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AN ORDER DROP THEOREM

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A drop theorem on ordered metric spaces is established from the (pre) order version of Ekeland's variational principle in Turinici [An St UAIC Iaşi (Math), 36 (1990), 329-352]. The logical equivalence between these results is also discussed.

1. Introduction.

Let $(X, ||.||)$ be a Banach space. For each $a \in X$ and each nonempty subset C of X with $a \notin C$, put $D(a, C) = \{\lambda a + (1-\lambda)c; 0 \leq \lambda \leq 1, c \in C\}$; this will be referred to as the *drop* generated by a and C. Note that $D(a, c) = [a, c]$ (the *segment* with endpoints a and c), if $C = \{c\}$. The following result [known as the *drop theorem* (DT)] is our starting point.

Theorem 1. *Let the (nonempty) parts A, C of X be such that*

(1.1) *A is closed and C is closed, bounded, convex*

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(1.2)
$$\delta(A, C) := \inf\{||x - y||; x \in A, y \in C\} \geq \rho, \quad \textit{for a certain } \rho > 0.$$

Then, for each $a_0 \in A$, *there exists* $a \in A$ *with*

(1.3) $$a \in D(a_0, C) \quad (\textit{hence } a \in A \cap D(a_0, C))$$

(1.4) $$x \in A \cap D(a, C) \Rightarrow a = x (a \textit{ is } C\textit{-support for } A).$$

A first proof of Theorem 1 was given in the 1971 paper by Zabreiko and Krasnoselskii [19] (see also Daneš [7]), via Cantor's intersection theorem. Further, in his 1974 paper, Brøndsted [5] provided a different proof of the same, by using the Bishop-Phelps lemma [2]. This (cf. also Georgiev [10]) shows that Theorem 1 is logically reducible to the 1974 Ekeland's variational principle [9] (in short: (EVP)) which may be stated as:

Theorem 2. *Let* (M, d) *be a complete metric space and* $f : M \to R \cup \{\infty\}$, *some function with*

(1.5) $$f \textit{ is proper } (\mathrm{Dom}(f) \neq \emptyset), \quad \textit{bounded below } (f_* := \inf[f(M)] > -\infty)$$

(1.6) $$f \textit{ is lsc on } M: \quad [f \leq \tau] := \{x \in M; f(x) \leq \tau\} \textit{ is closed, } \forall \tau \in R.$$

Then, for each $a_0 \in \mathrm{Dom}(f)$ *there exists* $a \in \mathrm{Dom}(f)$ *with*

(1.7) $$d(a_0, a) \leq f(a_0) - f(a) \quad (\textit{ hence } f(a_0) \geq f(a))$$

(1.8) $$x \in M, d(a, x) \leq f(a) - f(x) \Rightarrow a = x (a \textit{ is } f\textit{-variational }).$$

The structural analogy between these statements is clear; because, roughly speaking, their conclusions (1.3)/(1.4) and (1.7)/(1.8) may be expressed as: each element is majorized by a "maximal" one. So, it is natural asking of to what extent is this retainable from a logical perspective; i.e.,

(1.9) (DT) and (EVP) are deductible from each other.

A first (positive) answer to this was obtained in 1985 by Daneš [8], via specific methods related to the geometry of the ambient Banach space.

Further, in his 1986 paper, Penot [12] established a second answer to the same, again via geometric methods. Note that an interesting conclusion of these developments is the following: (EVP) is logically equivalent with its continuous (modulo f) variant. So, further extensions of such facts are not without interest. Among these, the "relational" way is a promising one. Its starting point is the 1990 (pre)order extension of Ekeland's variational principle obtained by Turinici [19] (cf. Section 2); which seems to include it in a strict way (from a technical viewpoint). The next step is to look for an appropriate counterpart of the drop theorem; the answer to this will be provided in Section 3. Further, in Section 4, the converse question is considered; and the conclusion to be derived is the (pre)order counterpart of (1.9) above. Notice that, the proposed arguments are not reducible to the ones of the quoted authors (when this preorder is the trivial one). Finally, Section 5 is devoted to the (pre)order version of the "amended" Ekeland variational principle in Georgiev [10]. We provide two proofs for this result. The former of these is purely metrical; and consists in a direct application of the (pre)order Ekeland variational principle to our data. The latter of these is geometric in nature; and is related to our previous developments. Further aspects will be discussed elsewhere.

2. Order EVP in metric and normed spaces.

Let (M, d) be a metric space and $(\leq)$, some *preorder* (i.e.: reflexive and transitive relation) over it; the triplet $(M, d; \leq)$ will be referred to as a *preordered metric structure*. We say that the (nonempty) subset A of M is $(\leq)$-*closed* when the limit of each ascending sequence in A belongs to A. Further, call the underlying preorder $(\leq)$, *self-closed* if $M(x, \leq) := \{y \in M; x \leq y\}$ is $(\leq)$-closed for each $x \in M$. Finally, term the ambient space, $(\leq)$-*complete* when each ascending Cauchy sequence in M converges (in M). The following result [referred to as the (pre)order Ekeland variational principle; in short: (OEVP)] is now available (cf. Turinici [19]).

Theorem 3. *Let the preordered metric structure* $(M, d; \leq)$ *be such that* (M, d) *is* $(\leq)$*-complete and* $(\leq)$ *is self-closed. Further, let the function* $f : M \to R \cup \{\infty\}$ *be such that* (1.5) *is valid, as well as*

(2.1) $\qquad f$ *is* $(\leq) - lsc$ *on* $M : [f \leq \tau]$ *is* $(\leq)$*-closed,* $\forall \tau \in R$.

Then, for each $a_0 \in \mathrm{Dom}(f)$ *there exists* $a \in \mathrm{Dom}(f)$ *with*

$$a_0 \le a, d(a_0, a) \le f(a_0) - f(a) \ (\textit{hence } f(a_0) \ge f(a)) \tag{2.2}$$

$$a \le x, d(a, x) \le f(a) - f(x) \ \Rightarrow a = x \ (a \textit{ is } (\le, f)\textit{-variational}). \tag{2.3}$$

Some remarks are in order. The basic tool of the proof is the pseudometric maximality principle in Turinici [17]. But, the ordering principles in Brezis and Browder [4], Altman [1] or Kang and Park [11] are also working here; because (cf. Turinici [18]) all these are equivalent to each other. Further, Theorem 3 includes Theorem 2, to which it reduces when $(\le) = M^2$ (the *trivial* preorder on M). For the converse question, note that

$$f \text{ is } (\le)\text{-antitone} \quad (x \le y \Longrightarrow f(x) \ge f(y)) \tag{2.4}$$

is a particular case of (2.1) (under the self-closedness of $(\le)$); i.e., Theorem 3 is applicable to such functions. This, however, is no longer valid for Theorem 2. Indeed, let us take $M = R$ (endowed with the usual metric and order). Then, the function

$$f = -\ \mathrm{sgn}\ (\text{i.e.: } f(x) = -x/|x| \text{ if } x \ne 0 \text{ and } f(0) = 0)$$

is $(\le)$–antitone; hence $(\le)$-lsc as well (because $(\le)$ is self-closed); but not lsc, as it can be directly seen; and this proves the claim. In other words, a reduction of Theorem 3 to Theorem 2 is not possible *in such a way*. The question of this being realizable via different methods is open; we conjecture that a positive answer is eventually available.

The obtained (OEVP) is metrical in nature. However, it may be put into a normed framework (useful to the converse question formulated in Section 1). Precisely, the following "normed" version of Theorem 3 is available:

Theorem 4. *Let the Banach space* $(V, ||.||)$ *be given, as well as the self-closed preorder* $(\le)$ *over it. Further, let the (nonempty) part* N *of* V *and the function* $g : V \to R \cup \{-\infty\}$ *be such that (for a certain* $\rho > 0$*)*

$$N \ \textit{is} (\le)\textit{-closed and } \mathrm{diam}(N) \le \rho \tag{2.5}$$

$$g \textit{ is } (\le)\textit{-usc on } V \textit{ and } 0 = \inf[g(N)] \le \sup[g(N)] < \rho. \tag{2.6}$$

Then, for each $b_0 \in N$ *with* $g(b_0) = 0$ *there exists* $b \in N$ *with*

$$b \ge b_0, ||b - b_0|| \le g(b) - g(b_0) (= g(b)) \tag{2.7}$$

(2.8) $\quad ||x - b|| > g(x) - g(b), \quad$ *for all* $x \in N(b, \leq),\ x \neq b.$

For the moment, Theorem 4 is a logical consequence of Theorem 3. This follows by simply taking ($M = N$, d=the norm-induced metric) and also

$$(f : M \to R) : \quad f(x) = \sup[g(N)] - g(x), x \in M.$$

The remarkable fact to be added is that the converse inclusion also holds:

Proposition 1. *Under these conventions,*

(2.9) *Theorem 3 is deductible from Theorem 4.*

So (combining with the above) these results are deductible from each other.

Proof. Let the complete metric space (M, d) and the preorder $(\leq)$ over it be as in the premises of Theorem 3. Further, let $f : M \to R \cup \{\infty\}$ be some function as in (1.5) + (2.1); and $a_0 \in \mathrm{Dom}(f)$ be arbitrary fixed. Without loss of generality, one may assume that

(2.10) $\quad a_0 \leq x, \forall x \in M; \quad f(a_0) = \sup[f(M)].$

For, otherwise, the nonempty subset $M_0 = \{x \in M; a_0 \leq x, f(a_0) \geq f(x)\}$ (which fulfills (2.10)) is $(\leq)$-closed (hence $(\leq)$-complete) by the admitted hypothesis; and, from the conclusions (2.2) + (2.3) written for M_0, it clearly follows the same fact for M. Let $\rho > 0$ be taken in accordance with

(2.11) $\quad f(a_0) - f_* < \rho \quad (\text{i.e.}: \sup[f(M)] - \inf[f(M)] < \rho).$

Again without loss of generality, one may now assume that

(2.12) $\quad \mathrm{diam}(M) := \sup\{d(x, y); x, y \in M\} \leq \rho.$

For, if this fails, let us substitute d by the metric (fulfilling (2.12)) $e(x, y) = \min\{\rho, d(x, y)\}$, $x, y \in M$. Then, from (2.2) + (2.3) written in terms of e, it follows via (2.11) the validity of the same in terms of d; hence the claim.

We are now passing to the effective part of the argument. Let V stand for the class of all continuous bounded functions $\varphi : M \to R$. This is a linear space which becomes a Banach one, with respect to

$$||\varphi|| = \sup\{|\varphi(x)|; x \in M\}, \varphi \in V \ \text{(the } \textit{supremum} \text{ norm).}$$

For each $z \in M$, let $T(z)$ stand for the element in V introduced as

$$T(z)(x) = d(z, x) - d(a_0, x), \quad x \in M.$$

The canonical map $z \mapsto T(z)$ fulfills $d(z, w) = ||Tz - Tw||$, for all $z, w \in M$; so, it is an isometry between M and its image $N = T(M)$ in V. Let $(\leq)$ stand for the relation (over V)

$$x \leq y \text{ iff either } x, y \in N, T^{-1}(x) \leq T^{-1}(y) \text{ or } x, y \in V \backslash N, x = y.$$

Clearly, $(\leq)$ acts as a self-closed preorder on V. In addition to this, the working condition (2.12) and the properties of T give (2.5). Let $h : V \to R \cup \{\infty\}$ stands for the function

$$h(v) = f(T^{-1}(v)), v \in N; \quad h(v) = \infty, \text{ otherwise.} \tag{2.21}$$

By the same way as in Daneš [8] one gets that (cf. the working hypotheses (2.10) + (2.11) and the remarks above)

$$h \text{ is } (\leq)\text{- lsc on } V, \ \sup[h(N)] = h(T(a_0)) < \inf[h(N)] + \rho.$$

So, if we introduce the function

$$(g : V \to R \cup \{-\infty\}) : g(v) = h(T(a_0)) - h(v), v \in V,$$

it is clear that (2.6) holds. Summing up, Theorem 4 applies to our data. Hence, for the starting point $b_0 = T(a_0)$ in N (with $g(b_0) = 0$ as it can be directly seen) there exists $b = T(a)$ in N so that (2.7) + (2.8) be true. But, in this case, (2.2) + (2.3) are being retainable, by the very definition of $(\leq)$ and g. The proof is thereby complete. □

In particular, when $(\leq)$ is the trivial preorder on M, this result reduces to the one in Daneš [op. cit.]. Further aspects may be found in Penot [12].

3. Main results.

Let $(X, ||.||)$ be a Banach space; and $(\leq)$, some self-closed preorder over it. Further, let the (nonempty) parts A, C of X be such that (1.2) is fulfilled. The following result [referred to as the *(pre)order drop theorem* (ODT)] is available.

Theorem 5. *Suppose that (in addition)*

(3.1) *A is $(\leq)$-closed and C is closed, bounded, convex.*

Then, for each $a_0 \in A$*, there exists* $a \in A$ *with*

(3.2) $\quad a \geq a_0, a \in D(a_0, C) \quad$ (*hence* $a \in A(a_0, \leq) \cap D(a_0, C)$)

(3.3) $\quad x \in A(a, \leq) \cap D(a, C) \Rightarrow x = a$ (a *is* $(\leq, C)$*-support for* A).

In particular, when $(\leq)$ is the trivial preorder on X, this statement is nothing but the drop theorem (DT). But (cf. the remarks in Section 1) there is a logical equivalence between this last result and Ekeland's variational principle (EVP). So, it is natural to ask whether this equivalence can be extended to our preorder setting [involving (ODT) and (OEVP)]. The answer to this is affirmative; and the first half of it is precised in

Proposition 2. *Let the conditions above be in use. Then*

(3.4) $\quad$ *(ODT) is deductible from (OEVP).*

Proof. Let the parts A, C of X and the number $\rho > 0$ be as in the premises of (ODT); and take some a_0 in A. Denote

$$M = A(a_0, \leq) \cap D(a_0, C), d = \text{ the norm-induced metric.}$$

Let $(\preceq)$ stand for the preorder on M

$$(x, y \in M) \; x \preceq y \text{ iff } x \leq y \text{ and } y \in D(x, C).$$

It is clear, via (3.1) (and the self-closedness of $(\leq)$) that (M, d) is $(\preceq)$-complete and $(\preceq)$ is self-closed. Further, denote $f(x) = (\alpha/\rho)\delta(x, C), x \in M$; where

$$\alpha = \Delta(M, C) := \sup\{||x - y||; x \in M, y \in C\} \text{ (hence } \alpha \geq \rho).$$

By the nonexpansiveness of $u \mapsto \delta(u, C)$, f is continuous (hence $(\preceq)$-lsc) over M. Finally, let $x, y \in M$ be such that $x \preceq y$; i.e.,

$$x \leq y \text{ and } y = \lambda x + (1 - \lambda)c, \text{ for some } \lambda \in [0, 1], c \in C.$$

By the convexity of $u \mapsto \delta(u, C)$, one has $\delta(y, C) \leq \lambda\delta(x, C) \leq \delta(x, C)$; wherefrom

$$(1 - \lambda)\rho \leq (1 - \lambda)\delta(x, C) \leq \delta(x, C) - \delta(y, C).$$

On the other hand $||x - y|| = (1 - \lambda)||x - c|| \leq (1 - \lambda)\alpha$; so, combining these

(3.5) $\quad x, y \in M, x \preceq y \Longrightarrow x \leq y, ||x - y|| \leq f(x) - f(y).$

(Note that, as a direct consequence of this, $(\preceq)$ is antisymetric (hence an *order*) on M). Summing up, the (pre)order Ekeland variational principle (subsumed to Theorem 3) applies to $(M, d; \preceq)$ and f. It gives us, for the starting point $a_0 \in M$, some point $a \in M$ with the properties (2.2) + (2.3) (written for these data). The former of these yields (3.2) (by the definition of $(\leq)$). And the latter one gives (3.3) if we take (3.5) into account. Hence the conclusion. □

The following complement of this result is useful for us. Let again A, C be as in Theorem 5. Denote

$$C_\theta = X[C, \theta\rho] := \{x \in X : \delta(x, C) \leq \theta\rho\}, \theta \geq 0.$$

This is a family of closed bounded convex parts of X fulfilling

$$C_0 = C \quad \text{and } \theta \mapsto C_\theta \text{ increases with } \theta \tag{3.6}$$

$$\delta(A, C_\theta) \geq \rho(1-\theta) > 0, \quad \text{for each } \theta \in [0, 1[\tag{3.7}$$

$$C_\mu = X[C_\nu, (\mu - \nu)\rho], \quad \text{whenever } \mu \geq \nu. \tag{3.8}$$

Theorem 6. *Let μ be arbitrary fixed in $]0, 1[$. Then, for each a_0 in A, there exists a point $a(\mu)$ in A with*

$$\begin{aligned} &a(\mu) \geq a_0, a(\mu) \in D(a_0, C_\mu) \\ &(\textit{hence } a(\mu) \in A(a_0, \leq) \cap D(a_0, C_\mu)) \end{aligned} \tag{3.9}$$

$$A(a(\mu), \leq) \cap D(a(\mu), C_\nu) = \{a(\mu)\}, \forall \nu \in [0, \mu]. \tag{3.10}$$

Moreover, $\forall \nu \in]0, \mu[$ one has (with $\alpha(\mu, \nu) = \Delta(a(\mu), C_\nu)$)

$$(\mu - \nu)\rho\|x - a(\mu)\| \leq \alpha(\mu, \nu)\delta(x, A(a(\mu), \leq)), \forall x \in D(a(\mu), C_\nu). \tag{3.11}$$

Hence, in particular (with the same μ, ν as before)

$$(x_n) \subseteq D(a(\mu), C_\nu), \delta(x_n, A(a(\mu), \leq)) \to 0 \textit{ imply } x_n \to a(\mu). \tag{3.12}$$

Proof. By the admitted hypotheses and (3.7), Theorem 5 is applicable to (A, C_μ) and $\rho_\mu = \rho(1 - \mu)$. So, for the starting point a_0 in A, there exists a point $a(\mu) \in A$ with the properties (3.2) + (3.3) (written for $(C_\mu, a(\mu))$ in place of (C, a)). From this, (3.9) + (3.10) are clear; so, it remains to prove (3.11). Fix ν in $]0, \mu[$; and let $x \neq a(\mu)$ be some point in $D(a(\mu), C_\nu)$. Hence

$$x = \theta a(\mu) + (1-\theta)c, \quad \text{for some } \theta \in [0, 1[, \ c \in C_\nu; \tag{3.13}$$

in addition, $x \notin A(a(\mu), \leq)$ (by (3.10)). Without loss of generality, one may suppose that

$$(3.14) \qquad ||x - a(\mu)|| > \delta(x, A(a(\mu), \leq));$$

for, otherwise, combining with $(\mu-\nu)\rho \leq (1-\nu)\rho \leq \alpha(\mu, \nu)$ (cf. (3.7)), we are done. Let ϵ be taken in the open interval of the numbers in (3.14). There exists, by definition, some $x_A \in A(a(\mu), \leq)$ with

$$||x - a(\mu)|| > \epsilon > ||x - x_A||(> 0, \text{ in view of } x \notin A(a(\mu), \leq)).$$

This, and (3.10), tells us that $x_A \notin D(a(\mu), C_\mu)$. On the other hand, $x \in \mathrm{int}[D(a(\mu), C_\mu)]$; because $C_\nu \subseteq \mathrm{int}\ [C_\mu]$. So, necessarily

$$[x_A, x] \cap \mathrm{bd}[D(a(\mu), C_\mu)] \neq \emptyset \quad \text{(where "bd" = the boundary)} .$$

Let y be some point in this intersection. Note that $y \neq a(\mu)$; for, otherwise, $||x-a(\mu)|| \leq ||x-x_A|| < ||x-a(\mu)||$, contradiction. Denote further (with $\theta \in [0, 1[$ taken as in (3.13))

$$u = (y - \theta a(\mu))/(1-\theta) \ (\text{ hence } y = \theta a(\mu) + (1-\theta)u).$$

The case (=alternative) below is not acceptable

$$u \in X(C_\nu, (\mu - \nu)\rho) := \{x \in X; \delta(x, C_\nu) < (\mu - \nu)\rho\}.$$

For (in view of (3.8)) $u \in \mathrm{int}[C_\mu] \subseteq \mathrm{int}[D(a(\mu), C_\mu)]$; and this, combined with $a(\mu) \in D(a(\mu), C_\mu)$, gives $y \in \mathrm{int}[D(a(\mu), C_\mu)]$ (cf. Bourbaki [3, Ch 2, Sect 2]); in contradiction with the choice of y. Hence $\delta(u, C_\nu) \geq (\mu - \nu)\rho$; wherefrom

$$||x - y|| = (1-\theta)||u - c|| \geq (1-\theta)(\mu - \nu)\rho.$$

On the other hand, (3.13) and the definition of $\alpha(\mu, \nu)$ give

$$||x - a(\mu)|| = (1-\theta)||a(\mu) - c|| \leq (1-\theta)\alpha(\mu, \nu);$$

so, by simply combining with the above,

$$(\mu - \nu)\rho||x - a(\mu)|| \leq \alpha(\mu, \nu)||x - y|| \leq \alpha(\mu, \nu)||x - x_A|| \leq \alpha(\mu, \nu)\epsilon.$$

As ϵ was arbitrarily chosen with $||x-a(\mu)|| > \epsilon > \delta(x, A(a(\mu), \leq))$, this yields (3.11), by a limit process. □

Some remarks are in order. When A is bounded, the conclusions of Theorem 5 are valid (cf. Phelps [13]) even if C were unbounded;

precisely, when (3.1) is to be written as

(3.15) A is $(\leq)$-closed bounded and C is a closed convex cone.

This approach goes back to Bishop and Phelps [2]; see also Ursescu [20]. On the other hand, if X is reflexive and C is some closed ball in X, i.e.,

$$C = X[c, \sigma], \quad \text{for some } c \in X \text{ and some } \sigma > 0,$$

the regularity condition (1.2) may be relaxed as (under $(\leq) = X^2$)

(3.16) $\delta(A, C) = 0$, $A \cap C = \emptyset$ (cf. Rolewicz [16]).

So, we may presume that a similar conclusion is retainable in our preorder setting. Further aspects were developed in Browder [6].

4. The converse question.

We are now in position to discuss the converse to (3.4) implication (cf. Proposition 2). As precised there, a positive answer is (ultimately) available, by the formal analogies between Theorem 3 and Theorem 5. But, from a technical viewpoint, the situation is a bit more complicated, by the different nature of the concepts involved. This, among others, motivated the developments involving Theorem 4, which may be viewed as an attempt of transposing the metrical concepts of Theorem 2 in a normed setting. Their usefulness is to be judged from

Proposition 3. *Under the precised conventions,*

(4.1) *(OEVP) is deductible from (ODT).*

Hence (cf. Proposition 2) the order Ekeland variational principle and the order drop theorem are deductible from each other.

Proof. Let the Banach space $(V, ||.||)$, the self-closed preorder $(\leq)$ over it, the (nonempty) part N of V and the function $g : V \to R \cup \{-\infty\}$ be such that conditions (2.5) + (2.6) are fulfilled (for some $\rho > 0$). Without loss of generality, one may assume

(4.2) $$\gamma := \sup[g(N)] > 0 \quad (\text{hence } 0 < \gamma < \rho).$$

Fix also $b_0 \in N$ with $g(b_0) = 0$; and let the numbers ω, λ be introduced as

$$(4.3) \qquad 3 < \omega < 4 \quad (\text{hence } \gamma + \rho < (\omega - 1)\rho); \lambda := \frac{1}{\omega - 1}.$$

Take the product structure $X = V \times R$, endowed with the λ-maximum norm

$$||(x, \xi)|| = \max(\lambda||x||, |\xi|), \quad x \in V, \xi \in R$$

and the "product" preorder

$$(x, \xi) \leq (y, \eta) \quad \text{if and only if } x \leq y, \xi \leq \eta.$$

Clearly, $(X, ||.||)$ is a Banach space; and $(\leq)$ is self-closed over it (by the admitted hypotheses). Define further the subsets (of X)

$$(4.4) \qquad A = \{(x, \xi) \in N \times R_+; g(x) \geq \xi\}, \quad C = X[(b_0, \omega\rho), \rho].$$

These fulfill the regularity condition (3.1). Moreover, in view of

$$(4.5) \quad (y, \eta) \in C \Rightarrow |\eta - \omega\rho| \leq \rho \Rightarrow \gamma + \rho < (\omega - 1)\rho \leq \eta \leq (\omega + 1)\rho$$

one gets, for each $(x, \xi) \in A, (y, \eta) \in C$, the relation

$$||(x, \xi) - (y, \eta)|| \geq |\xi - \eta| = \eta - \xi \geq (\omega - 2)\rho > \rho;$$

wherefrom $\delta(A, C) \geq \rho$ (i.e., condition (1.2) holds too). Summing up, Theorem 5 is applicable to these data. It gives us, for the starting point $a_0 = (b_0, 0)$ in A, some point $a = (b, \beta)$ in A with the properties (3.2) + (3.3). We claim that $b \in N$ is our desired element. This may be shown under the lines below.

(i) By (3.2), there must be some ν in $[0, 1]$ and some (y, η) in C with $b - b_0 = \nu(y - b_0)$, $\beta = \nu\eta$. The choice of C now gives

$$\lambda||y - b_0|| \leq \rho \text{ (hence } \lambda||b - b_0|| = \lambda\nu||y - b_0|| \leq \nu\rho \leq \rho);$$

which shows that $(b, \omega\rho)$ also belongs to C. On the other hand, the same fact yields, via (4.5) (and the choice of λ)

$$(4.6) \qquad ||b - b_0|| = \frac{\beta}{\eta}||y - b_0|| \leq \frac{\beta(\omega - 1)\rho}{(\omega - 1)\rho} = \beta.$$

Finally, $\beta \leq g(b) \leq \gamma < \omega\rho$ (by the definition of A); wherefrom

$$(b, g(b)) = \mu(b, \beta) + (1 - \mu)(b, \omega\rho), \quad \text{for some } \mu \in [0, 1].$$

This, along with $(b, \beta) \le (b, g(b))$ yields, via (3.3)

$$(b, g(b)) \in A((b, \beta), \le) \cap D((b, \beta), C) = \{(b, \beta)\}.$$

Hence, necessarily, $g(b) = \beta$; and this gives (via (3.2) + (4.6))

$$b \ge b_0, ||b - b_0|| \le g(b) \ (= g(b) - g(b_0)); \text{ which is just (2.7).}$$

(ii) Suppose by contradiction that there would be some $z \in N(b, \le)$ distinct from b, such that (2.8) be false:

$$0 < ||z - b|| \le g(z) - g(b) \ (= g(z) - \beta). \tag{4.7}$$

Put for simplicity $\zeta = ||z - b|| + \beta$; note that

$$\beta < \zeta \le g(z) \text{ [hence } (z, \zeta) \in A((b, \beta), \le)]. \tag{4.8}$$

We claim that there may be found some number ϵ in $]0, 1[$ with the properties

$$\lambda||\frac{1}{\epsilon}(z - b) + b - b_0|| \le \rho, \quad |\frac{1}{\epsilon}(\zeta - \beta) + \beta - \omega\rho| \le \rho. \tag{4.9}$$

To verify this, note that the underlying relations are fulfilled as soon as

$$(1/\epsilon)(\zeta - \beta) + \beta = (\omega - 1)\rho; 0 < \epsilon < 1.$$

(The assertion follows at once from the definition of λ and ζ; we do not give details). This now forces us taking

$$\epsilon = (\zeta - \beta)/[(\omega - 1)\rho - \beta] \text{ (= the unique solution of that equation).}$$

That ϵ belongs to $]0, 1[$ follows directly via (4.7) and (4.8):

$$\beta < \zeta, \beta = g(b) \le \gamma < (\omega - 1)\rho \ \Rightarrow \epsilon > 0$$

$$\zeta \le g(z) \le \gamma < (\omega - 1)\rho \Rightarrow \epsilon < 1.$$

Moreover, ϵ fulfills (4.9) as above said; and this proves our claim. But then, the point $(y = \frac{1}{\epsilon}(z - b) + b, \ \eta = \frac{1}{\epsilon}(\zeta - \beta) + \beta)$ belongs to C. And, from $z - b = \epsilon(y - b)$, $\zeta - \beta = \epsilon(\eta - \beta)$, one derives, via (3.3) + (4.8)

$$(z, \zeta) \in A((b, \beta), \le) \cap D((b, \beta), C) = \{(b, \beta)\} \text{(hence } z = b, \zeta = \beta);$$

in contradiction with (4.7). Hence, this working assumption is unacceptable; and conclusion (2.8) follows. □

In particular, when $(\le)$ is the trivial preorder on M, this result is just the one in Daneš [8] (see also Penot [12]). But, we must underline that our techniques are quite distinct from the ones of these authors. Further aspects may be found in Georgiev [10] and Qiu [14].

5. Amended OEVP.

Let us now return to the framework of Section 2. The following local type result, referred to as the *amended* order Ekeland variational principle [in short: amended (OEVP)] may be stated.

Theorem 7. *Let preordered metric space $(M, d; \leq)$ be such that (M, d) is $(\leq)$-complete and $(\leq)$ is self-closed. Further, let the function $f : M \to R \cup \{\infty\}$ be such that (1.5) is valid, as well as (2.1). Then, for each $a_0 \in$ Dom(f) and $\epsilon > 0$ there exist $a = a(a_0, \epsilon) \in$ Dom(f) and $\lambda = \lambda(a_0, \epsilon)$ in $]0, 1[$ with*

$$(5.1) \qquad a_0 \leq a, d(a_0, a) \leq f(a_0) - f(a) + \epsilon$$

$$(5.2) \qquad d(a, x) - (f(a) - f(x)) > \lambda d(a, x), \quad \forall x \in M(a, \leq), x \neq a.$$

Hence, in particular, we have the conclusion

$$(5.3) \quad (x_n) \subseteq M(a, \leq), \; d(a, x_n) - (f(a) - f(x_n)) \to 0 \text{ imply } x_n \to a.$$

Proof. Let the number $\lambda = \lambda(a_0, \epsilon)$ be taken in accordance with

$$(5.4) \quad 0 < \lambda < 1 \text{ and } \frac{\lambda}{1-\lambda}(f(a_0) - f_*) < \epsilon, \text{ where } f_* = \inf[f(M)].$$

Further, let the metric $(x, y) \mapsto e(x, y)$ over M be introduced as

$$e(x, y) = (1 - \lambda)d(x, y), \; x, y \in M \quad \text{(in short: } e = (1 - \lambda)d).$$

By the admitted hypotheses, Theorem 3 is applicable to the preordered metric space $(M, e; \leq)$ and the same function f. Hence, for the starting point $a_0 \in$ Dom(f) there exists $a \in$ Dom(f) such that (2.2) + (2.3) be valid (with e in place of d). The latter of these is just (5.2). And, by the former one, we derive (cf. (5.4))

$$d(a_0, a) \leq \frac{1}{1-\lambda}(f(a_0) - f(a)) = f(a_0) - f(a) + \frac{\lambda}{1-\lambda}(f(a_0) - f(a))$$
$$\leq f(a_0) - f(a) + \frac{\lambda}{1-\lambda}(f(a_0) - f_*) < f(a_0) - f(a) + \epsilon;$$

wherefrom, (5.1) holds too. □

In particular, when $(\leq)$ is the trivial preorder on M, this result is nothing but the amended Ekeland variational principle (in short: amended (EVP)) due to Georgiev [10]. Now, the quoted result was established by means of the trivial preorder version of Theorem 6. So, it is expectable that such a device could also work in our preorder setting. As we shall see, a positive answer to this is available. For technical reasons it would be useful starting with the normed setting (of Theorem 4). Precisely, we have

Theorem 8. *Let the Banach space $(V, ||.||)$ be given, as well as the self-closed preorder $(\leq)$ over it. Further, let the (nonempty) part N of V and the function $g : V \to R \cup \{-\infty\}$ be such that* (2.5) + (2.6) *hold (for some $\rho > 0$). Then, for each $b_0 \in N$ with $g(b_0) = 0$ and each $\epsilon > 0$ there exist $b = b(b_0, \epsilon) \in N$ and $\tau = \tau(b_0, \epsilon)$ in $]0, 1[$ such that*

(5.5) $$b \geq b_0, ||b - b_0|| \leq g(b) - g(b_0) + \epsilon \ (= g(b) + \epsilon)$$

(5.6) $$\tau||x - b|| \leq g(b) + ||x - b|| - g(x), \quad \textit{for all } x \in N.$$

Hence, in particular, we have the conclusion

(5.7) $$(x_n) \subseteq N(b, \leq), g(b) + ||x_n - b|| - g(x_n) \to 0 \textit{ imply } x_n \to b.$$

Proof. Without loss of generality, one may assume that (4.2) is true. Fix a certain ω in $]3, 4[$; and introduce the functions

$$\varphi(t) = \frac{t}{\omega - t}, 0 < t < \omega; \quad \psi(t) = \frac{\varphi(1 + t)}{\varphi(1)} - 1, 0 < t < \omega - 1.$$

Further, let us consider the Cartesian product $X = V \times R$ endowed with the $\varphi(1)$-maximum norm and the "product" preorder of Proposition 3. Finally, let the subsets A, C of X be taken as in (4.4); note that (with the notations of Section 3) $C_\theta = X[(b_0, \omega\rho), (1 + \theta)\rho]$, for all $\theta \geq 0$. By the remarks in the quoted statement, conditions of Theorem 6 are holding for these data. So, given the starting point $a_0 = (b_0, 0)$ in A and μ in $]0, 1/4[$ (arbitrary for the moment) there exists $a(\mu) = (b, \beta)$ in A fulfilling (3.9) + (3.10) as well as: for each ν in $]0, \mu[$, the evaluation (3.11) is true. We now claim that b is our desired element (for a suitable choice of (μ, ν)). Th is will be shown in several steps.

(j) By (4.3) and the preliminary choice of μ,

$$\omega - \frac{5}{2} > 2\mu \quad \left(\text{hence } \omega - 1 - 2\mu > \frac{3}{2}\right).$$

This, by the Lagrange mean value theorem, yields (for all such μ)

$$\psi(\mu) \leq \frac{\mu\varphi'(1+\mu)}{\varphi(1)} = \frac{\omega(\omega-1)\mu}{(\omega-1-\mu)^2} \leq \frac{\omega\mu}{\omega-1-2\mu} < \frac{2}{3}\omega\mu. \tag{5.8}$$

(jj) By (3.9), there must be some θ in $[0,1[$ and some (y,η) in C_μ with $b - b_0 = \theta(y - b_0)$, $\beta = \theta\eta$. The representation above for C_μ imposes to (y,η) a couple of conditions like

$$\varphi(1)||y - b_0|| \leq (1+\mu)\rho, \quad |\eta - \omega\rho| \leq (1+\mu)\rho$$

$$(\text{hence} \quad (\omega - 1 - \mu)\rho \leq \eta \leq (\omega + 1 + \mu)\rho).$$

This, and (5.8), yields (via $\beta \leq g(b) < \rho$)

$$||b - b_0|| \leq \frac{\beta}{\eta}||y - b_0|| \leq \beta(1 + \psi(\mu)) < \beta + \frac{2}{3}\rho\omega\mu. \tag{5.9}$$

So, in order that (5.5) be true, we have to impose a condition like

$$\frac{2}{3}\rho\omega\mu < \epsilon \quad \left(\text{hence } 0 < \mu < \min\left\{\frac{1}{4}, \frac{3\epsilon}{2\rho\omega}\right\}\right). \tag{5.10}$$

(jjj) It remains now to discuss (5.6). Let the pair (μ, ν) be taken in accordance with (5.10) and

$$\mu < \omega - 3; \quad \frac{2}{3} < \frac{\nu}{\mu} < 1. \tag{5.11}$$

We show that, under these requirements, one has an evaluation like

$$(x, \beta + ||x - b||) \in D((b,\beta), C_\nu), \quad \text{for all } x \in N(b, \leq). \tag{5.12}$$

In fact, take some x in $N(b, \leq)$; without loss of generality one may assume that $x \neq b$. The desired conclusion is clearly obtainable from

$$(x, \beta + ||x - b||) = (b, \beta) + \theta(y - b, \eta - \beta),$$

$$\text{for some } \theta \text{ in }]0,1[\text{ and } (y,\eta) \text{ in } C_\nu.$$

This also reads (in an evident way)

$$\left(y = \frac{1}{\theta}(x - b) + b, \eta = \frac{1}{\theta}||x - b|| + \beta\right) \text{ is in } C_\nu; \quad \text{or, equivalently}$$

$$\varphi(1)\left\|\frac{1}{\theta}(x-b)+b-b_0\right\| \le (1+\nu)\rho, \tag{5.13}$$
$$\left|\frac{1}{\theta}\|x-b\|+\beta-\omega\rho\right| \le (1+\nu)\rho.$$

A reasonable way of satisfying the second requirement above is by solving

$$\frac{1}{\theta}\|x-b\|+\beta-\omega\rho = -(1+\nu)\rho;\ \text{hence } \theta = \frac{\|x-b\|}{(\omega-1-\nu)\rho-\beta}. \tag{5.14}$$

That $\theta > 0$ is clear, by the choice of ω (and $\beta < \rho$). We also have $\theta < 1$; for (in view of (2.5) and (5.11))

$$\|x-b\| \le \rho < (\omega-1-\mu)\rho-\beta < (\omega-1-\nu)\rho-\beta.$$

This value of θ is also appropriate for the first half of (5.13). Indeed, a sufficient condition for the underlying relation to hold is

$$\varphi(1)\left(\frac{1}{\theta}\|x-b\|+\|b-b_0\|\right) \le (1+\nu)\rho.$$

This (cf. (5.9) and (5.14)) is fulfilled as soon as

$$\varphi(1)[(\omega-1-\nu)\rho+\beta\psi(\mu)] \le (1+\nu)\rho;\ \text{or, equivalently, } \beta\psi(\mu) \le \omega\rho\nu.$$

But, the written condition is verified, by (5.8) and the choice (5.11) of (μ,ν); hence (5.12) is retainable. Note that, as an immediate consequence of this (and (3.10))

$$(x,\beta+\|x-b\|) \notin A((b,\beta),\le),\quad \text{for each } x \in N(b,\le),\ x \ne b. \tag{5.15}$$

(jv) Let $x \in N(b,\le)$ be arbitrary fixed. By (5.12), it follows that evaluation (3.11) is applicable to $(x,\beta+\|x-b\|)$ (and the couple (μ,ν) above). To give it an appropriate form, we start by noting that (cf. (2.5))

$$\alpha(\mu,\nu) = \Delta((b,\beta),C_\nu) \le \|(b,\beta)-(b_0,\omega\rho)\| + (1+\nu)\rho \le$$
$$\max\{\lambda\|b-b_0\|, |\beta-\omega\rho|\} + (1+\nu)\rho \le (\omega+1+\nu)\rho.$$

On the other hand, the distance in the left member of (3.11) is evaluated as (from $\lambda < 1$)

$$\|(x,\beta+\|x-b\|)-(b,\beta)\| = \max(\lambda\|x-b\|, \|x-b\|) = \|x-b\|.$$

For the (point to set) distance in the right member of the same, it will suffice considering the alternatives

$$(5.16)\quad g(x) \geq \beta [\text{hence } (x, g(x)) \in A((b, \beta), \leq)];\ \ g(x) < \beta (= g(b)).$$

In fact, the former of these yields (cf. (5.15))

$$d[(x, \beta + ||x - b||), A((b, \beta), \leq)] \leq ||(x, \beta + ||x - b||) - (x, g(x))|| =$$
$$= \beta + ||x - b|| - g(x) \leq g(b) + ||x - b|| - g(x).$$

This, added to the above, allows us deducing [via (3.11)]

$$(\mu - \nu)\rho ||x - b|| \leq (\omega + 1 + \nu)\rho (g(b) + ||x - b|| - g(x));$$

wherefrom (5.6) is clear, with $\tau = (\mu - \nu)/(\omega + 1 + \nu)$. The latter alternative of (5.16) gives us the same conclusion (in a trivial way). This ends the argument. □

Now, by simply remembering the way of translating Theorem 3 in terms of Theorem 4, one re-obtains Theorem 7 (in this geometric way). Note that, when $(\leq)$ is the trivial preorder in V, Theorem 8 reduces to the statement in Georgiev [10] proved via slightly different methods. Some interesting applications of such results to the generic well-posedness in the sense of Revalski [14] are available. These will be discussed in a separate paper.

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