An optimal scanning sensor activation policy for parameter estimation of distributed systems

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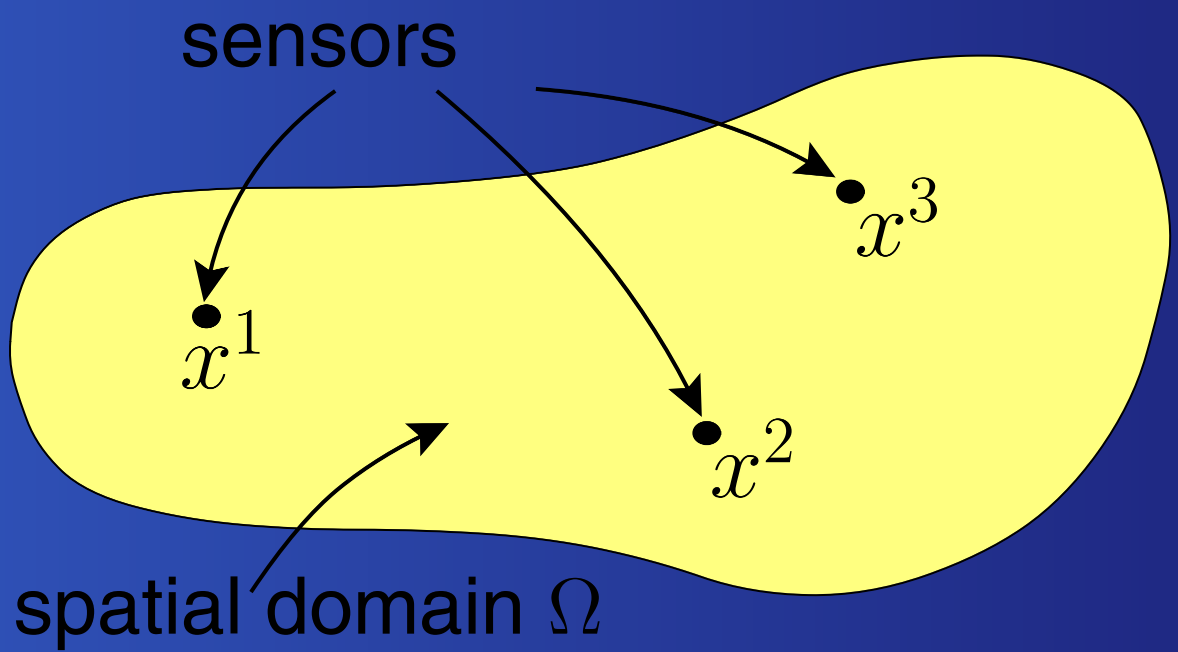
Spatiotemporal dynamics

Distributed parameter system—dynamic system whose state depends on both time and space; its model is known up to a vector of unknown parameters θ.

Observations—using sensors in order to estimate θ.

Subject of the talk



Problem: How to determine “optimal” sensor locations?

Motivations

- calibration of smog prediction models
- data assimilation in meteorology and oceanography
- smart material systems
- fault detection and isolation in DPSs
- groundwater resources management
- recovery of valuable minerals and hydrocarbon
- inspection in hazardous environments

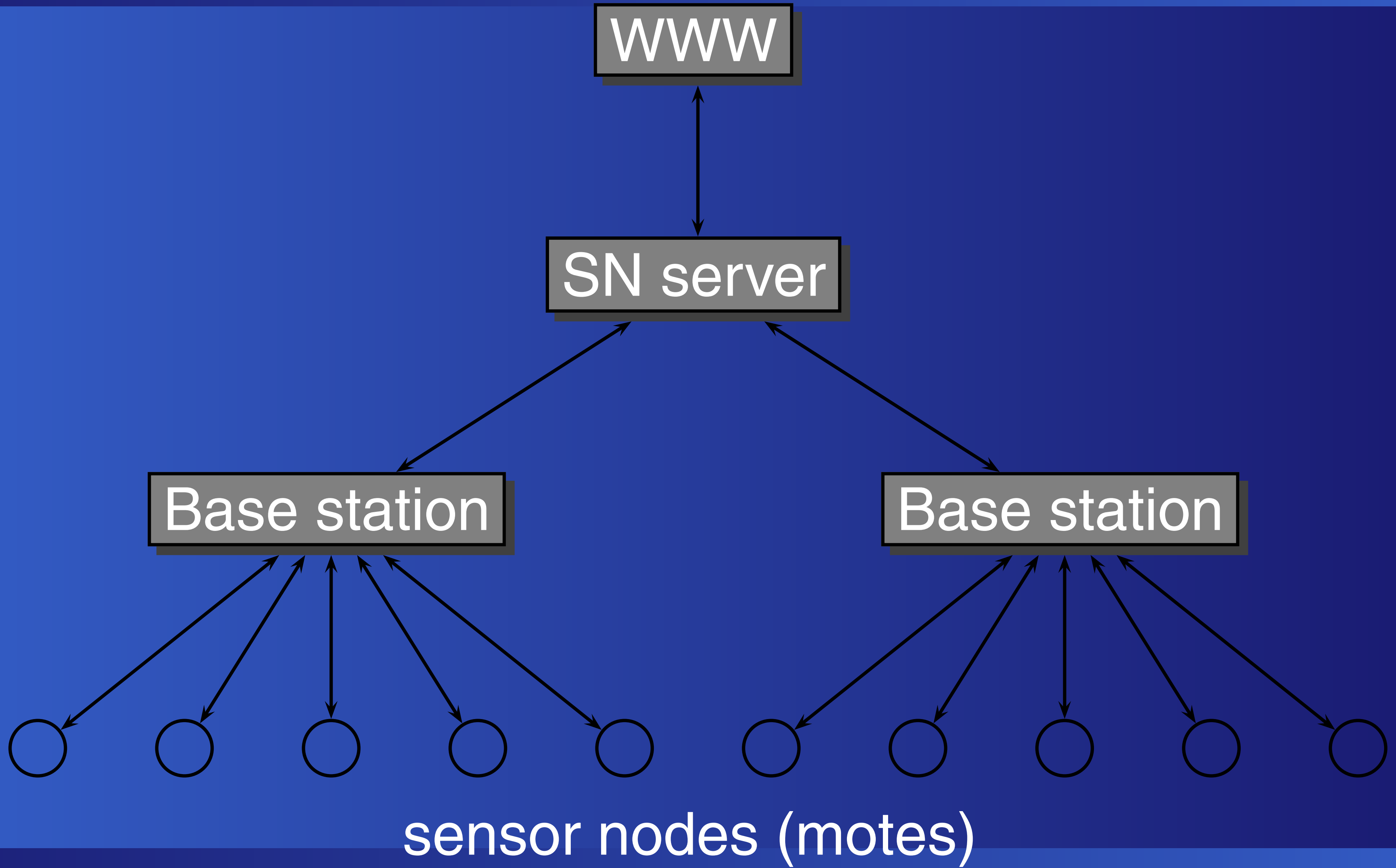
Technology: Wireless sensor networks

Sensor network—an array of sensors of diverse type interconnected by a communication network.

A large number of inexpensive, miniature and low-power SN nodes can be deployed throughout a physical space, providing dense sensing close to physical phenomena.

WSNs incorporate technologies from sensing, communication and computing.

Generic architecture



SN nodes

Motes must be low cost, low power (for long-term operation), automated (maintenance free), robust (to withstand errors and failures), and non-intrusive.

- Hardware: a microprocessor, data storage, sensors, AD-converters, a data transceiver (Bluetooth), controllers, and an energy source
- Software: TinyOS

They are already manufactured (Crossbow, Intel).

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Another problem: How to choose a minimal number of sensors?

Pros and cons

A reason for not using all the available sensors could be the reduction of the observation system complexity and the cost of operation.

- Another interpretation: mobile sensors.
- Best points are tracked.
- Technology makes it affordable.

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But:

- Lack of efficient algorithms.
- Combinatorial complexity excludes naive approaches.

Existing approaches

- Conversion to a non-linear problem of state estimation (Malebranche, 1988; Korbicz *et al.*, 1988);
- Employing random fields analysis (Kazimierczyk, 1989; Sun, 1994);
- Formulation in the spirit of optimum experimental design (Uspenskii & Fedorov, 1975; Quereshi *et al.*, 1980; Rafajłowicz, 1978–1995; Kammer, 1990; 1992; Sun, 1994; Point *et al.*, 1996; Vande Wouwer *et al.*, 1999; Patan, 2004; Uciński, 1999–2007).

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System description

Let $\Omega \subset \mathbb{R}^2$ be a region with boundary $\partial\Omega$.

$$\frac{\partial y}{\partial t} = \mathcal{F}\left(\boldsymbol{x}, t, y, \ \boldsymbol{\theta}\ \right) \quad \text{in } \Omega \times T,$$

subject to appropriate I & BCs, where

- $\boldsymbol{x}$ – spatial variable, t – time, $T = (0, t_f)$;
- $y = y(\boldsymbol{x}, t)$ – state variable;
- $\mathcal{F}$ – well-posed differential operator which involves spatial derivatives;
- $\boldsymbol{\theta} \in \mathbb{R}^m$ – vector of *unknown* parameters.

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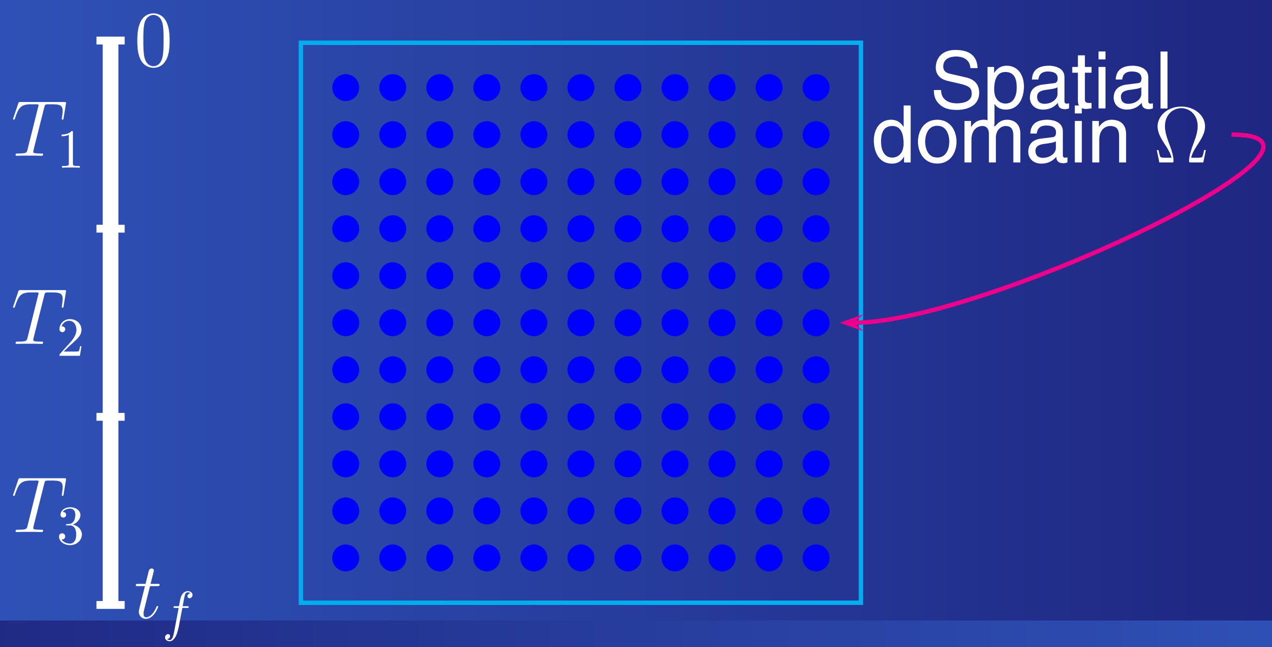
Scanning observations

Given I stationary located sensors, partition the observation horizon $T = (0, t_f)$ into subintervals T_k, $k = 1, \ldots, K$ and activate a best n-element subset of sensors over each T_k.



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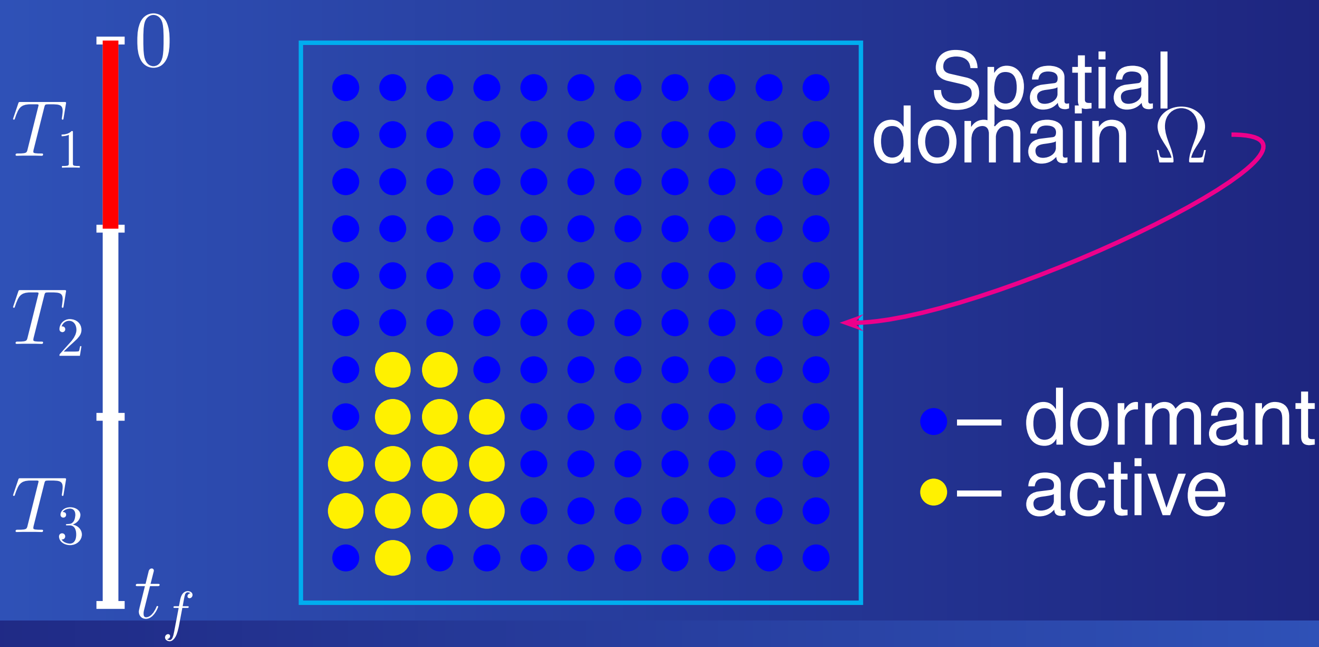

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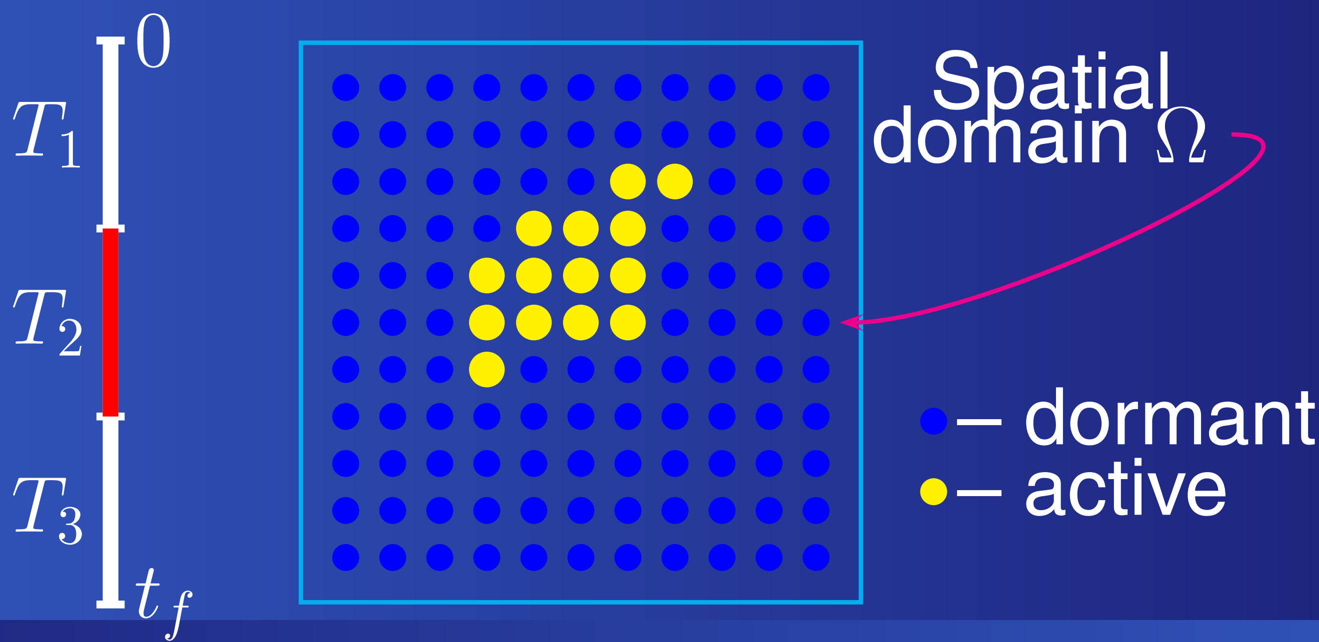

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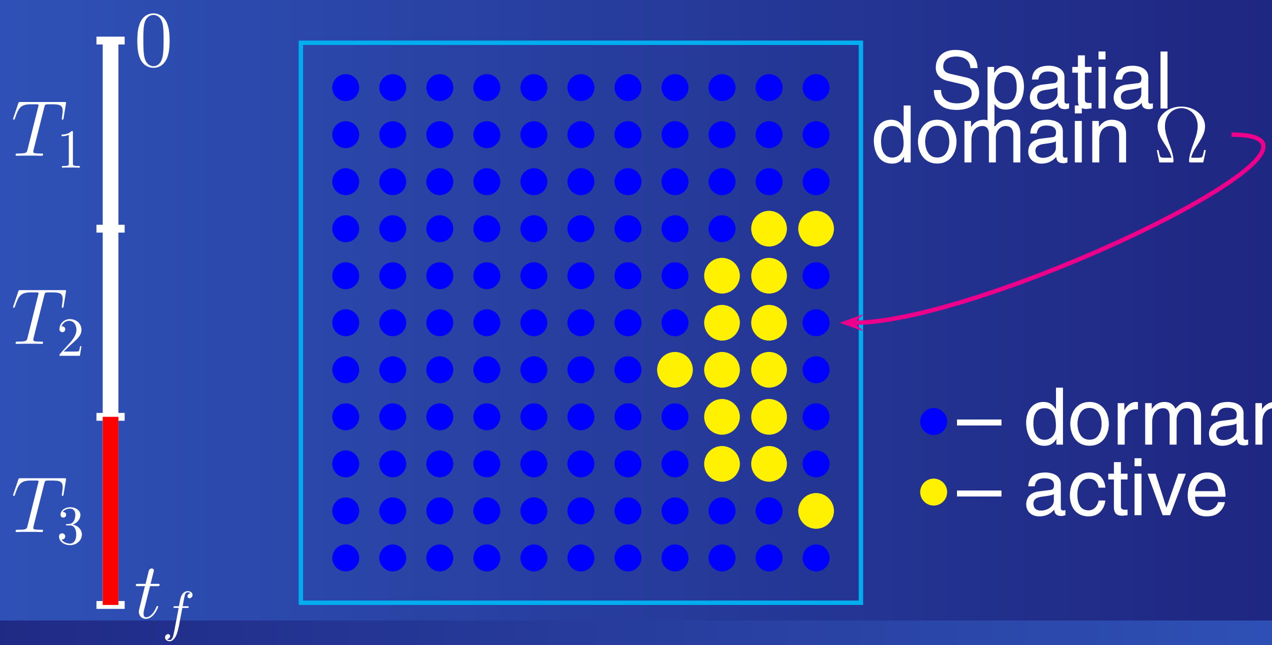

Output equation

At Stage k we have time subinterval T_k and

$$\boxed{z_{ki}(t) = v_{ki}\left[y(\boldsymbol{x}^i, t; \boldsymbol{\theta}) + \varepsilon(\boldsymbol{x}^i, t)\right]}$$

for $t \in T_k$ and $i = 1, \dots, I$, where $\varepsilon(\,\cdot\,,\,\cdot\,)$ – white Gaussian measurement noise,

$$v_{ki} = \begin{cases} 1 & \text{if the } i\text{-th sensor is active over } T_k \\ 0 & \text{otherwise} \end{cases}$$

Least-squares criterion

The LS estimate of $\boldsymbol{\theta}$ is the one which minimizes

$$\mathcal{J}(\boldsymbol{\theta}) = \sum_{i=1}^{I} \sum_{k=1}^{K} v_{ki} \int_{T_k} \Big[z_{ki}(t) - \hat{y}(\boldsymbol{x}^i, t; \boldsymbol{\theta}) \Big]^2 \, \mathrm{d}t$$

where $\hat{y}(\,\cdot\,,\,\cdot\,;\boldsymbol{\theta})$ stands for the solution to the state equation corresponding to a given value of $\boldsymbol{\theta}$.

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$$\operatorname{cov}\hat{\boldsymbol{\theta}} = E\Big\{(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta})^T\Big\} \succeq M^{-1}$$

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We have $\operatorname{cov}\hat{\boldsymbol{\theta}} = M^{-1}$ provided that an estimator is efficient. But what is M?

Fisher Information Matrix (FIM)

$$\boldsymbol{M}(\boldsymbol{v}) = \sum_{k=1}^{K} \sum_{i=1}^{I} v_{ki} \boldsymbol{M}_{ki}$$

$$\boldsymbol{M}_{ki} = \int_{T_k} \boldsymbol{g}(\boldsymbol{x}^i, t) \;\; \boldsymbol{g}^{\mathsf{T}}(\boldsymbol{x}^i, t) \;\, \mathrm{d}t$$

where $\boldsymbol{g}(\boldsymbol{x}^i, t) = \big(\nabla_{\boldsymbol{\theta}} y\big)(\boldsymbol{x}^i, t; \boldsymbol{\theta}^0)$ are sensitivity coefficients

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$$M_{ki} = \int_{T_k} g(x^i, t)\, g^{\top}(x^i, t)\, \mathrm{d}t$$

where $g(x^i, t) = \big(\nabla_{\theta} y\big)(x^i, t; \theta^0)$ are sensitivity coefficients

Ultimate formulation

Find $\boldsymbol{v} = (v_{11}, \dots, v_{1I}, \dots, v_{K1}, \dots, v_{KI})$ s.t.

$$\mathcal{P}(\boldsymbol{v}) = \log\det\left(\sum_{k=1}^{K}\sum_{i=1}^{I} v_{ki}\boldsymbol{M}_{ki}\right) \rightarrow \max$$

subject to the constraints

$$\sum_{i=1}^{I} v_{ki} = n, \qquad k = 1, \dots, K$$

$$v_{ki} \in \{0, 1\}, \quad i = 1, \dots, I, \quad k = 1, \dots, K$$

Branch-and-bound

Consider two sensors of which only one may be active, and two time stages. Any solution can be represented as a 2×2 matrix with binary entries

	$i = 1$	$i = 2$
$k = 1$	0	1
$k = 2$	1	0

i.e., rows correspond to consecutive time stages, and columns are associated with individual sensors.

Branch-and-bound

Thus we have four admissible solutions:

0	1
0	1

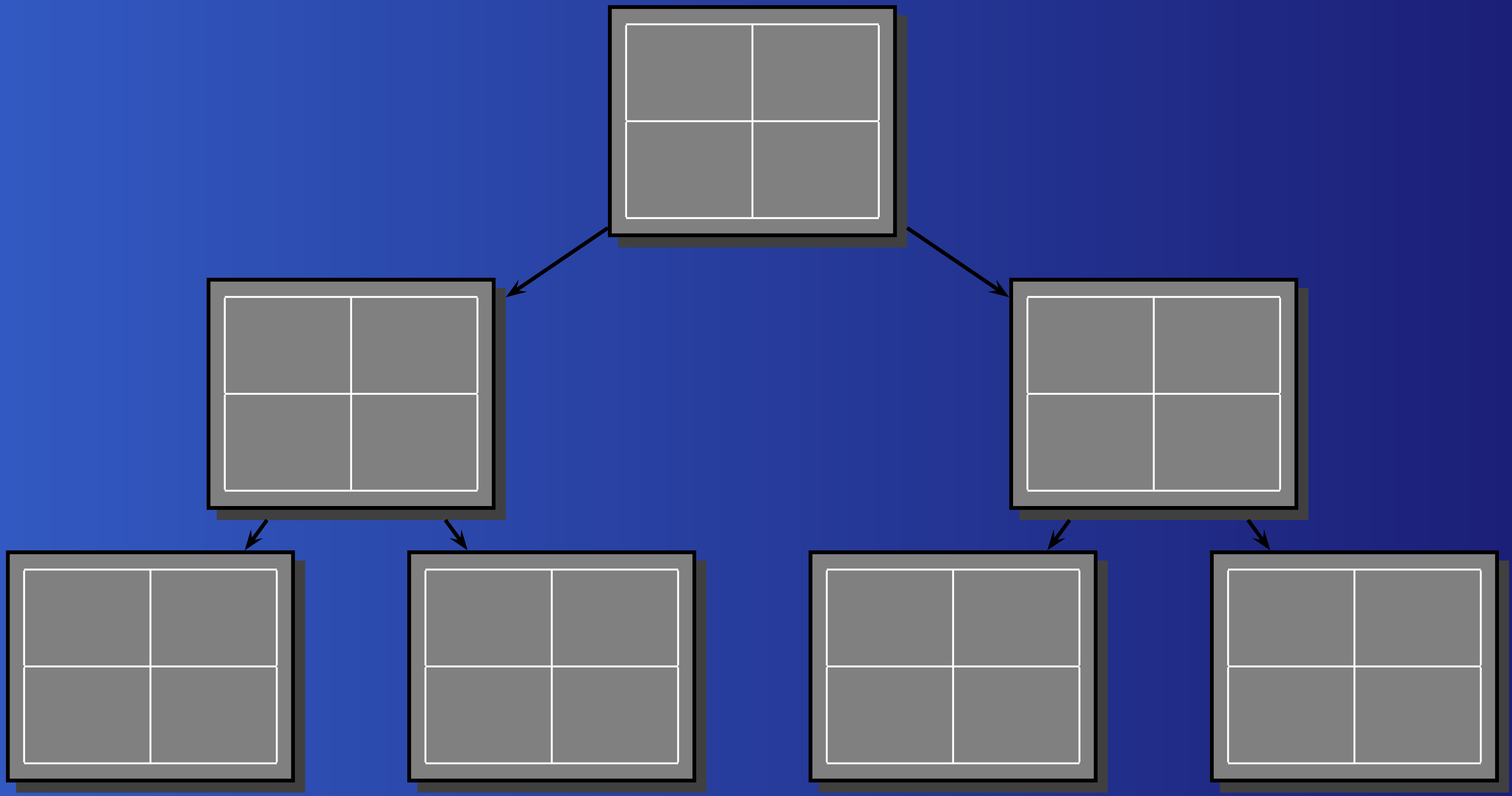
0	1
1	0

1	0
0	1

1	0
1	0

How to automatically enumerate them?

Branching



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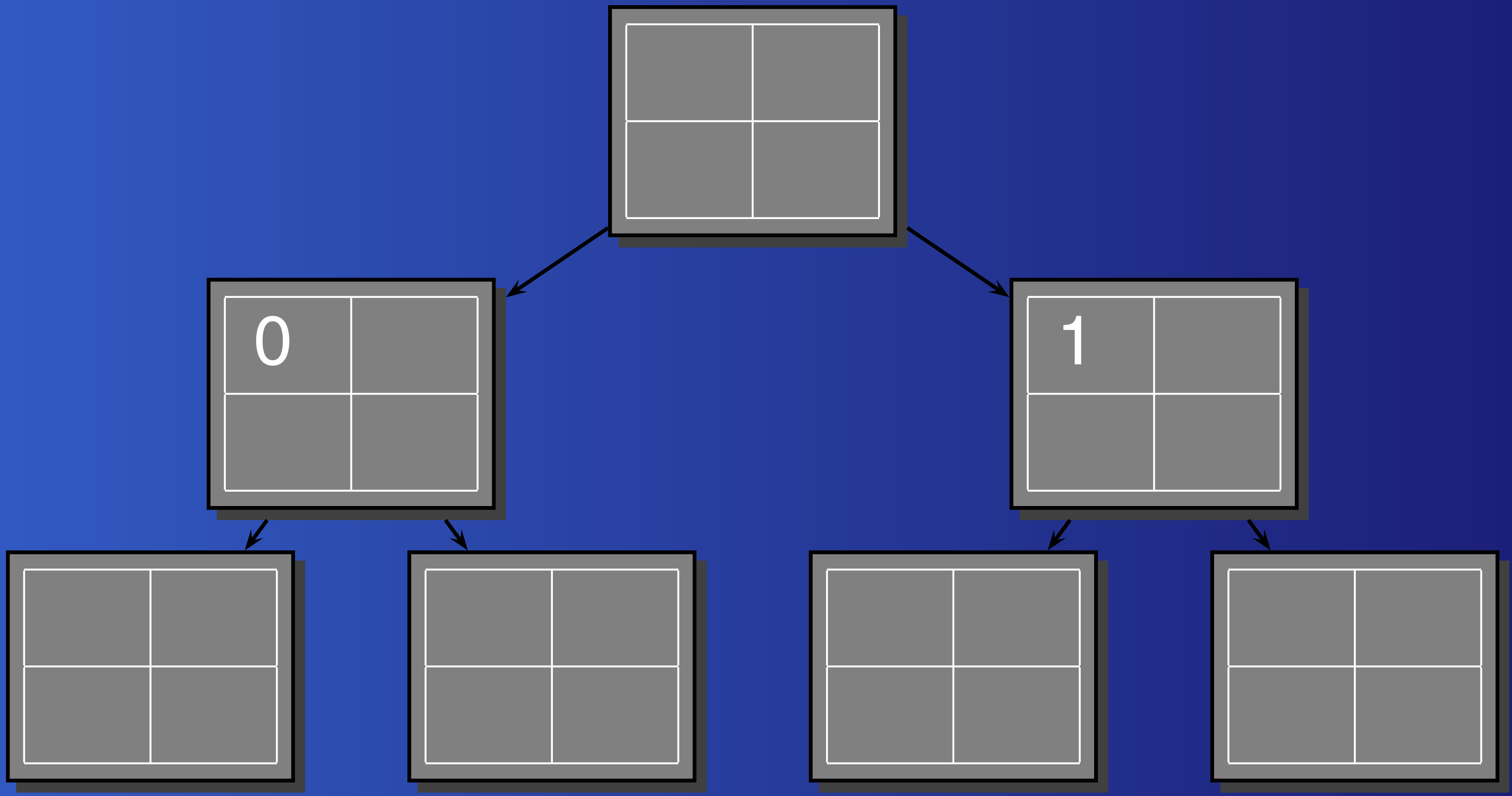

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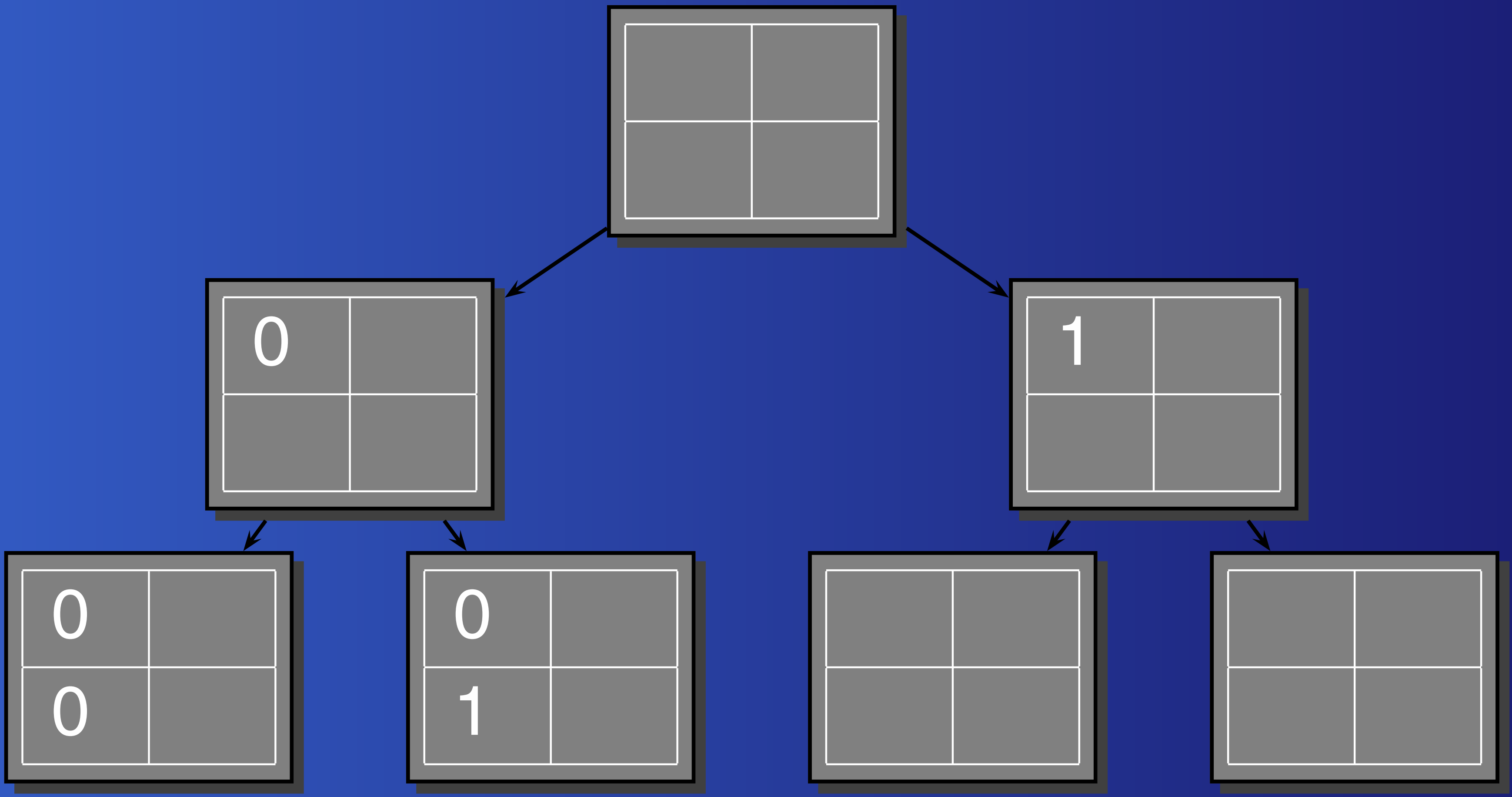

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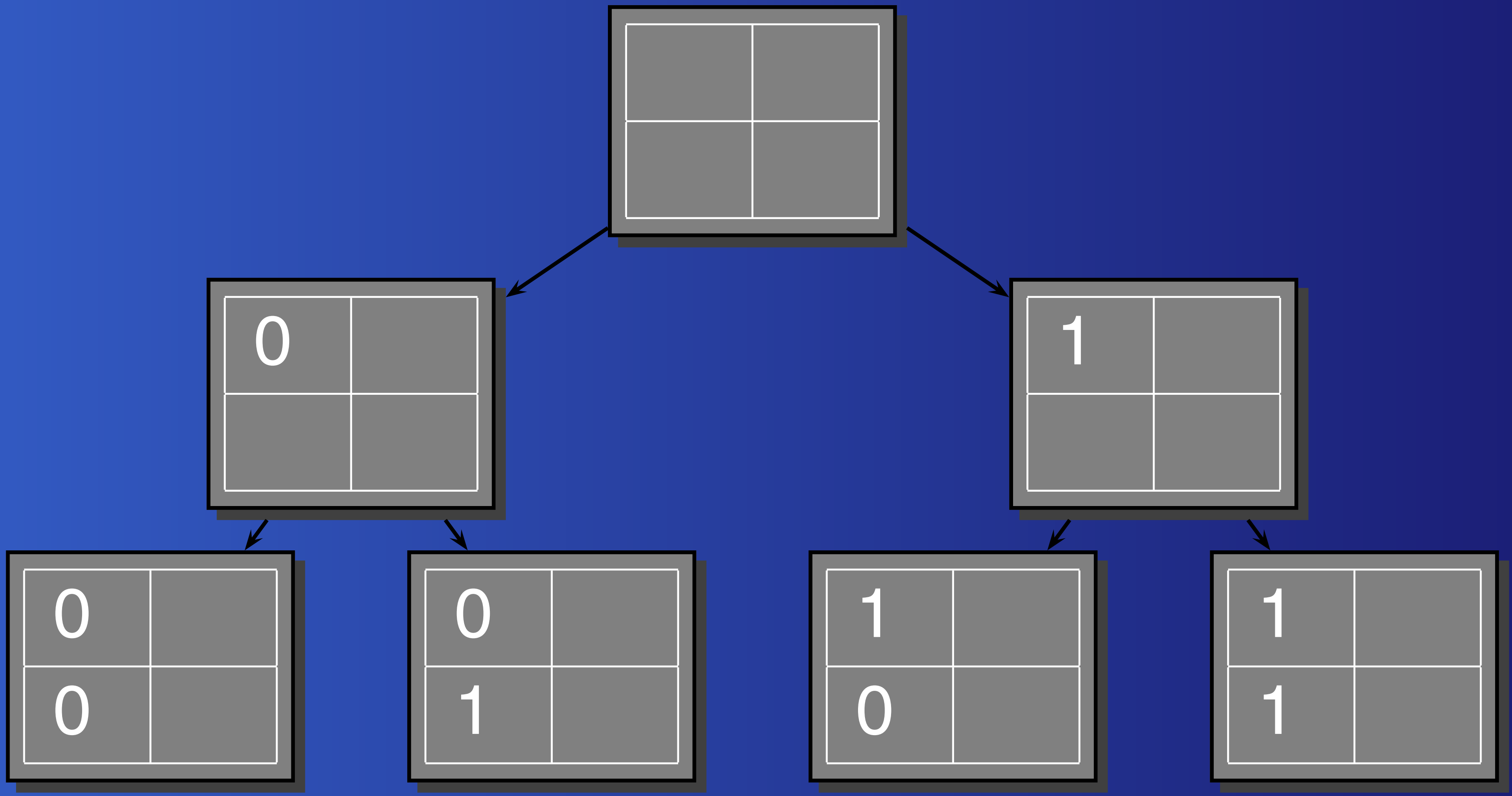

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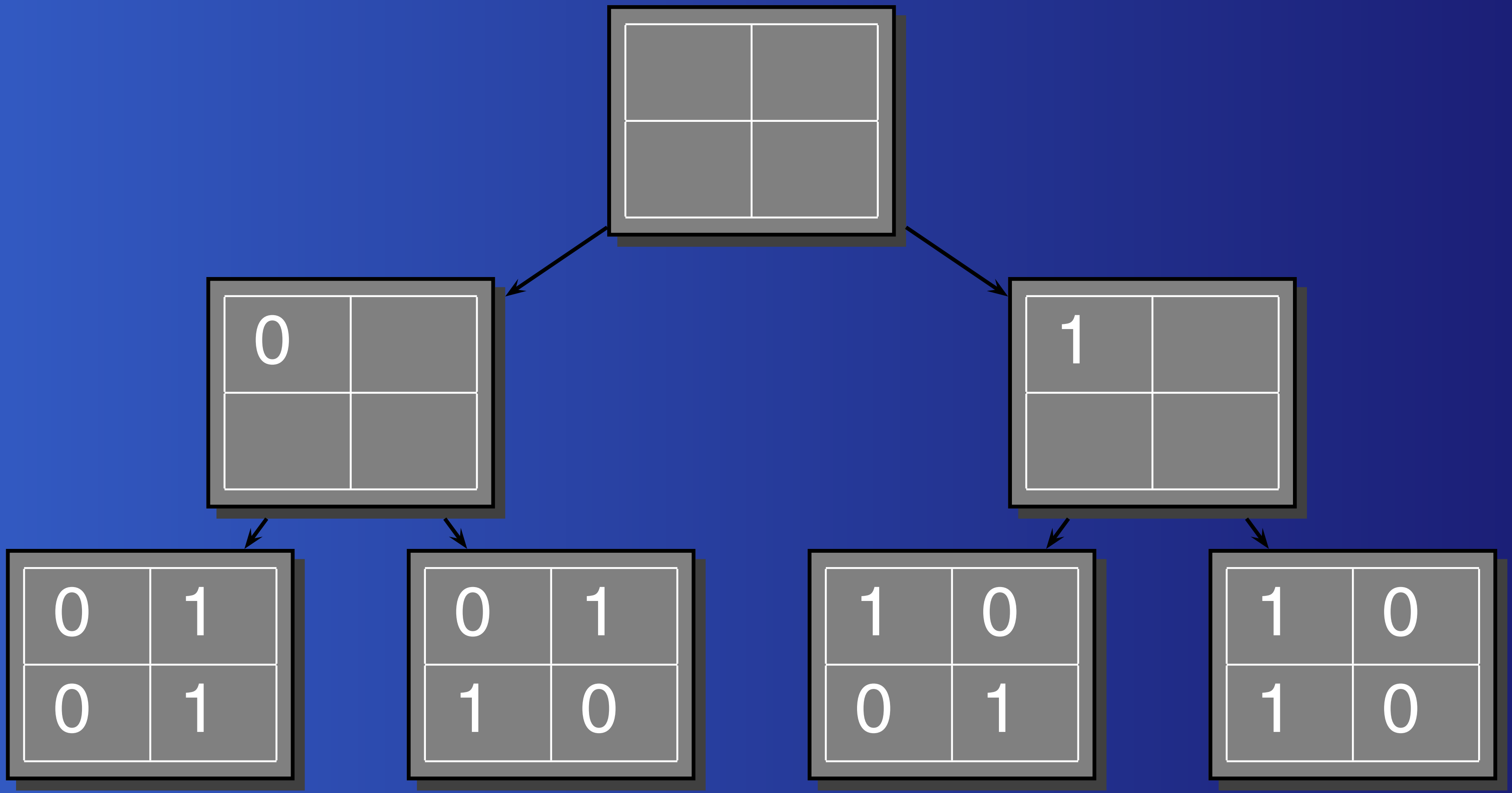
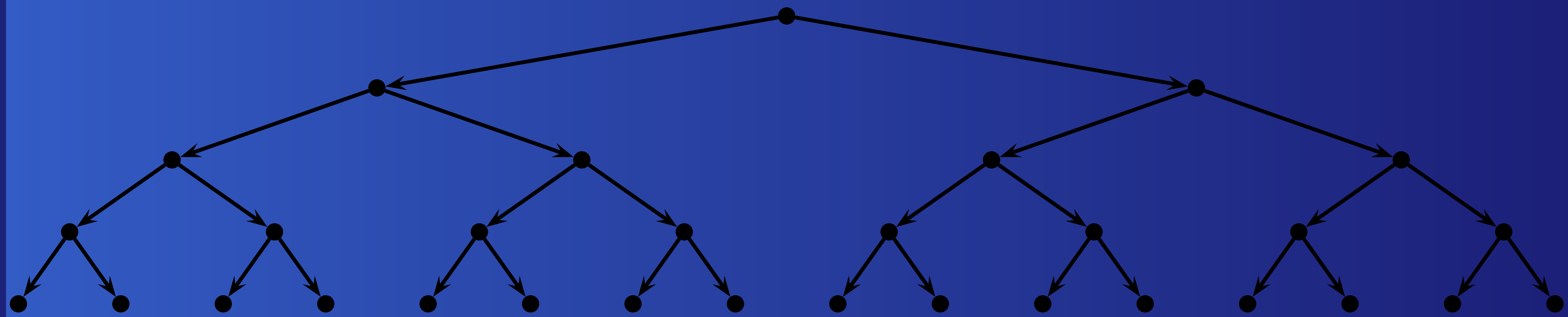


BB trees

Clearly, binary trees can be larger:



For example, for 1000 sites, 500 active sensors and 10 stages, the number of admissible solutions is $\binom{1000}{500}^{10} > 10^{2994}$.

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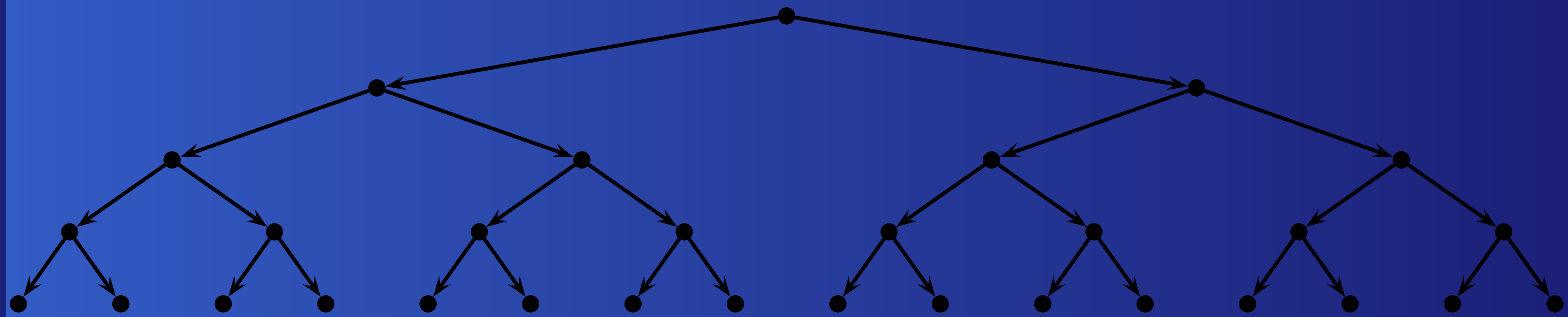

For example, for 1000 sites, 500 active sensors and 10 stages, the number of admissible solutions is $\binom{1000}{500}^{10} > 10^{2994}$. How to reduce such huge numbers?

Idea of bounding

Economize computations by eliminating subtrees that have no chance of containing an optimal solution.

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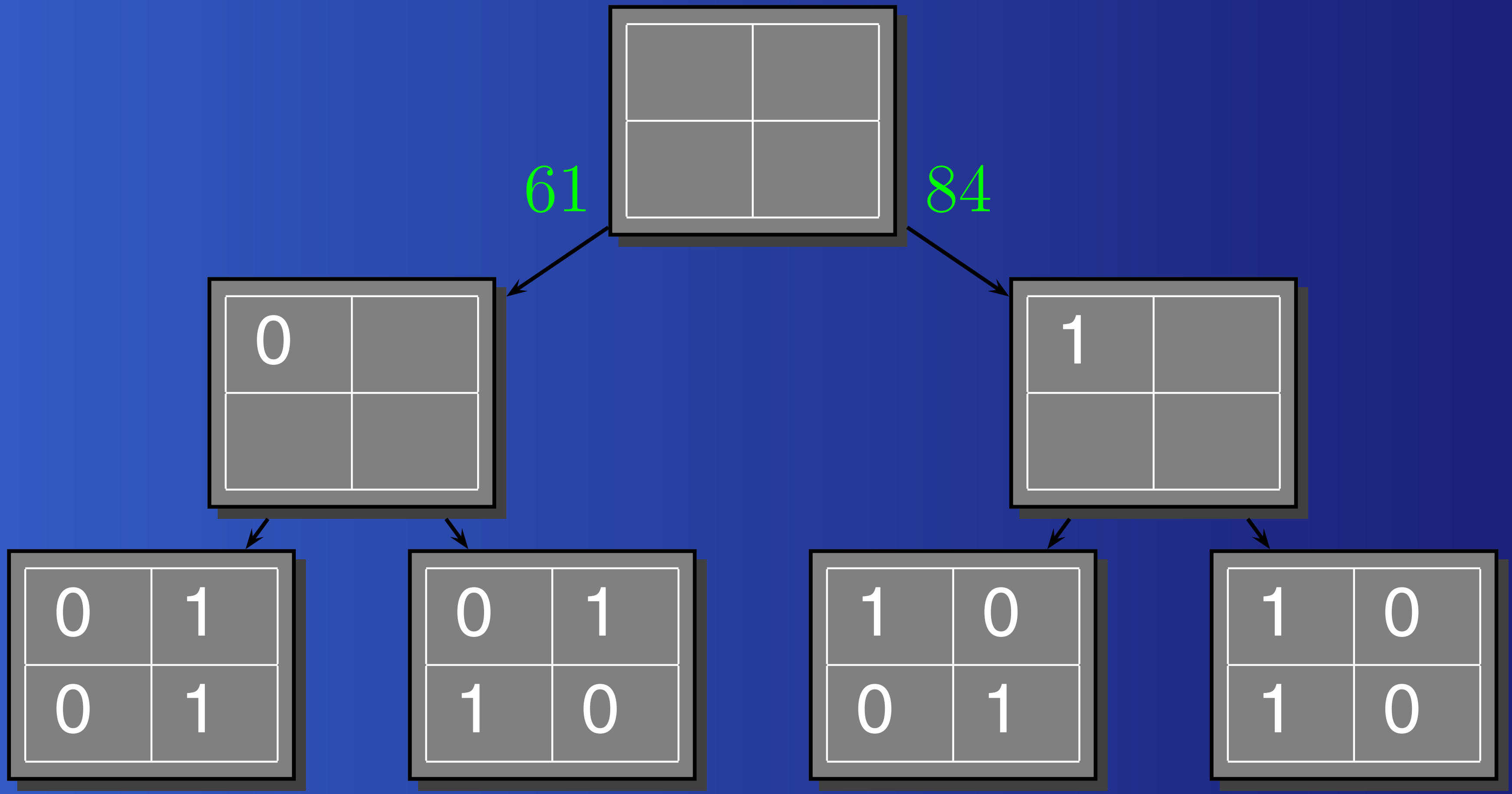
Moreover, assume that we also know a lower bound to the maximum value of the objective function over all admissible solutions.

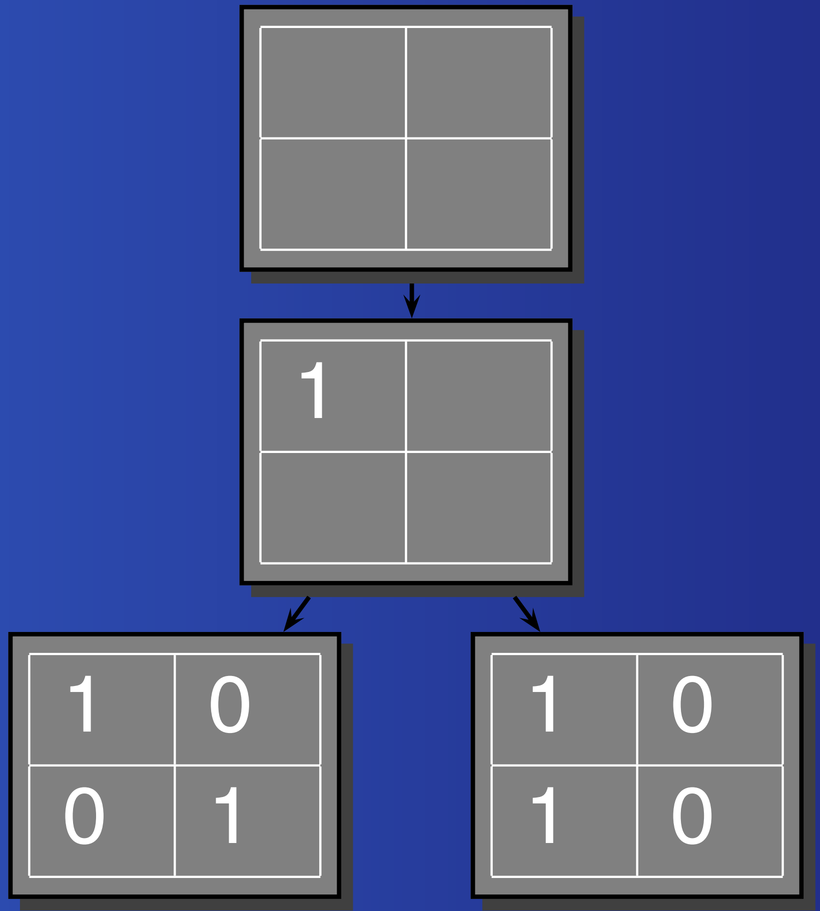
Bounding

Let $LOWER = 78$.



Pruning

We may thus discard the left subtree.



Finding upper bounds

Let

$$E = \underbrace{\{1,\dots,K\}}_{\text{time}} \times \underbrace{\{1,\dots,I\}}_{\text{space}}$$

For each BB node, define $E_0, E_1 \subset E$ s.t.

- $(k,i) \in E_0 \Rightarrow i$-th sensor is dormant over T_k,
- $(k,i) \in E_1 \Rightarrow i$-th sensor is active over T_k,
- $(k,i) \in E \setminus (E_0 \cup E_1) \Rightarrow$ the status of i-th sensor over T_k is not determined.

Relaxed problem

Find $\boldsymbol{v} = (v_{11}, \dots, v_{1I}, \dots, v_{K1}, \dots, v_{KI})$ s.t.

$$\mathcal{P}(\boldsymbol{v}) = \log\det\left(\sum_{k=1}^{K}\sum_{i=1}^{I} v_{ki}\boldsymbol{M}_{ki}\right) \longrightarrow \max$$

$$\sum_{i=1}^{I} v_{ki} = n, \qquad k = 1, \dots, K$$

$$v_{ki} = 0, \quad (k,i) \in E_0$$

$$v_{ki} = 1, \quad (k,i) \in E_1$$

$$0 \le v_{ki} \le 1, \quad (k,i) \in E \setminus (E_0 \cup E_1)$$

Conveniently altered formulation

Find $\boldsymbol{w} = (w_{1,1}, \dots, w_{1,q_1}, \dots, w_{L,1}, \dots, w_{L,q_L})$ s.t.

$$\mathcal{Q}(\boldsymbol{w}) = \log\det\left(\boldsymbol{A} + \sum_{l=1}^{L}\sum_{j=1}^{q_l} w_{lj}\boldsymbol{S}_{lj}\right) \to \max$$

$$\sum_{j=1}^{q_l} w_{lj} = r_l, \quad l = 1, \dots, L,$$

$$0 \le w_{lj} \le 1, \quad j = 1, \dots, q_l, \quad l = 1, \dots, L,$$

The admissible set is a polygon!

Optimality conditions

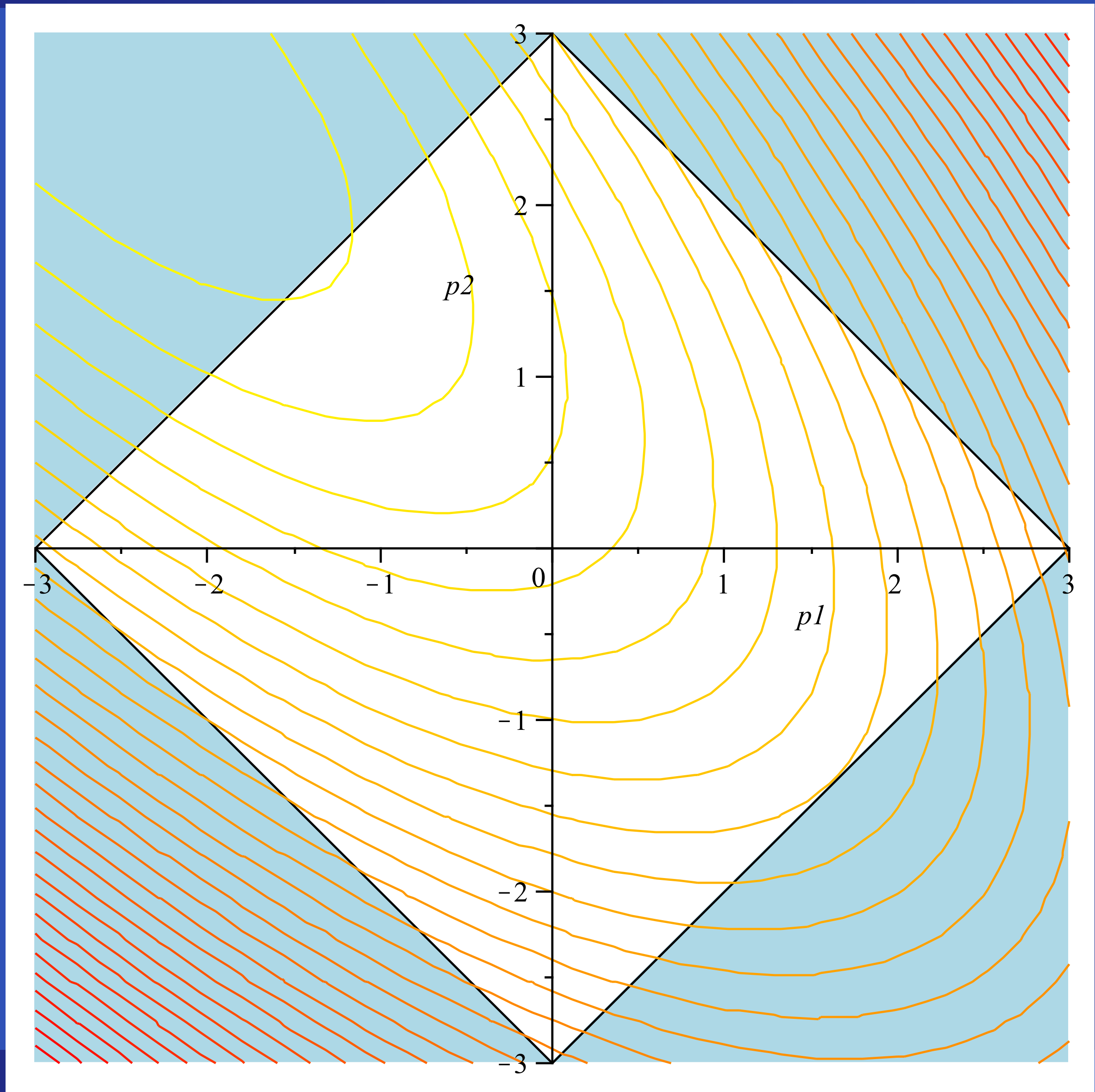
Proposition 1. *A vector* $\boldsymbol{w}^\star$ *is a global solution iff there exist numbers* $\lambda_l^\star$, $l = 1, \dots, L$ *such that*

$$\varphi(l, j, \boldsymbol{w}^\star) \begin{cases} \geq \lambda_l^\star & \text{if } w_{lj}^\star = 1 \\ = \lambda_l^\star & \text{if } 0 < w_{lj}^\star < 1 \\ \leq \lambda_l^\star & \text{if } w_{lj}^\star = 0 \end{cases}$$

where $\varphi(l, j, \boldsymbol{w}) = \operatorname{tr}\big[\boldsymbol{G}^{-1}(\boldsymbol{w})\boldsymbol{S}_{lj}\big]$,

$$\boldsymbol{G}(\boldsymbol{w}) = \boldsymbol{A} + \sum_{l=1}^{L}\sum_{j=1}^{q_l} w_{lj}\boldsymbol{S}_{lj}$$

Simplicial decomposition



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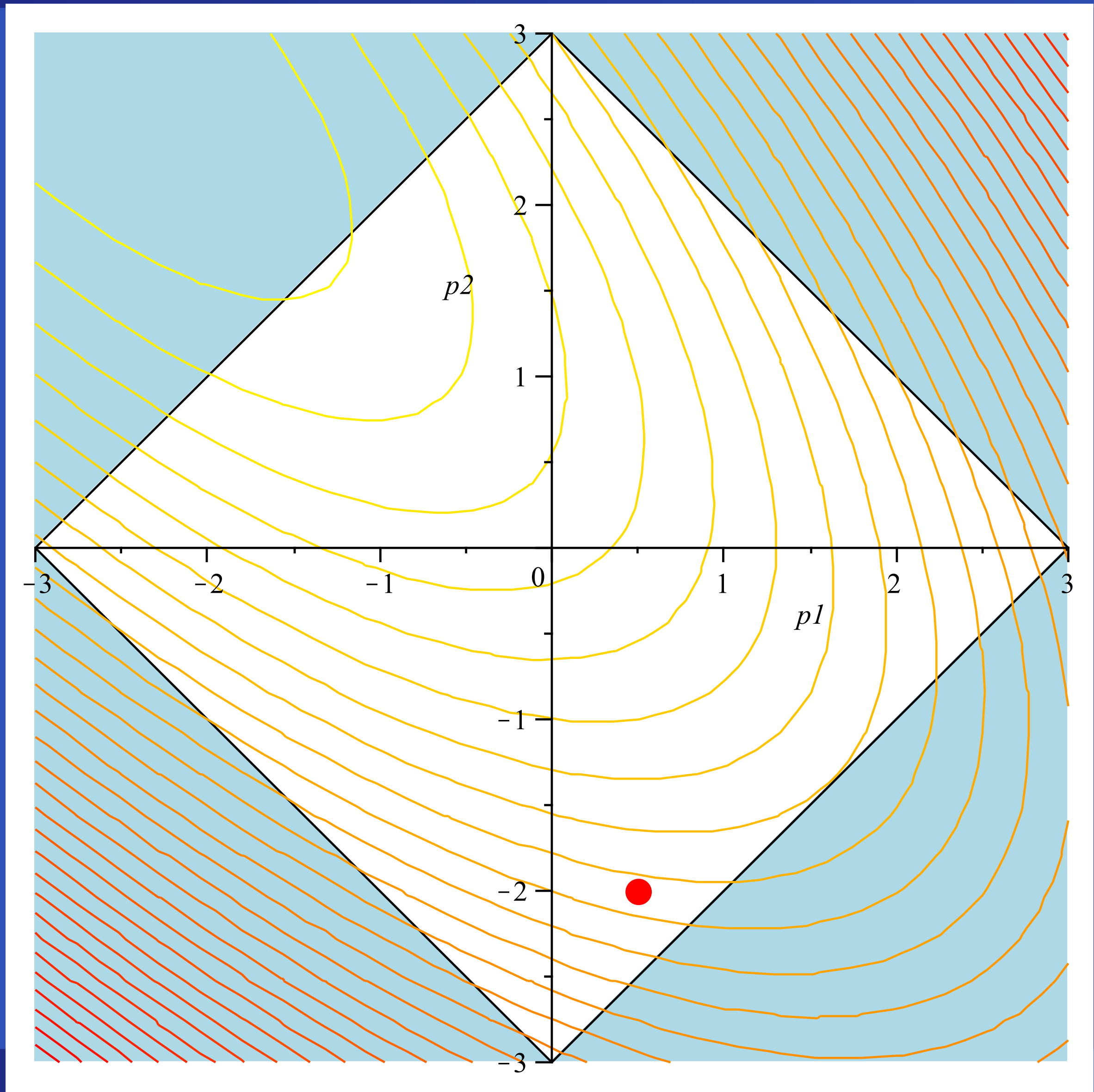

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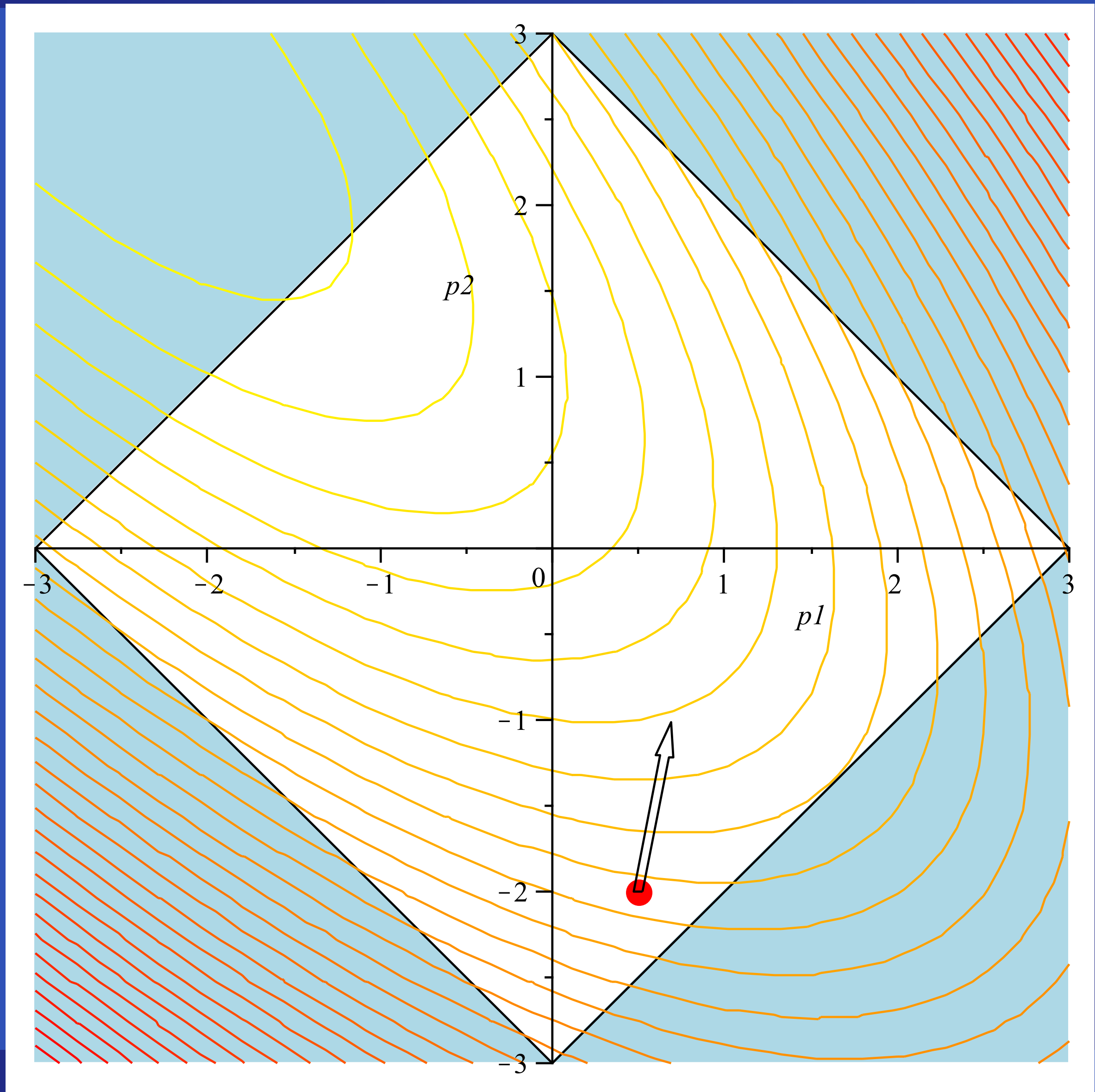

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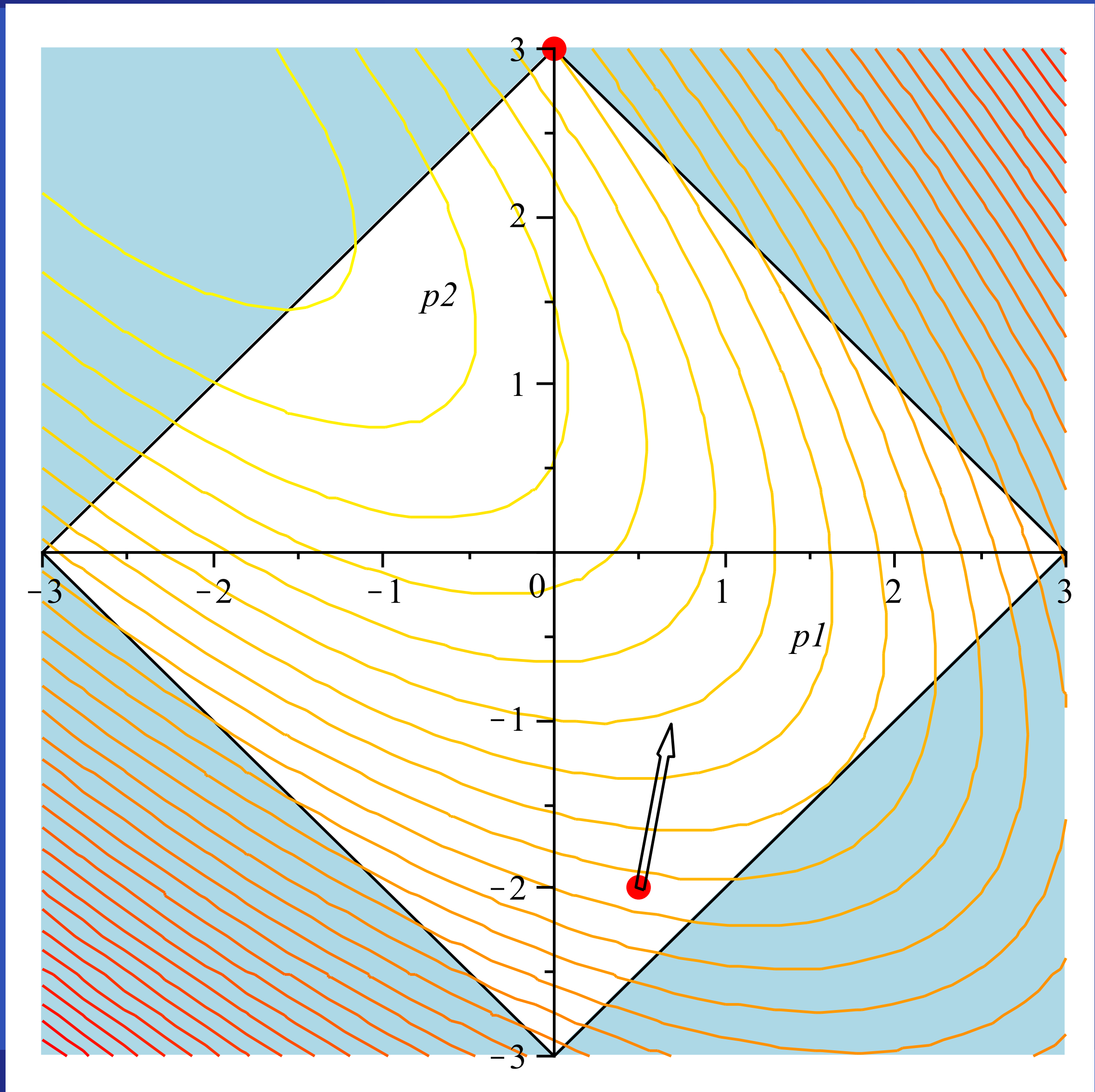

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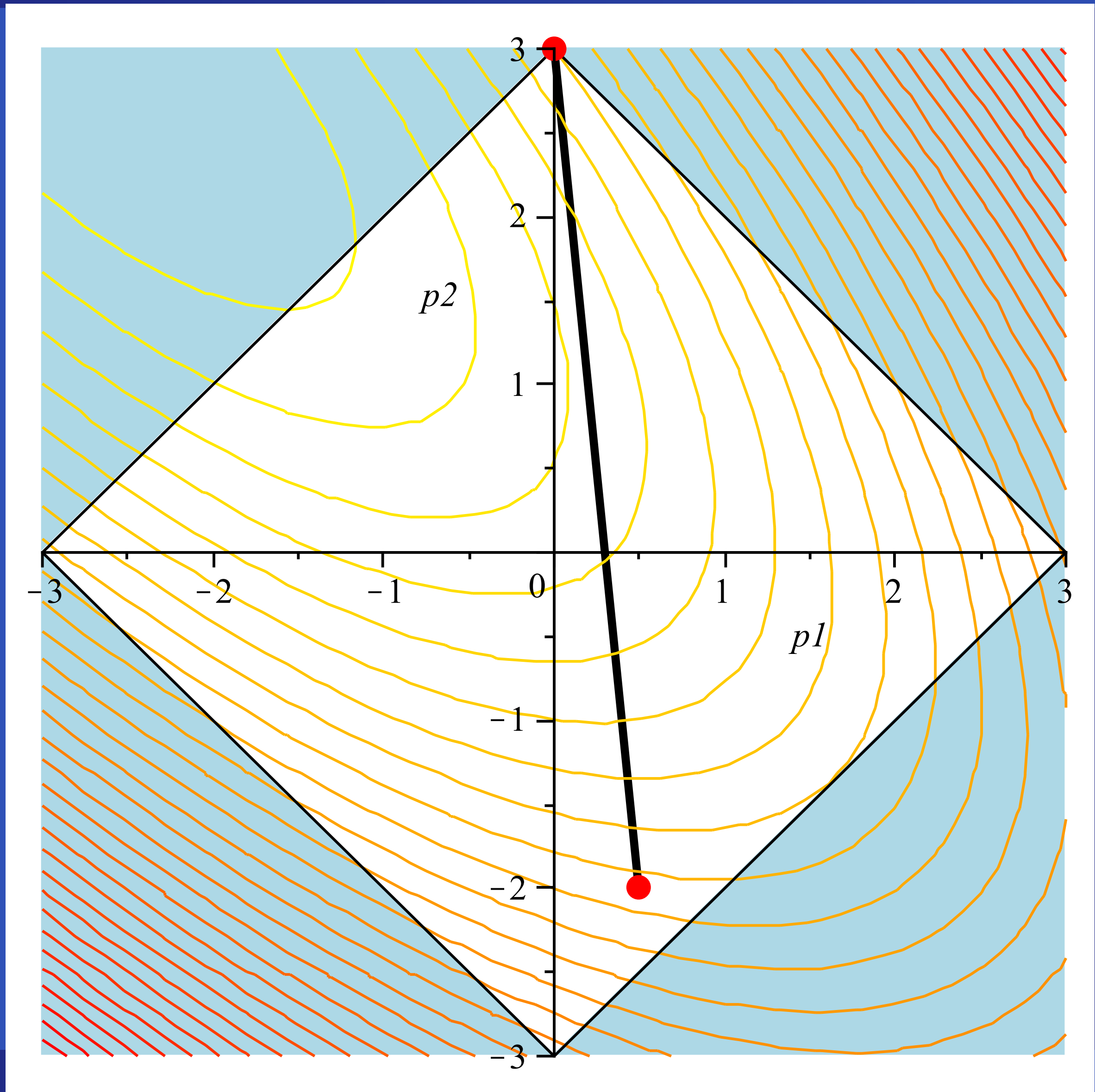

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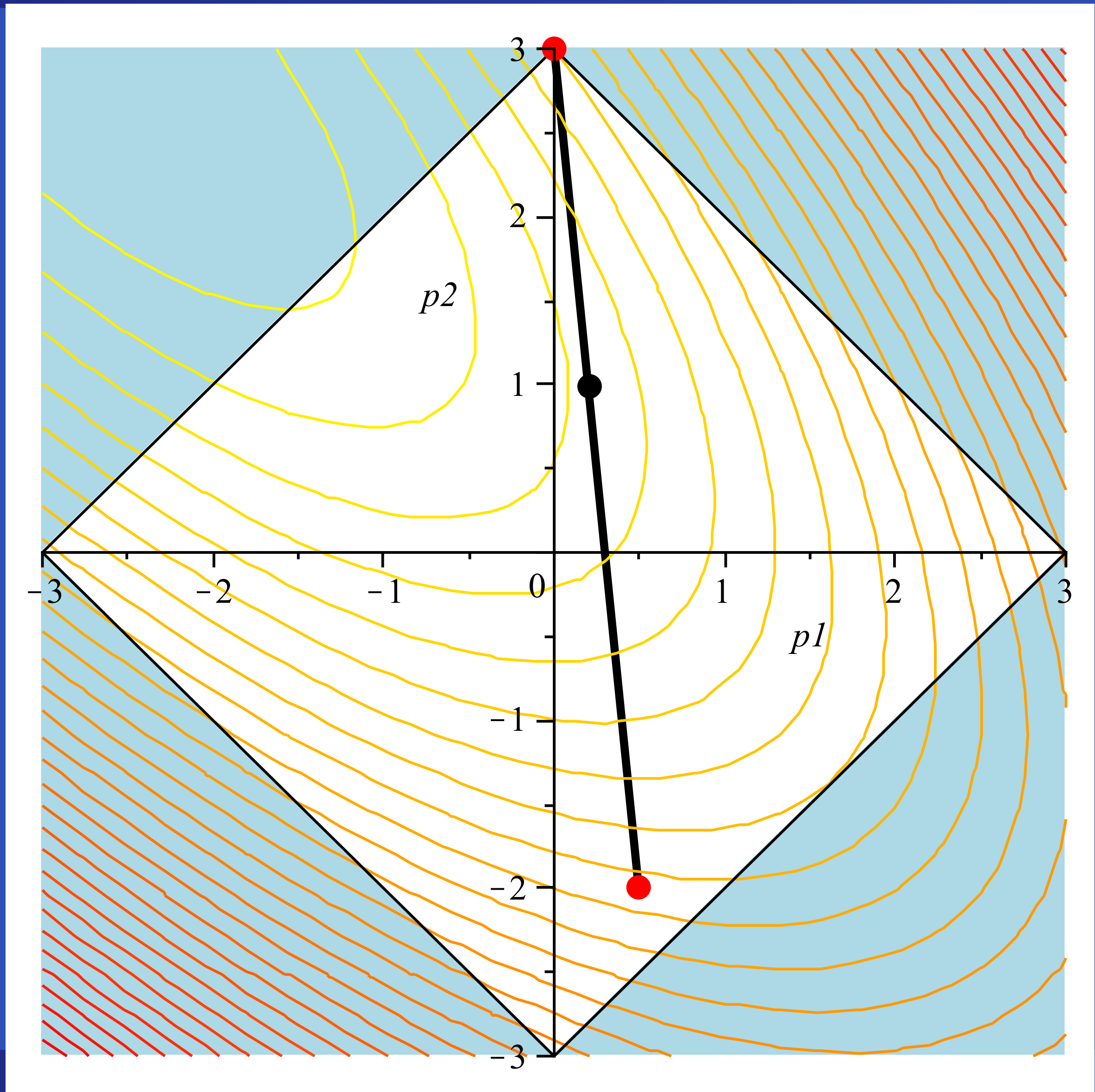

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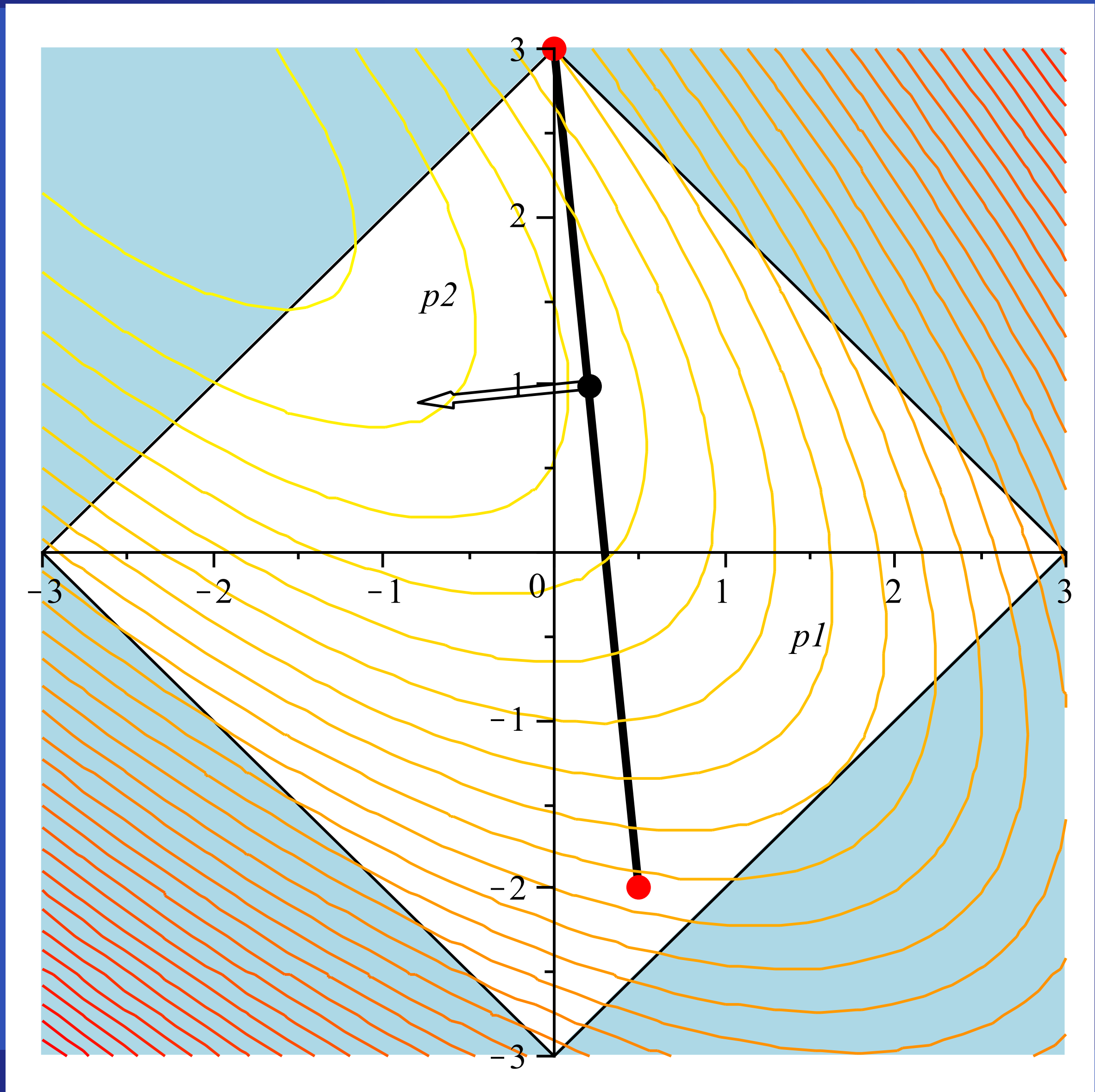

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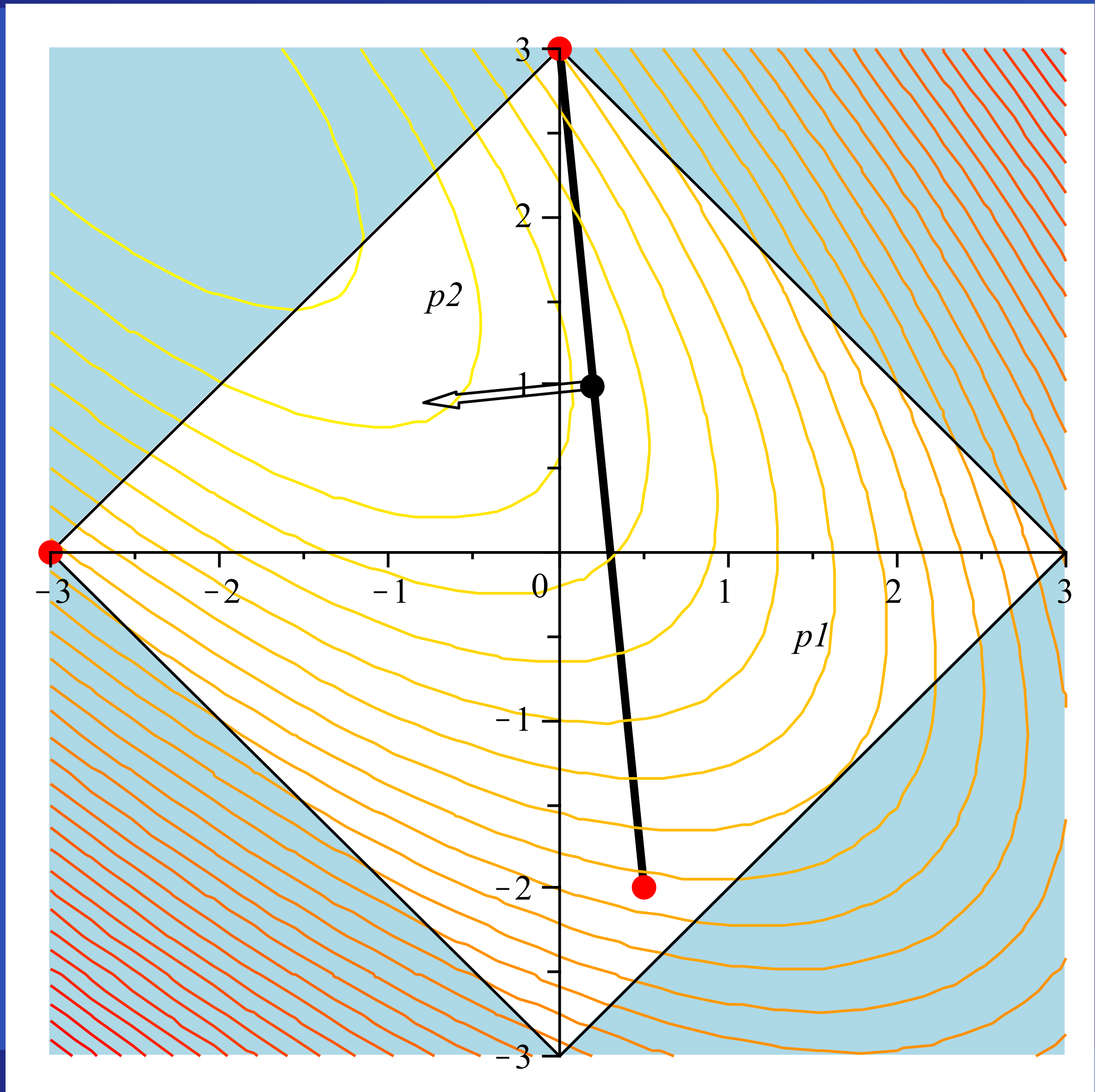

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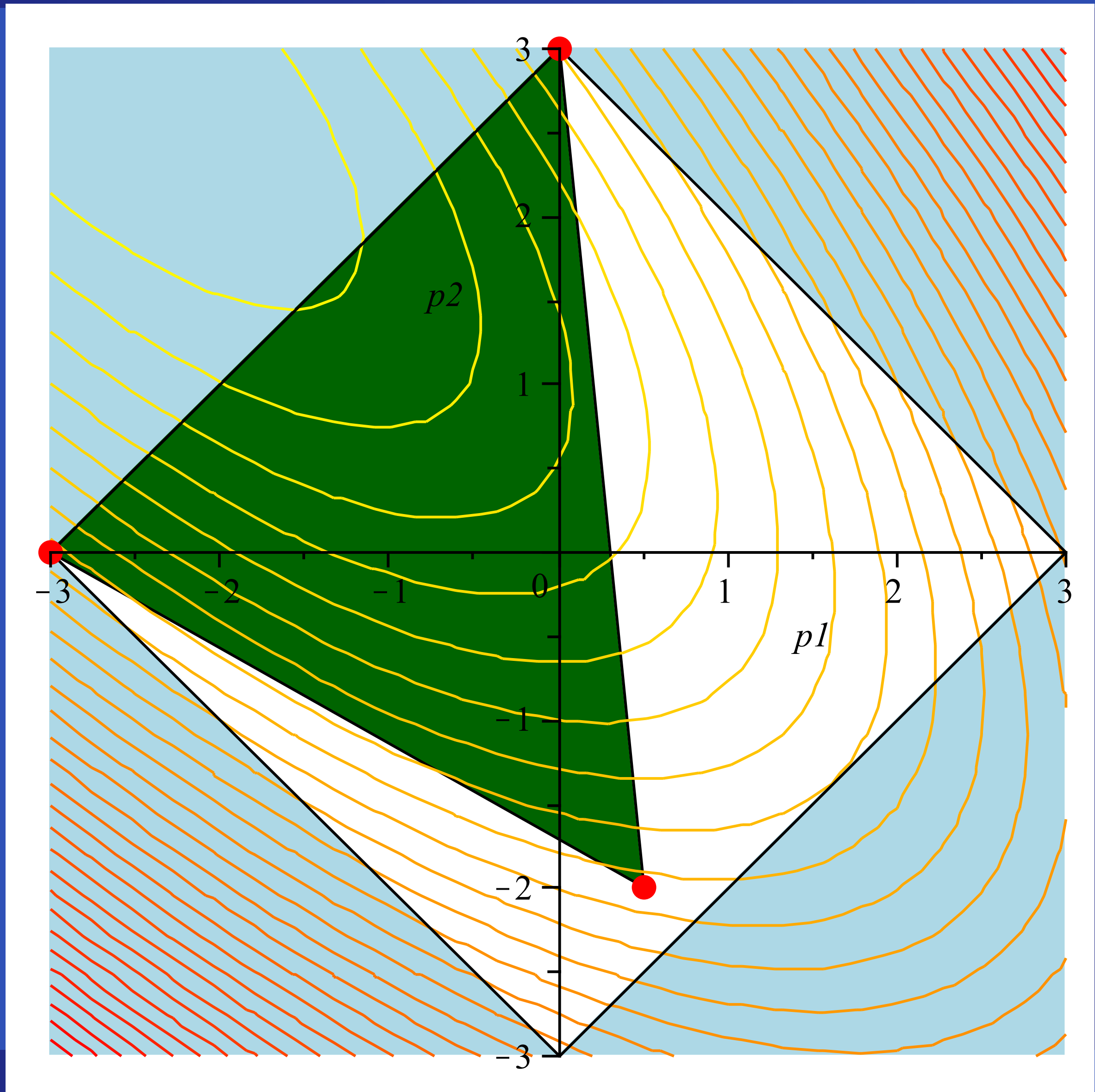

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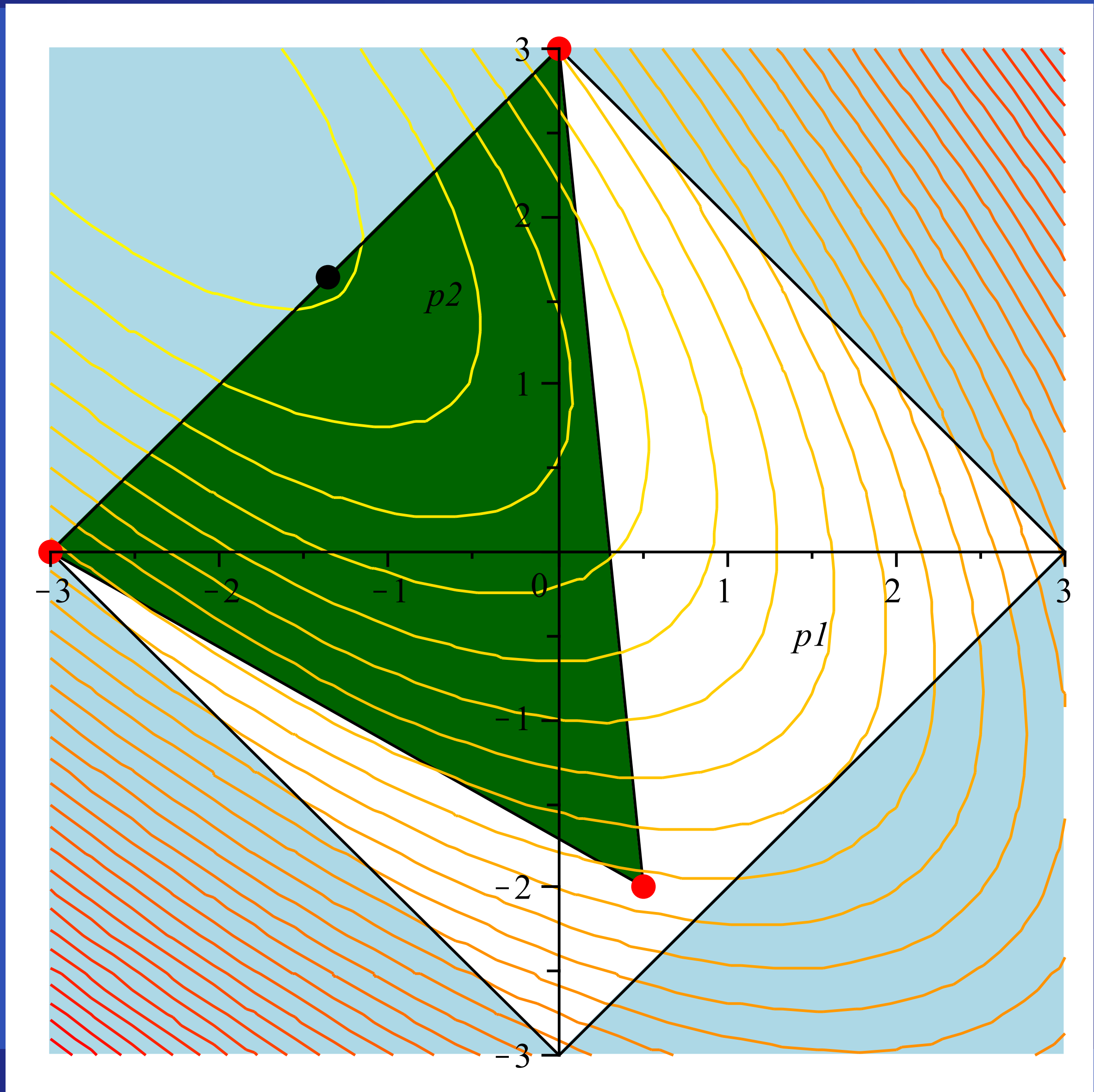

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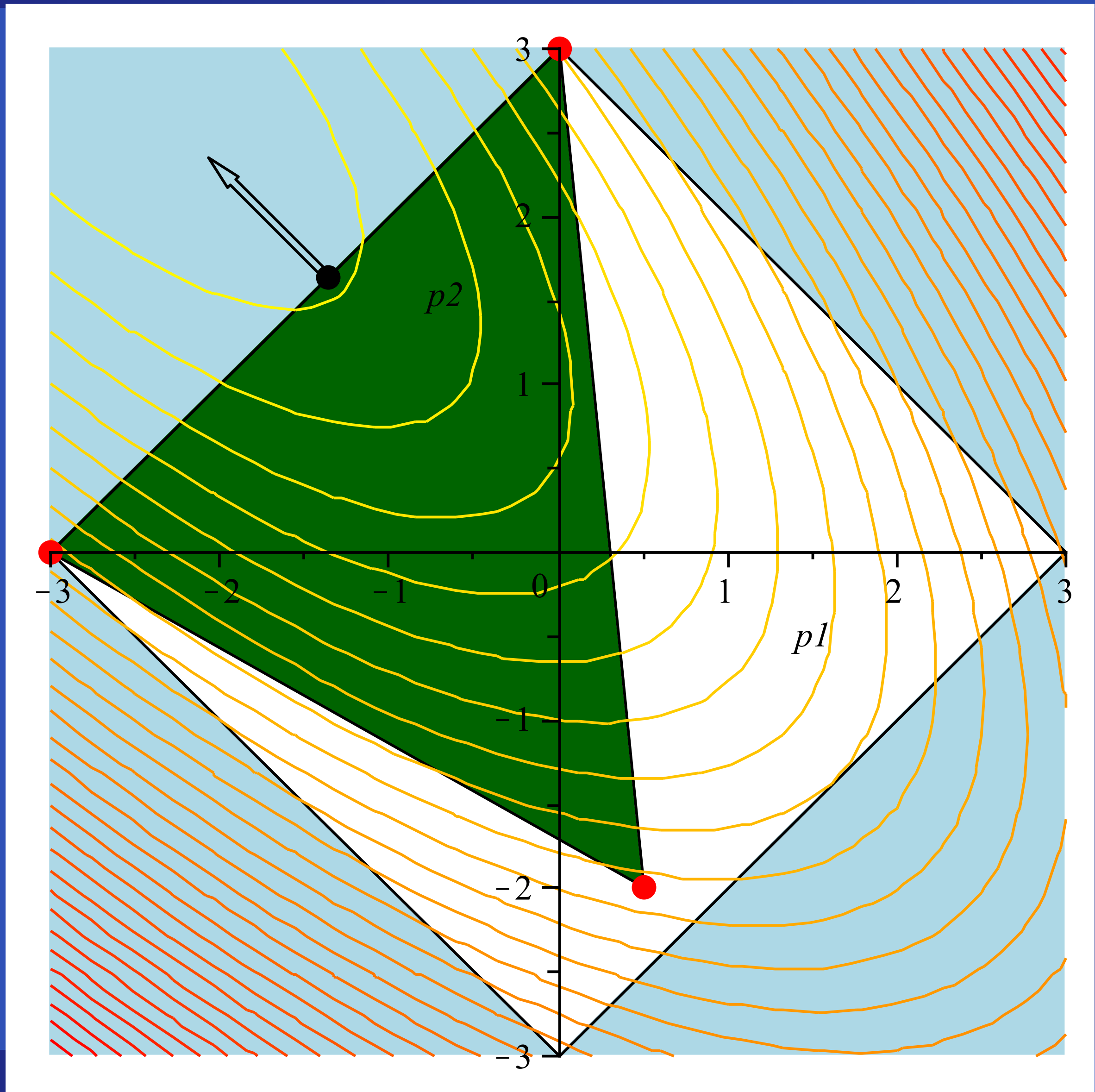


Simplicial decomposition

Step 0: (Initialization)

Set

$$\boldsymbol{w}^{(0)} = (\underbrace{r_1/q_1, \dots, r_1/q_1}_{q_1 \text{ times}}, \dots, \underbrace{r_L/q_L, \dots, r_L/q_L}_{q_L \text{ times}}).$$

and $Z^{(0)} = \{\boldsymbol{w}^{(0)}\}$. Select $0 < \epsilon \ll 1$, a parameter used in the stopping rule, and set $\tau = 0$.

Simplicial decomposition

Step 1: (Column generation subproblem)

Determine

$$z = \arg\max_{w \in W} \nabla Q(w^{(\tau)})^{\mathsf{T}} (w - w^{(\tau)}).$$

Step 2: (Termination check)

If $\nabla Q(w^{(\tau)})^{\mathsf{T}} (z - w^{(\tau)}) \leq \epsilon$, then STOP and $w^{(\tau)}$ is optimal. Otherwise, set $Z^{(\tau+1)} = Z^{(\tau)} \cup \{z\}$.

Simplicial decomposition

Step 3: (Restricted master problem)

Find

$$\boldsymbol{w}^{(\tau+1)} = \arg \max_{\boldsymbol{w} \in \mathrm{co}(Z^{(\tau+1)})} \mathcal{Q}(\boldsymbol{w})$$

and purge $Z^{(\tau+1)}$ of all extreme points with zero weight in the expression of $\boldsymbol{w}^{(\tau+1)}$ as a convex combination of elements in $Z^{(\tau+1)}$. Increment τ by one and go back to Step 1.

Restricted master problem

Find $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_s)$ s.t.

$$\mathcal{T}(\boldsymbol{\alpha}) = \log\det\left(\sum_{\ell=1}^{s} \alpha_\ell \boldsymbol{Q}_\ell\right) \longrightarrow \max$$

$$\sum_{\ell=1}^{s} \alpha_\ell = 1$$

$$\alpha_\ell \geq 0, \quad \ell = 1, \dots, s$$

Computer example

Consider the heat equation

$$\frac{\partial y(x,t)}{\partial t} = \frac{\partial}{\partial x_1}\left(\kappa(x)\frac{\partial y(x,t)}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\kappa(x)\frac{\partial y(x,t)}{\partial x_2}\right) + 20\exp\left(-50(x_1-t)^2\right), \quad (x,t)\in(0,1)^3,$$

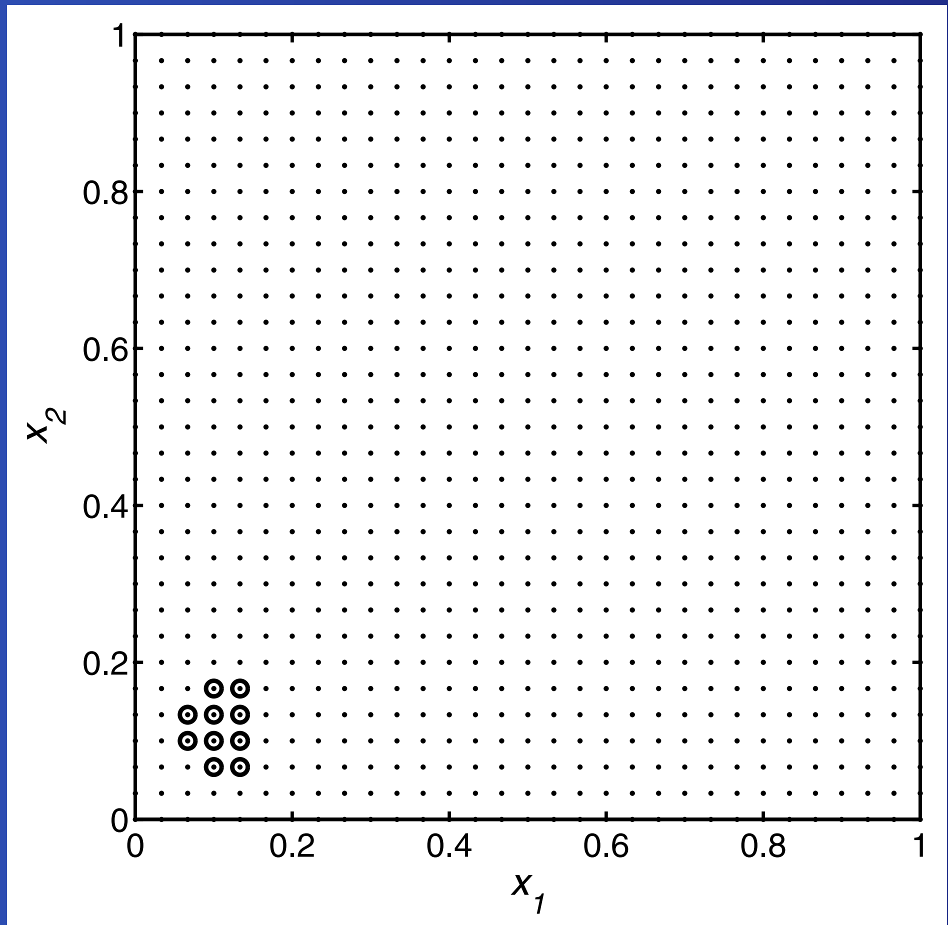
$$y(x,0)=0, \quad x\in\Omega$$

$$y(x,t)=0, \quad (x,t)\in\partial\Omega\times T$$

where $\kappa(x) = \theta_1 + \theta_2 x_1 + \theta_3 x_2$,
$\theta_1 = 0.1,\ \theta_2 = -0.05,\ \theta_3 = 0.2$

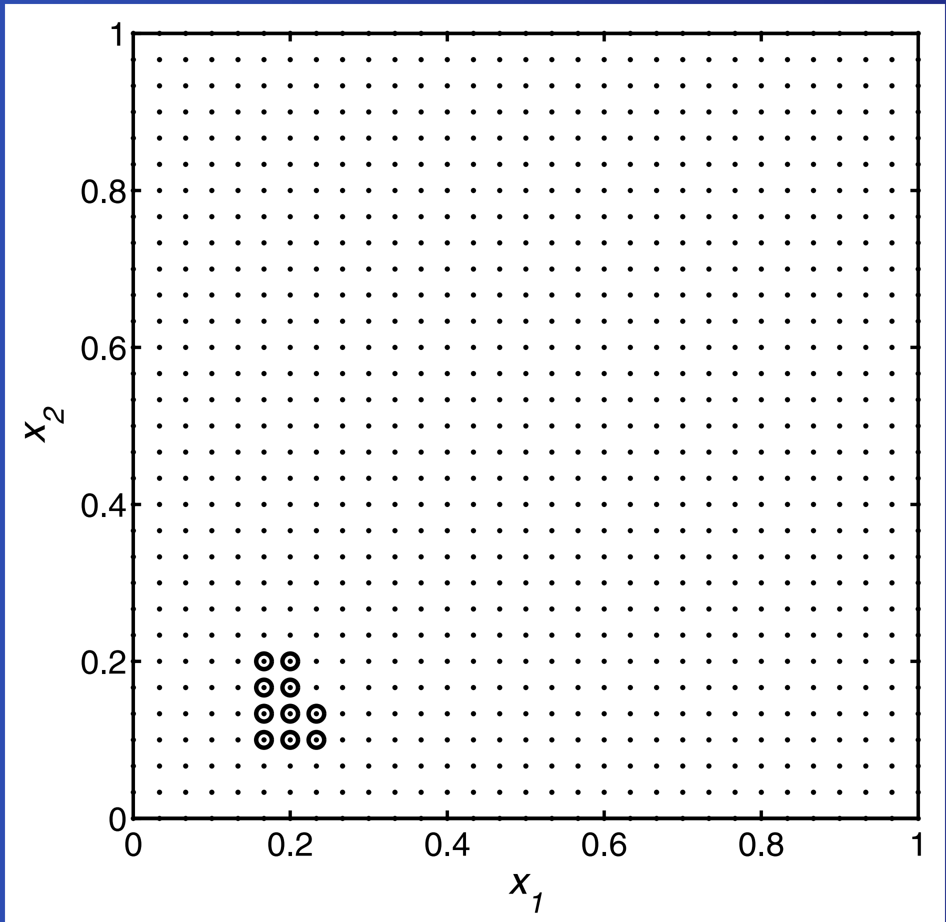
Typical behaviour

10 active sensors out of 900, Stage 1 of 6



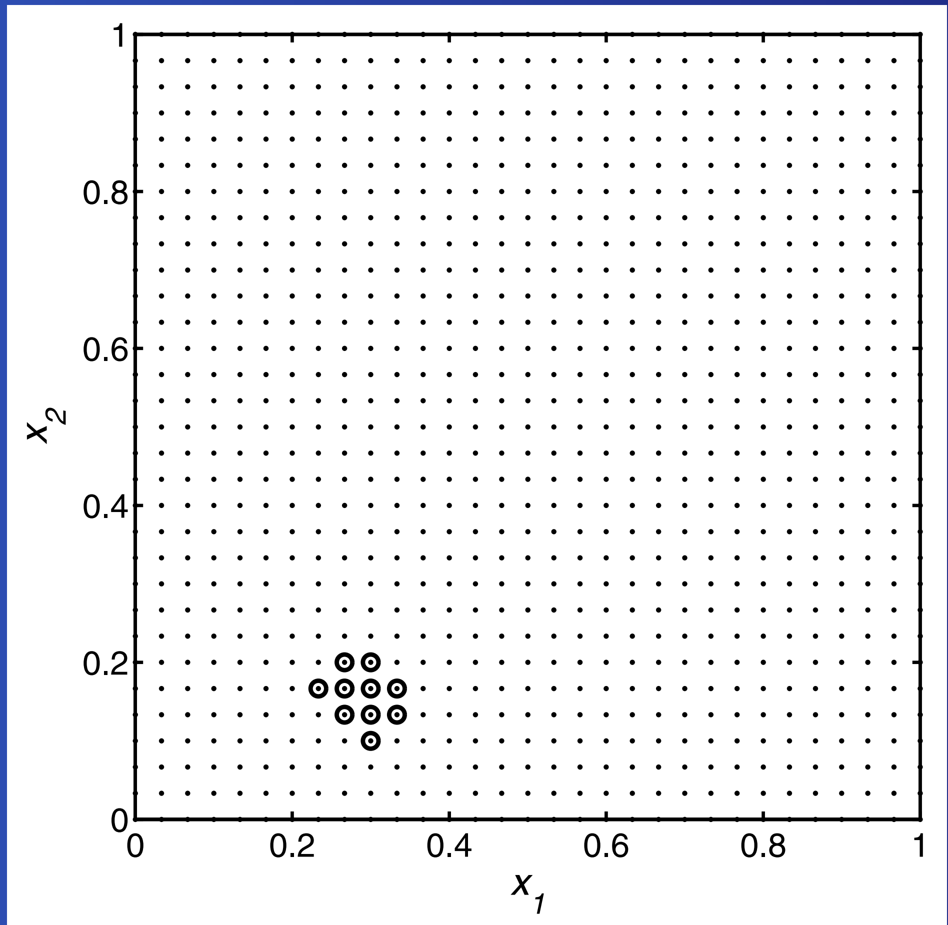
Typical behaviour

10 active sensors out of 900, Stage 2 of 6



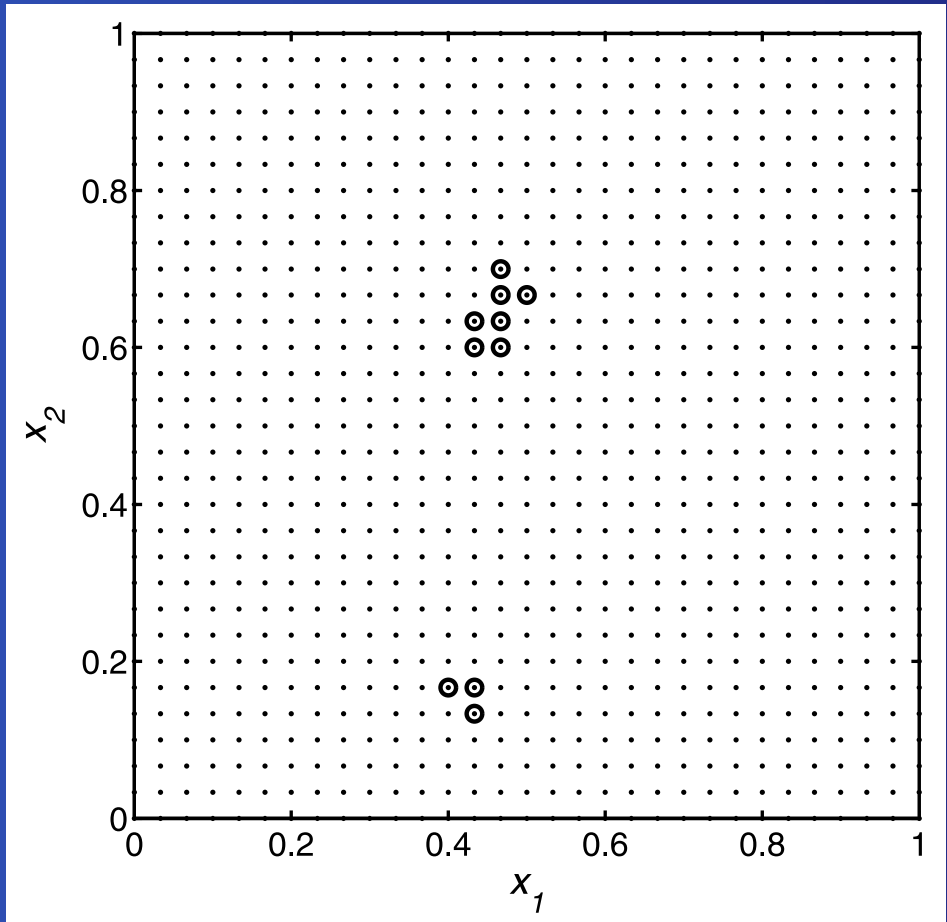
Typical behaviour

10 active sensors out of 900, Stage 3 of 6



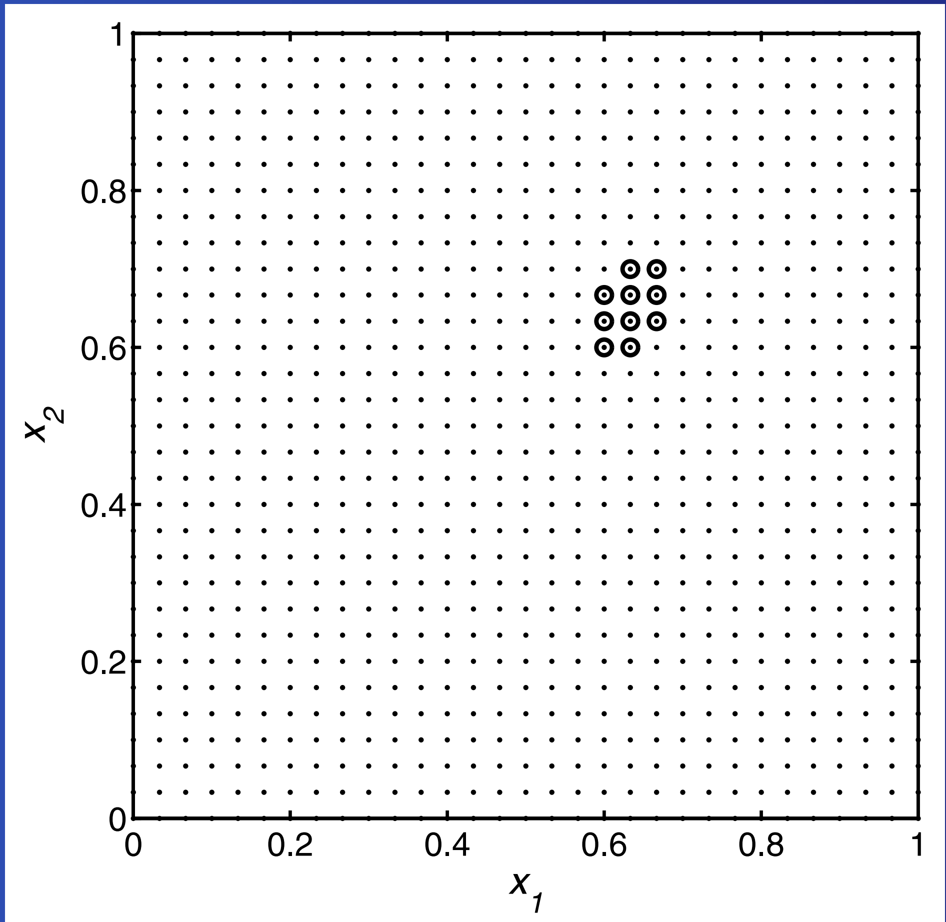
Typical behaviour

10 active sensors out of 900, Stage 4 of 6



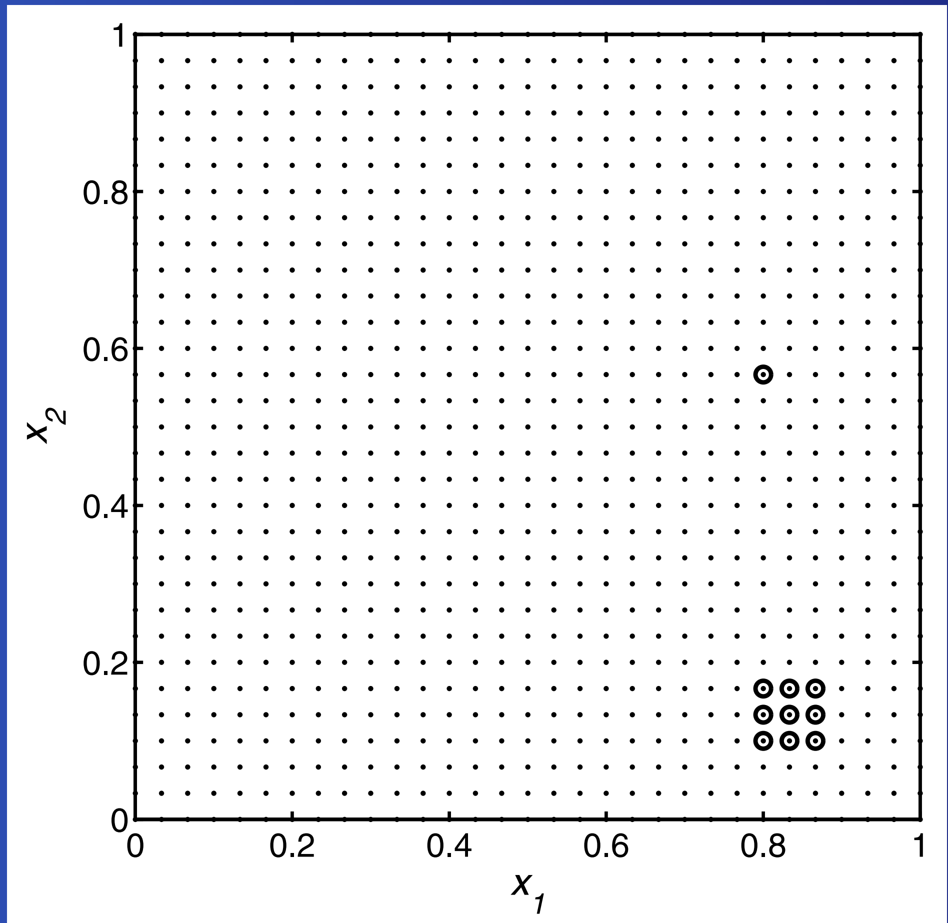
Typical behaviour

10 active sensors out of 900, Stage 5 of 6



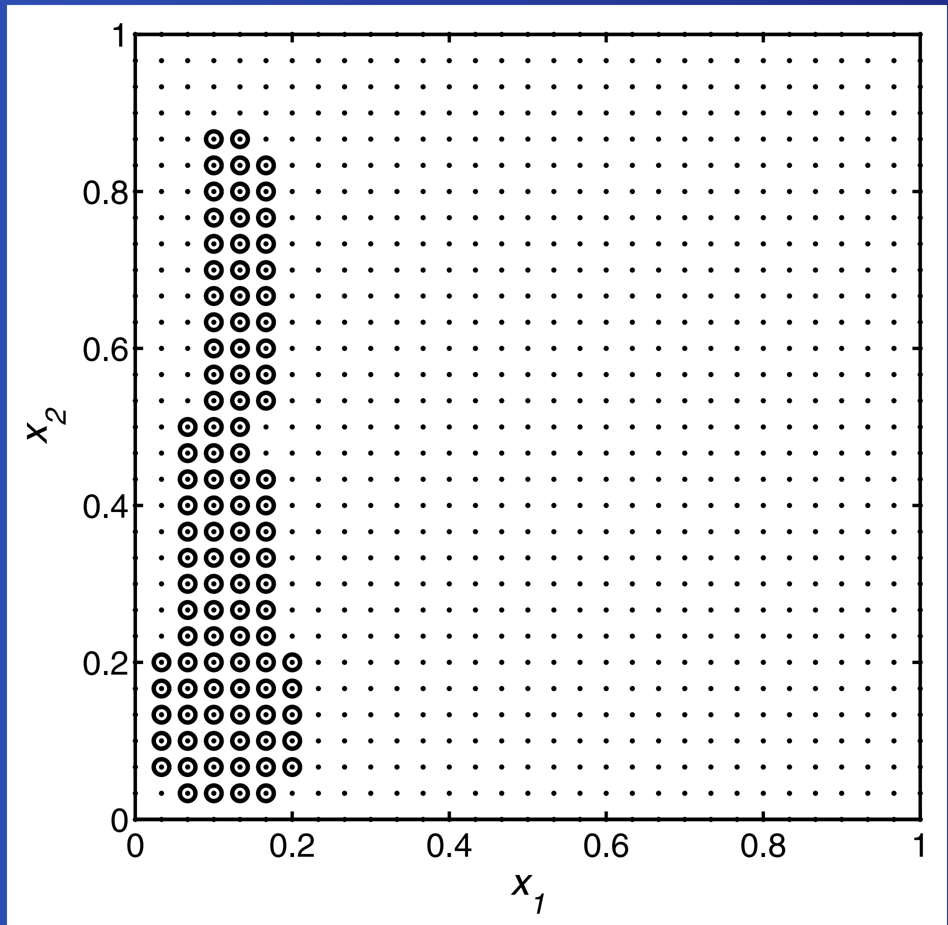
Typical behaviour

10 active sensors out of 900, Stage 6 of 6



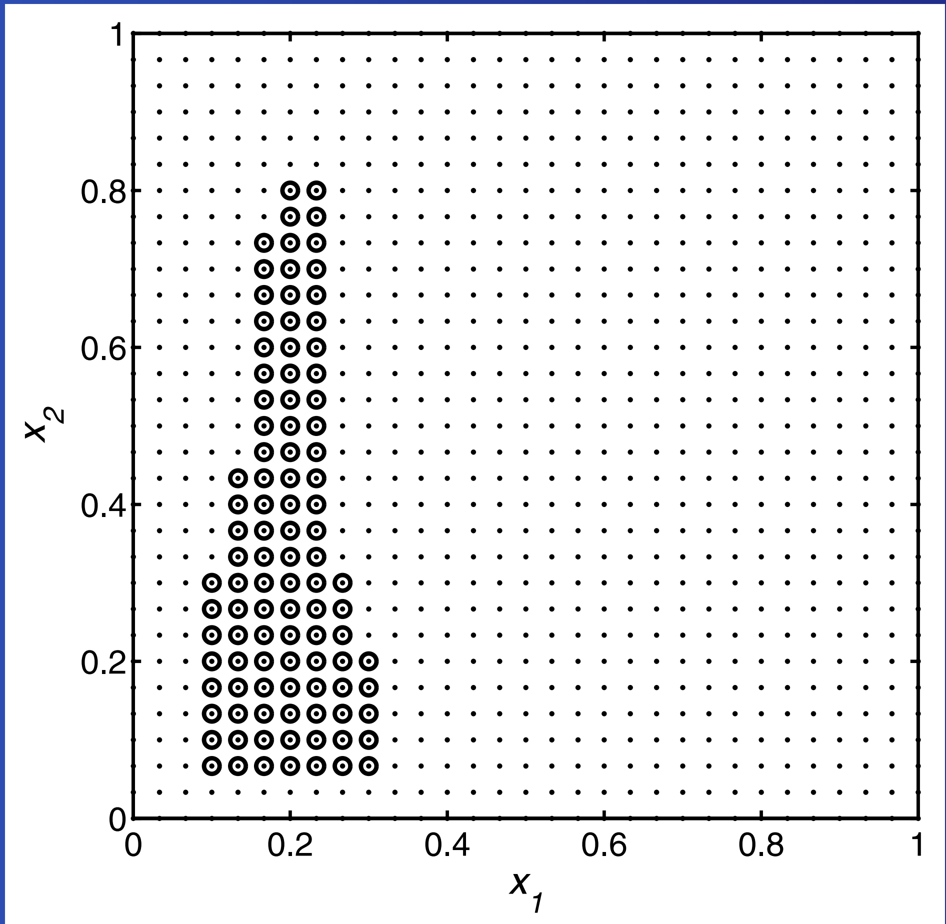
Typical behaviour

100 active sensors out of 900, Stage 1 of 6



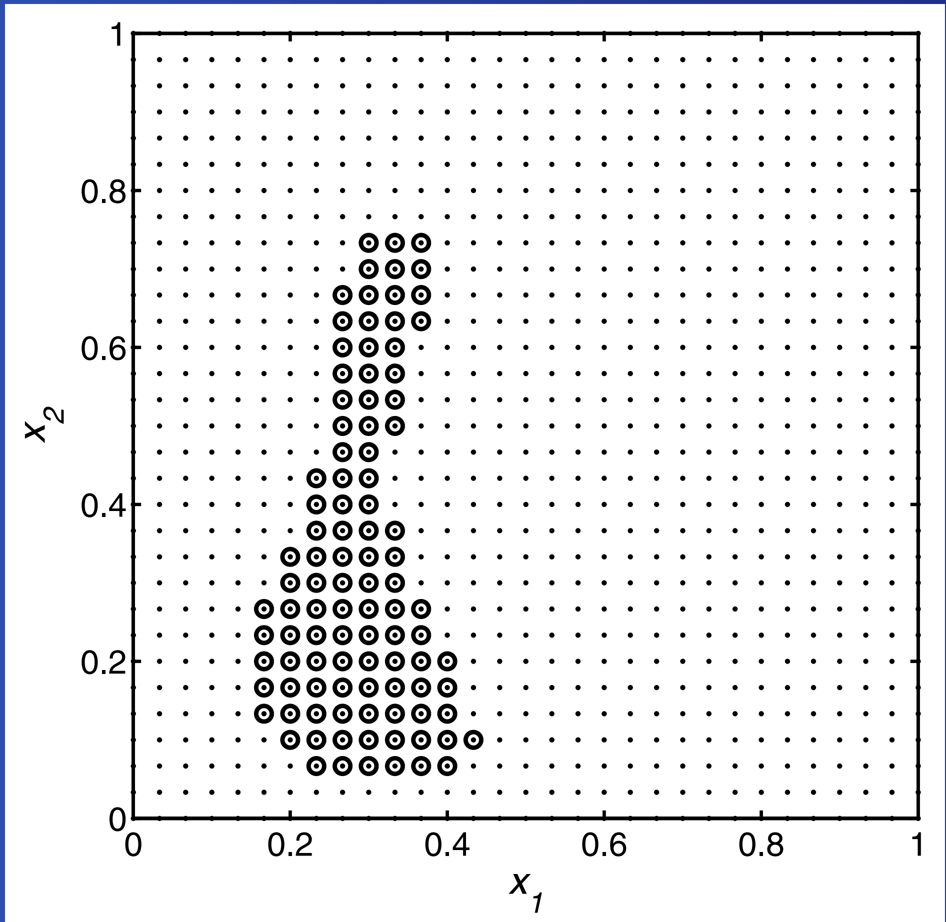
Typical behaviour

100 active sensors out of 900, Stage 2 of 6



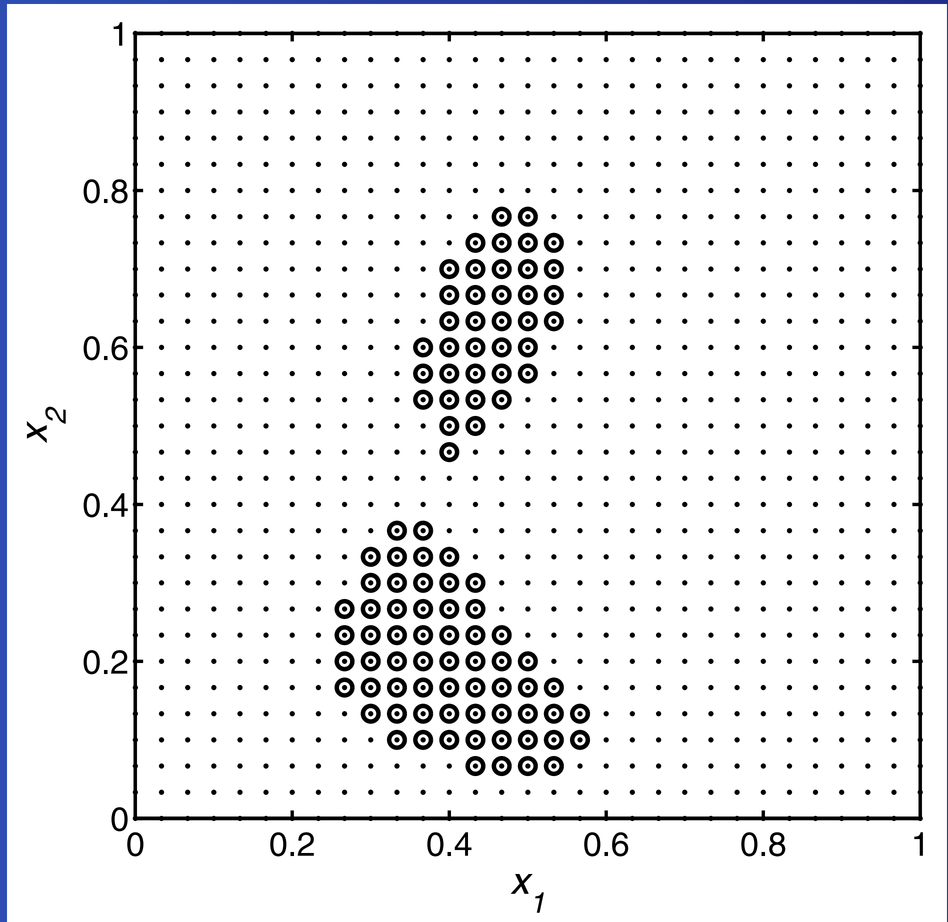
Typical behaviour

100 active sensors out of 900, Stage 3 of 6



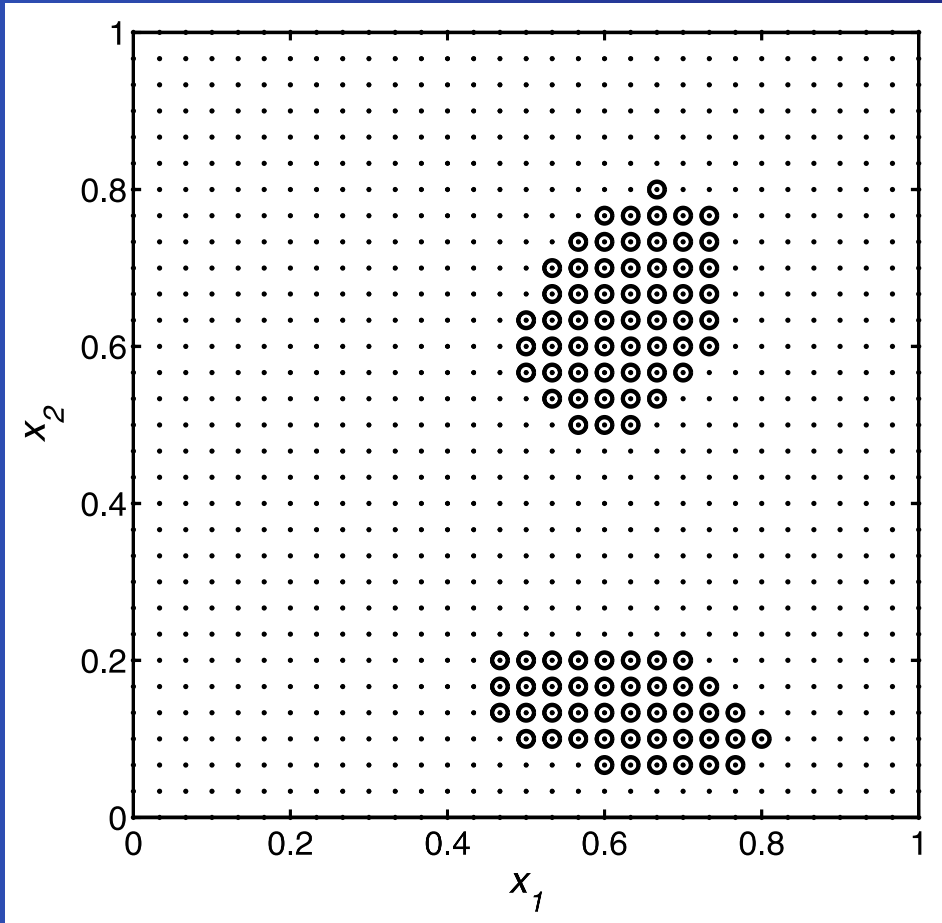
Typical behaviour

100 active sensors out of 900, Stage 4 of 6



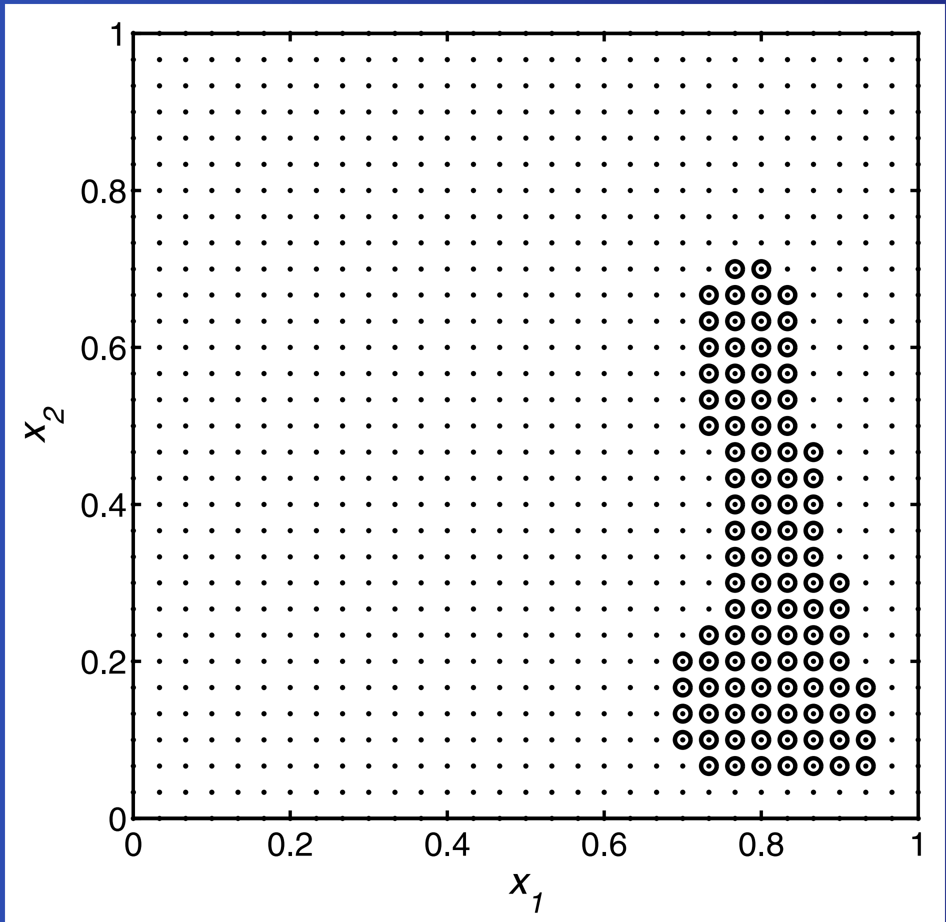
Typical behaviour

100 active sensors out of 900, Stage 5 of 6



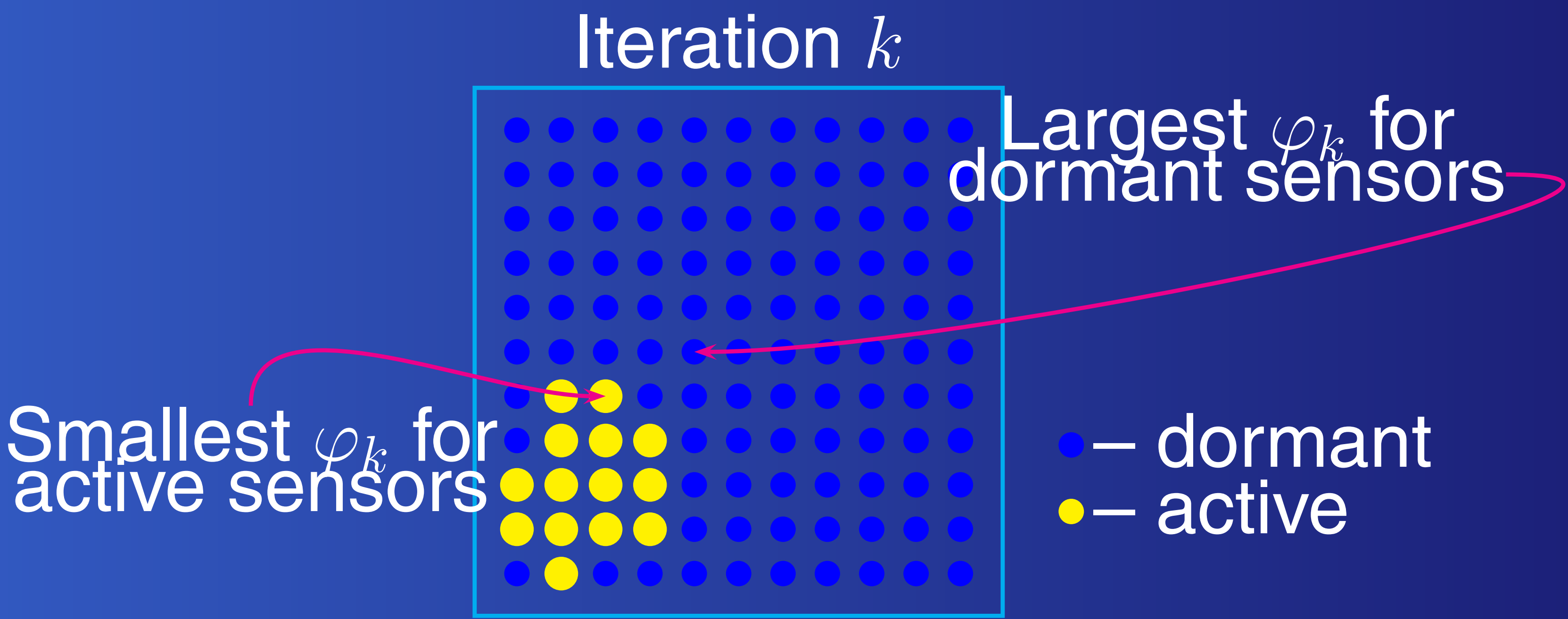
Typical behaviour

100 active sensors out of 900, Stage 6 of 6



Exchange algorithm as an alternative

The measurement points on T_k should thus coincide with maximum points of $\varphi_k(\,\cdot\,, \xi^\star)$. This forms a basis for a numerical algorithm of exchange type:



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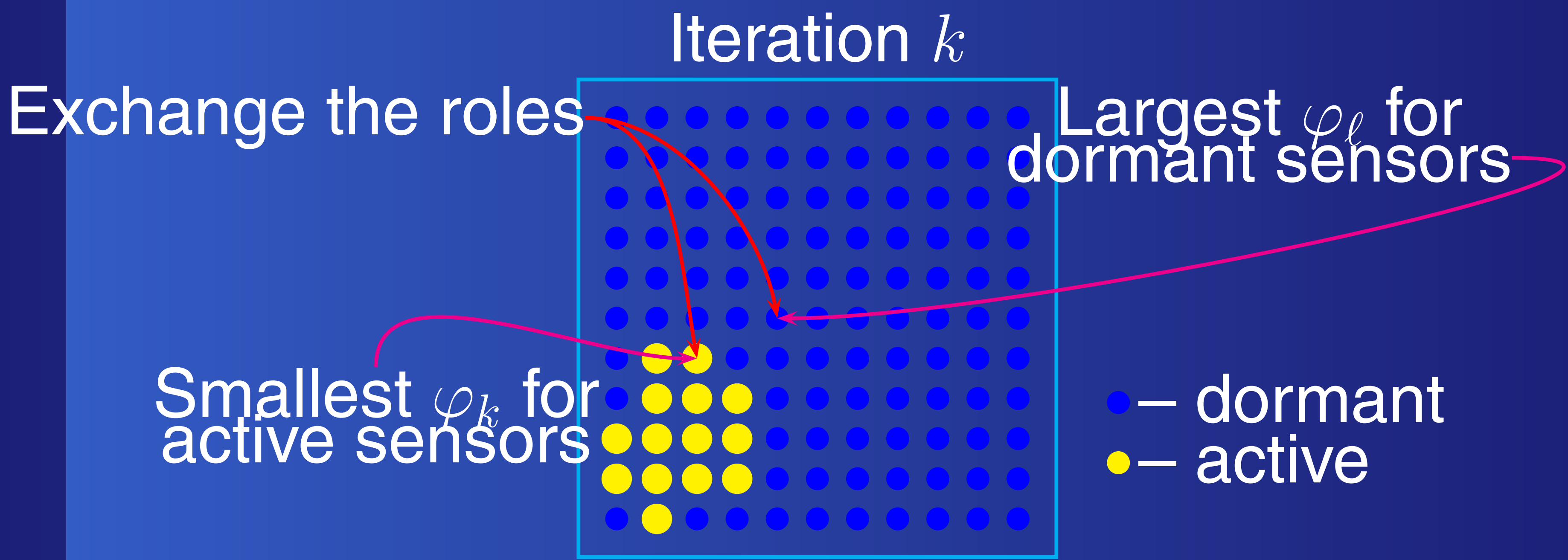

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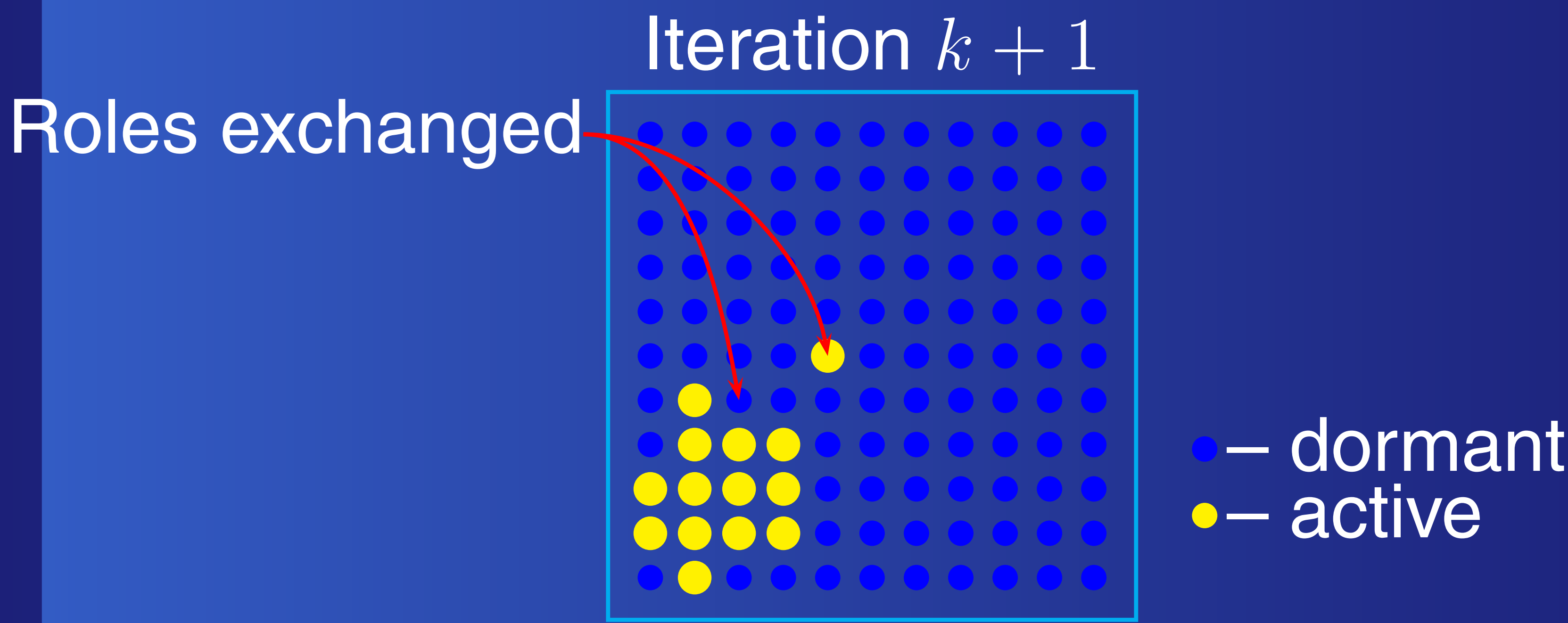

Conclusions

Summary of the contributions:

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Summary of the contributions:

- Reduces the problem to a guided branch-and-bound algorithm which can be easily implemented using existing components.
- Provides characterizations of optimal activations strategies for scanning sensors.
- Works well for both high and low numbers of sensors.
- Always produces a global optimum.

For the interested audience

Details beyond the talk are described in the book

D. Uciński (2005): *Optimal Measurement Methods for Distributed–Parameter System Identification*. — Boca Raton, FL: CRC Press, 392 p., 52 illus.

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Thank you!