  $DR_2^+(P)$ . Due to the similarity between the two parts, we only discuss the part using  $DR_1^+(P)$ .

### 4.1 Phase 1.A

We first sketch the outline and describe the main operation of the algorithm, then we give the analysis and some computational details.

#### 4.1.1 The Outline

We associate each  $r_a^+ \in DR_1^+(P)$  with  $v_a$ , and denote by  $R_C$  the set of those rays in  $DR_1^+(P)$  whose starting vertices are on a subchain  $C$  of  $bd(P)$ . The outline below sketches the divide-and-conquer strategies used by this phase.

**Input.** A subchain  $C$  of  $bd(P)$  with  $|C| = m$ ,  $R_C$ , and a positive integer  $d$ .

**Step a.1.** If  $m \leq d$ , then use one processor to perform the computation in  $O(m)$  time.

**Step a.2.** If  $d < m \leq d^6$ , then divide  $C$  into two subchains  $C_1$  and  $C_2$  of equal size, and recursively solve the two subproblems on  $(C_1, R_{C_1}, d)$  and  $(C_2, R_{C_2}, d)$ , in parallel. Then perform the computation for  $C$  and  $R_C$  using the output from the recursive calls on the two subproblems, with  $m/d$  processors and in  $O(\log m + (d \log d)^{1/2})$  time.

**Step a.3.** If  $m > d^6$ , then partition  $C$  into  $g = (m/d)^{1/3}$  subchains  $C_1, C_2, \dots, C_g$  of size  $m^{2/3}d^{1/3}$  each. Then recursively solve the  $g$  subproblems, in parallel. Finally, perform the computation for  $C$  and  $R_C$  using the output from the  $g$  recursive calls, with  $m/d$  processors and in  $O(\log m)$  time.

Observe that, if we could perform the various steps of the above outline within the claimed complexity bounds, then a procedure with such an outline would run in  $O(d + \log m)$  time with  $O(m/d)$  processors. Choosing  $d = \log m$ , then the time and processor complexities become  $O(\log m)$  and  $O(m/\log m)$ , respectively. Therefore, a call to the procedure with input  $(C, R_C, \log n)$ ,  $|C| = n$ , will take  $O(\log n)$  time using  $O(n/\log n)$  processors.

We must discuss what exactly is computed within the above outline. Let  $L(R_C)$  be the set of lines containing the rays in  $R_C$ . The following is computed in **Steps a.1–a.3**.

- (i) Compute the internal convex path  $ICP(C)$ .
- (ii) Build the data structure which stores the common intersection of the relevant half-planes of  $L(R_C)$  (e.g.,  $UPCI(L(R_C))$ ) (this data structure is needed for performing the lines-vs-segment tests).
- (iii) Do the lines-vs-segment tests to identify the bad edges on  $C$  (we may not be able to identify all the bad edges in this phase; some of them are left to **Phase 1.B**).

Since  $P$  is star-shaped, in each of the three steps, the internal convex paths for the subchains of  $bd(P)$  can be computed in the required complexity bounds by using the algorithm in [5]. By Lemma 5,  $R_C$  (thus  $L(R_C)$ ) is sorted by slopes. Hence the common intersection of the relevant half-planes for  $L(R_C)$  is also computable in the required complexity bounds (e.g., by the algorithms in [8, 5]). Therefore, we will focus on how to use the internal convex paths and the data structure for the lines-vs-segment tests returned from the recursive calls to identify the bad edges in this phase.

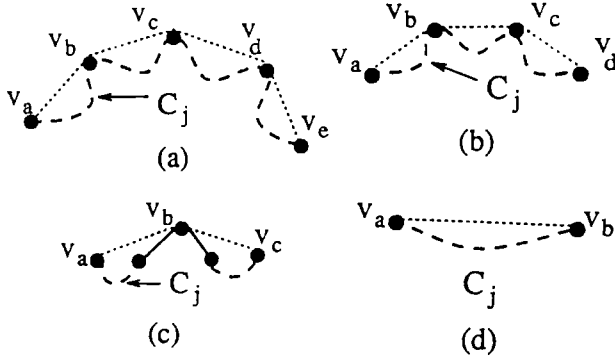


Figure 3: Illustrating the four cases of  $|ICP(C_j)|$ .

#### 4.1.2 The Main Operation

The following operation is crucial in the conquer stages of Steps a.2 and a.3 (say, in Step a.3): given  $ICP(C_s)$  and  $UPCI(L(R_{C_i}))$ ,  $s = 1, 2, \dots, g$ , determine the bad edges on  $C_j$  using  $R_{C_i}$ , for every pair of  $i$  and  $j$ ,  $1 \leq i < j \leq g$ . We classify  $ICP(C_j)$  into one of four possible cases according to its size, and show how to determine the bad edges on  $C_j$  by using  $R_{C_i}$  in each case. The classification is as follows.

**Case (a).**  $ICP(C_j)$  has more than 3 segments (see Fig. 3 (a)). Then all the edges on  $C_j$  are bad. In Fig. 3 (a), the edges on  $C_j$  are all contained in  $HE_{b-1}^+ \cup HE_{c+1}^-$ . Hence there is no need to use  $R_{C_i}$  to identify the bad edges on  $C_j$ .

**Case (b).**  $ICP(C_j)$  has exactly 3 segments (see Fig. 3 (b)). Then the edges on subchains  $bd_{ab}$  and  $bd_{cd}$  of  $C_j$  are all bad (cf. Fig. 3 (b)), because they are contained in  $HE_{b-1}^+ \cup HE_{c+1}^-$ . Furthermore, if  $\overline{v_b v_c}$  is not an edge of  $P$  (i.e.,  $c > b+1$ ), then all the edges on  $C_j$  are bad because the edges on subchain  $bd_{bc}$  are also contained in  $HE_{b-1}^+ \cup HE_{c+1}^-$ . The only edge on  $C_j$  that may not be bad is  $\overline{v_b v_c}$ , provided that  $c = b+1$ . Thus the problem in this case, when  $c = b+1$ , is that of finding whether or not  $\overline{v_b v_{b+1}}$  is bad with respect to  $R_{C_i}$ .

**Case (c).**  $ICP(C_j)$  has 2 segments (see Fig. 3 (c)). Then clearly all the edges on  $C_j$ , except the two edges  $\overline{v_{b-1} v_b}$  and  $\overline{v_b v_{b+1}}$ , are bad (cf. Fig. 3 (c)). We need to check  $\overline{v_{b-1} v_b}$  and  $\overline{v_b v_{b+1}}$  by using  $R_{C_i}$ .

**Case (d).**  $ICP(C_j)$  has 1 segment (see Fig. 4 (d)). If there is a ray  $r_i^+ \in R_{C_i}$  such that  $bd_{ab} \subseteq HE_i^+$ , then certainly all the edges on  $C_j$  are bad. Otherwise, we might have to “shoot” the rays of  $R_{C_i}$  onto  $C_j$  to find which of the edges on  $C_j$  are bad (this ray-shooting on  $C_j$  is to be done in Phase 1.B). Thus we need to check whether  $bd_{ab} \subseteq HE_i^+$  for a  $r_i^+ \in R_{C_i}$ . This check is done by testing segment  $\overline{v_a v_b}$  against the rays in  $R_{C_i}$ .

From the discussion above, it is clear that the main computation in Cases (a)–(d) is to test an edge of  $C_j$  (in Cases (b) and (c)) or a segment of  $ICP(C_j)$  (in Case (d)) against the rays in  $R_{C_i}$ , in order to find out whether the edge of  $C_j$  or  $C_j$  itself is bad with respect to  $R_{C_i}$ . We call such a test a *bad-segment test*.

We need some notations for describing the solution to the bad-segment tests. WLOG, let  $v_1$  be at the origin and  $e_1$  be on the positive  $x$ -axis (hence  $P - e_1$  is above  $l(e_1)$ ). The polar angle of a point  $p \in bd(P) - v_1$ , denoted by  $\alpha(p)$ , is the one from the positive  $x$ -axis counterclockwise to the ray starting at  $v_1$  and going through  $p$ . Since  $v_1$  is in the kernel of  $P$  and  $v_1$  is convex, it follows that  $0 \leq \alpha(p) < \pi$  for

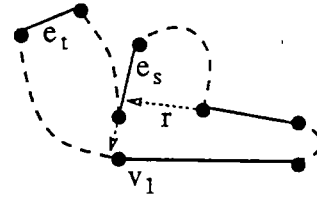


Figure 4: Illustrating the proof of Lemma 6.

each point  $p \in bd(P) - v_1$ , and that the polar angles of the points on  $bd(P) - v_1$ , from  $v_1$  counterclockwise to  $v_n$ , are in non-decreasing order. For each  $r_i^+ \in DR_1^+(P)$ ,  $h_i^+$  is on  $bd_{(i+2)1}$ . For a ray  $r_i^+ \in DR_1^+(P)$  and a segment  $s$ , we say  $s$  is properly contained in the *upper-right* (resp., *upper-left*) *quarter-plane* of  $r_i^+$  iff (i)  $s$  is contained in the intersection of the right (resp., left) half-plane of  $r_i^+$  and the left half-plane of the ray starting at  $v_1$  and going through  $v_{i+1}$ , and (ii)  $s$  does not intersect  $r_i^+$ . Observe that if  $e_i \in HE_i^+$ , then  $e_i$  is properly contained in the upper-right quarter-plane of  $r_i^+$ .

WLOG, we assume that for each ray  $r$  in  $R_{C_i}$ , the right half-plane of  $r$  is equal to the lower half-plane of line  $l(r)$ .

We would like to obtain the answer to the bad-segment test on  $R_{C_i}$  and a segment  $s$  (of  $C_j$  or  $ICP(C_j)$ ) by performing a lines-vs-segment test  $Test(L(R_{C_i}), s)$  (because we can handle  $Test(L(R_{C_i}), s)$  by using Lemma 4). In general, however, a bad-segment test cannot be answered by a lines-vs-segment test. For example, line  $l(r)$  intersecting a segment  $s'$  does not necessarily imply that ray  $r$  also intersects  $s'$ . Furthermore, even though the half-line defined by a ray  $r \in R_{C_i}$  does intersect segment  $s$  (of  $C_j$  or  $ICP(C_j)$ ),  $r$  may first-hit a point on some  $C_k$ ,  $i \leq k < j$ , before it intersects  $s$  (i.e.,  $C_k$  blocks  $r$  from reaching  $s$  if  $r$  is viewed as a beam of light emanating from its starting vertex). Therefore, even if the result of  $Test(L(R_{C_i}), s)$  does indicate that  $s$  is properly contained in the upper-right quarter-plane of  $r_i^+$  for a  $r_i^+ \in R_{C_i}$ , the rays of  $R_{C_i}$  may be totally blocked from  $C_j$ . This means that, in this situation, no edge of  $C_j$  truly belongs to  $HE_i^+$  for any  $r_i^+ \in R_{C_i}$ , and hence no edge of  $C_j$  is bad with respect to  $R_{C_i}$ . If we had to find out whether or not  $R_{C_i}$  is totally blocked from  $C_j$ , then for every  $k$ ,  $i \leq k < j$ , we might either do  $O(|ICP(C_k)|)$  bad-segment tests (for  $R_{C_i}$  and each segment of  $ICP(C_k)$ ), or “shoot” each ray of  $R_{C_i}$  on  $ICP(C_k)$  (by doing a binary search on  $ICP(C_k)$ ). Since we can have  $|ICP(C_k)|$  proportional to  $|C_k|$  and  $|R_{C_i}|$  proportional to  $|C_i|$ , either method would be too expensive to be performed within the desired complexity bounds. The next lemma saves us from doing these costly computations.

**Lemma 6** *If a ray  $r \in R_{C_i}$  first-hits some  $C_k$  at edge  $e_s$ ,  $i \leq k < j$ , and if edge  $e_w$  on  $C_j$  is properly contained in the upper-right quarter-plane of  $r$ , then there exists a vertex  $v_z$  on  $bd_{(s-1)(w-2)}$  such that  $e_w \in HE_z^+$ .*

**Proof.** Chain  $bd_{(s-1)w}$  must start in the right half-plane of  $r$ . It then intersects  $r$  on  $e_s$ , and eventually enters the right half-plane of  $r$  to join  $v_w$  (see Fig. 4). Since  $P$  is visible from  $v_1$ ,  $ICP(bd_{s,w})$  makes right turns only. Hence there must exist a vertex  $v_{z+1}$  on  $ICP(bd_{s,w})$  such that  $s-1 \leq z \leq w-2$  and  $e_w \in HE_z^+$ .  $\square$

Lemma 6 implies that if edge  $e_w$  of  $C_j$  is properly contained in the upper-right quarter-plane of a ray  $r \in R_{C_i}$ , then  $e_w$  is definitely a bad edge. Note that for any  $k'$  such that  $k' < i$  or  $k' > j$ ,  $C_{k'}$  cannot block the rays of  $R_{C_i}$  from  $C_j$  (by Lemma 1).

The next lemma justifies the use of the lines-vs-segment tests for the bad-segment tests.

**Lemma 7** For  $i < j$ , a bad-segment test on  $R_{C_i}$  and a segment  $s$  of  $C_j$  or  $ICP(C_j)$  can be done by using  $Test(L(R_{C_i}), s)$ .

**Proof.** Let  $r_i^+$  be a ray in  $R_{C_i}$ . Recall that by our assumption, the left half-plane of  $r_i^+$  is the upper half-plane of  $l(r_i^+)$ . The lemma holds if the following is true: (i)  $s$  is properly contained in the upper-right quarter-plane of  $r_i^+$  iff  $s$  is below  $l(r_i^+)$ , and (ii)  $s$  is properly contained in the upper-left quarter-plane of  $r_i^+$  iff  $s$  is above  $l(r_i^+)$ . We only give the proof for (i) (that for (ii) is very similar). If  $s$  is properly contained in the upper-right quarter-plane of  $r_i^+$ , then  $s$  is below  $l(r_i^+)$ . If  $s$  is below  $l(r_i^+)$ , then  $s$  is properly contained in the lower half-plane of  $l(r_i^+)$  (i.e., the right half-plane of  $r_i^+$ ) and certainly  $s \neq \overline{v_i v_{i+1}}$ . The facts that  $i < j$ , that  $s \neq \overline{v_i v_{i+1}}$ , that  $P$  is visible from  $v_1$ , and that  $v_1$  is convex, together imply that  $\alpha(v_{i+1}) \leq \alpha(p) < \pi$  for each point  $p$  on  $s$ . This means that  $s$  is to the left of the ray  $r$  starting at  $v_1$  and going through  $v_{i+1}$ , and thus  $s$  is in the left half-plane of  $r$ .  $\square$

If the result of  $Test(L(R_{C_i}), s)$  indicates that neither (i)  $s$  is below a line in  $L(R_{C_i})$  nor (ii)  $s$  is above all the lines in  $L(R_{C_i})$ , then there must be a ray  $r \in R_{C_i}$  such that the half-line defined by  $r$  intersects  $s$ . We need to distinguish two types of intersection between  $r$  and  $s$ .

Suppose that  $Test(L(R_{C_i}), s)$  indicates that neither (i) nor (ii) occurs. Let ray  $r$  in  $R_{C_i}$  intersect  $s$ . Let  $s = \overline{v_a v_b}$ ,  $a < b$ , and let  $r(s)$  be the ray starting at  $v_a$  and going through  $v_b$ . If the starting vertex of  $r$  is properly contained in the right half-plane of  $r(s)$ , then we say  $r$  pseudo-hits  $s$ ; otherwise,  $r$  does not pseudo-hit  $s$ . Furthermore,  $R_{C_i}$  is said to pseudo-hit  $s$  if (1) for each ray  $r'$  in  $R_{C_i}$ ,  $r'$  intersecting  $s$  implies that  $r'$  pseudo-hits  $s$ , and (2) there is at least one ray in  $R_{C_i}$  that pseudo-hits  $s$ ; otherwise, we say  $R_{C_i}$  does not pseudo-hit  $s$ . We distinguish the types of hits from  $R_{C_i}$  on  $s$  because only when  $R_{C_i}$  does not pseudo-hit  $s$  can the rays in  $R_{C_i}$  first-hit  $bd_{ab} - v_a$  (it is still possible that  $R_{C_i}$  is blocked from  $bd_{ab}$ ). If  $R_{C_i}$  pseudo-hits  $s$ , then clearly no ray in  $R_{C_i}$  can first-hit  $bd_{ab} - v_a$ .

We define the polar angle of  $s$  as  $\alpha(s) = \alpha(r(s))$  (with obvious meaning for  $\alpha(r(s))$ ). The following lemma characterizes the types of hits.

**Lemma 8** Suppose that a ray  $r_i^+$  of  $R_{C_i}$  intersects a segment  $s$  on  $ICP(C_j)$ ,  $i < j$ . Then  $r_i^+$  pseudo-hits  $s$  iff  $\alpha(r_i^+) > \alpha(s)$ .

**Proof.** Omitted. See the full paper.  $\square$

The next Lemma shows that the lines-vs-segment tests can be used to find out the type of hits from  $R_{C_i}$  on  $s$ .

**Lemma 9** The type of hits from  $R_{C_i}$  on a segment  $s$  of  $ICP(C_j)$ ,  $i < j$ , can be determined by using the lines-vs-segment tests in the same complexity bounds as those for  $Test(L(R_{C_i}), s)$ .

**Proof.** Omitted. See the full paper.  $\square$

If the result of  $Test(L(R_{C_i}), s)$  indicates that  $s$  is neither below a line in  $L(R_{C_i})$  nor above all the lines in  $L(R_{C_i})$ , then one of the following three situations occurs: (1) if  $s$  is an edge of  $C_j$ , then  $s$  is not bad with respect to  $R_{C_i}$ , or (2) if  $s$  is not an edge of  $C_j$  and  $R_{C_i}$  pseudo-hits  $s$ , then  $bd_{ab}$  is not bad with respect to  $R_{C_i}$  (where  $s = \overline{v_a v_b}$ ,  $a < b$ ), or (3) if  $s$  is not an edge of  $C_j$  and  $R_{C_i}$  does not pseudo-hit  $s$ , then  $|ICP(C_j)|$  may be Case (d) (which is to be handled in Phase 1.B).

### 4.1.3 Performing $Test(L(R_{C_i}), s)$ in Step a.3

We need to discuss how the lines-vs-segment tests are actually performed in the algorithm. We only discuss this for Step a.3 (Step a.2 is left to the full paper).

In Step a.3, there are  $O(g^2) = O((m/d)^{2/3})$  pairs of  $C_i$  and  $C_j$ ,  $i < j$ , where  $m = |C|$ . For each pair of  $C_i$  and  $C_j$ , we need to do  $Test(L(R_{C_i}), s)$  for  $O(1)$  segments  $s$  on  $ICP(C_j)$ , if  $ICP(C_j)$  is not of Case (a). There are totally  $O(m/d)$  processors available. Thus  $O(g) = O((m/d)^{1/3})$  processors are allocated to each pair of  $C_i$  and  $C_j$ . It suffices to show how to do one  $Test(L(R_{C_i}), s)$  using  $O(g)$  processors.

Our primary tool is Lemma 4. In Step a.3, to perform  $Test(L(R_{C_i}), s)$  in  $O(\log m)$  time, we need to achieve two things: (i) in performing the test, Lemma 4 is recursively applied only  $O(1)$  times, and (ii) the size of the test range, after (i) is done, is reduced to  $O((m/d)^{1/3})$  (so that the brute force method can take over).

In the conquer stage of Step a.3,  $UPCI(L(R_{C_i}))$  is available (stored in a rank tree), for each  $i = 1, 2, \dots, g$ , and we compute  $UPCI(L(R_C))$  from the  $UPCI(L(R_{C_i}))$ 's. Because  $L(R_C)$  is sorted by slopes, for each  $i$ ,  $bd(UPCI(L(R_{C_i}))) \cap bd(UPCI(L(R_C)))$  consists of at most one connected component, and  $bd(UPCI(L(R_{C_i}))) - bd(UPCI(L(R_C)))$  consists of at most two connected components. After  $UPCI(L(R_C))$  is computed, we still retain the (at most) two connected components of  $bd(UPCI(L(R_{C_i}))) - bd(UPCI(L(R_C)))$  in two separate rank trees. This structure can be readily maintained in every recursion level of Step a.3. Using this structure, it is easy to obtain the information about each  $UPCI(L(R_{C_i}))$  (either from  $UPCI(L(R_C))$  or from the two connected components of  $bd(UPCI(L(R_{C_i}))) - bd(UPCI(L(R_C)))$ ). Furthermore, for a subchain  $C_i^k$  of  $C_i$ , the information about  $UPCI(L(R_{C_i^k}))$  can be obtained from  $UPCI(L(R_{C_i}))$  and  $UPCI(L(R_{C_i^k})) - UPCI(L(R_{C_i}))$ . In general, the information about the common intersection of the half-planes at each level of the algorithm can be obtained from the information stored at its ancestor levels.

WLOG, we assume that in the recursive call on  $(C_i, R_{C_i}, d)$ ,  $C_i$  was partitioned into  $O((m/d)^{2/9})$  subchains (the case where  $C_i$  was as in Step a.2 can be easily taken care of within the desired complexity bounds). In performing  $Test(L(R_{C_i}), s)$ , we repeat the following two steps. (i) Apply Lemma 4 to  $Test(L(R_{C_i}), s)$  using  $O((m/d)^{1/3})$  processors. (Note that the information on  $UPCI(L(R_{C_i^k}))$ , for each subchain  $C_i^k$  of  $C_i$ , is obtained using the structure described in the previous paragraph.) (ii) Either we have found the answer to  $Test(L(R_{C_i}), s)$  in (i), or we have reduced the test to a unique subchain  $C_i^k$  of  $C_i$ , in which case we continue the test by repeating (i) for  $Test(L(R_{C_i^k}), s)$ .

The size of  $C$  is  $m$  at the current recursion level of the algorithm. The following can be easily proved by induction: at the  $i$ -th recursion level below,  $i \geq 1$ , the chain size is  $O(m^{f(i)} d^{1-f(i)})$  and the chain is partitioned into  $O((m/d)^{f(i)(1/3)})$  subchains (for the  $(i+1)$ -th level), where  $f(i) = (2/3)^i$ . We want to stop the recursive procedure for  $Test(L(R_{C_i}), s)$  when we reach the structure for the  $i$ -th level below (in the recursion tree of Phase 1.A) for some  $i \geq 1$ , such that the size of the chain at that level is  $\leq (m/d)^{1/3}$ . Hence we have

$$(m/d)^{1/3} \geq m^{f(i)} d^{1-f(i)},$$

which is equivalent to

$$m^{(1/3)-f(i)} \geq d^{(4/3)-f(i)}.$$



Choosing  $i$  to be 5, the inequality becomes

$$m^{49/243} \geq d^{1+(49/243)},$$

and it holds as long as  $m \geq d^6$ . Thus, the test range size is reduced to  $O((m/d)^{1/3})$  by going down at most *five* levels in the recursion tree of Phase 1.A.

#### 4.1.4 The Procedure for Step a.3

We only sketch the computation for Step a.3, as follows. (1) For every pair of  $C_i$  and  $C_j$  such that  $i < j$  and  $|ICP(C_j)|$  is not **Case (a)**, do  $Test(L(R_{C_i}), s)$  for each segment  $s$  on  $ICP(C_j)$ . (2) For every  $R_{C_i}$ , find all segments  $s$  such that  $s$  is tested in (1) and  $s$  is *not below* any line in  $L(R_{C_i})$ ; furthermore, among these segments  $s$ , find  $s_i = \overline{v_a v_b}$  such that the vertex indices  $a$  and  $b$  for  $s_i$  are no bigger than the vertex indices for any of these segments, where  $a < b$  (by Lemma 1,  $R_{C_i}$  cannot reach any of these segments other than  $s_i$ ). Let  $s_i$  be on  $ICP(C_{j'})$  for some  $j'$ . (3) If  $|ICP(C_{j'})|$  is **Cases (b) or (c)**, or  $s_i$  is above all the lines in  $L(R_{C_i})$ , or  $R_{C_i}$  pseudo-hits  $s_i$ , then the bad edges on the subchain of  $bd(P)$  from the last vertex of  $C_i$  counterclockwise to  $v_b$  (on  $C_{j'}$ ) can be determined; otherwise,  $|ICP(C_{j'})|$  is **Case (d)** and  $R_{C_i}$  does not pseudo-hit  $s_i$  (in this case, we will decide in **Phase 1.B** the bad edges on the subchain of  $bd(P)$  from  $C_i$  counterclockwise to  $C_{j'}$ ).

## 4.2 Phase 1.B

By using the structure built in Phase 1.A, this phase finishes the identification of the edges in  $HE^+(P)$ . **Phase 1.B** has the same algorithmic outline as **Phase 1.A**. Its computation follows the recursion tree of **Phase 1.A**, level by level, from the root down to the leaves.

The following operation is typical in this phase: identify the bad edges on  $C'$  by using  $R_C$ , where  $C \neq C'$  and  $|ICP(C')|$  is **Case (d)**. What we do is to shoot  $R_C$  onto each subchain  $C'_i$  of  $C'$ . Thus, we encounter one of the four cases at each  $ICP(C'_i)$ , and we may have to continue the recursive computation on a unique  $C'_j$  where  $|ICP(C'_j)|$  is again **Case (d)**.

We will not go into the details of this procedure, since these details are similar to those in **Phase 1.A**. But we would like to point out one thing that must be handled carefully in this top-down procedure, as follows. If one keeps using the same data structure for  $R_C$  (i.e.,  $UPCI(L(R_C))$ ) in the upcoming recursion levels below  $C'$ , then the same  $O(\log |C'|)$  time will be spent at each level. This will not give an  $O(\log n)$  time algorithm; instead, the time bound so resulted will be  $O(\log n \log \log n)$ . To avoid this inefficiency, what we do when shooting  $R_C$  onto  $C'$  is to partition  $C$  into  $g = \lceil |C|/d \rceil^{1/3}$  subchains  $C_1, C_2, \dots, C_g$  (as in **Phase 1.A**). Then, by using the structures for the  $UPCI(L(R_{C_k}))$ 's (rather than  $UPCI(L(R_C))$ ), every  $R_{C_k}$  is independently shot onto each of the  $C'_i$ 's. Note that each  $UPCI(L(R_{C_k}))$  can be recovered from  $UPCI(L(R_C))$  and from the (at most) two connected components of  $UPCI(L(R_{C_k})) - UPCI(L(R_C))$  in the desired complexity bounds.

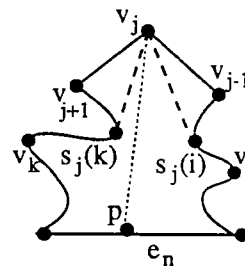


Figure 5: Illustrating Lemma 10.

## 5 Checking the Weak Visibility of a Polygon from an Edge

This section concerns the following problem: check whether or not a simple polygon  $P$  is weakly visible from a specified edge  $e$  of  $P$  (i.e., the case solved sequentially in [2, 13]). We show how to solve this problem in  $O(\log n)$  time using  $O(n/\log n)$  processors. Our solution consists of two parts: a preprocess and a two-phase procedure. WLOG, let  $e = e_n$ ,  $e_n$  be on the  $x$ -axis, and  $x(v_1) > x(v_n)$ . The preprocess reduces the problem to that of checking the weak visibility of a simple polygon from a *convex* edge. Also, the preprocess checks to make sure that for every  $i$ ,  $1 < i < n$ , none of the following local conditions is satisfied: (i)  $e$  does not intersect the left half-plane of  $r(e_i)$ , and (ii)  $\alpha(r(e_{i-1})) < \pi$ ,  $\alpha(r(e_i)) > \pi$ , and  $v_i$  is nonconvex, where  $r(e_i)$  is the ray starting at  $v_i$  and containing  $e_i$ . (For any  $i$ ,  $P$  is not weakly visible from  $e$  if either (i) or (ii) is satisfied.) The two-phase procedure handles the problem of checking the weak visibility of  $P$  from a convex edge (say,  $e_n$ ).

### 5.1 The Basic Idea

We still use  $ICP(bd_{ij})$  to represent the directed shortest path from  $v_i$  to  $v_j$  that goes through *only* the vertices of  $bd_{ij}$  (i.e., the computation of  $ICP(bd_{ij})$  is based only on  $bd_{ij}$  and disregards  $bd(P) - bd_{ij}$ ). Note that in general,  $ICP(bd_{ij})$  does not necessarily make consistent right turns for  $i < j$ ; furthermore,  $ICP(bd_{ij})$  may even intersect the exterior of  $P$  (because it can intersect  $bd(P) - bd_{ij}$ ).

The lemma below gives the basic idea for solving this weak visibility problem. For  $i < i'$  (resp.,  $i > i''$ ), let  $s_i(i')$  (resp.,  $s_i(i'')$ ) be the segment on  $ICP(bd_{i'i'})$  (resp.,  $ICP(bd_{i''i'})$ ) that contains  $v_i$ .

**Lemma 10** *If  $P$  is weakly visible from  $e_n$ , then for any  $i, j$ , and  $k$ ,  $1 \leq i < j < k \leq n$ , a scan of the interior angle of  $P$  at  $v_j$ , from edge  $e_{j-1}$  clockwise to edge  $e_j$ , encounters  $e_{j-1}$ ,  $s_j(i)$ ,  $s_j(k)$ , and  $e_j$ , in that order (cf. Fig. 5).*

**Proof.** Omitted. See the full paper.  $\square$

By Lemma 10, if  $s_j(i)$  and  $s_j(k)$  are not in correct order with  $e_{j-1}$  and  $e_j$  for some  $i, j$ , and  $k$ ,  $i < j < k$ , then  $P$  is not weakly visible from  $e$ . If they are in correct order, then there is a ray (say, the one starting at  $v_j$  and containing  $s_j(i)$ ) separating  $ICP(bd_{ij})$  from  $ICP(bd_{jk})$ . Let the ray starting at  $v_j$  and containing  $s_j(i)$  be denoted by  $r(s_j(i))$ . If  $r(s_j(i))$  separates  $ICP(bd_{ij})$  from  $ICP(bd_{jk})$ , then  $ICP(bd_{ik})$  can be computed efficiently from  $ICP(bd_{ij})$  and  $ICP(bd_{jk})$ . This is the idea used in the recursive algorithm.

It is known that if there is a ray (e.g.,  $r(s_j(1))$ ) separating  $ICP(bd_{1j})$  from  $ICP(bd_{jn})$  for every  $j$ ,  $1 < j < n$ , then  $P$  is weakly visible from  $e$  (e.g., see [12]). Our ultimate goal,

therefore, is to compute  $s_j(1)$  and  $s_j(n)$  for every  $j$  and check their relative order based on Lemma 10. Clearly, our main problem is to compute all the  $s_j(1)$ 's and  $s_j(n)$ 's.

The algorithm for computing the  $s_j(1)$ 's and  $s_j(n)$ 's consists of two phases. **Phase 2.A** uses Lemma 10 to compute the internal convex paths on certain subchains of  $bd(P)$  (hence at every recursion level, either it succeeds in computing  $ICP(C)$  for a subchain  $C$  of  $bd(P)$ , or it concludes that  $P$  is not weakly visible from  $e$ ). After the first phase is completed, if  $P$  is still not known to be non-weakly visible from  $e$ , then the algorithm proceeds with **Phase 2.B**. For every  $j$ , **Phase 2.B** constructs  $ICP(bd_{1j})$  and  $ICP(bd_{jn})$  (by using the internal convex paths obtained in **Phase 2.A**) and reports  $s_j(1)$  and  $s_j(n)$ .

We need to characterize  $ICP(bd_{ij})$  before going to the two-phase algorithm. For  $bd_{ij}$ ,  $i < j$ , we say  $ICP(bd_{ij})$  is in order if the vertices of  $ICP(bd_{ij})$  are in increasing order of vertex indices in the walk of  $ICP(bd_{ij})$  from  $v_i$  to  $v_j$ . For a segment  $s = \overline{v_a v_b}$ ,  $a < b$ , let  $r(s)$  be the ray starting at  $v_a$  and containing  $s$ . The following lemmas enable us to compute the internal convex paths efficiently.

**Lemma 11** *Suppose  $P$  is not known to be non-weakly visible from  $e$ . For  $i < j$ , if  $ICP(bd_{ij})$  is in order and makes only right turns, then for two consecutive segments  $\overline{v_a v_b}$  and  $\overline{v_b v_c}$  on  $ICP(bd_{ij})$ ,  $a < b < c$ ,  $\alpha(r(\overline{v_b v_c})) < \alpha(r(\overline{v_a v_b}))$ .*

**Proof.** If  $\alpha(r(\overline{v_a v_b})) \geq \pi$ , then the lemma is obviously true because  $ICP(bd_{ij})$  is in order and makes only right turns. If  $\alpha(r(\overline{v_a v_b})) < \pi$ , then either  $\alpha(r(\overline{v_b v_c})) < \alpha(r(\overline{v_a v_b}))$  or  $\alpha(r(\overline{v_b v_c})) > \alpha(r(\overline{v_a v_b})) + \pi$  (since  $ICP(bd_{ij})$  makes only right turns). But  $\alpha(r(\overline{v_b v_c})) > \alpha(r(\overline{v_a v_b})) + \pi$  and  $ICP(bd_{ij})$  being in order imply that the local condition (ii) tested in the preprocess is satisfied at  $v_b$ , a contradiction.  $\square$

**Corollary 1** *Suppose  $P$  is not known to be non-weakly visible from  $e$ . For  $i < j$ , if  $ICP(bd_{ij})$  is in order and makes only right turns, then the polar angles of the segments on  $ICP(bd_{ij})$  are in sorted order.*

**Proof.** An immediate consequence of Lemma 11.  $\square$

**Lemma 12** *Suppose  $P$  is not known to be non-weakly visible from  $e$ . For  $i, j$ , and  $k$ ,  $1 \leq i < j < k \leq n$ , if both  $ICP(bd_{ij})$  and  $ICP(bd_{jk})$  are in order and make only right turns, then  $ICP(bd_{ik})$  is in order and makes only right turns.*

**Proof.** Omitted. See the full paper.  $\square$

**Lemma 13** *Suppose  $P$  is not known to be non-weakly visible from  $e$ . For  $i < j$ , if  $ICP(bd_{ij})$  is in order and makes only right turns, then  $ICP(bd_{ij})$  can be partitioned into at most two subpaths  $C'$  and  $C''$  such that  $C'$  (resp.,  $C''$ ) is completely on the convex hull of  $C'$  (resp.,  $C''$ ).*

**Proof.** Let  $l$  be the horizontal line tangent to  $ICP(bd_{ij})$  (let  $l$  touch  $ICP(bd_{ij})$  at  $v_a$ ), such that  $ICP(bd_{ij}) - v_a$  is above  $l$ . By Corollary 1, the polar angles of the segments on  $ICP(bd_{ia})$  (resp.,  $ICP(bd_{aj})$ ) are all  $\geq \pi$  (resp.,  $\leq \pi$ ). Hence,  $ICP(bd_{ia})$  (resp.,  $ICP(bd_{aj})$ ) is completely on the convex hull of  $ICP(bd_{ia})$  (resp.,  $ICP(bd_{aj})$ ).  $\square$

## 5.2 Phase 2.A

This phase consists of three steps: **Step c.1**, **Step c.2**, and **Step c.3**. Its algorithmic outline and recurrences for the time and processor complexities are similar to those of **Phase 1.A**. Given input  $(C, |C|, d)$  to each recursive

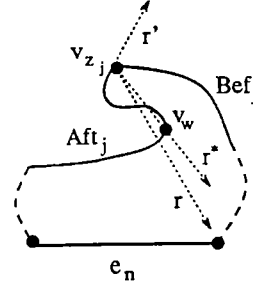


Figure 6: An example for illustrating Lemma 14.

call, where  $C$  is a subchain of  $bd_{1n}$ , either an answer “no” (indicating that  $P$  is already known to be non-weakly visible from  $e$ ) or  $ICP(C)$  is returned. All internal convex paths in this phase are stored in rank trees. Here we only discuss the computation in **Step c.3**.

In **Step c.3**, a chain  $C = bd_{st}$  is partitioned into  $g = (m/d)^{1/3}$  subchains  $C_1, C_2, \dots, C_g$ , where  $|C| = m$  and  $s < t$ . There are  $O(g^3)$  processors available. After the recursive calls for the  $C_i$ 's, if  $P$  is not known to be non-weakly visible from  $e$ , then  $ICP(C_1), ICP(C_2), \dots, ICP(C_g)$  are all available. Let  $v_z$  be the common vertex of  $C_j$  and  $C_{j+1}$ , for each  $j = 1, 2, \dots, g-1$ . For each  $v_z$ , let  $Bef_j$  (resp.,  $Aft_j$ ) denote the subchain of  $C$  before (resp., after)  $v_z$ , that is,  $Bef_j = bd_{s z_j}$  (resp.,  $Aft_j = bd_{z_j t}$ ). One important operation in this step is to find  $s_{z_j}(s)$  and  $s_{z_j}(t)$  for each  $v_z$ . This is because if  $s_{z_j}(s)$  and  $s_{z_j}(t)$  are in correct order with  $e_{z_j-1}$  and  $e_{z_j}$  (cf. Lemma 10) for each  $v_z$ , then there is a ray separating every pair of  $ICP(C_i)$  and  $ICP(C_j)$ , which makes the computation of  $ICP(C)$  possible. Note that  $ICP(Bef_j)$  and  $ICP(Aft_j)$  are in general not explicitly available for the computation of  $s_{z_j}(s)$  and  $s_{z_j}(t)$ . We only explain how to use  $Aft_j$  to compute  $s_{z_j}(t)$  (the case for  $s_{z_j}(s)$  is similar).

The following lemma is useful in computing the  $s_{z_j}(s)$ 's.

**Lemma 14** *Suppose that  $P$  is not known to be non-weakly visible from  $e$  and that every  $ICP(C_k)$  is in order and makes only right turns. For a  $v_z$ , let  $r$  be the ray starting at  $v_z$  and going through  $v_1$ , and let  $r'$  be the ray on line  $l(\overline{v_z v_n})$  which starts at  $v_z$  and does not contain  $v_n$  (see Fig. 6). For any  $i \in \{j+1, j+2, \dots, g\}$ , if the intersection of the left half-plane  $lfp(r)$  of  $r$  and the right half-plane  $rtp(r')$  of  $r'$  intersects  $ICP(C_i) - v_z$ , then  $P$  is not weakly visible from  $e$ .*

**Proof.** Omitted. See the full paper.  $\square$

For example, in Fig. 6,  $ICP(C_i) - v_z$  intersects  $lfp(r) \cap rtp(r')$  in such a way that a vertex  $v_w$  on  $ICP(C_i) - v_z$  is contained in  $lfp(r) \cap rtp(r')$ . Let  $r^*$  be the ray starting at  $v_z$  and going through  $v_w$ . Then  $e_n$  does not intersect the left half-plane of  $r^*$ . Hence  $P$  is not weakly visible from  $e_n$ .

Observe that in **Phase 2.A**, if  $P$  is not known to be non-weakly visible from  $e$ , then all the  $ICP(C_k)$ 's have the properties stated in Lemmas 11 and 12 (that all the  $ICP(C_k)$ 's are in order and make only right turns can be easily proved by induction by using Lemma 12). Therefore, we can check whether each  $ICP(C_i) - v_z$  intersects  $lfp(r) \cap rtp(r')$  in  $O(\log m)$  time using  $O(g)$  processors.

We compute the  $s_{z_j}(t)$ 's as follows. We first do the checking based on Lemma 14 for every  $v_z$  (in  $O(\log m)$  time using  $O(g^2)$  processors). Suppose  $P$  is still not known to be non-weakly visible from  $e$ , then we compute the common tangent between  $v_z$  and  $ICP(C_i)$  for each  $C_i \subseteq Aft_j$ , by

using the ray starting at  $v_{z_j}$  and containing  $\overline{v_1 v_{z_j}}$  as a ray separating  $v_{z_j}$  from  $ICP(C_i)$ . This is easily computed in  $O(\log m)$  time using  $O(g)$  processors for each  $C_i$ . Suppose among the  $O(g)$  common tangents so obtained, the tangent between  $v_{z_j}$  and  $ICP(C_{i'})$  is the one first encountered if we use a ray originating at  $v_{z_j}$  to scan around  $v_{z_j}$  clockwise, by starting at  $\overline{v_1 v_{z_j}}$ . Then  $s_{z_j}(t)$  is the common tangent  $\overline{v_{z_j} v_b}$  between  $v_{z_j}$  and  $ICP(C_{i'})$  ( $v_b$  is on  $ICP(C_{i'})$ ). Therefore, each  $s_{z_j}(t)$  is computable in  $O(\log m)$  time using  $O(g^2)$  processors.

Given  $s_{z_j}(s)$  and  $s_{z_j}(t)$  for each  $v_{z_j}$ , if  $s_{z_j}(s)$  and  $s_{z_j}(t)$  are in correct order with  $e_{z_{j-1}}$  and  $e_{z_j}$  (cf. Lemma 10), then the ray  $r(s_{z_j}(s))$  (starting at  $v_{z_j}$  and containing  $s_{z_j}(s)$ ) separates  $Bef_j$  from  $Aft_j$ , and hence it separates  $ICP(C_i)$  from  $ICP(C_k)$  for each  $C_i \subseteq Bef_j$  and each  $C_k \subseteq Aft_j$ . After  $r(s_{z_j}(s))$  is available, we can compute the common tangent between each pair of  $ICP(C_i)$  and  $ICP(C_k)$  in  $O(\log m)$  time using  $O(g)$  processors (by using [8, 5]).

Next, using the  $O(g^2)$  common tangents so obtained, we build a complete binary tree of internal convex paths whose leaves are associated with  $C_1, C_2, \dots, C_g$ , respectively. The root of the tree is associated with  $bd_{st} = C$  and it stores  $ICP(bd_{st})$  (in a rank tree).

The complete binary tree of internal convex paths above is denoted by  $T_C$ . The root of  $T_C$  is denoted by  $root(T_C)$  and the left (resp., right) child of an internal node  $u$  of  $T_C$  is denoted by  $lch(u)$  (resp.,  $rch(u)$ ). The information associated with each node of  $T_C$  is as follows.  $root(T_C)$  is associated with  $bd_{st}$  and it stores  $ICP(bd_{st})$ . For an internal node  $u$ , the subchain of  $C$  associated with  $u$  is the union of the subchains associated with the descendent leaves of  $u$ . Suppose that the subchain associated with  $u$  is  $bd_{ac}$  and the subchains associated with  $lch(u)$  and  $rch(u)$  are respectively  $bd_{ab}$  and  $bd_{bc}$ ,  $a < b < c$ . Observe that  $ICP(bd_{ab}) - ICP(bd_{ac})$  (resp.,  $ICP(bd_{bc}) - ICP(bd_{ac})$ ) consists of at most one connected component. The information stored at  $lch(u)$  (resp.,  $rch(u)$ ) is  $ICP(bd_{ab}) - ICP(bd_{ac})$  (resp.,  $ICP(bd_{bc}) - ICP(bd_{ac})$ ), represented by a rank tree. The height of  $T_C$  is  $O(\log m)$ . This tree structure has been used in [9] for triangulating a one-sided monotone polygon (in [9], each node of the tree stores a portion of the convex hull for its associated subchain; see [9] for the definitions and the details). The construction of  $T_C$  is done by an algorithm in [9, 5], in  $O(\log m)$  time using  $O(g^3)$  processors.

In total, the procedure for Step c.3 requires  $O(\log m)$  time and  $O(g^3)$  processors.

### 5.3 Phase 2.B

At the root of the recursion tree of Phase 2.A, if  $P$  has been decided to be non-weakly visible from  $\epsilon$ , then the algorithm stops. Otherwise, we proceed with Phase 2.B to compute  $s_j(1)$  and  $s_j(n)$  for every  $j$ . WLOG, we just show how to compute the  $s_j(1)$ 's (the computation for the  $s_j(n)$ 's is similar).

In Phase 2.A, we have constructed a complete binary tree of the internal convex paths (cf. Step c.3). We denote this tree by  $T$ .  $T$  has  $O(n/d) = O(n/\log n)$  nodes. The root of  $T$  stores  $ICP(bd_{1n})$ . Each non-root node uses a rank tree to store at most one connected portion of the internal convex path for the subchain of  $bd_{1n}$  associated with that node. Hence there are totally  $O(n/\log n)$  internal convex paths stored at the nodes of  $T$ .

The algorithmic structure of Phase 2.B is the same as the structure of tree  $T$ . The procedure follows a top-down paradigm. It starts at  $root(T)$ , then goes to the two children

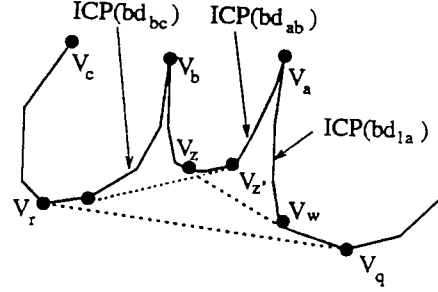


Figure 7: Illustrating Lemma 15.

of  $root(T)$ , and so on, level by level, until the leaves of  $T$  are reached. Because the height of  $T$  is  $O(\log n)$ , we need to process each level of  $T$  (in most part of the procedure) in  $O(1)$  time, in order to achieve an  $O(\log n)$  time algorithm.

Let  $u \neq root(T)$  be an internal node of  $T$ . Let the chains associated with  $u$ ,  $lch(u)$ , and  $rch(u)$  be respectively  $bd_{ac}$ ,  $bd_{ab}$ , and  $bd_{bc}$ ,  $1 < a < b < c$ . The main computation of Phase 2.B is based on the lemma below. WLOG, we assume that up to the level of  $u$ ,  $P$  is not known to be non-weakly visible from  $\epsilon$ .

**Lemma 15** *Let the common tangent between  $ICP(bd_{1a})$  and  $ICP(bd_{ac})$  touch  $ICP(bd_{1a})$  at  $v_q$  and  $ICP(bd_{ac})$  at  $v_r$ , the common tangent between  $ICP(bd_{1a})$  and  $ICP(bd_{ab})$  touch  $ICP(bd_{1a})$  at  $v_w$  and  $ICP(bd_{ab})$  at  $v_z$ , and the common tangent between  $ICP(bd_{ab})$  and  $ICP(bd_{bc})$  touch  $ICP(bd_{ab})$  at  $v_{z'}$  (see Fig. 7). Then the common tangent between  $ICP(bd_{1a})$  and a vertex on  $bd_{ab}$  touches  $ICP(bd_{1a})$  on  $ICP(bd_{wa})$  (with  $w \geq q$ ), and the common tangent between  $ICP(bd_{1b})$  and a vertex on  $bd_{bc}$  touches  $ICP(bd_{1b})$  either on  $ICP(bd_{qb})$  ( $= ICP(bd_{qw}) \cup \overline{v_w v_z} \cup ICP(bd_{zb})$ ) (when  $v_r \in ICP(bd_{bc})$ ) or on  $ICP(bd_{z'b})$  (when  $v_r \in ICP(bd_{ab})$ ).*

**Proof.** Omitted. See the full paper.  $\square$

We call  $ICP(bd_{qa})$  in Lemma 15 the left internal convex path to the chain (i.e.,  $bd_{ac}$ ) associated with  $u$  and denote it by  $LICP_u$ . Likewise,  $LICP_{lch(u)} = ICP(bd_{wa})$  and  $LICP_{rch(u)} = ICP(bd_{qb})$  (when  $v_r \in ICP(bd_{bc})$ ). Observe that  $LICP_{lch(u)}$  and  $LICP_{rch(u)}$  are disjoint except possibly at  $v_w$ . For convenience, we also let  $ICP_u$  denote  $ICP(bd_{ac})$ .

The computation based on Lemma 15 involves the following main operations: (i) computing the common tangent between two internal convex paths, (ii) splitting an internal convex path into two subpaths, and (iii) combining two internal convex paths together to form another path. We need to perform each of these operations in  $O(1)$  time for most part of the algorithm. Obviously, an appropriate data structure for representing the internal convex paths is essential in this process.

We can no longer use rank trees to represent the internal convex paths because of their logarithmic heights. Instead, we represent the internal convex path stored at each node of  $T$  by an array. The conversion of the rank tree representation into the array representation for the internal convex paths is easy to do in  $O(\log n)$  time using  $O(n/\log n)$  processors. We henceforth assume that the internal convex path stored at each node of  $T$  is represented by an array.

The computation on a node  $u$  of  $T$  involves  $ICP_u$  and  $LICP_u$ , which need to be represented in such a way that enables us to compute their common tangent in  $O(1)$  time.

Recall that  $lch(u)$  (resp.,  $rch(u)$ ) stores the portion of  $ICP_{lch(u)}$  (resp.,  $ICP_{rch(u)}$ ) that is not on  $ICP_u$ . Only  $root(T)$  has its internal convex path, i.e.,  $ICP(bd_{1n})$ , stored in a single array.  $lch(root(T))$  may have one portion

of  $ICP(bd_{1(n/2)})$  stored in  $root(T)$ . The left child of  $lch(root(T))$  may have one portion of  $ICP(bd_{1(n/4)})$  stored in  $lch(root(T))$ , which may again have a portion stored in  $root(T)$ . In general, a node  $u$  may have a portion of  $ICP_u$  stored in each of its ancestors in  $T$ . That is,  $ICP_u$  is obtained from the  $O(\log n)$  arrays stored at its ancestors. Therefore, we represent  $ICP_u$  by using  $O(\log n)$  subarrays.

Let  $ICP_u$  be represented by  $A_u(1), A_u(2), \dots, A_u(k)$ , in order, where each  $A_u(i)$  is a subarray of an array stored at an ancestor of  $u$ . Each  $A_u(i)$  is specified by two pointers, one pointing to the first element of  $A_u(i)$  and the other pointing to the last element of  $A_u(i)$ . Suppose those  $2k$  pointers are available in the beginning of the computation at  $u$ , where  $u$  is an internal node of  $T$ . In the process at  $u$ , we split  $ICP_u$  at  $v_{z'}$ , where  $v_{z'}$  (resp.,  $v_{z''}$ ) on  $ICP_{lch(u)}$  (resp.,  $ICP_{rch(u)}$ ) is the endpoint of the common tangent between  $ICP_{lch(u)}$  and  $ICP_{rch(u)}$ , as follows. Let  $v_{z'}$  be contained in  $A_u(i)$  for some  $i$ . Then  $A_u(i)$  is split at  $v_{z'}$  into two subarrays  $A'_u$  and  $A''_u$ , such that  $v_{z'}$  is the last element of  $A'_u$  and  $v_{z''}$  is the first element of  $A''_u$ . Let the representation of  $ICP_{lch(u)}$  be the union of  $ICP_{lch(u)} \cap ICP_u$  (represented by  $O(\log n)$  pieces from  $u$ ) and  $ICP_{lch(u)} - ICP_u$  (one single piece stored at  $lch(u)$ ). That is,  $ICP_{lch(u)}$  is represented by  $A_u(1), A_u(2), \dots, A_u(i-1), A'_u$ , and  $B_{lch(u)}$ , in order, where  $B_{lch(u)}$  is the array representing  $ICP_{lch(u)} - ICP_u$ . The similar thing is done for  $ICP_{rch(u)}$ .

We associate with  $ICP_u$   $k$  size parameters  $size_u(1), size_u(2), \dots, size_u(k)$ , where  $size_u(i) = |A_u(1)| + \dots + |A_u(i)|$ . Using the size parameters, we can quickly access the  $j$ -th vertex on  $ICP_u$  for any  $j$  (say, in  $O(1)$  time using  $O((\log n)^{1/2})$  processors). When  $ICP_u$  is split to form  $ICP_{lch(u)}$  and  $ICP_{rch(u)}$ , the size parameters for the representation of  $ICP_{rch(u)} \cap ICP_u$  can be easily updated (in  $O(1)$  time using  $O(\log n)$  processors) because we just need to subtract/add a same number from/to all the parameters in the list for  $rch(u)$  and then add a new parameter (for  $ICP_{rch(u)} - ICP_u$ ) to the beginning of the list. The update on the parameters for  $lch(u)$  is even easier (only a new term is added to the end of the parameter list).

It can be shown that  $LICP_u$  is also represented by  $O(\log n)$  subarrays. Hence the "split" and "combine" operations on  $LICP_u$  are also the same as on  $ICP_u$ .

With these representations for the  $ICP_u$ 's and  $LICP_u$ 's, the common tangent between every pair of  $ICP_u$  and  $LICP_u$ , at each level of  $T$  (in most part of the algorithm), can be computed in  $O(1)$  time and  $O(n/\log n)$  processors by using [1].

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