ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 14, nº 3 (1987), p. 355-387
<http://www.numdam.org/item?id=ASNSP_1987_4_14_3_355_0>

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An Optimization Problem in Heat Conduction

N.E. AGUILERA (*) - L.A. CAFFARELLI (**) - J. SPRUCK (**)

In this paper we discuss a classical optimization problem in heat conduction that may briefly be described as follows (see for instance [D].) Given a surface $\partial\Omega$ in R^3, with prescribed temperature distribution ϕ, we want to surround $\partial\Omega$ with a prescribed volume of insulating material so as to minimize the loss of heat in a stationary situation.

Mathematically, we seek a function u, (the temperature in Ω), which is harmonic along the insulating material and vanishes outside of it. The quantity to be minimized (the flow of heat) is proportional to u_ν along $\partial\Omega$, or, what amounts to the same thing, to the total mass of Δu.

Our problem was inspired by the papers [A-C] and [A-A-C], which treat the case $\phi \equiv$ constant temperature distribution on $\partial\Omega$. The quantity to be minimized then reduces to the Dirichlet integral. The case of prescribed ϕ presents several new difficulties due to the fact that the free boundary condition now has a non-local character. This forces us to use non-local perturbations to study the free boundary. Furthermore, once initial regularity of the free boundary is established, the free boundary condition being non-local presents a novel type of higher regularity in the spirit of [K-N-S].

In order to overcome the difficulties of the problem, we make essential use of powerful new results of Jerison and Kenig ([J- K]) on the behavior of harmonic functions in domains with some irregular geometry, and also a monotonicity formula introduced in [A-C-F]. These are used to establish the Hölder continuity of u_ν along the (reduced) free boundary.

Although the problem presented here is for bounded Ω, there are no major changes in solving it for unbounded Ω, perhaps a more interesting geometry from the physical point of view.

It seems that the natural framework for our approach would be to minimize, instead of the flux of heat, a general monotone operator $A(u_\nu)$, like $\int u_\nu^p$ or $\inf u_\nu$ along $\partial\Omega$. We hope to expand this idea in future research

(*) Supported by CONICET, Argentina.
(**) Partially supported by N.S.F.
Pervenuto alla Redazione il 14 maggio 1986.

since these problems arise in questions of domain optimization for electrostatic configurations.

An outline of the paper is as follows. In Section 1 we formulate the variational problem for the temperature u and in Section 2 we show the existence of a weak solution to the variational problem. Section 3 contains the preliminary regularity properties of the minimizer u in the spirit of [A-C]. In particular, it is shown (Corollary 3.9) that u is Lipschitz in $\overline{\Omega}$, regular near $\partial\Omega$ and

$$\int_{\Omega} \Delta u dx = \int_{\partial\Omega} u_\nu d\sigma$$

Denoting $U = \{x \in \Omega : u(x) > 0\}$ and $F = \Omega - U$, we prove preliminary regularity properties of the free boundary ∂F in Corollary 3.21. Section 4 contains a proof that U is a non-tangentially accessible domain in the sense of [J-K]. In section 5 we make non-local perturbations to show that u_ν is Hölder continuous on the "reduced boundary" R of ∂F (Theorem 5.4). In the process we derive the so-called free boundary relations (Theorem 5.5 and 5.5′). We then go on to show in Section 7 that R is analytic.

1. - Statement of the problem

Suppose we are given

1) a bounded domain Ω in R^n whose boundary, $\partial\Omega$, is smooth
2) a function ϕ defined on $\partial\Omega$ positive and smooth and
3) a number $\mu, 0 < \mu < |\Omega|$ (where $|\Omega|$ denotes the Lebesgue measure of Ω).

Our purpose is to study the problem:

(P) Minimize

$$J(v) = \int_{\Omega} \Delta v dx$$

among all functions v that satisfy

i) $v \in H^1(\Omega)$, $v \geq 0$ in Ω and $v = \phi$ on $\partial\Omega$ (in the $H^1(\Omega)$ sense).
ii) $\Delta v \geq 0$ in the distribution sense in Ω.
iii) $|\{x \in \Omega : v(x) > 0\}| \leq \mu$.

We recall that if v satisfies ii) then we can interpret $\Delta v dx$ as a non-negative measure, so that $J(v)$ makes sense.

We will show that a solution to P exists and we will study regularity properties of the solution and the corresponding free boundary, i.e. the boundary of the set where the solution vanishes.

However, we will not directly solve this problem but use a penalized version, showing later (in section 6) that for small values of the penalization parameter we have a solution to P.

The penalized problem is stated as follows:

(P_ε) Minimize

$$J_\varepsilon(v) = \int_\Omega \Delta v \mathrm{d}x + f_\varepsilon(|\{x \in \Omega : v(x) > 0\}|)$$

among all functions v satisfying i) and ii) above where f_ε is defined as

$$f_\varepsilon(t) = \begin{cases} \mu + \frac{1}{\varepsilon}(t-\mu) & \text{if } t \geq \mu \\ \mu + \varepsilon(t-\mu) & \text{if } t \leq \mu \end{cases}$$

and $\varepsilon > 0$ is a small number.

2. - Existence of a solution to problem P_ε

It is not difficult to obtain an a priori bound on the H^1 norm of competing functions:

LEMMA 2.1 *Let v satisfy properties* (1.1) i, ii. *Then*

$$\int_\Omega |\nabla v|^2 \mathrm{d}x \leq \int_\Omega |\nabla w|^2 \mathrm{d}x + MJ(v)$$

where w is the harmonic function in Ω with boundary values ϕ, and $M = \sup_{\partial\Omega} \phi$.

PROOF. By the maximum principle $0 \leq v \leq w \leq M$. Since

$$\int_\Omega [\nabla v \cdot \nabla(v-w) + (v-w)\Delta v] \mathrm{d}x = 0$$

$$\int_\Omega \nabla w \cdot \nabla(w-v) \mathrm{d}x = 0$$

we have

$$\int_\Omega |\nabla v|^2 \mathrm{d}x = \int_\Omega |\nabla w|^2 \mathrm{d}x + \int_\Omega (w-v)\Delta v \mathrm{d}x \leq \int_\Omega |\nabla w|^2 \mathrm{d}x + MJ(v)$$

as required.

Since the function w of Lemma 2.1 satisfies (1.1) i, ii problem P_ε is well posed and if we set

$$d_\varepsilon = \inf J_\varepsilon(v)$$

among all admissible functions v for P_ε, then $d_\varepsilon < \infty$. Using lemma 2.1 we can find a minimizing sequence $\{v_k\}$ of admissible functions and a function $u_\varepsilon \in H^1(\Omega)$ so that

$$(2.1)\qquad \begin{aligned} &J_\varepsilon(v_k) \searrow d_\varepsilon \\ &v_k \to u_\varepsilon \text{ weakly in } H^1(\Omega), \text{ strongly in } L^2(\Omega) \text{ and} \\ &\text{almost everywhere in } \Omega. \end{aligned}$$

We conclude from (2.1) that $u_\varepsilon \geq 0$, $u_\varepsilon = \phi$ on $\partial\Omega$ (in the sense of H^1) and $\Delta u_\varepsilon \geq 0$ in the distribution sense. Moreover,

$$(2.2)\qquad \begin{aligned} &\int_\Omega |\nabla u_\varepsilon|^2 \mathrm{d}x \leq \underline{\lim} \int_\Omega |\nabla v_k|^2 \mathrm{d}x \\ &|\{x \in \Omega : u_\varepsilon(x) > 0\}| \leq \underline{\lim} |\{x \in \Omega : v_k(x) > 0\}| \end{aligned}$$

by the well known semicontinuity properties of these functionals. It follows from (2.2) that

THEOREM 2.1. *Given $\varepsilon > 0$, there exists a solution u_ε to problem P_ε, i.e*

$$d_\varepsilon = \min J_\varepsilon(v) = J_\varepsilon(u_\varepsilon).$$

3. - Regularity properties of solutions to P_ε

In this section we derive regularity properties of minimizers analogous to those in [A-C,§4] using similar techniques.

We use the notation $B(x,r)$ for the open ball of radius r and center x and $\overline{B}(x,r)$ for its closure.

We will keep $\varepsilon > 0$ fixed and write, for simplicity, $u = u_\varepsilon$.

LEMMA 3.1. *For $x \in \Omega$ and $r > 0$ such that $\overline{B}(x,r) \subset \Omega$, the average*

$$\fint_{B(x,r)} u \mathrm{d}y = \frac{1}{|B(x,r)|} \int_{B(x,r)} u \mathrm{d}y$$

is a increasing function of r for fixed x.

PROOF. Let $\{\phi_k\}$ be a sequence of $C_0^\infty(R^n)$ functions such that as $k \to \infty$, ϕ_k converges to the Dirac measure supported at the origin, in the sense of distributions. Then $u * \phi_k$, the convolution of u and ϕ_k which is defined in an appropriate subset of Ω, is a smooth subharmonic function, i.e. for $r_2 \geq r_1 > 0$

$$\fint_{B(x,r_2)} u_* \phi_k \mathrm{d}y \geq \fint_{B(x,r_1)} u_* \phi_k \mathrm{d}y.$$

Since $u * \phi_k \to u$ locally in $L^1(\Omega)$, the lemma follows.

The previous lemma allows us to redefine u on a set of Lebesgue measure zero so that

$$\fint_{B(x,r)} u\mathrm{d}\sigma = \frac{1}{\int_{\partial B(x,r)} \mathrm{d}\sigma} \cdot \int_{\partial B(x,r)} u\mathrm{d}\sigma \to u(x)$$

as $r \searrow 0$, for any $x \in \Omega$.

LEMMA 3.2 *Let O be a smooth bounded domain. Let ψ be a smooth function defined in $\overline{O}$, which is positive in O, harmonic near ∂O and vanishing on ∂O. Then for any $v \in H^1(O)$ subharmonic (in the distribution sense)*

$$\int_O \Delta v \mathrm{d}x = \lim_{\delta \to 0^+} - \int_{\partial O_\delta} v \frac{\partial \psi_\delta}{\partial \nu} \mathrm{d}\sigma$$

where $\psi_\delta = \frac{1}{\delta}\min(\psi, \delta)$, $O_\delta = \{x \in \Omega : 0 < \psi_\delta < 1\}$ and ν is the outward normal. In the boundary integral, v is interpreted as its trace on ∂O_δ.

PROOF. Since $\psi_\delta \nearrow 1$ as $\delta \searrow 0$, we have

$$\int_O \Delta v \mathrm{d}x = \lim_{\delta \to 0^+} \int_O \psi_\delta \Delta v \mathrm{d}x. \tag{3.1}$$

But

$$\begin{aligned} \int_O \psi_\delta \Delta v \mathrm{d}x &= - \int_O \nabla \psi_\delta \cdot \nabla v \mathrm{d}x = - \int_{O_\delta} \nabla \psi_\delta \nabla v \mathrm{d}x \\ &= - \int_{\partial O_\delta} v \frac{\partial \psi}{\partial \nu} \mathrm{d}\sigma + \int_{O_\delta} v \Delta \psi_\delta \mathrm{d}x. \end{aligned} \tag{3.2}$$

The combination (3.1) (3.2) proves the lemma since $\Delta \psi_\delta = 0$ in O_δ for δ small enough.

COROLLARY 3.3. *Let $B = B(x,r)$ and let $v \in H^1(B)$ be subharmonic in B. Then*

$$\int_B \Delta v \mathrm{d}x = \lim_{h \to 0^+} \frac{1}{h} \int_{|y|=1} [v(x+ry) - v(x+(r-h)y)] \mathrm{d}\sigma(y). \tag{3.3}$$

PROOF. In the previous lemma we let $B = O$ and $\psi = \log \frac{r}{\rho}$ if $n = 2$, $\psi = \frac{1}{\rho^{n-2}} - \frac{1}{r^{n-2}}$ if $n \geq 3$, (where $\rho = \rho(y) = |y - x|$) near ∂B and ψ smooth in B. A simple change of variables gives (3.3).

LEMMA 3.4. *Let O and ψ be as in lemma 3.2. Let $v \in H^1(O)$ be a non-negative subharmonic function, and choose a representative so that*

$$\int_{B(x,r)} v\mathrm{d}y \searrow v(x) \text{ as } r \searrow 0$$

everywhere in 0. *Then $\int_O v\Delta v\mathrm{d}x$ has a meaning and*

$$\int_O v\Delta v\mathrm{d}x + \int_O |\nabla v|^2\mathrm{d}x = \lim_{\delta\to 0^+} - \int_{\partial O_\delta} \frac{v^2}{2}\frac{\partial}{\partial\nu}\psi_\delta\mathrm{d}\sigma.$$

PROOF. For any compact subset K of Ω, $\int_K v\Delta v\mathrm{d}x$ has a meaning since v can be approximated by a decreasing sequence of smooth functions. We have

$$\int_O v\Delta v\mathrm{d}x = \lim_{\delta\to 0^+}\int_O v\psi_\delta\Delta v\mathrm{d}x$$

and, as in lemma 3.2,

$$\begin{aligned}\int_O v\psi_\delta\Delta v\mathrm{d}x = \int_O \nabla(v\psi_\delta)\cdot\nabla v\mathrm{d}x = \\ -\int_O \psi_\delta|\nabla v|^2\mathrm{d}x - \int_O v\nabla\psi_\delta\cdot\nabla v\mathrm{d}x = \\ -\int_O \psi_\delta|\nabla v|^2\mathrm{d}x - \frac{1}{2}\int_{O_\delta}\nabla v^2\cdot\nabla\psi_\delta\mathrm{d}x = \\ -\int_O \psi_\delta|\nabla v|^2\mathrm{d}x - \frac{1}{2}\int_{\partial O_\delta} v^2\frac{\partial\psi}{\partial\nu}\mathrm{d}\sigma\end{aligned}$$

and the lemma follows.

We will also need the following result from [A-C]

LEMMA 3.5. *Suppose $v \in H^1(\Omega)$ is a non-negative semicontinuous function. There exists a constant $c > 0$, depending only on dimension, such that whenever $\overline{B}(x,r) \subset \Omega$,*

$$\Big(\frac{1}{r}\int_{\partial B(x,r)} v\mathrm{d}\sigma\Big)^2\cdot|\{y\in B(x,r) : v(y) = 0\}| \le c\int_{B(x,r)} |\nabla v - h|^2\mathrm{d}y$$

where h is the harmonic function in $B(x,r)$ taking boundary values equal to v on $\partial B(x,r)$.

We can now prove

THEOREM 3.6. *There exists a constant $M_\epsilon > 0$ such that if u is a solution to P_ϵ and $\overline{B}(z,r) \subset \Omega$ with*

$$\frac{1}{r} \int_{\partial B(z,r)} u d\sigma \geq M_\epsilon$$

then $B(z,r) \in \{y \in \Omega : u(y) > 0\}$ and u is harmonic in $B(z,r)$.

PROOF. Let $B = B(x,r)$ and let v be the solution of the variational problem

$$\int_\Omega |\nabla v|^2 dx \to \min$$

among functions in $H^1(\Omega)$ which equal ϕ on $\partial\Omega$ and are *nonpositive* on $\{x \in \Omega : u(x) = 0\} \backslash B$. Such a solution exists since u itself is a competing function. The minimizer v has the following properties:

a. $\Delta v \geq 0$ (this follows by taking arbitrary $\delta > 0$ and non-negative test functions η, and comparing v and $v - \delta\eta$)

(3.4) b. $v \geq 0$ (compare v and $\max(v, -\delta)$ for $\delta > 0$), and actually $v \geq u$ (compare v and $v + \delta(u - v)_+$ for $\delta > 0$)

c. $v \leq M = \sup_{\partial\Omega} \phi$.

It follows that v is admissible for the problem P_ϵ and so

$$J_\epsilon(u) \leq J_\epsilon(v).$$

We now use lemma 3.2 with $O = \Omega$ and a suitable ψ to obtain

$$\int_\Omega \Delta u dx - \int_\Omega \Delta v dx = \lim_{\delta \to 0+} \int_{\partial\Omega - \delta} (v - h)\frac{\partial\psi}{\partial\nu} d\sigma. \tag{3.5}$$

Since $u = v$ on $\partial\Omega$ and $\frac{\partial\psi}{\partial\nu} \geq 0$ on $\partial\Omega_\delta \cap \Omega$,

$$\int_{\partial\Omega_\delta} (v^2 - u^2)\frac{\partial\psi}{\partial\nu} d\sigma = \int_{\partial\Omega_\delta} (v + u)(v - u)\frac{\partial\psi}{\partial\nu} d\sigma \leq 2M \int_{\partial\Omega_\delta} (v - u)\frac{\partial\psi}{\partial\nu} d\sigma. \tag{3.6}$$

Using (3.5) (3.6) and lemma 3.4 we obtain

$$\int_\Omega \Delta u dx - \int_\Omega \Delta v dx \geq \frac{1}{M} \lim_{\delta \to 0+} \frac{1}{2} \int_{\partial\Omega_\delta} (v^2 - u^2)\frac{\partial\psi}{\partial\nu} d\sigma \tag{3.7}$$

$$\geq \frac{1}{M}\left[\int_\Omega u\Delta v dx - \int_\Omega v\Delta v dx + \int_\Omega (|\nabla v|^2) dx\right]$$

But $\int_\Omega u\Delta u dx \geq 0$ and $\int_\Omega v\Delta v dx = 0$ (since for any $\eta \in C_0^\infty(\Omega)$), $\eta \geq 0$ and $|\lambda|$ small, the function $v + \lambda\eta v$ is nonpositive on $\{x \in \Omega : u(y) = 0\} \setminus B$ and takes the same values as v on $\partial\Omega$, so that

$$\int_\Omega |\nabla v|^2 dx \leq \int_\Omega |\nabla v|^2 dx + 2\lambda \int_\Omega \nabla v\eta \nabla v dx + \lambda^2 \int_\Omega |\nabla v\eta|^2 dx$$

and since λ is arbitrary

$$0 = \int_\Omega \nabla v\eta \cdot \nabla v dx = -\int_\Omega \eta v \Delta v dx.$$

Letting $\eta \to 1$ gives $\int_\Omega v\Delta v dx = 0$. Thus from (3.7) follows

$$\int_\Omega \Delta u dx - \int_\Omega \Delta v dx \geq \frac{1}{M}\int_\Omega (|\nabla u|^2 - |\nabla v|^2) dx. \tag{3.8}$$

Consider now the harmonic function h in B which equals u on ∂B, and extend h by u outside B. Then $h \in H^1(\Omega)$, $h \geq 0$, $\Delta h \geq 0$ so that h is admissible for both P_ε and as a competitor against v. It follows that

$$u \leq h \leq v \leq M$$

and from (3.8)

$$\begin{aligned}\int_\Omega \Delta u dx - \int_\Omega \Delta v dx &\geq \frac{1}{M}\int_\Omega (|\nabla u|^2 - |\nabla h|^2) dx \\ &= \frac{1}{M}\int_B (|\nabla u|^2 - |\nabla h|^2) dx = \frac{1}{M}\int_B |\nabla(u-h)|^2 dx.\end{aligned} \tag{3.9}$$

Since $J_\varepsilon(u) \leq J_\varepsilon(v)$ we obtain from (3.9)

$$\frac{1}{M}\int_B |\nabla(u-h)|^2 dx \leq \frac{1}{\varepsilon}|\{x \in B : u(x) = 0\}|. \tag{3.10}$$

Using lemma 3.5 (with v replaced by u) and (3.10) we obtain

$$\frac{1}{M}\int_B |\nabla(u-h)|^2 dx \leq \frac{1}{\varepsilon}\left(\frac{1}{r}\int_{\partial B} u d\sigma\right)^{-2}\int_B |\nabla(u-h)|^2 dx \tag{3.11}$$

and so if $\frac{1}{r}\int_{\partial B} u d\sigma > (\frac{M}{\varepsilon})^{1/2} = M_\varepsilon$, we must have

$$\int_B |\nabla(u-h)|^2 dx = 0.$$

This proves u is harmonic in B and thus $u > 0$ in B.

We will now show that the zeros of u stay away $\partial\Omega$ using a result whose proof is similar to that of lemma 3.5 and can be found in [A-A-C].

LEMMA 3.7. *There exists a positive constant c, depending on the smoothness of ϕ and $\partial\Omega$ but not on ε, such that for any $\delta > 0$ small enough,*

$$\frac{1}{\delta^2}|\{x \in \Omega : u(x) = 0\} \cap D_\delta| \leq c \int_{D_\delta} |\nabla(u - h)|^2 \mathrm{d}x$$

where $D_\delta = \{y \in \Omega : d(y, \partial\Omega) < \delta\}$ and h is the harmonic function in D_δ taking boundary values u on ∂D_δ.

As a consequence we have

THEOREM 3.8. *With the notation of the previous lemma, there exists a $\delta = \delta(\varepsilon)$ such that*

$$D_\delta \subset \{x \in \Omega : u(x) > 0\}.$$

PROOF. If h is as in lemma 3.7, but extended to Ω by u outside D_δ, we have that

$$0 \leq u \leq h \leq M = \sup_{\partial\Omega} \phi.$$

Let us find v minimizing $\int_\Omega |\nabla v|^2 \mathrm{d}x$ among $v \in H^1(\Omega)$ with $v = \phi$ on $\partial\Omega$ and $v \leq 0$ on $\{x \in \Omega : u(x) = 0\} \setminus D_\delta$. Then as in Theorem 3.6,

$$\int_\Omega \Delta u \mathrm{d}x - \int_\Omega \Delta v \mathrm{d}x \geq \frac{1}{M} \int_\Omega (|\nabla u|^2 - |\nabla v|^2) \mathrm{d}x \geq \frac{1}{M} \int_{D_\delta} (|\nabla u|^2 - |\nabla h|^2) \mathrm{d}x$$

and the proof follows as in that theorem using now lemma 3.7 in place of lemma 3.5.

Combining Theorems 3.6 and 3.8 we can state

COROLLARY 3.9. *The set $\{x \in \Omega : u(x) > 0\}$ is open, u is harmonic there and u is Lipschitz continuous in $\overline{\Omega}$. Also, u is regular near $\partial\Omega$ and*

$$\int_\Omega \Delta u \mathrm{d}x = \int_{\partial\Omega} u_\nu \mathrm{d}\sigma.$$

In the sequel, we will denote by U the set $\{x \in \Omega : u(x) > 0\}$ and by F the set $\Omega - U = \{x \in \Omega : u(x) = 0\}$. We can now state a "non-degeneracy" property of u, similar to that in [A-C]:

THEOREM 3.10. *For $0 < \tau < 1$, there exists a constant $m_\varepsilon(\tau)$ such that if $B(x,r) \subset \Omega$ and*

$$\frac{1}{r} \int_{\partial B(x,r)} u d\sigma \leq m_\varepsilon(\tau)$$

then $B(x,\tau r) \subset F$.

PROOF. Let us denote by v the function minimizing $\int_\Omega |\nabla v|^2 dx$ in $H^1(\Omega)$ subject to the constraints $v = \phi$ on $\partial\Omega$ and $v \leq 0$ on $\overline{B}(x,\tau r) \cup F$. Then $\Delta v \geq 0$, $0 \leq v \leq u$ and $\int_\Omega v\Delta v dx = 0$. Since v competes with u in problem P_ε, we find

$$\int_\Omega \Delta v dy - \int_\Omega \Delta u dy \geq \varepsilon |U \cap B(x,\tau r)|. \tag{3.12}$$

Also

$$\int_\Omega \Delta v dy - \int_\Omega \Delta u dy \leq \frac{1}{m} \int_\Omega (|\nabla v|^2 - |\nabla u|^2 dy \tag{3.13}$$

where $m = \inf_{\partial\Omega} \phi$. This follows as in the proof of Theorem 3.6.

Now define

$$g(\rho) = \begin{cases} \log \dfrac{\rho}{\tau r} & \text{if } n = 2 \\ \dfrac{1}{(\tau r)^{n-2}} - \dfrac{1}{\rho^{n-2}} & \text{if } n \geq 3 \end{cases}$$

and for y in $B(x, \sqrt{\tau r})$

$$h(y) = \min\{u(y), \frac{s}{g(\sqrt{\tau r})} \max\{g(\rho), 0\}\} \tag{3.14}$$

where $s = \max_{\overline{B}(x,\tau r)} u$ and $\rho = \rho(y) = |y - x|$. Extending h by u outside $B(x, \sqrt{\tau r})$, we see that $h = 0$ in $F \cup \overline{B}(x,\tau r)$ and $h = u$ on $\partial\Omega$. Therefore h competes with v and so from (3.12) (3.13) we get

$$\varepsilon |U \cap B(x,\tau r)| \leq \frac{1}{m} \int_{B(x,\sqrt{\tau r})} (|\nabla h|^2 - |\nabla u|^2) dy. \tag{3.15}$$

Since $h \equiv 0$ on $B(x,\tau r)$ we may rewrite (3.9) as

$$\int_{B(x,\tau r)} |\nabla u|^2 dy + m\varepsilon |U \cap B(x,\tau r)| \leq \int_{B(x,\sqrt{\tau r}) - B(x,\tau r)} (|\nabla h|^2 - |\nabla u|^2) dy. \tag{3.16}$$

Since $|\nabla h|^2 - |\nabla u|^2 = -2\nabla h \nabla(u-h) - |\nabla(u-h)|^2$ we may estimate the right hand side of (3.16) by

$$-2 \int_{B(x,\sqrt{\tau r})} \nabla(\max\{u-h,0\})\nabla h dy = 2 \int_{\partial B(x,\tau r)} u\nabla h \cdot \nu d\sigma \leq \frac{c(n,\tau)}{r} s \int_{\partial B(x,\tau r)} u d\sigma$$

by the definition of h (see (3.14)). Here $c(n,\tau)$ is constant depending only on the dimension n and τ. Therefore

$$\int_{B(x,\tau r)} |\nabla u|^2 dy + m_\varepsilon |U \cap B(x,\tau r| \leq \frac{c(n,\tau)}{r} s \int_{B(x,\tau r)} u d\sigma. \tag{3.17}$$

On the other hand,

$$\begin{aligned} \int_{\partial B(x,\tau r)} u d\sigma &\leq c(n)(\frac{1}{\tau r} \int_{B(x,\tau r)} u dy + \int_{B(x,\tau r)} |\nabla u| dy \\ &\leq c(n)\left((\frac{s}{\tau r}+1)|U \cap B(x,\tau r)| + \int_{B(x,\tau r)} |\nabla u|^2 dy \right). \end{aligned} \tag{3.18}$$

Since u is subharmonic,

$$s = \max_{B\sqrt{\tau r}} u \leq \bar{c}(n,\tau) \int_{\partial B_r} u d\sigma \tag{3.19}$$

It follows from (3.17) (3.18) (3.19) that if

$$\frac{1}{r} \int_{\partial B(x,r)} u d\sigma \leq m_\varepsilon(\tau)$$

and $m_\varepsilon(\tau)$ is sufficiently small depending on n, ε, τ, then necessarily $u \equiv 0$ on $B(x,\tau r)$ proving the theorem.

As a consequence ∂F has the following properties:

COROLLARY 3.11. *There exists a constant c_ε, $0 < c_\varepsilon < 1$, such that for any $x \in \partial F$ and $r > 0$ with $B(x,r) \subset \Omega$,*

$$c_\varepsilon \leq \frac{|FB(x,r)|}{|B(x,r)|} \leq 1 - c_\varepsilon \tag{3.20}$$

Moreover, given $\delta > 0$ and $x \in \partial F$ with $B(x,\delta) \subset \Omega$ there is a point $y \in U$ so that $|y-x| < \delta$, $u(y) > c\delta$ and in fact $B(y, c'\delta) \subset U$ where c, c' are positive constants depending on u_ε.

PROOF. By Theorems 3.6, 3.8, 3.10, if $x \in \partial F$, $B(x,r) \subset \Omega$ and $0 < \tau < 1$, we always have

$$m_\varepsilon(\tau) \leq \frac{1}{r} \int_{\partial B(x,r)} u d\sigma \leq M_\varepsilon$$

so that for fixed $r > 0$ we may find $y \in \partial B(x, r/2)$ such that $u(y) \geq m_\varepsilon(\frac{1}{2})\frac{r}{2}$. If we now choose τ small,

$$\frac{1}{\tau r} \int_{B(y,\tau r)} u d\sigma = \frac{1}{\tau r} u(y) \geq \frac{1}{2\tau} m_\varepsilon(\frac{1}{2}) \geq M_\varepsilon$$

and we obtain $|y - x| < r$, $u(y) \geq cr$ and $B(y, c'r) \subset U$.

Turning now to the density property (3.20), the previous argument gives an upper bound for

$$\frac{|B(x,r) \cap F}{|B(x,r)|}.$$

To show the lower bound, let us use the notations of Theorem 3.6. Then by (3.10)

$$\int_{B(x,r)} |\nabla(u-h)|^2 dy \leq \frac{M}{\varepsilon} |F \cap B(x,r)|. \tag{3.21}$$

By Poisson's integral formula, we have for $|y - x| < \tau R$, $0 < \tau < 1$

$$h(y) \leq (1 - c(n,\tau)) \int_{\partial B(x,r)} u d\sigma.$$

Also $u(y) = |u(y) - u(x)| \leq K\tau r$ where K is the Lipschitz norm of u (Theorem 3.8). By Theorem 3.10 we may conclude

$$h(y) - u(x) \geq (1 - c(n,\tau)) \int_{\partial B(x,r)} u d\sigma - K\tau r \geq [(1 - c(n,\tau)) m_\varepsilon(\tau) - K\tau] r.$$

It should be clear that $c(n,\tau) \to 0$ as $\tau \to 0$ and $m_\varepsilon(\tau)$ increases as τ decreases, so that for small τ

$$h(y) - u(y) \geq cr, \; y \in B(x, \tau r) \tag{3.22}$$

where c depends on u. Using Poincare's inequality

$$\frac{c_n}{r^2} \int_{B(x,r)} |u - h|^2 dy \leq \int_{B(x,r)} |\nabla(u-h)|^2 dy$$

and (3.21) (3.22) we obtain

$$\frac{c_n}{r^2}c^2r^2|B(x,\tau r)| \leq \frac{M}{\varepsilon}|F \cap B(x,r)|$$

which is the remaining bound.

We can therefore apply the results in [A-C, §4] to obtain:

COROLLARY 3.21 i) *If $H^{n-1}(A)$ denotes the $(n-1)$ dimensional Hausdorff measure of the set A, we have*

$$H^{n-1}(\partial F) < \infty$$

Moreover, for some positive constants $c_\varepsilon, C_\varepsilon$

$$c - \varepsilon r^{n-1} \leq H^{n-1}(F \cap B(x,r)) \leq C_\varepsilon r^n$$

for all $x \in \partial F$ and $r > 0$, $B(x,r) \subset \Omega$.

ii) *There exists a Borel function $q = q_\varepsilon$ such that for any $\eta \in C_0^\infty(\Omega)$*

$$-\int_\Omega \nabla u \cdot \nabla \eta dx = \int_{\partial F} \eta q dH^{n-1}$$

iii) *There exists positive constants c_ε and C_ε such that*

$$c_\varepsilon \leq q(x) \leq C_\varepsilon$$

for H^{n-1} almost all points x in ∂F.

iv) *For H^{n-1} almost all points in ∂F, an outward normal $\nu = \nu(x)$ to ∂F is defined and furthermore*

$$u(x+y) = q(x) y \cdot \nu^+ + O(y)$$

where $\frac{O(y)}{|y|} \to 0$ as $|y| \to 0$. This allows us to define $q(x) = u_\nu(x)$ at those points.

4. - Regularity of the free boundary

In this section we show that $U = \{x \in \Omega : u(x) > 0\}$ is a non-tangentially accessible domain and can therefore apply the results of [J-K].

Let us recall that $u = u_\varepsilon$ in a solution to the problem P_ε, where $\varepsilon > 0$ is fixed but small. We introduce the notation

$$F_a = \{x \in \Omega : u(x) \leq a\}$$

and

$$U_a = \{x \in \Omega : u(x) > a\}$$

for any $a \geq 0$.

THEOREM 4.1. *There exist constants $\lambda = \lambda_\varepsilon > 1$ and $\sigma = \sigma_\varepsilon < 1$ such that whenever $u(x) > 0$, $\delta = d(x,F)$ and $B(x,\delta) \subset \Omega$, then if $\delta_1 = d(x, F_{\frac{1}{2}u(x)})$, we have*

1) $\sigma\delta \le \delta_1 \le \delta$
2) *For some $y \in \partial B(x,\delta_1)$, $u(y) \ge \lambda v(x)$*

PROOF. Let us denote by m the constant $m_\varepsilon(\frac{1}{2})$ appearing in Theorem 3.10 and by K the Lipschitz norm of u (recall Theorem 3.8). If we define σ by

$$\sigma = \frac{1}{2}\frac{m}{K}$$

then we have

$$m\delta < u(x) \le K\delta$$

and for $|y - x| < \sigma\delta$, $u(y) > \frac{1}{2}u(x)$ which proves 1).

Let us now take $z \in F_{1/2u(x)}$ such that $\delta_1 = |x - z|$. If $|y - z| < \frac{1}{2}\sigma\delta$,

$$u(y) \le \frac{1}{2}u(x) + \frac{1}{2}\sigma K\delta \le \frac{3}{4}u(x).$$

Since u is harmonic in $B(x,\delta)$, we have with $B_1 = B(x,\delta_1)$ and $B = B(z, \frac{1}{2}\sigma\delta)$, that

$$u(x)\int_{\partial B_1} d\sigma = \int_{(\partial B_1)\cap B} u d\sigma + \int_{(\partial B_1)\setminus B} u d\sigma \le \frac{3}{4}u(x)\int_{(\partial B_1)\cap B} d\sigma + \int_{(\partial B_1)\setminus B} u d\sigma.$$

Let us denote by α the quotient

$$\alpha = \frac{\int_{(\partial B_1)\setminus B} d\sigma}{\int_{\partial B_1} d\sigma}.$$

Then $0 < \alpha < 1$ and

$$\left(\frac{1}{\alpha} - \frac{3}{4}\left(\frac{1-\alpha}{\alpha}\right)\right)u(x) \le \frac{1}{\int_{(\partial B_1)\setminus B} d\sigma} \cdot \int_{(\partial B_1)\setminus B} u d\sigma = u(y). \tag{4.1}$$

The constant on the left of (4.1) is $\frac{1+3\alpha}{4\alpha} = \frac{1}{4\alpha} + \frac{3}{4}$, a decreasing function of α. Now $\delta_1 \le \delta$ and $\sigma < 1$, so α is bounded above by

$$\beta = \frac{1}{\int_{\partial B_1} d\sigma}\int_{(\partial B_1)\setminus B(z,\frac{1}{2}\sigma\delta_1)} d\sigma$$

which is a number less than 1, depending on the dimension and the constant σ but not on x, z, δ or δ_1. We therefore obtain 2) with $\gamma = \frac{1}{4\beta} + \frac{3}{4} > 1$.

LEMMA 4.2. *There exists constants $\gamma = \gamma_\varepsilon$ and $\tau = \tau_\varepsilon$ such that whenever $x_0 \in F$, $x \in B(x_0, \frac{r}{2}) \cap U$ and A is a connected component of $U_{\frac{1}{2}u(x)} \cap B(x_0, r)$ containing x, then*

1) *For some $y_0 \in B(x_0, r)$, $B(y_0, \gamma r) \subset A$*
2) $\int_A |\nabla u| \rho^{2-n} dy \geq \tau r^2$ *where* $\rho = \rho(y) = |y - x_0|$.

PROOF. Starting with $x_1 = x$, we use Theorem 4.1 to inductively define a sequence of points $x_1, x_2, \dots, x_k, x_{k+1}$ so that for $j = 1, \dots, k$,

$$\text{i) } |x_{j+1} - x_j| = \delta_j = d(x_j, F_{\frac{1}{2}u(x_j)})$$
$$\text{ii) } B(x_j, \delta_j) \subset U_{\frac{1}{2}u(x_1)}$$
$$\text{iii) } u(x_{j+1}) > \lambda u(x_j)$$

By iii) and the Lipschitz character of u, we know that we cannot continue this process indefinitely without stepping out of $B(x_0, r)$, so we stop at the first k for which $B(x_{k+1}, \delta_{k+1}) \not\subset B(x_0, r)$. Notice also that by ii) and iii), and since $\delta_1 \leq \frac{r}{2}$, $B(x_j, \delta_j) \subset A$ for $j = 1, \dots, k$. By Theorem 4.1 we know that

$$\delta_j < d(x_j, F) \leq \frac{1}{\sigma} \delta_j$$

and therefore, with the notation of that theorem,

$$m\delta_j < u(x_j) < \frac{K}{\sigma} \delta_j.$$

Now since $u(x_{j+\ell}) \geq \lambda^\ell u(x_j)$, we must have (recall that $\sigma = \frac{1}{2}\frac{m}{K}$)

$$\delta_j < \frac{1}{m} u(x_j) \leq \frac{1}{m} \lambda^{-\ell} u(x_{j+\ell}) \leq \frac{1}{2\sigma^2} \lambda^{-\ell} \delta_{j+\ell}$$

so

$$|x_k - x_0| \leq |x_k - x_1| + x_1 - x_0| \leq \sum_{j=1}^{k-1} \delta_j + \frac{r}{2} \leq$$
$$\leq \frac{1}{2\sigma^2} \delta_k \sum_{\ell=1}^{k-1} \lambda^{-\ell} + \frac{r}{2} \leq \frac{1}{2\sigma^2(\lambda - 1)} \delta_k + \frac{r}{2}$$

since $\lambda > 1$.

Also $\delta_{k+1} \leq 2\delta k$ and $B(x_{k+1}, \delta_{k+1}) \not\subset B(x_0, r)$ and hence

$$r \leq |x_{k+1} - x_0| + \delta_{k+1} \leq |x_k - x_0| + 3\delta_k \leq \frac{r}{2} + c\delta_k$$

from which we derive 1) with $y_0 = x_k$ and suitable γ.

Now we are going to use 1) in order to obtain 2). Since $B(y_0, \gamma_x) \subset U \cap B(x_0, r)$, we must have

a) $m\gamma r \leq u(y_0)$

b) If $|y - y_0| \leq \frac{1}{2}\sigma\gamma r$ then $u(y) \geq \frac{3}{4}u(y_0)$.

Introducing polar coordinates centered at x_0, we may describe any point in the convex hull of $B(y_0, \frac{1}{2}\sigma\gamma r) \cup \{x_0\}$ by $\rho y'$ where $0 \leq \rho \leq \rho_0(y')$ and y' varies over a subset Σ of the unit sphere ($|y'| = 1$) For each $y' \in \Sigma$, consider $\rho_1(y')$ so that the segment $\underline{\rho y'}$ is inside A for $\rho_1 < \rho < \rho_0$ and either $\rho_1 = \sigma r$ or both $\rho_1 > \sigma r$ and $\rho_1 y' \in \overline{A}$.

In either case we have

$$u(\rho y') \leq \max\{\frac{1}{2}u(\alpha_1), \frac{1}{2}u(y_0)\} = \frac{1}{2}u(y_0)$$

and therefore

$$\frac{1}{4}u(y_0) \leq u(\rho_0 y') - u(\rho_1 y') \leq \int_{\rho_1}^{\rho_0} |\nabla u(\rho y')| \mathrm{d}\rho$$

and since $\rho_1 \geq \sigma r$

$$\frac{1}{4}\sigma r u(y_0) \leq \int_{\rho_1}^{\rho_0} |\nabla u(\rho y')| \rho \mathrm{d}y.$$

Integrating this inequality for $y' \in \Sigma$ we obtain

$$\frac{1}{4}\sigma r u(y_0) \int_{\Sigma} \mathrm{d}y' \leq \int_{A} |\nabla u(y)| \rho^{2-n} \mathrm{d}y.$$

We now notice that the integral on the left is bounded below by a constant depending only on the dimension and the quantity $\gamma\sigma$. Also $u(y_0) \geq m\gamma r$, so that for some $\tau > 0$ we have 2).

In [A-C-F] the following lemma is proved, although the statement there differs slightly from ours.

LEMMA 4.3. *Let v be a continuous function defined on $B = B(x_0, R)$. Suppose that v is harmonic in the open set $\{x \in B : v(x) \neq 0\}$. Let A_1 and A_2 be two different connected components in B of the set $\{x \in B : v(x) \neq 0\}$ and define for $0 < r < R$*

$$\phi(x) = \left(\frac{1}{r^2} \int_{B(x_0,r)\cap A_1} |\nabla v|^2 \rho^{2-n} \mathrm{d}x)(\frac{1}{r^2} \int_{B(x_0,r)\cap A_2} |\nabla v|^2 \rho^{2-n} \mathrm{d}x \right)$$

where $\rho = \rho(x) = |x - x_0|$. Then

1) *ϕ is a non-decreasing function of r*

2) *For almost all $r, 0 < r < R$, $\phi'(r) \geq \frac{1}{r} g(a)\phi(r)$ where*

$$a = a(r) = \frac{H^{n-1}((\partial B(x_0, r)) \setminus (A_1 \cup A_2))}{H^{n-1}(\partial B(x_0, r))}$$

and $g(a)$ is a convex and increasing function of a taking positive values for positive a.

We will also need the following result

LEMMA 4.4. *With the notation of the previous lemma, suppose in addition that for some constant $c' > 0$ and any $r, 0 < r < R$*

$$|B(x_0, r) \setminus (A_1 \cup A_2)| \geq c'|B(x_0, r)|.$$

Then for some positive β depending only on the dimension and the constant c'

$$\frac{\phi(r)}{r^\beta}$$

is a non-decreasing function of r.

PROOF. We are assuming now that for $0 < r < R$

$$\frac{1}{r^n} \int_0^r a(s) s^{n-1} ds \geq c'$$

where $a(s)$ was defined in the previous lemma. Since $0 \leq a \leq 1$, for small κ $0 < \kappa < 1$ we must have

$$c' \leq \frac{1}{r^n} \int_{\kappa r}^r a(s) s^{n-1} ds + \frac{1}{r^n} \int_0^{\kappa r} a(s) s^{n-1} ds \leq \frac{1}{r^n} \int_{\kappa r}^r a(s) s^{n-1} ds + c_n \kappa^n$$

or

$$(c' - c_n \kappa^n) \leq \frac{1}{r^n} \int_{\kappa r}^r a(s) s^{n-1} ds.$$

Let us fix κ so that $c' = 2c_n \kappa^n$. Then for some other constant $c > 0$, and for any r we must have

$$c \leq \frac{1}{r^n} \int_{\kappa r}^r a(s) s^{n-1} ds \leq \int_{\kappa r}^r a(s) \frac{ds}{s}.$$

We want to show that for some β, if $0 < r_1 < r_2 < R$ then $r_1^{-\beta}\phi(x_1) \leq r_2^{-\beta}\phi(r_2)$. So let us consider r_1 and r_2 fixed and assume that $\phi(r_1) \neq 0$.

We may choose k so that

$$\kappa^{k+1} < \frac{r_1}{r_2} < \kappa^k$$

i.e. $k \cong \log(\frac{r_2}{r_1})$.

Then

$$\int_{r_1}^{r_2} a(s)\frac{ds}{s} \geq \sum_{j=0}^{k-1} \int_{\kappa^{j+1}r_2}^{\kappa^j} a(s)\frac{ds}{s} \geq ck \geq c\log(\frac{r_2}{r_1}).$$

Since the function g in Lemma 4.3 is increasing,

$$g(\frac{1}{\log(\frac{r_2}{r_1})}\int_{r_1}^{r_2} a(s)\frac{ds}{s}) \leq \frac{1}{\log(\frac{r_2}{r_1})}\int_{r_1}^{r_2} g(a(s))\frac{ds}{s}$$

and so

$$\beta\log(\frac{r_2}{r_1}) \leq \int_{r_1}^{r_2} g(a(s))\frac{ds}{s}.$$

Using now that $\phi'(s) \geq \frac{g(a(s))}{s}\phi(s)$ we have

$$\log(\frac{\phi(r_2)}{\phi(r_1)}) \geq \int_{r_1}^{r_2} \frac{\phi'(s)}{\phi(s)}ds \leq \int_{r_1}^{r_2} g(a(s))\frac{ds}{s} \geq \log((\frac{r_2}{r_1})^\beta)$$

proving the lemma.

We would like to use now results concerning the boundary behavior of non-tangentially accessible domains [J-K], so let us first introduce the pertinent definition and results:

DEFINITION 4.5. Suppose k is fixed positive constant and let D be an open set in R^n. Then D satisfies a *Harnack chain condition* if for any $\delta > 0$ and any points $x, y \in D$ such that $|x-y| < c\delta$ and $B(x,\delta)$, $B(y,\delta)$ are contained in D, we can find points $x_1, x_2, \dots, x_\ell$ for which

1) $B_i = B(x_i, \delta) \subset D$ for $i = 1, \dots, \ell$
2) $x_i, x_\ell = y$ and $B_i \cap B_{i+1} \neq \phi$ for $i = 1, \dots, \ell - 1$
3) ℓ (the length of the chain) may depend on c but not on δ.

DEFINITION 4.6. A bounded open set $D \subset R^n$ is a *non-tangentially accessible domain* if there exist $M > 0$ and $r_0 > 0$ such that

1) For any $x \in D$ and any $r, 0 < r < r_0$, there exists $y \in D$ such that $k^{-1}r < |x-y| < r$ and $B(y, k^{-1}r) \subset D$.
2) The Lebesgue density of $R^n \setminus D$ at any of its points is bounded below uniformly by a positive constant, i.e. there exists $\kappa > 0$ such that for $x \in D$,

$$\overline{\lim}_{r\to 0}\frac{|B(x,r)\setminus D|}{|B(x,r)|} \leq \kappa.$$

THEOREM 4.7. *Let D be a non-tangentially accessible domain and let V be an open set, V and D contained in R^n. For any compact set $K, K \subset V$ there exists a constant $\alpha > 0$ such that for any positive harmonic functions v and w which vanish continuously on $(\partial D) \cap V$, the quotient $\frac{v}{w}$ is a Hölder continuous function of order α in $K \cap |\partial D$. In particular for any $x_0 \in K \cap \partial D$ the limit*

$$\lim_{\substack{x \to x_0 \\ x \in D}} \frac{v(x)}{w(x)}$$

exists.

We can now state one of the most important results in this section:

THEOREM 4.8. *Let $u = u_\varepsilon$ be a solution to the problem P_ε. Then the set $U = \{x \in \Omega : u(x) > 0\}$ is a non-tangentially accessible domain.*

PROOF. Since $\partial\Omega$ is smooth, by Theorem 3.8 we see that it is enough to study the properties of ∂F where $F = \{x \in \Omega : u(x) = 0\}$. By Corollary 3.11 we also know that the theorem will be proved if we can show that U satisfies the Harnack chain condition.

Suppose then that x_1 and x_2 are such that for some $\tilde{c} > 0$ and $\delta > 0$ we have

1) $|x_1 - x_2| < \tilde{c}\delta$
2) $B(x, \delta) \subset U$, $B(x_2, \delta) \subset U$.

Suppose now that, without loss of generality, $d(x_1, F) \leq d(x_2, F) = \delta_0$. If $\delta_0 \geq 2\tilde{c}\delta$ then $x_1 \in B(x_2, \tilde{c}\delta) \subset U$ and we can easily find the required chain. Let us consider then only the case $\delta_0 < 2\tilde{c}\delta$.

Let $x_0 \in F$ be such that $|x_0 - x_2| = \delta_0$ and let $r_0 = 4\tilde{c}\delta$. Then for $R \geq r_0$, x_1 and x_2 are in $B(x_0, \frac{R}{2})$. Let $d = \frac{1}{2}\min\{u(x_1), u(x_2)\}$.

We will presently show that if $R \geq cr_0$, where c may depend on u but not on x_1, x_2, then the connected components A_i of $B(x_0, R) \cap U_d$ which contain x_i, $i = 1, 2$, are actually the same. (Recall that $U_d = \{x \in \Omega : u(x) > d\}$).

Let us suppose that $A_1 \neq A_2$ and let us use Lemmas 4.3 and 4.4 with $v = (u - d)_+$. We see that because of Corollary 3.11, for some exponent $\beta > 0$ depending on u, but not on the particular points or radius,

$$r^{-\beta}\phi(r)$$

is non-decreasing, where ϕ is defined in lemma 4.3. We now apply Lemma 4.2 and Schwartz's inequality to obtain

$$\phi(r) \geq r^2$$

if $r \geq r_0$.

Since u is Lipschitz, say with constant K, we also have the bound

$$\phi(r) \leq cK^4 = c$$

and so

$$\tau^2 r_0^{-\beta} < r_0^{-\beta}\phi(r_0) \le R^{-\beta}\phi(R) \le cR^{-\beta}$$

or

$$R < cr_0.$$

Hence we must have $A_1 = A_2$. Since A_1 is open and connected we may find a curve Γ inside A_1 having x_1 and x_2 as end point. For each $y \in \Gamma$ we know that

$$u(y) > D = \frac{1}{2}\min\{u(x_1), u(x_2)\} \ge \frac{1}{2}m\delta$$

where m is the constant of Lemma 4.1. Therefore if $y \in \Gamma$, $d(y, F) > d/m$.

Let $\rho = \frac{1}{2}\frac{m}{K}\delta$, so that if $y \in \Gamma$ and $|x-y| < \rho$ then $u(x) > 0$. Since

$$\Gamma \in \bigcup_{y\in\Gamma} B(y,\rho)$$

we may find a sequence $y_1, \dots, y_\ell$ of points in Γ such that $\Gamma \subset \bigcup_{i=1}^{\ell} B(y_i,\rho)$, and we may further ask that no y in Γ belong to more than $c(n)$ of the balls $B(y_i,\rho)$.

Furthermore, since $\rho = \frac{1}{2}\frac{m}{K}\delta$, $r_0 = 4\tilde{c}\delta$ and $y_i \in B(x_0, cr_0)$ with c depending on u, ℓ must be bounded by a constant depending only on the dimension, the function u and the constant $\tilde{c}$, but not on x_1, x_2 or δ.

Using Theorem 4.7 we can state

COROLLARY 4.9. *Let u be a solution to the problem P_ε. Let $U = \{x \in \Omega : u(x) > 0\}$. Then*

a) *A (negative) Green's function for the Dirichlet problem exists in U, let us denote it by $\tilde{G}$.*

b) *There exists an exponent $\alpha > 0$ such that for any fixed $y \in U$ the quotient*

$$\frac{\tilde{G}(x,y)}{u(x)}$$

is a C^α function of x, for x away from y, taking values

$$\frac{\tilde{G}_\nu(x,y)}{u_\nu(x)}$$

at the regularity points of ∂U where the normal ν is defined. (Recall Corollary (3.12)).

c) *For any smooth function ψ we have*

$$\psi(y) = \int_{\partial U} \tilde{G}_\nu(x,y)\psi(x)dH^{n-1}(x) + \int_U \tilde{G}(x,y)\Delta\psi(x)dx$$

where ν denotes the normal pointing to the outside of U.

COROLLARY 4.10. *Let h be harmonic in U, $h = 1$ on $\partial\Omega$, $h = 0$ on ∂F. Then*

a) $c_\varepsilon u \leq h \leq C_\varepsilon u$ *for positive constants c_ε, C_ε depending on u*

b) *The quotient $\frac{h}{u}$ is a C^α function on $\overline{U}$, taking values $\frac{h_\nu}{u_\nu}$ at the regular points of ∂F.*

5. - Regularity of the normal derivative and the free boundary conditions

In this section we will show, using suitable perturbations of the free boundary, that the normal derivatives u_ν of a solution to P_ε, is Hölder continuous on most of the free boundary (Theorem 5.4). In the course of the proof, we will also derive the so-called "free boundary conditions" (Theorem 5.5) that will be used in section 7 to prove higher regularity.

Recall that $\varepsilon > 0$ is fixed and small, $u = u_\varepsilon$, $u_\nu = q$ as defined in Corollary 3.12, $U = \{x \in \Omega : u(x) > 0\}$ and $F = \Omega - U$.

To fix the ideas, consider a function ψ defined on R^n such that

1) $\psi(x)$ depends only on $|x|$,
2) $\psi(r)$ is non-increasing,
3) $\psi(r) \equiv 1$ if $r < \frac{1}{4}$, $\psi(r) \equiv 0$, if $r > \frac{1}{2}$,
4) $\psi \in C^\infty(R^n)$.

We denote by I the integral, $I = \int_{\{x:x_n=0\}} \psi(x)d\sigma$, where x is in R^n: $x = (x_1, \ldots, x_{n-1}, x_n)$.

Consider for δ small and positive the domains

$$
\begin{aligned}
&D = \{x \in R^n : x_n > 0, |x| < 1\} \\
(5.1)\qquad &D^+ = \{y \in R^n : y = x - \delta\psi(x)e_n \text{ for some } x \in D\} \\
&D^- = \{y \in R^n : y = x + \delta\psi(x)e_n \text{ for some } x \in D\}
\end{aligned}
$$

where $e_n = (0, \ldots, 0, 1)$.

The following lemma is a variant of the Hadamard variational formula; we will prove it using "interior variations" (see e.g. [G, Chapter 15]).

LEMMA 5.1. *Let v denote the harmonic function in D^+ (respectively D^-) taking boundary values x_n on $|x| = 1$ and zero otherwise. Then as $\delta \searrow 0$*

$$\frac{1}{\delta} \int_{D^+\cap\{x:x_n=0\}} v d\sigma \longrightarrow I$$

(respectively $\frac{1}{\delta} \int_{(\partial D^-)\cap D} x_n v_\nu dv \to I$ where v_ν is the inward normal derivative at ∂D^-).

PROOF. Let $y = x - \delta\psi(x)e_n$. If δ is small enough, the transformation $x \to y$ is a smooth change of variables. Let $v^*(x) = v(y(x))$. Then (see

[G, Chapter 15])

$$L_\delta(v^*) = 0 \text{ in } D$$

where

$$_\delta = \sum_k \frac{\partial}{\partial x_k}(\sum_\ell A_{k\ell}\frac{\partial}{\partial x_\ell})$$

and

$$A_{k\ell} = |\frac{\partial y}{\partial x}| \sum_j \frac{\partial x_k}{\partial y_j}\frac{\partial x_\ell}{\partial y_j}$$

Since v^* and x_n vanish for $x_n = 0$, we may use Green's theorem for the operator L_δ to obtain

$$\int_D (v^* L_\delta(x_n) - x_n L_\delta(v^*))\mathrm{d}x = \int_{(\partial D)\cap\{x:x_n>0\}} (v^*\frac{\partial x_n}{\partial \nu} - x_n \frac{\partial v^*}{\partial \nu})\mathrm{d}\sigma$$

where ν is the outward normal to ∂D.

Since $L_\delta v^* = 0$ and $v^* = v$ near $\partial D \cap \{x : x_n > 0\}$, we can rewrite this equation as

$$\int_D v^* L_\delta(x_n)\mathrm{d}x = \int_{(\partial D)\cap\{x:x_n>0\}} (v\frac{\partial x_n}{\partial \nu} - x_n \frac{\partial v}{\partial \nu})\mathrm{d}\sigma.$$

Since x_n and v are harmonic in D , we find

$$\int_D v^* L_\delta(x_n)\mathrm{d}x = \int_{\partial D\cap\{x:x_n>0\}} v\mathrm{d}\sigma.$$

On the other hand,

$$L_\delta(x_n) = \delta\Delta\psi + 0(\delta^2)$$

and v, v^* converge uniformly to x_n in D, as $\delta \to 0$ by the maximum principle. We conclude that

$$\int_{D^+\cap\{x:x_n=0\}} v\mathrm{d}\sigma = \delta \int_D x_n \Delta\psi \mathrm{d}x + 0(\delta^2).$$

Finally, by Green's theorem

$$\begin{aligned}\int_D x_n\Delta\psi\mathrm{d}x = \int_D (x_n\Delta\psi - \psi\Delta x_n)\mathrm{d}x &= \int_{\partial D} (x_n\psi_\nu - \psi(x_n)_\nu)\mathrm{d}\sigma \\ &= \int_{\{x:x_n=0\}} \psi\mathrm{d}\sigma = I\end{aligned}$$

proving the result for D^+. The proof of the corresponding result for D^- follows the same pattern.

We will denote by R a subset of ∂F consisting of points for which iii) and iv) of Corollary 3.12 apply and for which, furthemore

$$\frac{1}{r^{n-1}} \int_{(\partial F)\cap B(x,r)} |\nu(y) - \nu(x)| dH^{n-1}(y) \to 0$$

as $r \to 0$. We know that R can be chosen so that $H^{n-1}(\partial F \setminus R) = 0$.

For $x \in R$, it is possible to find now a function $\phi = \phi(r)$ defined for $r > 0$ so that ϕ is non-decreasing, and if $\nu = \nu(x)$ is the outward normal direction to F at x

(5.2)

a) $|u(x+y) - u_\nu(x)(y \cdot \nu)^+| \leq \phi(r)$ if $|y| \leq r$

b) If $y \in B(x,r)$ and either $y \cdot \nu < 0$ and $u(y) > 0$, or $y \cdot \nu > 0$ and $u(y) = 0$, then $|y \cdot \nu| \leq \phi(r)$

c) $\displaystyle \frac{1}{r^{n-1}} \int_{(\partial F)\cap B(x,r)} |\nu(y) - \nu(x)| dH^{n-1}(y) \leq \frac{1}{r}\phi(r)$

d) $\displaystyle \frac{1}{r^{n-1}} \int_{(\partial F)\cap B(x,r)} |u_\nu(y) - u_\nu(x)| dH^{n-1} \leq \frac{1}{r}\phi(r)$

e) $\displaystyle \frac{1}{r}\phi(r) \to 0$ as $r \to 0$

Suppose now that $x \in R$ and $r > 0$. Without loss of generality we may assume $x = 0$ and $\nu = \nu(x) = e_n$. We define the sets

$$(5.3) \qquad \begin{aligned} D^+(x,r) &= \{y : \frac{y}{r} - 2\frac{\phi(r)}{r}e_n \in D^+\} \\ D^-(x,r) &= \{y : \frac{y}{r} + 2\frac{\phi(r)}{r}e_n \in D^-\} \end{aligned}$$

where D^+ and D^- have the same meaning as in (5.1) and we take $\delta = \delta(r) = (\frac{\phi(r)}{r})^{1/2}$. Note that

$$\frac{1}{\delta}\frac{\phi(r)}{r} = \delta \to 0 \text{ as } r \to 0$$

LEMMA 5.2. *Let w be the harmonic function in $S = (D^+(x,r) \cup U) \cap B(x,r)$, taking boundary values u in $(\partial S) \cap \partial B(x,r)$ and zero otherwise. Then as $r \to 0$,*

$$\frac{1}{\delta}\frac{1}{r^n} \int_{S\cap\partial U} w dH^{n-1} \to I u_\nu(x)$$

PROOF. Since $u(y) \simeq u_\nu(x)y_n^+$ with error bounded by $\phi(r)$ (recall (5.2)), we have by regularity theory that $w(y) \simeq ru_\nu(x)v(\frac{y}{r})$ with error bounded by a constant (depending on the dimension n and $\psi(x)$) times $\phi(r)$, where v is the function defined in Lemma 5.1. Furthermore, since $\frac{1}{\delta}\frac{\phi(r)}{r} \to 0$ and $H^{n-1}(B(x,r) \cap \partial U)$ is bounded by Cr^{n-1} (Corollary 3.12), we only have to study the limits as $r \to 0$ of

$$\frac{1}{\delta}\frac{1}{r^{n-1}} \int_{S\cap\partial U} v(\frac{y}{r})dH^{n-1}(y)$$

By dilations, and Lemma 5.1

$$\frac{1}{\delta}\frac{1}{r^{n-1}} \int_{B(x,r)\cap\{y:y_n=0\}} v(\frac{y}{r})\mathrm{d}\sigma \to I$$

Now $|\nabla v|$ is bounded depending only on ψ and the Hausdorff distance from $S \cap \partial F$ to $B(x,r) \cap \{y : y_n = 0\}$ is bounded by $\phi(r)$ (condition 5.2b).

Using that U had finite perimeter, we may "integrate by parts" to obtain

$$\left| \int_{S\cap\partial U} \nu_n(y)v(\frac{y}{r})dH^{n-1}(y) - \int_{B(x,r)\cap\{y:y_n=0\}} v(y/r)dH^{n-1}(y)\right| \leq cr^{n-1}\frac{\phi(r)}{r}$$

where $\nu_n(y) = e_n \cdot \nu(y)$. Since ν is differentiable in the Lebesgue sense with respect to the $n-1$ dimensional Hausdorff measure restricted to ∂U (5.2c), the result follows.

Similarly the following lemma can be proved:

LEMMA 5.3. *Let w denote the harmonic function in $S = U \cap D^-(x,r)$ taking boundary values u in $(\partial S) \cap \partial B$ and zero otherwise. Then as $r \to 0$,*

$$\frac{1}{\delta r^n} \int_{U\cap\partial D^-(x,r)} uw_\nu dH^{n-1} \to Iu_\nu^2(x)$$

where ν is the inward normal to $D^-(x,r)$.

We now state and prove the principal result of this section

THEOREM 5.4. *u_ν is a Hölder continuous function on R.*

PROOF. Let x_1, x_2 be two points in R. Our plan is to perturb ∂F so as to obtain a bound for $|u_\nu(x_1) - u_\nu(x_2)|$ in terms of $|x_1 - x_2|^\alpha$, where α is the exponent of Corollaries 4.9 and 4.10.

Associated to x_1, x_2 we have functions ϕ_1, ϕ_2 defined in (5.2); without loss of generality we may suppose $\phi_1 = \phi_2 = \phi$. Suppose then that $0 < r < \frac{1}{10}|x_1 - x_2|$, $\phi(r) < 1$ and consider the sets $D^+(x_1,r)$, $D^-(x_2,r)$ defined by (5.3).

We denote by v, v_1, v_2 respectively, the continuous functions defined in $\overline{\Omega}$ by

$$
\begin{aligned}
\Delta v &= 0 \text{ in } A_0 = ((U \cup D^+(x_1,r)) \setminus B(x_2,r)) \cup (U \cap D^-(x_2,r)) \\
v &= u \text{ on } \partial\Omega \\
v &\equiv 0 \text{ on } \Omega \setminus A_0 \\
\Delta v_1 &= 0 \text{ in } A_1 = U \cup D^+(x_1,r) \\
v_1 &= u \text{ on } \partial\Omega \\
{}_1 &\equiv 0 \text{ in } \Omega \setminus A_1 \\
\Delta v_2 &= 0 \text{ in } A_2 = (U \setminus B(x_2,r)) \cup (U \cap D^-(x_2,r)) \\
v_2 &= u \text{ on } \partial\Omega \\
v_2 &\equiv 0 \text{ in } \Omega \setminus A_2
\end{aligned}
\tag{5.4}
$$

By the maximum principle, $v_2 \le v$, $u \le v_1$, and by the representation formula of corollary 4.9:

$$v_1(x) - u(x) = \int_{\Gamma_1} \tilde{G}(x,y) v_1(y) dH^{n-1}(y) \tag{5.5}$$

$$_2(x) - u(x) = -\int_{\Gamma_2} \tilde{G}(x,y)(v_2)_\nu(y) dH^{n-1}(y) \tag{5.6}$$

where $\Gamma_1 = D^+(x_1,r) \cap (\partial U)$, $\Gamma_2 = B(x_2,r) \cap (\partial(U) \setminus D^-(x_2,r))$ and ν is the outward normal.

Basic to our analysis is the harmonic function h in U with $h = 0$ on ∂F and $h = 1$ on $\partial\Omega$ (see Corollary 4.10). In terms of h we may express (here ν is the outward normal)

$$\int_{\partial\Omega} (v_\nu - u_\nu) d\sigma = \int_{\Gamma_1} h_\nu dH^{n-1} - \int_{\Gamma_2} h v_\nu dH^{n-1}. \tag{5.7}$$

We will estimate from above, the terms on the right-hand side of (5.7).

From (5.6), for $x \in B(x_1,r) \cap U$, $y \in B(x_2,r) \cap U$

$$
\begin{aligned}
v_2(x) = u(x) &\ge -\sup \left|\frac{\tilde{G}(x,y)}{u(y)}\right| \cdot \int_{\Gamma_2} u(v_2)_\nu dH^{n-1} \\
&\ge -c \int_{\Gamma_2} u(v_2)_\nu dH^{n-1}
\end{aligned}
\tag{5.8}
$$

by Corollary 4.9 (x_1 and x_2 will be kept fixed, as $r \to 0$).

If w_2 denotes the harmonic function of Lemma 5.3 (with $x = x_2$), we have $v_2 < w_2$ in $U \cap D^-(x_2,r)$. Therefore, $(v_2)_\nu \le (w_2)_\nu$ on Γ_2, so from (5.8) and Lemma 5.3 we obtain

$$v_2(x) - u(x) \geq -cI\delta r^n u_\nu^2(x_2) \geq -c\delta r^n.$$

In particular, if w_1 denotes the harmonic function of Lemma 5.2 (with $x = x_1$), we have

$$v \geq w_1 - c\delta r^n \text{ in } D^+(x_1, r) \cup (U \cap B(x_1, r))$$

since $v \geq v_2$ and $w_1 = u$ on $(\partial B(x_1, r)) \cap U$. Therefore,

$$\begin{aligned}\int_{\Gamma_1} h_\nu v dH^{n-1} &\leq \int_1 h_\nu w_1 dH^{n-1} - c\delta r^n \int_{\Gamma_1} h_\nu dH^{n-1} \\ &= -h_\nu(x_1) u_\nu(x_1) I\delta r^n + \int_{\Gamma_1} (h_\mu(x) - h_\nu(x_1) w_1 dH^{n-1} + 0(\delta r^n).\end{aligned}$$

From the proof of Lemma 5.2 we see that $w_1 \leq c\phi(r)$ on Γ_1, so that

$$\int_{\Gamma_1} h_\nu v dH^{n-1} \leq -h_\nu(x_1) u_\nu(x_1) I\delta r^n + 0(\delta r^n). \tag{5.9}$$

We now estimate $\int_{\Gamma_2} hv_\nu dH^{n-1}$ from below.

Since $v(x) \leq v_1(x) \leq M = \sup_{\partial\Omega} \phi$, we obtain from (5.5) that

$$v(x) < u(x) + cr^{n-1} \text{ in } U \cap D^-(x_2, r).$$

It follows that

$$(x) < w_2(x) + cr^{n-1}\tilde{w}(x) \text{ in } U \cap D^-(x_2, r)$$

where w_2 denotes the harmonic function of Lemma 5.3 (with $x = x_2$) and $\tilde{w}$ is a non-negative harmonic function in $S = U \cap D^-(x_2, r)$ taking smooth non-negative boundary values equal to 1 on $\partial S \cap \partial B(x_2, r)$ and 0 on $(\partial S) \cap B(x_2, r/2)$. Then $v_\nu \geq (w_2)_\nu + cr^{n-1}\tilde{w}_\nu$ on Γ_2 by the maximum principle and so

$$\begin{aligned}\int_{\Gamma_2} hv_\nu dH^{n-1} &\geq \int_{\Gamma_2} h(w_2)_\nu dH^{n-1} + cr^{n-1} \int_{\Gamma_2} h\tilde{w}_\nu dH^{n-1} \\ &= \int_{\Gamma_2} h(w_2)_\nu dH^{n-1} + 0(\delta r^n).\end{aligned} \tag{5.10}$$

Using Corollary 4.10 we may write

$$h = \frac{h_\nu(x_2)}{u_\nu(x_2)} u + O(r^\alpha) u$$

Inserting this into (5.10) and using Lemma 5.3 gives

$$\int_{\Gamma_2} hv_\nu dH^{n-1} \geq -h_\nu(x_2)u_\nu(x_2)I\delta r^n + o(\delta r^n) \tag{5.11}$$

combining (5.7) (5.9) (5.11) gives

$$\int_{\partial\Omega} (v_\nu - u_\nu)\mathrm{d}\sigma \leq -I\delta r^n(h_\nu u_\nu(x_1) - h_\nu u_\nu(x_2)) + 0(\delta r^n).$$

Since the volume added to U with $D^+(x_1,r)$ is $I\delta r^n$ with error $O(r^{n-1}\phi(r)) = 0(\delta r^n)$, and the volume taken away from U with $D^-(x_2,r)$ is $I\delta r^{n-1}$ with the same error, we conclude

$$0 \leq J_\varepsilon(v) - J_\varepsilon(u) \leq I\delta r^n(h_\nu u_\nu(x_1) - h_\nu u_\nu(x_2)) + o(\delta r^n).$$

Dividing by δr^n and letting $r \to 0$ gives

$$h_\nu u_\nu(x_1) - h_\nu u_\nu(x_2) \geq 0.$$

Reversing the roles of x_1, x_2, we conclude

$$h_\nu u_\nu \equiv \text{ constant on } R. \tag{5.12}$$

To prove Theorem 5.4 we write

$$\begin{aligned} 0 &= h_\nu u_\nu(x_1) - h_\nu u_\nu(x_2) \\ &= \frac{h_\nu}{u_\nu}(x_1)(u_\nu^2(x_1) - u_\nu^2(x_2)) + u_\nu^2(x_2)(\frac{h_\nu}{u_\nu}(x_1) - \frac{h_\nu}{u_\nu}(x_2)). \end{aligned}$$

Recalling corollaries 4.10 and 3.12 we find

$$|u_\nu(x_1) - u_\nu(x_2)| \leq C|x_1 - x_2|^\alpha.$$

It follows from [A-C] that ∂F is a $C^{1,\alpha}$ surface in a neighborhood of any point in R. The conclusion (5.12) is important enough to state as a separate

THEOREM 5.5. *Let u_ε be a solution to P_ε and let h be harmonic in U with $h = 1$ on $\partial\Omega$ and $h = 0$ on ∂F. Then*

$$u_\nu h_\nu \equiv \textit{ constant on } R.$$

We can also restate Theorem 5.5 in an equivalent but possibly more glamorous form:

THEOREM 5.5'. *Let $G(x,y)$ be the (negative) Green's function for Laplace's equation in Ω. There exists a constant A such that for any interior point y of F*

$$\int_{\partial F} \frac{G(x,y)}{u_\nu(x)} dH^{n-1}(x) = A.$$

It is also possible to prove

THEOREM 5.6. *Let $x_0 \in R$ and consider a suitable variation of ∂F near x_0. If U^* denotes the set obtained from U and v is the harmonic function in U^* taking values u on $\partial\Omega$ and 0 in $\Omega - U^*$, then*

$$\int_{\partial\Omega} v_\nu - u_\nu)\mathrm{d}\sigma = -h_\nu u_\nu(x_0)(|U^*| - |U|) + o(|U^*| - |U|).$$

6. - Relation between solutions of P and P_ε

So far we have considered $\varepsilon > 0$ small but fixed. Since we will not make further perturbations of the free boundary, this is a good place to show that for ε small enough a solution u to the problem P_ε is actually a solution to P, although we could have obtained this result earlier. We only need to show that for ε small enough, $|U| = \mu$ where μ is the prescribed volume.

LEMMA 6.1. *With the notation of the previous sections, there exists a positive constant c independent of ε such that*

$$\inf_R u_\nu \le c.$$

PROOF. Let K be a compact subset of Ω with smooth boundary so that if $\mathcal{A} = \Omega - K$, $|\mathcal{A}| = \mu$. Let $w \in H^1(\Omega)$ satisfy

1) $w = u = \phi$ in $\partial\Omega$
2) $w > 0$ and $\Delta w = 0$ in $\mathcal{A}$
3) $w = 0$ in K.

Then w is an admissible function for both P and P_ε and therefore

$$\int_{\partial F} u_\nu dH^{n-1} + f_\varepsilon(|U|) = J_\varepsilon(u) \le J_\varepsilon(w) = \int_{\partial\Omega} w_\nu \mathrm{d}\sigma + \mu.$$

The right-hand side is independent of ε where as both terms on the left are positive (if $\varepsilon < 1$), so for some $c > 0$, independent of ε,

$$\int_{\partial F} u_\nu dH^{n-1} \leq c$$

and

$$f_\varepsilon(|U|) \leq c.$$

The last inequality implies that

$$|U| \leq c_\varepsilon + \mu$$

so that if ε is chosen small

$$|U| \leq \frac{|\Omega| + \mu}{2}$$

and

$$|F| = |\Omega - U| \geq \frac{|\Omega| - \mu}{2}.$$

The isoperimetric inequality shows that $H^{n-1}(\partial F) \geq c_1$, for some c_1 independent of ε and therefore

$$c_1 \inf_R u_\nu \leq \int_{\partial F} u_\nu dH^{n-1} \leq c$$

proving the lemma.

COROLLARY 6.2. *If ε is small enough then $|U| \leq \mu$.*

PROOF. Let us suppose that $|U| > \mu$ and choose a point $x_0 \in R$ so that, by the previous lemma, $u_\nu(x_0) \leq c$. Let us decrease the size of U perturbing the free boundary near x_0. Using the notation and result of Theorem 5.6 we have

$$\int_{\partial\Omega} (v_\nu - u_\nu) d\sigma = u_\nu(x_0) h_\nu(x_0)(|U| - |U^*|) + o(|U^*| - |U|).$$

Now the maximum principle yields $nh_\nu \leq u_\nu \leq Mh_\nu$ on R where $m = \min_{\partial\Omega} u$, $M = \max_{\partial\Omega} u$ and therefore

$$0 \leq J_\varepsilon(v) - J_\varepsilon(u) \leq c(|U| - |U^*|) + \frac{1}{\varepsilon}(|U^*| - |U|) + o(|U^*| - |U|)$$

giving an upper bound for $\frac{1}{\varepsilon}$.

We now prove the converse inequality:

LEMMA 6.3. *If $\varepsilon > 0$ is small enough then $|U| \geq \mu$.*

PROOF. Let us suppose that $|U| < \mu$ and let d be the distance from F to

$\partial\Omega$. Let $x^* \in \partial F$ and $Y^* \in \partial\Omega$ be such that $|y^* - x^*| = d$, and let $z^* = \frac{x^* + y^*}{2}$. Using a suitable barrier it is not difficult to show that $u(z) \geq c$ for $z \in B(z^*, \frac{d}{4})$ where c is indipendent of ε since $d \leq$ diam Ω. If z is the point in the segment $[x^*, z^*]$ at distance $\delta/2$ from z^* and δ is small, we have, by Lemma 3.5,

$$c|F \cap B(z, \frac{d+\delta}{2})| \leq (\frac{2u(z)}{d+\delta})^2 |F \cap B(z, \frac{\delta + d}{2})| \leq c \int_{B(z, \frac{d+\delta}{2})} (|\nabla u|^2 - |\nabla w|^2) dx$$

where w is the harmonic function in $B(z, \frac{d+\delta}{2})$ taking boundary values u.

If v is the harmonic function in $U \cap B(z, \frac{d+\delta}{2})$ with boundary values u, the techniques in the proof of Theorem 3.6 show that

$$\int_{\partial\Omega} (u_\nu - v_\nu) d\sigma \geq c \int_{B(z, \frac{d+\delta}{2})} (|\nabla u|^2 - |\nabla w|^2) \geq c|F \cap B(z, \frac{d+\delta}{2})|.$$

Since $J_\varepsilon(u) \leq J_\varepsilon(v)$ we have (recall we assume $|U| < \mu$)

$$\int_{\partial\Omega} (u_\nu - v_\nu) d\sigma \leq \varepsilon |F \cap B(z, \frac{d+\delta}{2})|$$

from which the lemma follows.

Summing up the two previous results we obtain

THEOREM 6.4. *If ε is small enough, depending on the data, then any solution to problem P_ε is a solution to problem P.*

7. - Analyticity of the free boundary

In section 5, we have shown that there is a subset R of the free boundary ∂F ($H^{n-1}(\partial F \setminus R) = 0$) so that near any point of R, ∂F can be represented as a $C^{1,\alpha}$ graph Γ. Using the "free boundary condition" Theorem 5.5, we now show that Γ is analytic.

Introduce coordinates $x = (x_1, \dots, x_n) = (x', x_n)$ with the origin $0 \in \Gamma$, so that the positive x_n direction is normal to Γ at 0, and points into $U = \{u > 0\}$. We will work locally in a small ball B about 0, and let $B^+ = B \cap U$. Our free boundary condition says that there is positive harmonic function h in B^+ satisfying $h = 0$, $h_\nu \cdot u_\nu \equiv c$ on Γ.

THEOREM 7.1. *Let u, h be positive harmonic functions in B^+ satisfying $u = h = 0$, $u_\nu h_\nu \equiv c$ on Γ, $\Gamma \in C^{1,\alpha}$. Then Γ is analytic.*

PROOF. We follow the method of [K-N-S]. For $x \in B^+ \cap \Gamma$, let $y = (x', u)$. The map $x \to y$ is then locally $1-1$. We shrink B^+ (if necessary) and transform B^+ into a region U in $\{y_n \geq 0\}$ and Γ into a flat boundary $S \subset \{y_n = 0\}$. The associated "partial Legendre transform" is $x_n = \psi(y)$, which defines the inverse mapping. Then

$$u_n = \frac{1}{\psi_n}, u_\alpha = -\frac{\psi_\alpha}{\psi_n}, \ \alpha < n \tag{7.1}$$
$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha} - \frac{\psi_\alpha}{\psi_n}\frac{\partial}{\partial y_n}, \frac{\partial}{\partial x_n} = \frac{1}{\psi_n}\frac{\partial}{\partial y_n}.$$

Define $\phi(y)$ by $\phi(y) = u(x)$, $y \in U$: Using (7.1), we find that ψ, ϕ satisfy the system

$$\Delta u \equiv -\frac{\psi_{nn}}{\psi_n^3} + \sum_{\alpha<n}(-(\frac{\psi_\alpha}{\psi_n})_n + \frac{\psi_\alpha}{\psi_n}(\frac{\psi_\alpha}{\psi_n})_n) = 0 \text{ in } U \tag{7.2}$$

$$\Delta h \equiv \frac{1}{\psi_n}(\frac{\phi_n}{\psi_n})_n + \sum_{\alpha<n}((\phi_\alpha - \frac{\psi_\alpha}{\psi_n}\phi_n)_\alpha - \frac{\psi_\alpha}{\psi_n}(\phi_\alpha - -\frac{\psi_\alpha}{\psi_n}\phi_n)_n) = 0 \tag{7.3}$$

To find boundary conditions for ϕ, ψ we observe that since

$$x_n = \psi(y_1, \ldots, y_{n-1}, 0)$$

parametrizes Γ,

$$\nu = \frac{(-\psi_1, \ldots - \psi_{n-1}, 1)}{\sqrt{1 + \sum_{\alpha<n} \psi_\alpha^2}}.$$

Hence

$$u_\nu = \sum_{\alpha<n}(\frac{-\psi_\alpha}{\psi_n})(-\psi_\alpha) + \frac{1}{\psi_n} = \frac{1}{\psi_n}\sqrt{1 + \sum_{\alpha<n}\psi_\alpha^2}$$
$$h_\nu = \sum_{\alpha<n}(\phi - \frac{\psi_\alpha}{\psi_n}\phi_n)(-\psi_\alpha) + \frac{\phi_n}{\psi_n} = \frac{\phi_n}{\psi_n}\sqrt{1 + \sum_{\alpha<n}\psi_\alpha^2}.$$

Therefore ϕ, ψ satisfy the boundary conditions

$$\phi = 0, \ \frac{\phi_n}{\psi_n^2}(1 + \sum_{\alpha<n}\psi_\alpha^2) \equiv c \text{ on } S. \tag{7.4}$$

We claim that the system (7.2) (7.3) (7.4) is elliptic and coercive in the sense of Agmon-Douglas-Nirenberg (with the obvious choice of weights). To see this, we compute the principal part of the linearization at $y = 0$, using that $\psi_\alpha(0) = 0$, $\alpha < n$. We find:

$$L\overline{\psi} = 0 \tag{7.5}$$
$$L(\overline{\phi} - \frac{\phi_n(0)}{\psi_n(0)}\overline{\psi}) = 0$$

where $L = \frac{1}{\psi_n(0)^2}\partial_n^2 + \sum_{\alpha<n} \partial_\alpha^2$, and

$$\overline{\phi} = 0,\ \overline{\phi}_n - 2\frac{\phi_n(0)}{\psi_n(0)}\overline{\psi}_n = 0 \text{ on } y_n = 0. \tag{7.6}$$

The system (7.5) is clearly elliptic since it has uncoupled into $L\overline{\psi} = 0$, $L\overline{\phi} = 0$. It is also coercive, for $\overline{\phi} = 0$ is coercive for $L\overline{\phi} = 0$, and $\overline{\psi}_n = 0$ is coercive for $L\overline{\psi} = 0$.

To establish the analyticity of $\Gamma : x_n = \psi(y', 0)$ we need to satisfy the initial regularity assumptions to apply [M, Th. 6.8.2]. For a nonlinear system of the type (7.2) (7.3) (7.4), it is normally required that ϕ, $\psi \in C^2(U \cup S)$. However, our system can be written in divergence form:

$$\frac{(1 + \sum_{\alpha<n} \psi_\alpha^2)_n}{\psi_n^2} + \sum_{\alpha<n}(\frac{\psi_\alpha}{\psi_n}) = 0$$

$$(\frac{\phi_n}{\psi_n})_n + \sum_{\alpha<n}((\phi_\alpha - \frac{\psi_\alpha}{\psi_n}\phi_n)\psi_n)_\alpha + 0.$$

Using the divergence structure, one may show that if ϕ, ψ are initially $C^{1,\alpha}(U \cup S)$ (this is the case here since $\Gamma \in C^{1,\alpha}$), then ϕ, ψ are in $C^{2,\alpha}(U \cup S)$ so that Morrey's theorem can be applied (for more details see [K-N-S. pp. 112-113]) and the analyticity of Γ follows.

Acknowledgement

The second author would like to thank the hospitality of Mittag-Leffler Institute where this research was partially developed.

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