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An Optimized Symmetric 8-Step Semi-Embedded Predictor-Corrector Method for IVPs with Oscillating Solutions

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Abstract: In this paper we present a new optimized symmetric eight-step semi-embedded predictor-corrector method (SEPCM) with minimal phase-lag. The method is based on the symmetric multistep method of Quinlan-Tremaine [1], with eight steps and eighth algebraic order and is constructed to solve IVPs with oscillating solutions. We compare the new method to some recently constructed optimized methods and other methods from the literature. We measure the efficiency of the methods and conclude that the new method with minimal phase-lag is the most efficient of all the compared methods and for all the problems solved.

Keywords: Orbital problems, phase-lag, initial value problems, oscillating solution,symmetric, multistep, predictor-corrector, semi-embedded.

1. Introduction

We study the numerical integration of special second-order periodic initial-value problems of the form

$$y''(x) = f(x,y),\ \ y(x_0) = y_0\ \ and\ \ y'(x_0) = y'_0 \quad (1)$$

with an oscillatory solutions. These ordinary differential equations are of second order in which the derivative y' does not appear explicitly (see for numerical methods for these problems [1] - [29] and references therein).

2. Phase-lag analysis of symmetric multistep methods

For the numerical solution of the above initial value problem (1), multistep methods of the form

$$\sum_{i=0}^{m} a_i y_{n+i} = h^2 \sum_{i=0}^{m} b_i f(x_{n+i}, y_{n+i}) \quad (2)$$

with m steps can be used over the equally spaced intervals $\{x_i\}_{i=0}^{m} \in [a,b]$ and $h = |x_{i+1} - x_i|,\ \ i = 0(1)m-1$. If the method is symmetric then $a_i = a_{m-i}$ and $b_i = b_{m-i},\ \ i = 0(1)\lfloor \frac{m}{2} \rfloor$.

Method (2) is associated with the operator

$$L(x) = \sum_{i=0}^{m} a_i u(x+ih) - h^2 \sum_{i=0}^{m} b_i u''(x+ih) \quad (3)$$

where $u \in C^2$.

Definition 1. The multistep method (2) is called algebraic of order p if the associated linear operator L vanishes for any linear combination of the linearly independent functions $1,\ x,\ x^2,\ \ldots,\ x^{p+1}$.

When a symmetric $2k$-step method, that is for $i = -k(1)k$, is applied to the scalar test equation

$$y'' = -\omega^2 y \quad (4)$$

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a difference equation of the form

$$A_k(v)y_{n+k} + ... + A_1(v)y_{n+1} + A_0(v)y_n + A_1(v)y_{n-1} + ... + A_k(v)y_{n-k} = 0 \quad (5)$$

is obtained, where $v = \omega h$, h is the step length and $A_0(v)$, $A_1(v), \ldots, A_k(v)$ are polynomials of v.

The characteristic equation associated with (5) is

$$A_k(v)s^k + ... + A_1(v)s + A_0(v) + A_1(v)s^{-1} + ... + A_k(v)s^{-k} = 0 \quad (6)$$

From Lambert and Watson (1976) we have the following definitions:

Definition 2.A symmetric $2k$-step method with characteristic equation given by (6) is said to have an interval of periodicity $(0, v_0^2)$ if, for all $v \in (0, v_0^2)$, the roots $s_i, i = 1(1)2k$ of Eq. (6) satisfy:

$$s_1 = e^{i\theta(v)},\ s_2 = e^{-i\theta(v)},\ and\ |s_i| \leq 1,\ i = 3(1)2k \quad (7)$$

where $\theta(v)$ is a real function of v.

Definition 3. For any method corresponding to the characteristic equation (6) the phase-lag is defined as the leading term in the expansion of

$$t = v - \theta(v) \quad (8)$$

Then if the quantity $t = O(v^{q+1})\ as\ v \to \infty$, the order of phase-lag is q.

Theorem 1.[12] *The symmetric $2k$-step method with characteristic equation given by (6) has phase-lag order q and phase-lag constant c given by*

$$-cv^{q+2} + O(v^{q+4}) = \frac{2A_k(v)\cos(kv) + ... + 2A_j(v)\cos(jv) + ... + A_0(v)}{2k^2A_k(v) + ... + 2j^2A_j(v) + ... + 2A_1(v)} \quad (9)$$

The formula proposed from the above theorem gives us a direct method to calculate the phase-lag of any symmetric $2k$- step method.

In our case, the symmetric 8-step method has phase-lag order q and phase-lag constant c given by:

$$-cv^{q+2} + O(v^{q+4}) = \frac{T_0}{T_1} \quad (10)$$

$$T_0 = 2\,A_4(v)\cos(4v) + 2\,A_3(v)\cos(3v) + 2\,A_2(v)\cos(2v) + 2\,A_1(v)\cos(v) + A_0(v)$$

$$T_1 = 32A_4(v) + 18A_3(v) + 8A_2(v) + 2A_1(v)$$

3. Construction of the new optimized semi-embedded predictor-corrector method

From the form (2) and without loss of generality we assume $a_m = 1$ and we can write

$$y_{n+m} + \sum_{i=0}^{m-1} a_i y_{n+i} = h^2 \sum_{i=0}^{m} b_i f(x_{n+i}, y_{n+i}), \quad (11)$$

finally we get

$$y_{n+m} = -\sum_{i=0}^{m-1} a_i y_{n+i} + h^2 \sum_{i=0}^{m} b_i f(x_{n+i}, y_{n+i}) \quad (12)$$

If the method is symmetric then $a_i = a_{m-i}$ and $b_i = b_{m-i},\ \ i = 0(1)\lfloor \frac{m}{2} \rfloor$.

3.1. The explicit method phase-lag order infinite (phase-fitted)

From the form (12) with $m = 8$ and $b_8 = 0$ we get the form of the explicit symmetric eight-step methods:

$$y_4 = -\Big(y_{-4} + a_3\,(y_3 + y_{-3}) + a_2\,(y_2 + y_{-2}) + a_1\,(y_1 + y_{-1}) + a_0 y_0\Big) + h^2\Big(b_3\,(f_3 + f_{-3}) + b_2\,(f_2 + f_{-2}) + b_1\,(f_1 + f_{-1}) + b_0 f_0\Big). \quad (13)$$

The characteristic equation (6) becomes

$$A_4(v)s^4 + A_3(v)s^3 + A_2(v)s^2 + A_1(v)s + A_0(v) + A_1(v)s^{-1} + A_2(v)s^{-2} + A_3(v)s^{-3} + A_4(v)s^{-4} = 0 \quad (14)$$

where

$$A_i(v) = a_i + v^2\,b_i,\ i = 0(1)4,\ a_4 = 1,\ b_4 = 0. \quad (15)$$

From (13) with
$a_3 = -2,\ a_2 = 2,\ a_1 = -1,\ a_0 = 0,$

$b_3 = \frac{17671}{12096},\ b_2 = -\frac{23622}{12096},\ b_1 = \frac{61449}{12096},\ b_0 = -\frac{50516}{12096},$

$y_i = y(x + ih)$ and $f_i = f(x + ih, y(x + ih))$, where h is the step length,
we obtain the symmetric multistep method of Quinlan-Tremaine [1], with eight steps and eighth algebraic order.

This method has an interval of periodicity $(0, v_0^2)$ where $v_0^2 = 0.52$.

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From (13) and by keeping the same a_i coefficients and by nullifying the phase-lag, we get:

$$a_3 = -2,\ a_2 = 2,\ a_1 = -1,\ a_0 = 0,$$
$$b_0 = -20b_3 + \frac{601}{24},\ b_1 = 15b_3 - \frac{101}{6},$$
$$b_2 = -6b_3 + \frac{109}{16},$$
$$b_3 = \frac{1}{96\,v^2\,(\cos(v)-1)^3}$$
$$\Big(-192\,(\cos(v))^4 + 192\,(\cos(v))^3$$
$$+\left(96 - 327\,v^2\right)(\cos(v))^2$$
$$+\left(-120 + 404\,v^2\right)\cos(v) - 137\,v^2 + 24\Big) \quad (16)$$

$y_i = y(x+ih)$, $f_i = f(x+ih, y(x+ih))$ where $v = \omega\,h$, ω is the frequency and h is the step length. For small values of v the above formulae are subject to
heavy cancelations. In this case the following Taylor series expansion must be used:

$$b_3 = \frac{17671}{12096} - \frac{45767}{725760}v^2 + \frac{164627}{47900160}v^4$$
$$-\frac{520367}{15850598400}v^6 + \frac{76873}{89669099520}v^8$$
$$-\frac{9190171}{32011868852864000}v^{10} - \frac{6662921}{34060628114472960}v^{12}$$
$$-\frac{2866814089}{2043637686868377600000}v^{14}$$
$$-\frac{10228341391}{16921320047270166528000}v^{16}$$
$$-\frac{1074205110763}{48394975335192676270080000}v^{18}$$
$$-\frac{1485941749021}{2032588964078092403343360000}v^{20} \quad (17)$$

where $v = \omega\,h$, ω is the frequency and h is the step length.

The explicit symmetric multistep method (16) has an interval of periodicity $(0, v_0^2)$ where $v_0^2 = 0.64$.

In order to find the Local Truncation Error(LTE), we express $y_{\pm i}$, $i = 1(1)4$ and $f_{\pm j}$, $j = 0(1)4$ via Taylor series and we substitute in (16). Based on this procedure we obtain the following expansion for the LTE:

$$L.T.E. = \frac{45767}{725760}\,h^{10}\left(y_n^{(10)} + y_n^{(8)}\omega^2\right) + O(h^{12}) \quad (18)$$

The above optimized explicit symmetric multistep method (16) has eight steps, eight algebraic order and infinite order of phase-lag (phase-fitted) (see [5]).

3.2. The implicit method (corrector)

From (12) for $m = 8$, we get the form of the symmetric implicit eight-step method:

$$y_4 = -\Big(y_{-4} + \alpha_3\,(y_3 + y_{-3}) + \alpha_2\,(y_2 + y_{-2})$$
$$+\alpha_1\,(y_1 + y_{-1}) + \alpha_0 y_0\Big)$$
$$+h^2\Big(\beta_4\,(f_4 + f_{-4}) + \beta_3\,(f_3 + f_{-3})$$
$$+\beta_2\,(f_2 + f_{-2}) + \beta_1\,(f_1 + f_{-1}) + \beta_0 f_0\Big). \quad (19)$$

The characteristic equation (6) becomes

$$A_4(v)s^4 + A_3(v)s^3 + A_2(v)s^2 + A_1(v)s + A_0(v)$$
$$+A_1(v)s^{-1} + A_2(v)s^{-2} + A_3(v)s^{-3} + A_4(v)s^{-4} = 0 \quad (20)$$

where $A_i(v) = \alpha_i + v^2\,\beta_i, \qquad i = 0(1)4, \qquad \alpha_4 = 1.$

From (19) and by keeping the same a_i coefficients ($\alpha_i = a_i, i = 0(1)4$, where $a_4 = 1$, $a_3 = -2$, $a_2 = 2$, $a_1 = -1$, $a_0 = 0$) we satisfy as many algebraic equations as possible.
After achieving 10th algebraic order we obtain the implicit symmetric multistep method:

$$\alpha_4 = 1,\ \alpha_3 = -2,\ \alpha_2 = 2,\ \alpha_1 = -1,\ \alpha_0 = 0$$
$$\beta_0 = \frac{17273}{72576},\ \beta_1 = \frac{280997}{181440},\ \beta_2 = -\frac{33961}{181440},$$
$$\beta_3 = \frac{173531}{181440},\ \beta_4 = \frac{45767}{725760} \quad (21)$$

with eight steps and tenth algebraic order.

The implicit symmetric multistep method (21) has an interval of periodicity $(0, v_0^2)$ where $v_0^2 = 2.39$.

The local truncation error of the above method is given by:

$$L.T.E. = -\frac{58061}{31933440}\,h^{12}y_n^{(12)} + O(h^{14}) \quad (22)$$

We note that the symmetric multistep method of Quinlan-Tremaine [1], with eight steps and eighth algebraic order, derives from (21) if we take $\beta_4 = 0$ (see [6]).

3.3. The new predictor-corrector method

From J.D. Lambert (1991) we have that the general k-step predictor-corrector or PC pair is:

$$\left.\begin{array}{l} \sum_{j=0}^{m} a_j^*\,y_{n+j} = h\,\sum_{j=0}^{m-1} b_j^*\,f_{n+j} \\ \sum_{j=0}^{m} a_j\,y_{n+j} = h\,\sum_{j=0}^{m} b_j\,f_{n+j} \end{array}\right\} \quad (23)$$

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Let the predictor and corrector defined by (23) have orders p^* and p respectively. The order of a PC method depend on the gap between p^* and p and on λ, the number of times the corrector is called. If $p^* < p$ and $\lambda =< p - p^* - 1$, the order of the PC method is $p^* + \lambda (< p)$ [3] .

We consider the pair of linear multistep methods:

$$\left.\begin{array}{l} \sum_{i=0}^{m} a_i y_{n+i} = h^2 \sum_{i=0}^{m-1} b_i(v) f(x_{n+i}, y_{n+i}) \\ \sum_{i=0}^{m} a_i y_{n+i} = h^2 \sum_{i=0}^{m} \beta_i f(x_{n+i}, y_{n+i}) \end{array}\right\}$$

where $|a_0| + |b_0(v)| \neq 0$, $|a_0| + |\beta_0| \neq 0$, $v = \omega\, h$, ω is the frequency and h is the step length.

Without loss of generality we assume $a_m = 1$ and we can write

$$\left.\begin{array}{l} y_{n+m} + \sum_{i=0}^{m-1} a_i y_{n+i} = h^2 \sum_{i=0}^{m-1} b_i(v) f(x_{n+i}, y_{n+i}) \\ y_{n+m} + \sum_{i=0}^{m-1} a_i y_{n+i} = h^2 \Big(\beta_m f(x_{n+m}, y_{n+m}) \\ \qquad + \sum_{i=0}^{m-1} \beta_i f(x_{n+i}, y_{n+i}) \Big) \end{array}\right\}$$

and we have

$$\left.\begin{array}{l} y_{n+m} = -\sum_{i=0}^{m-1} a_i y_{n+i} + h^2 \sum_{i=0}^{m-1} b_i f(x_{n+i}, y_{n+i}) \\ y_{n+m} = -\sum_{i=0}^{m-1} a_i y_{n+i} + h^2 \beta_m f(x_{n+m}, y_{n+m}) \\ \qquad + h^2 \sum_{i=0}^{m-1} \beta_i f(x_{n+i}, y_{n+i}) \end{array}\right\}$$

If we call $A_n = -\sum_{i=0}^{m-1} a_i y_{n+i}$, we can write

$$\left.\begin{array}{l} y_{n+m} = A_n + h^2 \sum_{i=0}^{m-1} b_i(v) f(x_{n+i}, y_{n+i}) \\ y_{n+m} = A_n + h^2 \beta_m f(x_{n+m}, y_{n+m}) \\ \qquad + h^2 \sum_{i=0}^{m-1} \beta_i f(x_{n+i}, y_{n+i}) \end{array}\right\}$$

From this pair, a new predictor-corrector (PC) pair form, is formally defined as follows:

$$\left.\begin{array}{l} y^*_{n+m} = A_n + h^2 \sum_{i=0}^{m-1} b_i(v) f(x_{n+i}, y_{n+i}) \\ y_{n+m} = A_n + h^2 \beta_m f(x_{n+m}, y^*_{n+m}) \\ \qquad + h^2 \sum_{i=0}^{m-1} \beta_i f(x_{n+i}, y_{n+i}) \end{array}\right\} \quad (24)$$

where $A_n = -\sum_{i=0}^{m-1} a_i y_{n+i}$, $|a_0| + |b_0(v)| \neq 0$, $|a_0| + |\beta_0| \neq 0$, $v = \omega\, h$, ω is the frequency and h is the step length.

We call the above method Semi-Embedded Predictor-Corrector Method (SEPCM), in the sense that a part of the predictor method is contained in the corrector method.

If the method is symmetric then $a_i = a_{m-i}$, $b_i(v) = b_{m-i}(v)$ and $\beta_i = \beta_{m-i}$, $i = 0(1)\lfloor \frac{m}{2} \rfloor$.

From (24), (16) and (21) a new symmetric eight-step semi-embedded predictor-corrector method (SEPCM) with minimal phase-lag obtained:

$$\left.\begin{array}{l} A = -y_{-4} + 2(y_3 + y_{-3}) - 2(y_2 + y_{-2}) + (y_1 + y_{-1}) \\ y^*_4 = A + h^2 \Big(b_3(v)(f_3 + f_{-3}) + \big(\frac{109}{16} - 6\, b_3(v)\big) \\ \qquad (f_2 + f_{-2}) + \big(15\, b_3(v) - \frac{101}{6}\big)(f_1 + f_{-1}) \\ \qquad + \big(\frac{601}{24} - 20\, b_3(v)\big) f_0 \Big) \\ y_4 = A + h^2 \Big(\frac{45767}{725760}(f^*_4 + f_{-4}) + \frac{173531}{181440}(f_3 + f_{-3}) \\ \qquad - \frac{33961}{181440}(f_2 + f_{-2}) + \frac{280997}{181440}(f_1 + f_{-1}) + \frac{17273}{72576} f_0 \Big) \end{array}\right\} \quad (25)$$

where

$$b_3(v) = \frac{T_2}{96\, v^2 \left(\cos(v) - 1\right)^3}, \quad (26)$$

where $T_2 = -192\, (\cos(v))^4 + 192\, (\cos(v))^3 + \big(96 - 327\, v^2\big) (\cos(v))^2 + \big(-120 + 404\, v^2\big) \cos(v) - 137\, v^2 + 24$ and $y_i = y(x + ih)$, $f_i = f(x + ih, y(x + ih))$, $v = \omega\, h$, ω is the frequency and h is the step length.

For small values of v the following Taylor series expansions must be used:

$$\begin{aligned} b_3(v) = {} & \frac{17671}{12096} - \frac{45767}{725760} v^2 + \frac{164627}{47900160} v^4 \\ & - \frac{520367}{15850598400} v^6 + \frac{76873}{89669099520} v^8 \\ & - \frac{9190171}{3201186852864000} v^{10} \\ & - \frac{6662921}{340606281144729600} v^{12} \\ & - \frac{2866814089}{2043637686868377600000} v^{14} \\ & - \frac{10228341391}{16921320047270166528000} v^{16} \\ & - \frac{1074205110763}{48394975335192676270080000} v^{18} \\ & - \frac{1485941749021}{20325889640780924033433600000} v^{20}, \end{aligned} \quad (27)$$

where $v = \omega\, h$, ω is the frequency and h is the step length.

The characteristic equation (6) becomes

$$\begin{aligned} & A_4(v) s^4 + A_3(v) s^3 + A_2(v) s^2 + A_1(v) s + A_0(v) \\ & + A_1(v) s^{-1} + A_2(v) s^{-2} + A_3(v) s^{-3} + A_4(v) s^{-4} = 0 \end{aligned} \quad (28)$$

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where

$$A_i(v) = \alpha_i + v^2(\beta_i - a_i\,\beta_4) - v^4\,b_i(v)\,\beta_4,$$
$$i = 0(1)4, \qquad \alpha_4 = a_4 = 1, \qquad b_4(v) = 0. \tag{29}$$

The new optimized symmetric eight-step semi-embedded predictor-corrector method (25) has an interval of periodicity $(0, v_0^2)$ where $v_0^2 = 1.307$.
The local truncation error of the above method is given by:

$$L.T.E. = \Big(\frac{12506213339}{5794003353600}\,y_n^{(12)} + \frac{2094618289}{526727577600}\,y_n^{(10)}\omega^2\Big)h^{12} + O(h^{14}) \tag{30}$$

The new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) (25) has eight steps, tenth algebraic order and tenth order of phase-lag.

4. Numerical results

4.1. The problems

The efficiency of the new optimized symmetric semi-embedded eight-step predictor-corrector method will be measured through the integration of four initial value problems with oscillating solution.

4.1.1. Duffing Equation

$$y'' = -y - y^3 + 0.002\cos(1.01\,t),$$

with $y(0) = 0.200426728067, y'(0) = 0,\ t \in [0, 1000\,\pi]$.
Theoretical solution:

$$y(t) = 0.200179477536\cos(1.01\,t) + 2.46946143\cdot 10^{-4}\cos(3.03\,t) + 3.04014\cdot 10^{-7}\cos(5.05\,t) + 3.74\cdot 10^{-10}\cos(7.07\,t) + \ldots \tag{31}$$

Estimated frequency: $w = 1$.

4.1.2. Nonlinear Equation

$y'' = -100\,y + \sin(y)$, with $y(0) = 0, y'(0) = 1,\ t \in [0, 20\,\pi]$.

The theoretical solution is not known, but we use $y(20\,\pi) = 3.92823991\ \cdot\ 10^{-4}$.
Estimated frequency: $w = 10$.

4.1.3. Orbital Problem by Stiefel and Bettis

The "almost" periodic orbital problem studied by [4] can be described by

$$y'' + y = 0.001\,e^{i\,x}, \quad y(0) = 1, \quad y'(0) = 0.9995\,i, \quad y \in \mathcal{C}, \tag{32}$$

or equivalently by

$$u'' + u = 0.001\,\cos(x), \quad u(0) = 1, \quad u'(0) = 0,$$
$$v'' + v = 0.001\,\sin(x), \quad v(0) = 0, \quad v'(0) = 0.9995. \tag{33}$$

The theoretical solution of the problem (32) is given below:

$$y(x) = u(x) + i\,v(x), \quad u, v \in \mathcal{R}$$
$$u(x) = \cos(x) + 0.0005\,x\,\sin(x),$$
$$v(x) = \sin(x) - 0.0005\,x\,\cos(x).$$

The system of equations (33) has been solved for $x \in [0, 1000\,\pi]$.
Estimated frequency: $w = 1$.

4.1.4. Two-dimensional Kepler problem (Two-Body Problem)

$$y'' = -\frac{y}{(y^2+z^2)^{\frac{3}{2}}},\ z'' = -\frac{z}{(y^2+z^2)^{\frac{3}{2}}}, \tag{34}$$

with $y(0) = 1 - e,\ y'(0) = 0,\ z(0) = 0,\ z'(0) = \sqrt{\frac{1+e}{1-e}},\ t \in [0, 1000\,\pi]$, where e is the eccentricity.

The theoretical solution of this problem is given below:
$y(t) = \cos(u) - e$
$z(t) = \sqrt{1-e^2}\,\sin(u)$.
where u can be found by solving the equation $u - e\ sin(u) - t = 0$.

We used the estimation $w = \frac{1}{(y^2+z^2)^{\frac{3}{4}}}$ as frequency of the problem.

4.2. The methods

We have used several multistep methods for the integration of the four test problems. These are:

- –The new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) with tenth algebraic order and minimal phase-lag (25) (New SEPCM 8ste)
- –The symmetric 10-step method of Quinlan-Tremaine of order ten [1] (QT10)
- –The optimized symmetric 8-step method (16) of order eight [5] (QT8 Phase-fitted)

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–The symmetric 8-step method of Quinlan-Tremaine of order eight [1] (QT8)
–The 10-stage exponentially-fitted method of Simos and Aguiar of algebraic order nine [9] (SA 10Ste EF)
–The symmetric 6-step method of Jenkins of order six [7] (Jenkins 6ste)
–The 3-step, 3-stage exponentially-fitted predictor-corrector method (EF2) of Psihoyios and Simos of algebraic order five [10] (PS 3Ste EF2)
–The 2-step, 3-stage exponentially-fitted predictor-corrector method of Simos and Williams of algebraic order six [8] (SW 3Ste 3Sta EF PC)
–The 4-step predictor-corrector method Adams-Bashforth - Moulton of order four (ABM4)
–The 4-step predictor-corrector method Milne-Simpson of order four (Milne Simpson).

4.3. Comparison

We present the **accuracy** of the tested methods expressed by the $-\log_{10}$(max. error over interval) or $-\log_{10}$(error at the end point), depending on whether we know the theoretical solution or not, versus the CPU time.
In Table 1 we see the comparison of the new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) (25) and the multistep symmetric method of Quinlan-Tremaine [1] with eight steps for all the problems solved.

In Figure 1 we see the results for the Duffing equation, in Figure 2 the results for the Nonlinear equation, in Figure 3 the results for the Stiefel-Bettis almost periodic problem, and in Figures 4 and 5 the results for the two-dimensional Kepler problem (Two-body problem) for eccentricities e=0.0156 and e=0.6.
Among all the methods used, the new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) with tenth algebraic order is the most efficient.
The interval of periodicity of the new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) with tenth algebraic order is about 2.5 times larger than the multistep symmetric method of Quinlan-Tremaine with eight steps and eighth algebraic order and about two times larger than the optimized symmetric 8-step method (16) with eight steps and eighth algebraic order.
The new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) with tenth algebraic order can achieve the required accuracy with a step-size eight times larger than the multistep symmetric method of Quinlan-Tremaine with eight steps and eighth algebraic order for the Duffing equation and with a step-size two times larger than the multistep symmetric method of Quinlan-Tremaine with eight steps and eighth algebraic order for all the other problems solved.
An interesting remark is that the new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) with tenth algebraic order, is more efficient than the multistep symmetric method of Quinlan-Tremaine, with ten steps and tenth algebraic order.

5. Conclusions

We have constructed a new predictor-corrector (PC) pair form for the numerical integration of second-order initial-value problems (1), for which we know the frequency. From the form (24) we have developed a new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) (25).
The new method has algebraic order ten and it can be used to solve numerically, related initial-value problems with oscillatory solutions, for which we know the frequency.
The new semi-embedded predictor-corrector pair form (24) has the advantage that reduces the computational expense if the additions on the factor $\sum_{i=0}^{m-1} a_i y_{n+i}$, are done twice.
We have applied the new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) with tenth algebraic order (25) along with a group of several methods from the literature to four oscillatory problems. We concluded that that the new optimized eight-step symmetric semi-embedded predictor-corrector method (SEPCM) with tenth algebraic order (25) is highly efficient compared to other methods.

TABLE 1. Comparison of the new SEPCM 8-step and Quinlan-Trimaine 8-step method for all the problems solved

Test Problem	Method	Accuracy (digits)	CPU Time	Step Length	Maximum Error
Duffing's Equation					
	Quinlan-Tremaine 8step	10.73977737	9.503988356	0.02	1.82063E-11
	New SEPCM 8-step	10.7168827	5.796875	0.16	1.91919E-11
Nonlinear Equation					
	Quinlan-Tremaine 8step	11.63199947	2.3125	0.003867188	2.33346E-12
	New SEPCM 8-step	11.34144029	1.359375	0.007734375	4.55575E-12
Stiefel - Bettis					
	Quinlan-Tremaine 8step	11.58931711	26.46875	0.02	2.57E-12
	New SEPCM 8-step	12.00917043	15.578125	0.04	9.79E-13
Two-Body Problem					
Eccentricity 0.0156	Quinlan-Tremaine 8step	8.780099323	21.015625	0.0309375	1.65921E-09
	New SEPCM 8-step	8.525251233	12.03125	0.061875	2.98366E-09
Eccentricity 0.6	Quinlan-Tremaine 8step	7.282026611	172.859375	0.003867185	5.22364E-08
	New SEPCM 8-step	7.282411794	98.296875	0.00773437	5.21901E-08

Table 1 Comparison for CPU time and Step Length

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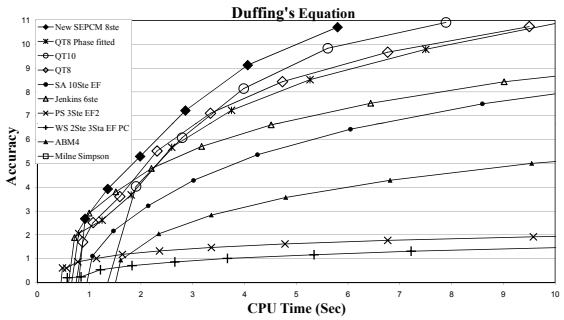

Figure 1 Efficiency for the Duffing equation

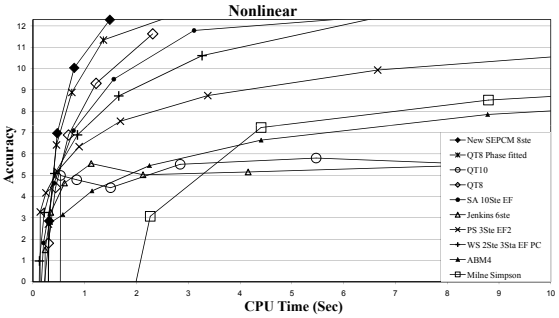

Figure 2 Efficiency for the Nonlinear equation

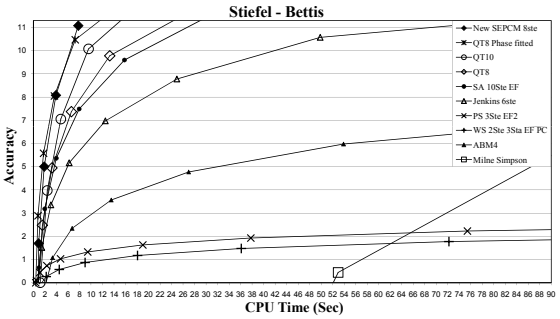

Figure 3 Efficiency for the Orbital Problem by Stiefel and Bettis

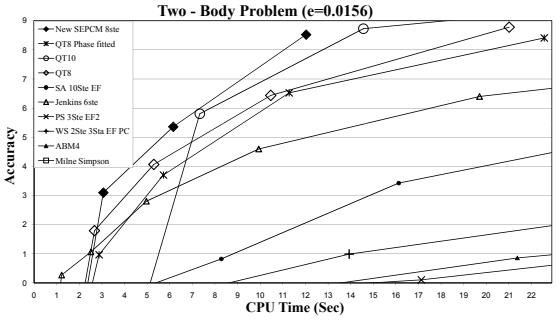

Figure 4 Efficiency for the Two-body problem using eccentricity e =0.0156

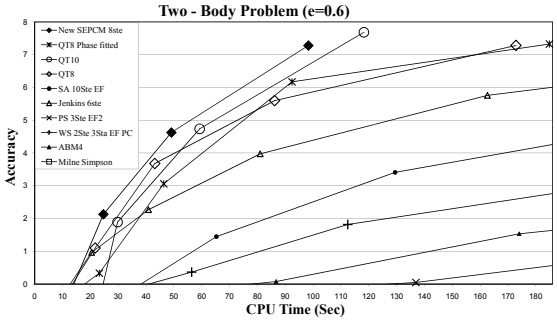

Figure 5 Efficiency for the Two-body problem using eccentricity e =0.6

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