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AN OPTIMIZATION PROBLEM WITH FREE BOUNDARY GOVERNED BY A DEGENERATE QUASILINEAR OPERATOR

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Abstract. In this paper we study the existence, regularity and geometric properties of an optimal configuration to a free boundary optimization problem governed by the p-Laplacian operator.

1. Introduction

Let D be a bounded smooth domain in $\mathbb{R}^n$ and φ a positive function defined on it (the temperature distribution of the body D). A classical minimization problem in heat conduction asks for the best way of insulating the body D, with a prescribed amount of insulating material in a stationary situation. This model also designs problems in electrostatics, potential flow in fluid mechanics among others.

The mathematical description of this problem is as follows: given a fixed number $\gamma > 0$ (quantity of insulation material), for each domain Ω surrounding D, such that $|\Omega \setminus D| = \gamma$, we consider the potential u associated to the configuration Ω, i.e., the harmonic function in $\Omega \setminus D$, taking boundary data equal to φ on ∂D and 0 on $\partial\Omega$. The flow of heat (quantity to be minimized), corresponding to the configuration Ω, is given by a nonlocal

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monotone operator, given by

$$J(\Omega) := \int_{\partial D} \Gamma\big(x, u_\mu(x)\big) dS,$$

where μ is the inward normal vector defined on ∂D. The function $\Gamma \colon \partial D \times \mathbb{R} \to \mathbb{R}$ is assumed to be convex and increasing on u_μ and continuous on x. Important examples are, $\Gamma(t) = t$ (classical heat conduction problem), $\Gamma(t) = t^p$ (optimal configurations in electrostatics), $\Gamma(x,t) = \max\{t, C(x)\}$ (problems in the material sciences).

In [20], the second author studied the problem of minimizing $J(\Omega)$ among all configurations Ω such that, say, $|\Omega \setminus D| = 1$, where $|A|$ is the volume of the set A. This optimization problem with linear heat flux, i.e., $\Gamma(x,t) = t$, was studied in [1] and [4]. Qualitative geometric properties of the free boundary, namely symmetry, uniqueness and full regularity of the free boundary, were explored in [21].

In this present paper, we turn our attention to this problem when we allow the temperature itself to deform the medium. We assume the influence of the temperature distribution on the medium is proportional to the magnitude of its gradient. These considerations lead us to study this optimization problem when temperature distribution is governed by the p-Laplacian. In other words, the variational problem we are interested in is, for $1 < p < \infty$

$$\text{Minimize } \left\{ \begin{array}{c} J(u) := \displaystyle\int_{\partial D} \Gamma(x, u_\mu) dS \;\Big|\; u \colon D^C \to \mathbb{R},\ u = \varphi \text{ on } \partial D, \\ \Delta_p u = 0 \text{ in } \{u > 0\} \text{ and } |\{u > 0\}| = 1. \end{array} \right\} \tag{1.1}$$

Here, $\Delta_p u := \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big)$. Several new difficulties appear when dealing with the nonlinear operator Δ_p. One of its main difficulties lies in the fact that the p-Laplacian is not uniformly elliptic.

In analogy with the linear case for the Laplacian operator, in this paper we shall restrict ourselves to the heat flux given by $\Gamma(x,t) = t^{p-1}$. In our physical considerations, we will assume the body to be insulated has much smaller volume than the quantity of insulation material. This leads us to consider a constant temperature distribution, say, $\varphi \equiv 1$. With these assumptions, problem (1.1) can be reformulated in terms of the following equivalent version of it:

$$\text{Minimize } \left\{ \begin{array}{c} J(u) := \displaystyle\int_{D^C} |\nabla u|^p dx \;\Big|\; u \colon D^C \to \mathbb{R},\ u = \varphi \text{ on } \partial D, \\ \Delta_p u = 0 \text{ in } \{u > 0\} \text{ and } |\{u > 0\}| = 1. \end{array} \right\} \tag{1.2}$$

In problem (1.2), we shall assume φ to be positive and, say, in $W^{1,p}(D_\delta) \cap C(\overline{D_\delta})$, for some small tubular neighborhood of ∂D. Our computations could be done assuming less but notice, however, that when $\varphi \equiv 1$, problem (1.2), is equivalent to problem (1.1) with $\Gamma(x,t) = t^{p-1}$: our initial physical motivation. In this paper we will only deal with problem (1.2). We hope to turn our attention to problem (1.1) in its full generality in future research.

From the mathematical point of view, our approach is motivated by recent advances on the free boundary regularity theory for minimum problems with a variable domain of integration involving degenerate quasilinear operators. Namely, D. Danielli and A. Petrosyan, in [10], have recently extended the celebrated work of H. Alt and L. Caffarelli [2], for the p-Laplacian operator. Our regularity results will rely on suitable modifications of the arguments in [10]. Furthermore, we shall establish a free boundary condition that will relate our optimization problem with Bernoulli-type problems, similar to the ones studied in [18], [12] and [10].

2. Mathematical fundamentals of the physical problem

In this section, we shall introduce the main mathematical tools we shall use throughout the whole paper. Throughout the article, $1 < p < \infty$ and $\Delta_p u$ stands for the p-Laplacian operator

$$\Delta_p u := \operatorname{div}\big(|\nabla|^{p-2}\nabla u\big).$$

Let U be a domain in $\mathbb{R}^N$. Let us recall that for any $\xi \in W^{1,p}(U)$, $\Delta_p \xi \in \big[W_0^{1,p}(U)\big]^*$ and

$$\langle \Delta_p \xi, v\rangle = \int_U |\nabla \xi|^{p-2}\nabla\xi \cdot \nabla v dx.$$

Problem (1.2) presents several difficulties from the mathematical point of view. Our strategy will be to study a penalized version of it, which is sort of a weak formulation of the problem. The idea is to grapple with the difficulty of volume constraint, which is very unstable under limits and makes perturbation arguments quite hard.

From now on, we denote by V the following set

$$V := \left\{u \in W^{1,p}(D^C) : u = \varphi \text{ on } \partial D\right\}.$$

The penalized problem is stated as follows: Let $\varepsilon > 0$ be fixed. We consider the function

$$f_\varepsilon := \begin{cases} 1 + \frac{1}{\varepsilon}(t-1) & \text{if } t \geq 1 \\ 1 + \varepsilon(t-1) & \text{otherwise.} \end{cases}$$

We then define the penalized functional as

$$J_\varepsilon(u) := \int_{D^C} |\nabla u|^p dx + f_\varepsilon\big(|\{u > 0\}|\big). \tag{2.1}$$

For the moment, we shall be interested in the following minimization problem

$$\min_{u \in V} J_\varepsilon(u). \tag{2.2}$$

For latter use, given a γ-bilipschitz function f we consider the similar penalized problem of (2.2) given by the functional:

$$J_f := \int_{D^C} |\nabla u|^p dx + f\big(|\{u > 0\}|\big). \tag{2.3}$$

3. Properties of solutions of problem (2.2)

In this section we shall derive existence of a minimizer for the penalized problem as well as some important nondegeneracy conditions, such as optimal regularity and linear growth away from the free boundary. The proof of Theorem 3.1 will be developed throughout this section. At the end, we shall be able to state a representation theorem that will be crucial to study further regularity properties of the free boundary.

Theorem 3.1. *For each $\varepsilon > 0$ fixed, there exists a minimizer $u_\varepsilon \in V$ for the functional J_ε. Furthermore*

(1) $u_\varepsilon \geq 0$.

(2) $\Delta_p u_\varepsilon$ *is a nonnegative Radon measure supported on $\partial\{u_\varepsilon > 0\}$. In particular,*

$$\Delta_p u_\varepsilon = 0 \text{ in } \{x \in D^C : u_\varepsilon(x) > 0\}.$$

(3) *Any minimizer u_ε of problem (2.2) is Lipschitz continuous and for any compact $\mathcal{K} \subset D^C$ there exists a constant $K = K(\mathcal{K}, p, n, \varepsilon, D)$ such that $\|u_\varepsilon\|_{Lip(\mathcal{K})} \leq K$.*

(4) *The function u_ε grows linearly away from the free boundary; i.e., for any compact $\mathcal{K} \subset D^C$, there exist positive constants c, C, depending on dimension, $\mathcal{K}$, p, D and ε, such that*

$$c\,dist\big(x, \partial\{u_\varepsilon > 0\}\big) \leq u_\varepsilon(x) \leq C\,dist\big(x, \partial\{u_\varepsilon > 0\}\big), \ \forall x \in \mathcal{K}.$$

(5) *The free boundary is uniformly dense; i.e., for any compact $\mathcal{K} \subset D^C$ fixed, there exist a constant $c = c(\mathcal{K}, p, n, \varepsilon, D)$ such that $c < \frac{|B_r \cap \{u>0\}|}{|B_r|} \leq 1 - c$, for any ball $B_r = B_r(x)$ centered at some point $x \in \partial\{u_\varepsilon > 0\} \cap \mathcal{K}$.*

Proof. The existence of a minimizer u_ε for problem (2.2) follows easily from the fact that, for any minimizing sequence u_ε^k, we may assume

- $\nabla u_\varepsilon^k \rightharpoonup \nabla u_\varepsilon$ in $L^p(D^C)$
- $u_\varepsilon^k \to u_\varepsilon$ almost everywhere in D^C,

for some $u_\varepsilon \in V$. Thus,

$$\int_{D^C} |\nabla u_\varepsilon|^p dx \leq \liminf_{k\to\infty} \int_{D^C} |\nabla u_\varepsilon^k|^p dx \quad \text{and} \quad |\{u_\varepsilon > 0\}| \leq \liminf_{k\to\infty} |\{u_\varepsilon^k > 0\}|.$$

Since f_ε is a continuous and increasing function, we obtain

$$J_\varepsilon(u_\varepsilon) \leq \liminf_{k\to\infty} J(u_\varepsilon^k).$$

Clearly, $u_\varepsilon \geq 0$, otherwise, $J_\varepsilon\big(\ (u_\varepsilon)^+\big) < J_\varepsilon\big(u_\varepsilon\big)$.

Observe that if u_ε is a minimizer, then $J(u_\varepsilon) \leq J(u_\varepsilon - \epsilon\eta)$, for every $\epsilon > 0$ and nonnegative $\eta \in C_0^\infty(D^c)$. Since $\{u_\varepsilon > \epsilon\eta\} \subset \{u\varepsilon > 0\}$ and f is increasing, we have that $f\big(\{u_\varepsilon > \epsilon\eta\}\big) \leq f\big(\{u_\varepsilon > 0\}\big)$ and consequently $\Delta_p u_\varepsilon$ is a nonnegative Radon measure supported on $\partial\{u_\varepsilon > 0\}$.

Now, we explain the main track and the necessary changes in order to obtain items (3), (4) and (5). We will follow the lines of [10] establishing a sequence of lemmas that are analogous to those in this refereed work. Instead of proving these lemmas with all the details, we shall restrict ourselves to enunciating them and sketching their proofs by pointing out the necessary modifications and referring to [10] for further details.

Up to the end of this section, we fix some γ-bilipschitz function f. Consider u a minimizer of the Problem (2.3) in some ball B. Denote by v the solution of the Dirichlet problem

$$\left\{ \begin{array}{rcll} \Delta_p v & = & 0 & \text{in } B \\ v & = & u & \text{on } \partial B. \end{array} \right. \tag{3.1}$$

Following the beginning of Section 3 in [10] we obtain:

Lemma 3.2. *There exists a constant $C = C(n,p,\gamma)$ such that*

$$\begin{gathered} \int_B |\nabla(u-v)|^p \leq C|\{u>0\}\cap B| \text{ for } p \geq 2 \text{ and} \\ \int_B |\nabla(u-v)|^p \leq C|\{u>0\}\cap B|^{\frac{p}{2}}\Big(\int_B |\nabla u|^p\Big)^{1-\frac{p}{2}} \text{ for } 1<p\leq 2. \end{gathered} \tag{3.2}$$

Moreover, the constant C goes to zero when γ goes to zero.

The next lemma is the analog of Lemma 3.1 in [10], with a similar proof.

Lemma 3.3. *Let f be a given γ- Lipschitz function and u be a bounded minimizer of Problem* (2.3) *in B_1. Then, $u \in C^\alpha$ in $B_{\frac{7}{8}}$ for some $\alpha = \alpha(n,p) \in (0,1)$ and $\|u\|_{C^\alpha(B_{\frac{7}{8}})} \le C(n,p,\|u\|_{L^\infty(B_1)},\gamma)$.*

Following Lemma 3.2 in [10], we obtain

Lemma 3.4. *Let u be a bounded local minimizer of Problem* (2.3) *in B_1 with $u(0) = 0$. Then, there exists a constant $C = C(n,p,\gamma)$ such that*

$$\|u\|_{L^\infty(B_{\frac{1}{4}})} \le C.$$

Proof. Assume, by contradiction, there exists a sequence u_k of bounded local minimizers of Problem (2.3) in B_1 with $u_k(0) = 0$ and $\max_{B_{\frac{1}{4}}} u_k > k$. In this case, we define: $d_k(x) = d(x, \partial\{u_k > 0\})$ and $\mathcal{O}_k = \{x \in B_1; d_k(x) \le \frac{1-\|x\|}{3}\}$. Observe that, since $u_k(0) = 0$, $d_k(x) < \|x\|$ for every $x \in B_1$. On the other hand, $(1-\|x\|)/3 \ge 1/4$ for every $x \in B_{1/4}$. From this inequality we conclude that $B_{1/4} \subset \mathcal{O}_k$. Now, define

$$m_k := \max_{\mathcal{O}_k}(1-\|x\|)u_k(x) \ge \tfrac{3}{4}\max_{B_{1/4}} u_k(x) > \tfrac{3}{4}k.$$

Consider any maximum point $x_k \in \mathcal{O}_k$ of the function $(1-\|x\|)u_k(x)$ and observe that

$$u_k(x_k) = \frac{m_k}{1-\|x\|} > \tfrac{3}{4}k. \tag{3.3}$$

Denote by y_k any point in $\partial\{u_k > 0\}$ such that $d_k(x_k) = \|y_k - x_k\|$ and define $\delta_k := \|x_k - y_k\|$. Since $x_k \in \mathcal{O}_k$, we have that $\delta_k \le \frac{(1-\|x\|)}{3}$. Then, for every $z \in B_{2\delta_k}(y_k)$:

$$\|z\| \le \|y_k\| + 2\delta_k \le \|x_k\| + 3\delta_k \le \|x_k\| + (1-\|x_k\|) \le 1.$$

From this inequality, we have that $B_{2\delta_k}(y_k) \subset B_1$. Now, we claim that $B_{\delta_k/2}(y_k) \subset \mathcal{O}_k$. In fact, if $z \in B_{\delta_k/2}(y_k)$ we have that:

$$d_k(z) \le \frac{\delta_k}{2} \le \frac{1-\|y_k\| + \frac{\delta_k}{2}}{3} \le \frac{(1-\|z\|)}{3},$$

where the second inequality is a consequence of $\|y_k\| + \delta_k \le 1$. Moreover, for $z \in B_{\delta/2}(y_k)$

$$(1-\|z\|) \ge (1-\|x_k\|) - \|x_k - z\| \ge (1-\|x_k\|) - \frac{3}{2}\delta_k \ge \frac{(1-\|x_k\|)}{2}.$$

As a consequence of this inequality,

$$\max_{\overline{B}_{\delta/2}(y_k)} u_k \le 2u_k(x_k). \tag{3.4}$$

By the definition of δ_k, we have that $B_{\delta_k}(x_k) \subset \{u_k > 0\}$. Recall from item (2) of Theorem 3.1 that $\Delta_p u_k = 0$ in $B_{\delta_k}(x_k)$. By the Harnack inequality for p-harmonic functions, we may conclude that there exists a constant $c = c(n,p) > 0$ such that

$$\min_{\overline{B}_{3\delta/4}(x_k)} u_k \geq c u_k(x_k). \tag{3.5}$$

In particular,

$$\max_{\overline{B}_{\delta/4}(y_k)} u_k \geq c u_k(x_k). \tag{3.6}$$

Consider the following scaling of u_k

$$w_k(x) = \frac{u_k(y_k + \frac{\delta}{2}x)}{u_k(x_k)}, \text{ for } x \in B_1. \tag{3.7}$$

Observe that since u is a local minimizer of Problem (2.3), w_k is a minimizer of the analogous problem replacing f_ϵ with $f_k = f_\epsilon / u_k(x_k)$. In other words, w_k is a local minimizer of

$$J_k(w) = \int_{B_{3/4}} |\nabla w|^p + f_k(|\{w > 0\}|).$$

Now, we denote by v_k the solution of $\Delta_p v_k = 0$ in $B_{3/4}$ and $v_k - w_k \in W_0^{1,p}(B_{3/4})$. Initially we observe that $\|f_k\|_{Lip}$ converges uniformly to zero, as $k \to \infty$. Since w_k is a minimizer of J_k, by Lemma 3.2 we guarantee the existence of a sequence of positive numbers C_k, that converges to zero as $k \to \infty$, such that

$$\int_{B_{3/4}} |\nabla(w_k - v_k)|^p \leq C_k. \tag{3.8}$$

By estimates (3.4) and (3.6) we have that

$$\max_{\overline{B}_1} w_k \leq 2, \ \max_{\overline{B}_{1/2}} w_k \geq c \text{ and } w_k(0) = 0.$$

Now notice that w_k is uniformly bounded, thus from Lemma 3.3 we conclude w_k and v_k are uniformly C^α in $B_{5/8}$. By compactness, we may assume (passing to a subsequence, if necessary) $w_k \to w_0$ and $v_k \to v_0$ uniformly on $B_{5/8}$. Equation (3.8) implies that $w_0 = v_0 + K$ in $B_{5/8}$. Since $v_k \to v_0$, we have that $\Delta_p v_0 = 0$ and this implies that $\Delta_p w_0 = 0$. By the strong minimum principle, we have that $w_0 = 0$ in $B_{5/8}$, because $w_0 \geq 0$ and $w_0(0) = 0$. It is a contradiction with $\max_{B_{1/2}} w_0 > c > 0$. This finishes the proof. $\square$

Lipschitz continuity now follows as in the uniformly elliptic case, see for instance, Theorem 2.3 in [3]. At this moment, the proofs of item (4) and (5) are similar to the proofs of Corollary 4.3 and Theorem 4.4 in [10] respectively.

The necessary minor modifications are similar to the ones treated in the proof of Lemma 3.4 and therefore we will skip the details. □

In the same spirit as [2], D. Danielli and A. Petrosyan provided in [10] a representation theorem which Theorem 3.1 puts our minimizers u_ε under the hypotheses of Theorem 3.1. The next theorem will play an important role in the investigation of fine regularity properties of the free boundary.

Theorem 3.5. *Let u_ε be a minimizer of Problem* (2.2). *Then*

(1) $\mathcal{H}^{n-1}\big(\mathcal{K} \cap \partial\{u_\varepsilon > 0\}\big) < \infty$ *for every compact set* $\mathcal{K} \subset D^C$.

(2) *There exists a Borel function q_ε such that*

$$\Delta_p u_\varepsilon = q_\varepsilon \mathcal{H}^{n-1} \lfloor \partial\{u_\varepsilon > 0\};$$

that is, for any $\psi \in C_0^\infty(D^C)$, there holds

$$-\int_{D^C} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla \psi dx = \int_{\{u_\varepsilon > 0\}} \psi q_\varepsilon d\mathcal{H}^{n-1}.$$

(3) *For any compact set $\mathcal{K} \subset D^C$, there exist positive constants c, C such that $c \le q_\varepsilon \le C$ and*

$$c r^{n-1} \le \mathcal{H}^{n-1}\big(B_r(x) \cap \partial\{u_\varepsilon > 0\}\big) \le C r^{n-1},$$

for every ball $B_x(r) \subset \mathcal{K}$ with $x \in \partial\{u_\varepsilon > 0\}$.

(4) *For $\mathcal{H}^{n-1}$ almost all points in $\partial\{u_\varepsilon > 0\}$, an outward normal $\nu = \nu(x)$ is defined and furthermore*

$$u_\varepsilon(x + y) = q_\varepsilon(x)(y \cdot \nu)^+ + o(y),$$

where $\frac{o(y)}{|y|} \to 0$ as $|y| \to 0$. This allows us to define $q_\varepsilon(x) = (u_\varepsilon)_\nu(x)$ at those points.

(5) $\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\} \setminus \partial_{\text{red}}\{u_\varepsilon > 0\}) = 0$.

4. A GEOMETRIC-MEASURE OF HADAMARD'S VARIATIONAL FORMULA AND THE FREE BOUNDARY CONDITION

In this section we suggest a geometric-measure version of the well-known Hadamard's variational formula (see [15]) to deduce the free boundary condition of Problem (2.2). Our approach is inspired by [1]. Roughly speaking, given two points in the reduced free boundary, say x_1 and x_2, the idea is to make an inward perturbation around x_1, an outward perturbation around x_2 in such a way that we do not disturb very much the original volume and then compare the optimal configuration to the perturbed one in terms of the functional J_ε. Here are the details.

Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative C^∞ function supported in $[0,1]$, with, say, $\int \rho(t)dt = 1$. Hereafter, we shall write $u = u_\varepsilon$ and fix two points x_1 and x_2 in the reduced free boundary $\partial_{\mathrm{red}}\{u > 0\}$. For any $0 < r < \frac{\mathrm{dist}(x_1,x_2)}{100}$, and $\lambda > 0$, we consider the vector field

$$P_r(x) := \begin{cases} x + \lambda r\rho\Big(\frac{|x-x_1|}{r}\Big)\nu(x_1) & x \in B_r(x_1) \\ x - \lambda r\rho\Big(\frac{|x-x_2|}{r}\Big)\nu(x_2) & x \in B_r(x_2) \\ x & \text{elsewhere.} \end{cases} \tag{4.1}$$

If υ is any vector in $\mathbb{R}^n$, from direct computation, we see that

$$DP_r(x)\cdot\upsilon = \upsilon + (-1)^{i+1}\Big\{\lambda\rho'\Big(\frac{|x-x_i|}{r}\Big)\frac{\langle x-x_i,\upsilon\rangle}{|x-x_i|}\Big\}\nu(x_i) \ \text{ in } B_r(x_i). \tag{4.2}$$

Notice that, if λ is small enough, P_r is a diffeomorphism that maps $B_r(x_i)$ onto itself. Indeed, if $\lambda\sup_{[0,1]}\rho'(t) < 1$, P_r is a local injective diffeomorphism. Now, if $\lambda\rho(t) \le 1-t$, for $0 \le t \le 1$,

$$|P_r(x) - x_i| \le |x-x_i| + \lambda r\rho\Big(\frac{|x-x_i|}{r}\Big) \le r,$$

for any $x \in B_r(x_i)$. Finally, notice that $P_r\big|_{\partial B_r(x_i)} = Id$, therefore P_r has to be onto.

For each $r > 0$ small enough, we will consider the r-perturbed configuration, v_r implicitly defined by

$$v_r(P_r(x)) = u(x). \tag{4.3}$$

The idea is to compare our optimal configuration $\{u > 0\}$ to its perturbation $\{v_r > 0\}$ in terms of the penalized Problem (2.2). An important geometric measure of information we shall use is the blow-up limit. For any $r > 0$ small enough and $i = 1,2$, consider the blow-up sequence, $u_r^i\colon B_1(0) \to \mathbb{R}$, given by

$$u_r^i(y) := \tfrac{1}{r}u\big(x_i + ry\big).$$

From the blow-up analysis, we know, the set $B_1 \cap \{u_r^i > 0\}$ approaches $\{y \in B_1 : \langle y,\nu(x_i)\rangle < 0\}$, as $r \to 0$. Let us compute the change on the volume of the perturbation. More specifically, making use of the change of variables theorem, we obtain

$$\frac{|(B_r(x_i)\cap\{v_r > 0\})|}{r^n} = \frac{1}{r^n}\int_{B_r(x_i)\cap\{v_r>0\}} dx = \int_{B_1\cap\{v_r(x_i+ry)>0\}} dy \tag{4.4}$$

$$= \int_{B_1\cap\{u_r^i>0\}} \det\left(DP_r(x_i+ry)\right) dy$$

$$\longrightarrow \int_{B_1\cap\{\langle y,\nu(x_i)\rangle<0\}} 1+(-1)^{i+1}\lambda\rho'(|y|)\Big\langle \frac{y}{|y|},\nu(x_i)\Big\rangle dy,$$

as $r\to 0$. Notice that there exists a constant $C(\rho)$ so that, for any unit vector $\nu\in\mathbb{S}^{n-1}$, there holds

$$C(\rho)\equiv \int_{B_1\cap\{\langle y,\nu\rangle<0\}} \rho'(|y|)\Big\langle \frac{y}{|y|},\nu\Big\rangle dy. \tag{4.5}$$

A similar computation shows that

$$\frac{|(B_r(x_i)\cap\{u>0\})|}{r^n}\longrightarrow \int_{B_1\cap\{\langle y,\nu(x_i)\rangle<0\}} dy, \tag{4.6}$$

as $r\to 0$. Combining (4.4), (4.5) and (4.6), we conclude

$$\frac{(|\{v_r>0\}|)-(|\{u>0\}|)}{r^n}\longrightarrow 0, \tag{4.7}$$

as $r\to 0$. From the Lipschitz continuity of the penalization f_ε, we obtain

$$f_\varepsilon\Big((|\{v_r>0\}|)\Big)-f_\varepsilon\Big(|(\{u>0\}|)\Big)\le \frac{1}{\varepsilon}o(r^n). \tag{4.8}$$

Now we shall turn our attention to the differential of the perturbation on the p-Dirichlet integral. Initially we observe that

$$\frac{1}{r^n}\int_{B_r(x_i)}|\nabla u(x)|^p dx=\int_{B_1}|\nabla u_r^i(y)|^p dy=\int_{B_1\cap\{u_r^i>0\}}|\nabla u_r^i(y)|^p d. \tag{4.9}$$

Now, applying twice the change of variables theorem, taking into account that P_r maps $B_r(x_i)$ diffeomorphically onto itself,

$$\begin{aligned}\frac{1}{r^n}\int_{B_r(x_i)}|\nabla v_r(x)|^p dx &= \frac{1}{r^n}\int_{B_r(x_i)}|DP_r(P_r^{-1}(x))^{-1}\cdot\nabla u(P_r^{-1}(x))|^p dx\\ &= \frac{1}{r^n}\int_{B_r(x_i)}|DP_r(y)^{-1}\cdot\nabla u(y)|^p|\det\big(DP_r(y)\big)|dy\\ &= \int_{B_1\cap\{u_r^i>0\}}|DP_r(x_i+rz)^{-1}\cdot\nabla u_r^i(z)|^p|\det\big(DP_r(x_i+rz)\big)|dz.\end{aligned} \tag{4.10}$$

Now, from (4.2), using the fact that for any matrix A, with $|A|<1$, we have $\big(Id+A\big)^{-1}=Id+\sum_{i=1}^{\infty}(-1)^iA^i$, we have

$$DP_r(x_i+rz)^{-1}\cdot\nabla u_r^i(z)=\nabla u_r^i(z)-(1)^{i+1}\lambda\frac{\rho'(|z|)}{|z|}\langle z,\nabla u_r^i(z)\rangle\nu(x_i)+o(\lambda). \tag{4.11}$$

On the other hand,

$$|\det\big(DP_r(x_i+rz)\big)| = 1 + (-1)^{i+1}\lambda\frac{\rho'(|z|)}{|z|}\langle z,\nu(x_i)\rangle. \tag{4.12}$$

Combining (4.9), (4.10), (4.11) and (4.12), we obtain

$$\begin{aligned}
&\frac{1}{r^n}\int_{B_r(x_i)}|\nabla v_r(x)|^p - |\nabla u(x)|^p dx \qquad (4.13)\\
&= (-1)^{i+1}\lambda\int_{B_1\cap\{u_r^i>0\}}|\nabla u_r^i(z)|^p\frac{\rho'(|z|)}{|z|}\langle z,\nu(x_i)\rangle dz\\
&+ (-1)^i\lambda\int_{B_1\cap\{u_r^i>0\}}p|\nabla u_r^i(z)|^{p-2}\frac{\rho'(|z|)}{|z|}\langle z,\nabla u_r^i(z)\rangle\langle\nabla u_r^i(z),\nu(x_i)\rangle dz + o(\lambda).
\end{aligned}$$

Again, from the blow-up analysis (see [10]), for each $\delta>0$, we know

$$\nabla u_r^i \to q(x_i)\nu(x_i),$$

uniformly in $B_1\cap\{\langle y,\nu(x_i)\rangle < -\delta\}$. Therefore, by r-uniform Lipschitz continuity of u_r^i, we have

$$\nabla u_r^i \to -q(x_i)\nu(x_i)\chi_{B_1\cap\{\langle y,\nu(x_i)\rangle<0\}}, \tag{4.14}$$

in $L^p(B_1)$. Letting $r\to 0$ in (4.13), we find

$$\begin{aligned}
&\frac{1}{r^n}\int_{B_r(x_i)}|\nabla v_r(x)|^p - |\nabla u(x)|^p dx \longrightarrow\\
&(-1)^{i+1}(p-1)\lambda\big(q(x_i)\big)^p\int_{B_1\cap\{u_r^i>0\}}\frac{\rho'(|z|)}{|z|}\langle z,\nu(x_i)\rangle dz + o(\lambda).
\end{aligned} \tag{4.15}$$

Notice that

$$\operatorname{div}\big(\rho(|z|)\big) = \frac{\rho'(|z|)}{|z|}\langle z,\nu(x_i)\rangle.$$

Thus, from the divergence theorem and the blow-up analysis,

$$\int_{B_1\cap\{u_r^i>0\}}\frac{\rho'(|z|)}{|z|}\langle z,\nu(x_i)\rangle dz \to \int_{B_1\cap\{\langle z,\nu(x_i)\rangle=0\}}\rho(|z|)d\mathcal{H}^{n-1}(z) = c(\rho). \tag{4.16}$$

Putting (4.15) and (4.16) together, we obtain

$$\int_{D^C}|\nabla v_r(x)|^p - |\nabla u(x)|^p dx = r^n\lambda(p-1)c(\rho)\big(q(x_1)^p - q(x_2)^p\big) + r^n o(\lambda). \tag{4.17}$$

From the minimality property of u, (4.8) and (4.17),

$$0 \le J_\varepsilon(v_r) - J_\varepsilon(u) \le r^n \lambda (p-1) c(\rho)\big(q(x_1)^p - q(x_2)^p\big) + r^n o(\lambda) + \frac{1}{\varepsilon} o(r^n). \tag{4.18}$$

Dividing (4.18) by r^n and letting $r \to 0$ we obtain

$$0 \le \lambda (p-1) c(\rho)\big(q(x_1)^p - q(x_2)^p\big) + o(\lambda). \tag{4.19}$$

Now dividing (4.19) by λ, letting $\lambda \to 0$, and afterwards reversing the places of x_1 and x_2, we finally obtain

$$q(x_1) = q(x_2). \tag{4.20}$$

Since x_1 and x_2 were taken arbitrarily in $\partial_{\mathrm{red}}\{u > 0\}$, we have proven

Theorem 4.1. *There exists a positive constant λ_ε such that*

$$q_\varepsilon \equiv \lambda_\varepsilon, \ \forall x \in \partial_{red}\{u_\varepsilon > 0\}.$$

It now follows from [10] that for each $\varepsilon > 0$ fixed, the reduced free boundary $\partial_{\mathrm{red}}\{u_\varepsilon > 0\}$ is a $C^{1,\alpha}$ smooth surface. Real analyticity of the reduced free boundary is then a consequence of [19]. It is worthwhile to point out that, for $n = 2$, a small variant of the main result in [11] assures full regularity of the free boundary $\{u_\varepsilon > 0\}$, as long as $p > 2 - \sigma$, for some universal constant σ.

In [6], [7] and [8], L. Caffarelli introduced and developed the, by now, well-known notion of viscosity solution of a given free boundary problem (for the Laplacian operator). It turns out that the notion of viscosity solution to a free boundary problem is rather weaker than the one we obtained in Theorem 4.1. We shall use the interpretation of our free boundary condition as in the viscosity sense as a geometric tool in the remaining sections.

Theorem 4.2 (Free boundary condition in the viscosity sense)**.** *Let $x_0 \in \partial\{u_\varepsilon > 0\}$ be a free boundary point. Suppose there exists a ball $B \subset \{u_\varepsilon > 0\}$ touching the free boundary; i.e., $\partial B \cap \partial\{u_\varepsilon > 0\} = \{x_0\}$. Then in B u has the asymptotic development*

$$u_\varepsilon(x) = \lambda_\varepsilon \langle x - x_o, \nu\rangle^+ + o(|x - x_0|), \tag{4.21}$$

where λ_ε is the positive constant provided in Theorem 4.1 *and ν is the unit normal vector to ∂B, pointing inward to $\{u_\varepsilon > 0\}$.*

Proof. Let $B = B_r(\xi) \subset \{u_\varepsilon > 0\}$ touch the free boundary at $\{x_0\}$. If follows by a small variant of Lemma A.1 in [12] (see also Lemma A1 in [8]) that, in B, u has the following asymptotic development

$$u_\varepsilon(x) = \theta \langle x - x_o, \nu\rangle + o(|x - x_0|), \tag{4.22}$$

for some $\theta > 0$. It remains to show, $\theta = \lambda_\varepsilon$. With no loss of generality we can assume $x_0 = 0$ and $\nu = e_1$. Consider then any blow-up sequence

$$u_\varepsilon^k(x) = \frac{1}{\rho_k} u_\varepsilon(\rho_k x), \tag{4.23}$$

as $\rho_k \to 0$. By Lipschitz continuity, we can assume u_ε^k converges over compact sets to a Lipschitz function u_ε^∞ defined on $\mathbb{R}^N$. Clearly, $u_\varepsilon^\infty \geq 0$ and $\Delta_p u_\varepsilon^\infty = 0$ in $\{u_\varepsilon^\infty > 0\}$. From (4.22), $u_\varepsilon^\infty(x) = \theta\langle x, e_1\rangle$, in $\{\langle x, e_1\rangle \geq 0\}$. We want to show $u_\varepsilon^\infty \equiv 0$ in $\{\langle x, e_1\rangle \leq 0\}$.

By a small modification of classical arguments, see for instance [2] or [14], we can show the limit of any blow-up sequence of u_ε is an absolute minimizer of J_ε in any ball. Thus any blow-up limit locally satisfies all the properties listed in Theorem 3.5. In particular, blow-ups have uniform positive zero sets; i.e., there exists a constant $c > 0$ such that

$$|\{u_\varepsilon^\infty = 0\} \cap B_R(0)| \geq cR^N, \forall R > 0. \tag{4.24}$$

We claim that (4.24) implies

$$u_\varepsilon^\infty(x_1, y) = o(x_1), \text{ as } x_1 \to 0^-. \tag{4.25}$$

Indeed, let $\ell := \limsup\{\partial_{x_1} u_\varepsilon^\infty(x_1, y) : x_1 \to 0^-, u_\varepsilon^\infty(x_1, y) > 0\}$. Take (a_n, y_n) to be a sequence such that $\partial_{x_1} u_\varepsilon^\infty(a_n, y_n) \to \ell$ and suppose by contradiction $\ell > 0$. Consider the blow-up sequence of u_ε^∞ with respect to $B_{a_n}(0, y_n)$. Again by Lipschitz continuity, up to a subsequence, we can assume the blow-up sequence converges to some function v. Arguing as in [10], Lemma 5.4, we obtain $v = -\ell x_1$ in $\{x_1 < 0\}$. However, we know that $v = \theta x_1$ in $\{x_1 > 0\}$ for some $\theta = \theta_v > 0$. This contradicts the uniform positive density of $\{v = 0\}$.

Our next step is to show that the error $o(x_1)$ in (4.24) depends only on the constant c in (4.24) and the Lipschitz norm, say L. In other words we will prove that if v is a Lipschitz nonnegative function on $\mathbb{R}^N$ satisfying (4.24), $\Delta_p v = 0$ in $\{v > 0\}$ and $\{x_1 > 0\} \subset \{v > 0\}$, then there exists a universal o so that (4.25) holds. For that, fix a $\varsigma = \mu c < 1$, for μ small. From (4.24), there exists a $Y \in B_1$ with $\langle Y, e_1\rangle < -\varsigma$ such that $v(Y) = 0$. By Lipschitz continuity, there is an r so that $B_r(Y) \subset \{x_1 < -\frac{1}{10}\varsigma\}$ and $v(x) \leq -\frac{L}{10}x_1$ in $B_r(Y)$. Let $\mathcal{B} := B_{|Y|}(0, Y') \cap \{x_1 < 0\}$, where $Y = (Y_1, Y')$ and consider an auxiliary function Θ satisfying

$$\begin{cases} \Delta_p\Theta = 0 & \text{in } \mathcal{B} \\ \Theta = -x_1 & \text{on } \partial\mathcal{B} \setminus B_r(Y) \\ \Theta = \frac{-x_1}{10} & \text{on } \partial\mathcal{B} \cap B_{\frac{r}{2}}(Y). \end{cases}$$

We choose Θ to satisfy $\frac{-x_1}{10} \leq \Theta \leq -x_1$ on $\partial\mathcal{B} \cap B_r(Y)$. By the comparison principle, $v \leq L\Theta$. Now, by the strong maximum principle (see for instance Theorem 6.5 in [17]), we know $0 < \Theta < -x_1$ in $\mathcal{B}$. Since $\Theta \equiv 0$ on $\overline{\mathcal{B}} \cap \{x_1 = 0\}$ we must have $\partial_1\Theta < 1$ on $\overline{\mathcal{B}} \cap \{x_1 = 0\}$. In fact, if for some $\chi \in \overline{\mathcal{B}} \cap \{x_1 = 0\}$ we had $\partial_1\Theta(\chi) = 1$, around such a point, $|\nabla\Theta|^{p-2}$ is a uniformly elliptic C^α matrix and we would reach a contradiction to the Hopf maximum principle applied to $\Theta + x_1$ (see for instance [16], notes of chapter 3 or [13]). Applying again the $C^{1,\alpha}$ regularity of Θ up to the boundary of $\mathcal{B}$, there exist $\kappa > 0$ and $\alpha < 1$, depending only on dimension and p, such that $\Theta(x) \leq -\alpha x_1,$ in $B_\kappa \cap \{x_1 < 0\}$, and thus

$$v(x) \leq -\alpha L x_1, \text{ in } B_\kappa \cap \{x_1 < 0\}. \tag{4.26}$$

Expression (4.26) implies that for any $R > 0$

$$\frac{v(Rx)}{R} \leq -\alpha L x_1, \text{ in } B_\kappa \cap \{x_1 < 0\}. \tag{4.27}$$

If we now apply a scaling induction argument to (4.27) we obtain

$$v(x) \leq -\alpha^j L x_1 \text{ in } B_{\kappa^j R} \cap \{x_1 < 0\}. \tag{4.28}$$

Since $\alpha < 1$ and $R > 0$ is arbitrary, (4.28) implies the error $o(x_1)$ in (4.24) is indeed uniform. Therefore, by rescaling u_ε^∞ as in (4.27), we finally conclude $u_\varepsilon^\infty(x) \equiv 0$ in $\{x_1 < 0\}$.

At this point we have proven $u_\varepsilon^\infty(x) = \theta\langle x, \nu\rangle^+$, for some $\theta > 0$. However it is possible to show that the crossing angle θ obtained by a blow-up sequence does not depend upon the free boundary point. Indeed, suppose

$$\frac{1}{\rho_k} u_\varepsilon(x_0 + \rho_k x) \to \theta_0\langle x, \nu_0\rangle^+ \text{ and } \frac{1}{\rho_k} u_\varepsilon(x_1 + \rho_k x) \to \theta_1\langle x, \nu_1\rangle^+ \tag{4.29}$$

with, say, $\theta_0 < \theta_1$. Then we can perform a corresponding perturbation argument as in (4.3) and obtain a function v_{ρ_k} with less energy than the minimizer u_ε which is a contradiction. Now, if we select a point $x_1 \in \partial_{\text{red}}\{u_\varepsilon > 0\}$, by $C^{1,\alpha}$ regularity of the free boundary around x_1, clearly, any blow-up sequence converges to $\lambda_\varepsilon\langle x, \nu\rangle^+$, thus $\theta = \lambda_\varepsilon$. □

5. Recovering the original problem

In this section we shall relate a solution to the penalized problem to a (possible) solution to our original problem. Roughly speaking the idea is that the function f_ε will charge a lot for those configurations that have a volume bigger than 1. We hope if the charge is too big, i.e., if $\varepsilon > 0$ is small enough, optimal configurations of Problem 2.2 will rather prefer to have volume 1 than paying for the penalization.

Lemma 5.1. *There exists a positive number $\delta = \delta_\varepsilon$, such that u_ε is positive for all points within a distance less than δ to the fixed boundary; i.e., $D_\delta = \{x \in D^C : dist(x, \partial D) < \delta\} \subset \{u_\varepsilon > 0\}$. In particular u_ε is continuous up to the fixed boundary ∂D.*

Proof. Let $X \in \partial D$ and consider $Y := X + \rho\mu(X)$, where $\mu(X)$ stands for the inward unit normal vector at X and $\rho = \rho(\partial D)$ is small enough so that $B := B_\rho(Y)$ touches ∂D at X. Consider v_δ to be the function satisfying

$$\begin{cases} \Delta_p v_\delta = 0 & \text{in} \quad B_{\rho+\delta}(Y) \setminus B_\rho(Y) \\ v_\delta = \inf\limits_{\partial D} \varphi & \text{on} \quad \partial B_\rho(Y) \\ v_\delta = 0 & \text{in} \quad B_{\rho+\delta}(Y)^C, \end{cases} \tag{5.1}$$

where δ is a small positive number to be chosen later. Let us call $C = B_{\rho+\delta}(Y) \cap D^C$ and $A := \{v_\delta > u_\varepsilon\} \cap C$. Our ultimate goal is to show that u_ε is positive in C, for some δ small enough. To this end, consider the function $M(x) := \max\{u_\varepsilon(x), v_\delta(x)\}$. Clearly, M competes with u_ε in problem (2.2). Thus,

$$\begin{aligned} &\int_A |\nabla v_\delta|^p dx + f_\varepsilon\big(|C| + |\{u_\varepsilon > 0\}| - |\{u_\varepsilon > 0\} \cap C|\big) \\ &\geq \int_A |\nabla u_\varepsilon|^p dx + f_\varepsilon\big(|\{u_\varepsilon > 0\}|\big). \end{aligned} \tag{5.2}$$

We now consider

$$m(x) := \begin{cases} v_\delta(x) & \text{in} \quad \big[B_{\rho+\delta}(Y) \setminus B_\rho(Y)\big] \cap D \\ \min\{u_\varepsilon(x), v_\delta(x)\} & \text{in} \quad \Big[\big[B_{\rho+\delta}(Y) \setminus B_\rho(Y)\big] \cap D\Big]^C. \end{cases}$$

We shall compare m and v_δ in terms of the functional

$$G_\mu(\phi) := \int_{B_{2\rho}(Y) \setminus B_\rho(Y)} \mu |\nabla \phi|^p + \frac{1}{\mu} \chi_{\{\phi > 0\}} dx. \tag{5.3}$$

Given μ small enough, there exists a $\delta = \delta(\mu)$ such that v_δ is a minimizer of G_μ, among all functions taking zero boundary data, say, on $\partial B_{2\rho}(Y)$ and $\inf\limits_{\partial D} \varphi$ on $\partial B_\rho(Y)$. Therefore,

$$\int_A \mu |\nabla u_\varepsilon|^p dx + \frac{1}{\mu}\big|\{u_\varepsilon > 0\} \cap A\big| \geq \int_A \mu |\nabla v_\delta|^p dx + \frac{1}{\mu}\big|A\big|. \tag{5.4}$$

Combining (5.2), (5.4) and the $\frac{1}{\varepsilon}$-Lipschitz continuity of f_ε, we obtain

$$\frac{\mu}{\varepsilon}\big(|C| - |\{u_\varepsilon > 0\} \cap C|\big) \tag{5.5}$$

$$\geq \mu\Big[f_\varepsilon\big(|C| + |\{u_\varepsilon > 0\} - |\{u_\varepsilon > 0\} \cap C|\big) - f_\varepsilon\big(|\{u_\varepsilon > 0\}|\big)\Big]$$
$$\geq \frac{1}{\mu}\Big[|A| - |\{u_\varepsilon > 0\} \cap A|\Big] = \frac{1}{\mu}\Big[|C| - |\{u_\varepsilon > 0\} \cap C|\Big].$$

Thus, if μ is taken small enough, this forces u_ε to be positive almost everywhere in C. Actually, $u_\varepsilon > 0$ everywhere in C, because of the uniformly density of the free boundary, Theorem 3.1, item (5). □

Lemma 5.2. *There exist positive constants c and C, independent of ε, such that $c \leq |\{u_\varepsilon > 0\}| \leq 1 + C\varepsilon$.*

Proof. Let $D^\star$ be any smooth domain containing D, so that $|D^\star \setminus D| = 1$. From the minimality of u_ε, we have

$$J_\varepsilon(u_\varepsilon) = \int_{D^C} |\nabla u_\varepsilon(x)|^p + f_\varepsilon\big(|\{u_\varepsilon > 0\}|\big) \leq J_\varepsilon(u^\star) = C, \tag{5.6}$$

where $u^\star$ is the p-harmonic function in $D^\star \setminus D$ taking boundary data equal to φ on ∂D and 0 on $\partial D^\star$. Thus

$$\tfrac{1}{\varepsilon}\big(|\{u_\varepsilon > 0\}| - 1\big) \leq f_\varepsilon\big(|\{u_\varepsilon > 0\}|\big) \leq C.$$

This proves the estimate from above. Let us turn our attention to the estimate from below. Expression (5.6), together with the Poincaré inequality, provides

$$\int_{D^C} |\nabla u_\varepsilon(x)|^p + |u_\varepsilon(x)|^p dx \leq C, \tag{5.7}$$

for some C independent of ε. Let D_δ be a tubular neighborhood of ∂D as in Lemma 5.1. From the mean value inequality, followed by Hölder's inequality and (5.7) we have

$$\begin{array}{rcl} \delta\varphi(x_0) & \leq & \displaystyle\int_0^\delta u\big(x_0 + t\mu(x_0)\big)dt + \int_0^\delta |\nabla u\big(x_0 + \overline{t}\mu(x_0)\big)|tdt \\ & \leq & C\Theta_{x_0}^{1/q}\delta^{1/q}\,(1+\delta)\,. \end{array} \tag{5.8}$$

Now we integrate (5.8) over ∂D and obtain

$$\int_{\partial D} \varphi dS \leq C(\delta)|\{u_\varepsilon > 0\} \cap D_\delta|^{1/q}. \tag{5.9}$$

Finally, from (5.9), there must exist a constant, independent of ε, so that $|\{u_\varepsilon > 0\}| \geq c$, as claimed. Notice that an estimate like (5.9) can be obtained by means of an integral argument. Indeed, from the fact that $\varphi \in W^{1,p}$ close to ∂D, we can apply the trace inequality and obtain $0 < \|\varphi\|_{L^q(\partial D)} \leq C|\{u_\varepsilon > 0\}|^{p-q}\|u\|_{W^{1,p}(D^C)} \leq C|\{u_\varepsilon > 0\}|^{p-q}$. □

Lemma 5.3. *There exists a positive constant C independent of ε, so that $\lambda_\varepsilon \leq C$, where λ_ε is the constant provided by Theorem* 4.1.

Proof. Applying the divergence theorem to the field $F_1 = u|\nabla u|^{p-2}\nabla u$, we have

$$\int_{D^C} |\nabla u_\varepsilon|^p dx = \int_{\partial D} \varphi|\nabla u|^{p-2}\partial_\mu u dS, \tag{5.10}$$

where μ is the outward unit vector in ∂D. If we apply the divergence theorem to the field $F_2 = |\nabla u|^{p-2}\nabla u$, we obtain

$$\lambda_\varepsilon^{p-1}\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\}) = \int_{\partial D} |\nabla u|^{p-2}\partial_\mu u dS. \tag{5.11}$$

The isoperimetric inequality gives a universal bound from below to $\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\})$; i.e, $\mathcal{H}^{n-1}(\partial\{u_\varepsilon > 0\}) \geq c$, for some c independent of ε. Combining this with (5.10) and (5.11), we obtain $\lambda_\varepsilon \leq C(D, \varphi)$, as claimed. $\square$

Lemma 5.4. *There exists a universal positive constant $c > 0$, such that $\lambda_\varepsilon \geq c$, for all $\varepsilon > 0$.*

Proof. Fix a $z_0 \in \partial D$ and select $z_1 = z_0 + \delta\mu(z_1)$, where $\delta > 0$ is small and $\mu(z_1)$ denotes the outward normal vector on ∂D. Consider the smooth family of domains $\Upsilon_t := B_{\delta+t}(z_1) \bigcap D^C$. Let t_ε denote the first t such that Υ_t touches $\partial\{u_\varepsilon > 0\}$, say $x_0 = \partial\Upsilon_{t_\varepsilon} \bigcap \partial\{u_\varepsilon > 0\}$. Define Ψ_ε to be a p-harmonic function in $\Upsilon_{t_\varepsilon} \setminus \Upsilon_0$, with the following boundary value data:

$$\Psi_\varepsilon\big|_{\partial\Upsilon_0} = \min_{\partial D} \varphi \quad \text{and} \quad \Psi_\varepsilon\big|_{\partial\Upsilon_{t_\varepsilon}} = 0.$$

By the maximum principle we have $u_\varepsilon \geq \Psi_\varepsilon$ in $\Upsilon_{t_\varepsilon} \setminus \Upsilon_0$. From Hopf's lemma (see for instance [22]) we also know there exists a constant $c > 0$ depending on ∂D and $\inf \varphi$, but independent of ε, such that

$$\Psi_{-\nu}(x_0) \geq c, \tag{5.12}$$

where ν denotes the outward unit normal vector of $B_{\delta+t_\varepsilon}(z_1)$, at x_0. Recall that, from Theorem 4.2, we have the following asymptotic development around x_0

$$\Psi(x) \leq u(x) = \lambda_\varepsilon\langle x - x_0, \nu\rangle^+ + o(|x - x_0|). \tag{5.13}$$

Dividing (5.13) by $|x - x_0|$, letting $x \to x_0$ and taking into account (5.12), we finally obtain $c \leq \lambda_\varepsilon$, as desired. $\square$

We are ready to show the main theorem of this section.

Theorem 5.5. *If ε is small enough, then any solution of Problem* (2.2) *is a solution of Problem* (1.2).

Proof. Let us initially suppose $|\{u_\varepsilon > 0\}| > 1$. In the same spirit as Section 4, consider an inward perturbation of the set $\{u_\varepsilon > 0\}$ with volume change V, in such a way that the set of positivity of the new function, $\widetilde{u}_\varepsilon$ is still bigger than 1. Thus,

$$f_\varepsilon(|\{\widetilde{u}_\varepsilon > 0\}|) - f_\varepsilon(|\{u_\varepsilon > 0\}|) = -\frac{1}{\varepsilon}V. \tag{5.14}$$

From (4.13) and Lemma 5.3, we have

$$\int_{D^C} |\nabla \widetilde{u}_\varepsilon|^p - |\nabla u_\varepsilon|^p = \lambda_\varepsilon^p V + o(V) \leq C^p V + o(V). \tag{5.15}$$

Using the fact that $J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(\widetilde{u}_\varepsilon)$, (5.14) and (5.15), we find

$$0 \leq C^p V + o(V) - \frac{1}{\varepsilon}V. \tag{5.16}$$

Finally, if we divide inequality (5.16) by V and let $V \to 0$, we obtain $\varepsilon > \frac{1}{C^p}$. If $|\{u_\varepsilon > 0\}| < 1$, we argue similarly, making an outward perturbation and using Lemma 5.4 to obtain another lower bound for ε. Thus, if ε is small enough, $|\{u_\varepsilon > 0\}|$ automatically adjusts to be equal to 1. □

6. Radial Symmetry

In this section we show a simple symmetry result of Problem (1.2). Indeed, we shall show the best way of insulating a uniformly heated spherical body is by a ball. Recall, when $\varphi \equiv$ Constant, Problem (1.2) is equivalent to our original physical optimization problem. Here is the theorem:

Theorem 6.1. *Let D be the unit ball and $\varphi \equiv 1$. Then Problem* (1.2) *has a unique solution and it is radially symmetric. In particular the free boundary is a sphere.*

Proof. Let $u = u_\varepsilon$ be a solution to Problem (1.2), with $D = B_1$ and $\varphi \equiv 1$. Denote $\Omega = \{u > 0\}$. Let B_{r_1} and B_{r_2} be the biggest ball inside $\Omega \setminus D$ and the smallest ball outside Ω, respectively. Let $y_1 \in \partial B_{r_1} \cap \partial\Omega$ and $y_2 \in \partial B_{r_2} \cap \partial\Omega$. Consider h_i, $i = 1, 2$, solutions of

$$\begin{cases} \Delta_p h_i &= 0 \text{ in } B_{r_i} \setminus B_1 \\ h_i &= 1 \text{ on } \partial B_1 \\ h_i &= 0 \text{ on } \partial B_{r_i}. \end{cases} \tag{6.1}$$

It is simple to show h_i is radially symmetric. Indeed, h_i is the unique minimizer of

$$E_p(f) := \int_{B_{r_i} \setminus B_1} |\nabla f(x)|^p dx,$$

among all functions $f \in W^{1,p}$ satisfying the according boundary data. For any orthonormal transformation $\mathcal{O} \in O(n)$, consider $h_i^{\mathcal{O}}(x) := h_i(\mathcal{O}x)$. Clearly, $h_i^{\mathcal{O}}$ has the same boundary data as h_i and furthermore,

$$E_p(h_i^{\mathcal{O}}) = \int_{B_{r_i}\setminus B_1} |\mathcal{O}^T \nabla h_i(\mathcal{O}x)|^p dx = E_p(h_i).$$

Thus $h_i^{\mathcal{O}} = h_i$. Since $\mathcal{O} \in O(n)$ was taken arbitrarily, h_i has to be radial.

In particular, the inward normal derivative of h_i over ∂B_{r_i} is a positive constant λ_i. Since $r_1 \leq r_2$, we have

$$\lambda_1 \geq \lambda_2. \tag{6.2}$$

Now, from the maximum principle, $h_1 \leq u \leq h_2$. Hence, from the free boundary condition in the viscosity sense, Theorem 4.2, we obtain

$$\lambda_2 = (h_2)_\nu(y_2) \leq \lambda_\varepsilon \leq (h_1)_\nu(y_1) = \lambda_1. \tag{6.3}$$

Combining (6.2) and (6.3), we conclude $\lambda_1 = \lambda_2$, and therefore, $r_1 = r_2$. This implies $\partial\Omega$ has to be a sphere of radius $r_1 = r_2$.

We have proven any solution to Problem (1.2), with $D = B_1$ and $\varphi \equiv 1$ is radially symmetric. Uniqueness now follows due to the volume constraint.

It is worthwhile to mention that this result can be obtained, as well, by the Schwarz rearrangement technique, see [21]. □

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