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An Order Theoretical Characterization of the Fourier Transform

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Dedicated to Prof. H.H. Schaefer on the occasion of his 60^{th} birthday

Most spaces of functions or measures on a locally compact group G carry two different orderings: a *pointwise ordering* and a *positive-definite ordering*. For example, on $L^1(G)$ the pointwise ordering is defined by the cone $L^1(G)_+ = \{f \in L^1(G) : f(t) \geqq 0$ for almost all $t \in G\}$ and a positive-definite ordering by the cone $L^1(G)_p = \overline{\text{co}}\{f^* * f : f \in L^1(G)\}$. In [1] and [2] we investigated these two orderings for the Fourier algebra $A(G)$, the Fourier-Stieltjes algebra $B(G)$ and for $L^1(G)$. The principal result was that each of these biordered spaces determines the group up to isomorphism.

In the present article we take up an idea of H.H. Schaefer's and discuss order properties of the Fourier transform $\mathscr{F}$. If G is abelian, then $\mathscr{F}$ is a biorder antiisomorphism from $L^1(G)$ onto $A(\hat{G})$ (where $\hat{G}$ denotes the dual group of G), i.e. $\mathscr{F}$ maps positive-definite functions onto pointwise positive functions and vice versa. So the following question arises: Given an arbitrary locally compact group G, can it happen that one finds a locally compact group $\hat{G}$ and a biorder antiisomorphism from $L^1(G)$ onto $A(\hat{G})$? By the results mentioned above, the group $\hat{G}$ would be uniquely determined by G, and one could consider $\hat{G}$ as the dual group of G. So the following result (Theorem 3.3) is not surprising. If such an order antiisomorphism F exists, then the group G is abelian, $\hat{G}$ is (up to an isomorphism) the dual group of G and F is the Fourier transform (up to a positive multiplicative constant).

Of course, one expects that a similar result holds for the Fourier-Stieltjes transform. And in fact, this will be proved in Sect. 6. But on the way, we (have to) show that $M(G)$ too carries the structure of a biordered space which is a complete isomorphism invariant of G (Sect. 4). This had been left open in [2] and demands most of the effort in this paper. The proof is given by reduction to the L^1-case. However, we observe that on $M(G)$ (and on $L^1(G)$) there are two different natural positive-definite orderings: one defined by the ordering inherited by the enveloping C^*-algebra (that is, by the cone $M(G)_p = \overline{\text{co}}\{\mu^* * \mu : \mu \in M(G)\}$), and one inherited by the left regular representation λ in $\mathscr{L}(L^2(G))$ (that is, by the cone $M(G)_{\lambda p} := \{\mu \in M(G) : \lambda(\mu) \in \mathscr{L}(L^2(G))$ is a positive-definite operator$\}$). We show that both of these positive-definite orderings lead to the desired result: the biordered spaces $(M(G), M(G)_+, M(G)_p)$ and $(M(G), M(G)_+, M(G)_{\lambda p})$ are both *complete* isomorphism invariants. For the first case, the corresponding result for $L^1(G)$ proved in [2] can be used, for the second we have to reconsider $L^1(G)$ with the positive-definite ordering defined by the left regular representation. This is done in Sect. 2.

1. Preliminaries

A. Ordered Vector Spaces

As a general reference we use [11, V] and [12]. Let E be a *real* or *complex* vector space. A subset C of E is called a *cone* if $C+C\subset C$ and $\mathbb{R}_+\cdot C\subset C$, and C is called a *proper cone* if in addition $C\cap(-C)=\{0\}$. A pair (E, C), where C is a proper cone in E, is called *ordered vector space*. On such a space an *ordering* is defined by $x\leqq y$ if and only if $y-x\in C$. An ordered vector space (E, C) is called a *vector lattice* if for all $x, y\in C-C$ there exists a least upper bound (then $C-C$ is a *real* vector lattice, see [11, II, Sect. 1]).

If E is a Banach space, we denote the dual space by E'. The dual cone C' of a closed proper cone C in E is defined by

$$C'=\{f\in E':\langle x,f\rangle\geqq 0 \quad \text{for all} \quad x\in C\}. \tag{1.1}$$

Suppose now that E is a *real* Banach space and C a closed cone in E. The following results are consequences of the Hahn-Banach theorem:

$$x\in C \quad \text{if and only if} \quad \langle x,f\rangle\geqq 0 \quad \text{for all} \quad f\in C'. \tag{1.2}$$

If C_1, C_2 are two closed cones in E, then

$$\begin{aligned} C_1\subset C_2 \quad &\text{if and only if} \quad C_2'\subset C_1'. \text{ In particular,}\\ C_1=C_2 \quad &\text{if and only if} \quad C_2'=C_1'. \end{aligned} \tag{1.3}$$

Let D be a cone in E'. Let $C=\{x\in E:\langle x,f\rangle\geqq 0 \text{ for all } f\in D\}$ be the *predual cone* of D. Then

$$C'=\bar{D}^{\sigma(E',E)}. \tag{1.4}$$

Let (E_1, C_1) and (E_2, C_2) be (real or complex) ordered vector spaces. A linear mapping $T: E_1\to E_2$ is called *positive* if $TC_1\subset C_2$. T is called *order isomorphism* if T is bijective and $TC_1=C_2$.

B. Involutive Banach Algebras and C-Algebras*

(See [5] as a general reference.) An involutive complex Banach algebra is defined according to [5, 1.2.1]. In particular, the involution is assumed to be isometric. $A_h:=\{x\in A: x=x^*\}$ is a real Banach space with dual $(A_h)'=(A')_h:=\{f\in A': \langle x^*,f\rangle=\langle x,f\rangle\ (x\in A)\}$. There is a canonical ordering on A defined by the cone $A_p:=\overline{\text{co}}\{x^*x: x\in A\}$ ($\overline{\text{co}}$ stands for closed convex hull). Then $A_p\subset A_h$, and if A has an approximate identity [5, B 29], then $A_p'\subset(A')_h$ [5, 2.1.5i]. The positive cone in a C^*-algebra $\mathfrak{A}$ is always denoted by $\mathfrak{A}_p$ (unfortunately, an unusual notation is necessary, since we frequently consider *two* different orderings); i.e.,

$$\mathfrak{A}_p=\overline{\text{co}}\{x^*x|x\in\mathfrak{A}\}=\{x\in\mathfrak{A}: x=x^*, \sigma(x)\subset\mathbb{R}_+\}. \tag{1.5}$$

The enveloping C^*-algebra of an involutive Banach algebra with approximate identity A is denoted by $C^*(A)$ [5, 2.7]. A_p is given by

$$A_p = A \cap C^*(A)_p . \tag{1.6}$$

[Clearly, $A \cap C^*(A)_p$ is a closed cone in A_h and $A_p \subset A \cap C^*(A)_p$. Let $\varphi \in A'_p$. Then it follows from [5, 2.7.5i] that $\varphi(A \cap C^*(A)_p) \subset \mathbb{R}_+$. So $A \cap C^*(A)_p \subset A_p$ by (1.3).] We will frequently use the following result.

$$\begin{aligned}&\text{The bidual } \mathfrak{A}'' \text{ of a } C^*\text{-algebra } \mathfrak{A} \text{ is a unital } C^*\text{-algebra;}\\ &\text{via the evaluation map one can identify } \mathfrak{A} \text{ with a subalgebra of } \mathfrak{A}''.\end{aligned} \tag{1.7}$$

Moreover $\mathfrak{A}_p = (\mathfrak{A}'')_p \cap \mathfrak{A}$. If $\mathfrak{A}$ has a unit e, then e is also the unit of $\mathfrak{A}''$ [14, III, Sect. 2].

C. Harmonic Analysis

(General reference: [5] and [6].) G_1, G_2 *denote locally compact groups, throughout.* We define the following function spaces:

$$\begin{aligned} C(G) &= \{f : G \to \mathbb{C} : f \text{ is continuous}\} \\ C^b(G) &= \{f \in C(G) : f \text{ is bounded}\} \\ C_0(G) &= \{f \in C(G) : f \text{ vanishes at infinity}\} \\ C_c(G) &= \{f \in C(G) : f \text{ has compact support}\} . \end{aligned}$$

For every complex valued function f on G we let $\bar{f}(x) = \overline{f(x)}$, $\check{f}(x) = f(x^{-1})$, $\tilde{f}(x) = \overline{f(x^{-1})}$ $(x \in G)$. $M(G)$ denotes the space of all bounded regular complex Borel measures on G. We frequently identify $M(G)$ with the dual space of $C_0(G)$. The space $M(G)$ is a unital involutive Banach algebra for convolution as multiplication and the involution $* : \mu \to \mu^*$ given by $\langle f, \mu^* \rangle = \overline{\langle \tilde{f}, \mu \rangle}$ $(f \in C_0(G))$. $L^1(G)$ is defined via the left Haar measure as usual. One can identify $L^1(G)$ with a closed bilateral ideal of $M(G)$. The involution restricted to $L^1(G)$ is given by $f^* = \Delta^{-1} \cdot \tilde{f}$ $(f \in L^1(G))$, where Δ denotes the modular function of G. Since $L^1(G)$ contains an approximate identity [5, 13.2.5], one can form the enveloping C^*-algebra, which is denoted by $C^*(G)$.

The *pointwise ordering* in $M(G)$ is defined by the cone $M(G)_+$ of all positive measures in the usual sense. Thus $M(G)_+$ is the dual cone of $C_0(G)_+ := \{f \in C_0(G) : f(t) \geq 0$ for all $t \in G\}$. The induced cone $L^1(G)_+$ on $L^1(G)$ given by $L^1(G)_+ = M(G)_+ \cap L^1(G) = \{f \in L^1(G) : f(t) \geq 0$ for almost all $t \in G\}$ defines the pointwise ordering on $L^1(G)$. The spaces $M(G)$ and $L^1(G)$ are vector lattices for the pointwise ordering (in fact, they are complex Banach lattices [12, II, Sect. 11]).

A *positive-definite* ordering on $L^1(G)$ is defined by its involutive Banach algebra structure. It is given by the cone

$$L^1(G)_p = \overline{\text{co}} \{f^* * f : f \in L^1(G)\} . \tag{1.7}$$

Its dual cone in $L^\infty(G)$ is

$$L^1(G)'_p = P(G) := \{f \in C^b(G) : f \text{ is positive definite}\} . \tag{1.8}$$

$P(G)$ is a closed cone in $C^b(G)$. Its linear hull

$$B(G) := \operatorname{span} P(G) \tag{1.9}$$

is called the Fourier-Stieltjes algebra. $B(G)$ is the dual space of $C^*(G)$. With the dual norm of $C^*(G)$ it is a Banach algebra for pointwise multiplication. If G is abelian, then $B(G) = \{\hat{\mu} : \mu \in M(\hat{G})\}$, where $\hat{\mu}$ denotes the Fourier-Stieltjes transform of $\mu \in M(\hat{G})$ and $\hat{G}$ the dual group of G. The closure of $C_c(G) \cap B(G)$ in $B(G)$

$$A(G) := \overline{C_c(G) \cap B(G)} \tag{1.10}$$

is a closed ideal in $B(G)$ and called the *Fourier algebra*. If G is abelian, then $A(G) = \{\hat{f} : f \in L^1(\hat{G})\}$.

The pointwise ordering on $B(G)$ and $A(G)$ is defined by the cones

$$B(G)_+ = \{u \in B(G) : u(t) \geqq 0 \quad \text{for all} \quad t \in G\}, \tag{1.11}$$

$$A(G)_+ = A(G) \cap B(G)_+ . \tag{1.12}$$

Note that

$$\|u\|_\infty \leqq \|u\| \qquad (u \in B(G)) \tag{1.13}$$

[where $\| \ \|_\infty$ denotes the uniform norm on $C^b(G)$], so that both cones are closed.

The *positive-definite ordering* on $B(G)$ (resp., $A(G)$) is defined by the cone $P(G)$ (resp., $A(G) \cap P(G)$).

We will frequently call a vector space with a *pointwise and* a *positive-definite* cone a *biordered space*. Examples are: $(L^1(G), L^1(G)_+, L^1(G)_p)$, $(A(G), A(G)_+, A(G) \cap P(G))$, $(B(G), B(G)_+, P(G))$.

A *biorder isomorphism* is a bijective linear mapping between two such spaces which preserves the pointwise and positive-definite ordering, i.e. which maps the pointwise positive cone onto the pointwise positive cone and the positive-definite cone onto the positive-definite cone. A *biorder anti-isomorphism* is a bijective linear mapping between two such spaces which reverses these two orderings (i.e. which maps the pointwise positive onto the positive-definite cone and the positive-definite onto the pointwise positive cone).

The left-regular representation will be denoted by λ; that is, $\lambda : G \to \mathscr{L}(L^2(G))$ is given by $(\lambda_t f)(s) = f(t^{-1}s)$ $(f \in L^2(G))$. It can be lifted to a representation of $M(G)$ – which we still denote by λ – by means of

$$\lambda(\mu) f = \mu * f \qquad (f \in L^2(G)). \tag{1.14}$$

The dual space of $A(G)$ can be identified with $VN(G)$, the von Neumann algebra generated by $\{\lambda_t : t \in G\}$ by means of the duality

$$\langle u, \lambda_t \rangle = u(t) \qquad (u \in A(G), t \in G). \tag{1.15}$$

Another characterization of $A(G)$ is the following (see [6, p. 218]):

$$A(G) = \{f * \tilde{g} : f \in L^2(G), g \in L^2(G)\}. \tag{1.16}$$

The duality (1.15) between $A(G)$ and $VN(G)$ is then given by

$$\langle (f * \tilde{g})^\vee, T \rangle = (Tf | g) \qquad (f, g \in L^2(G)) \tag{1.17}$$

[6, (3.11)].

The dual cone of $A(G)\cap P(G)$ is

$$(A(G)\cap P(G))'=VN(G)_p(=\{T\in VN(G):(Tf|f)\geqq 0 \quad \text{for all} \quad f\in L^2(G)\}). \tag{1.18}$$

2. Orderisomorphisms of Group Algebras

It has been proven in [2] that the biordered space $(L^1(G), L^1(G)_+, L^1(G)_p)$ is a complete isomorphism invariant for G. Here the positive-definite ordering is defined by the cone $L^1(G)_p:=\overline{\text{co}}\{f^* * f: f\in L^1(G)\}$. However, there exists another interesting positive-definite ordering on $L^1(G)$, which is defined by the cone

$$L^1(G)_{\lambda p}:=\{f\in L^1(G):\lambda(f)\in\mathscr{L}(L^2(G))_p\}. \tag{2.1}$$

Here $\lambda(f)$ is the operator on $L^2(G)$ given by $g\to f*g$ $(g\in L^2(G))$. Recall: $\mathscr{L}(L^2(G))_p=\{T\in\mathscr{L}(L^2(G)):(Tf|f)\geqq 0$ for all $f\in L^2(G)\}$. The purpose of this section is to prove that in the above statement, we can replace the cone $L^1(G)_p$ by $L^1(G)_{\lambda p}$; i.e. we will show that the biordered space $(L^1(G), L^1(G)_+, L^1(G)_{\lambda p})$ is also a complete invariant. The precise statement is the following.

Theorem 2.1. *Let $T: L^1(G_1)\to L^1(G_2)$ be a bijective, linear operator such that*

$$TL^1(G_1)_+=L^1(G_2)_+ \quad \text{and} \quad TL^1(G_1)_{\lambda p}=L^1(G_2)_{\lambda p}.$$

Then there exist a topological group isomorphism or anti-isomorphism $\beta: G_2\to G_1$ and a constant $d>0$ such that

$$(Tf)(t)=d\cdot f(\beta(t))\,(t\in G_2, f\in L^1(G_1)) \tag{2.2}$$

in the case that β is an isomorphism and

$$(Tf)(t)=d\cdot\Delta(t)^{-1}f(\beta(t))\,(t\in G_2, f\in L^1(G_1)) \tag{2.3}$$

if β is an anti-isomorphism, where Δ denotes the modular function of G_2.

We will now describe the cone $L^1(G)_{\lambda p}$ in more detail. If we denote by $C^*_\lambda(G):=\{\lambda(f): f\in L^1(G)\}^-\subset\mathscr{L}(L^2(G))$ the reduced enveloping C^*-algebra of $L^1(G)$, then we can identify $L^1(G)$ with a subalgebra of $C^*_\lambda(G)$. So we have

$$L^1(G)_{\lambda p}=C^*_\lambda(G)_p\cap L^1(G). \tag{2.4}$$

The dual space of $C^*_\lambda(G)$ can be identified with $B_\lambda(G)$, the space of all coefficient functions of those unitary representations of G which are weakly contained in the left-regular representation. $B_\lambda(G)$ is a closed ideal in $B(G)$ which contains $A(G)$ (see [6, 2.16]). The dual cone of $C^*_\lambda(G)_p$ in $B_\lambda(G)$ is

$$C^*_\lambda(G)'_p=P(G)\cap B_\lambda(G)=:P_\lambda(G) \tag{2.5}$$

(see [6, 2.6]). As a consequence of (2.5), the dual cone of $L^1(G)_{\lambda p}$ in $L^\infty(G)$ is

$$L^1(G)'_{\lambda p}=P_\lambda(G). \tag{2.6}$$

Proof of (2.6). Consider the real Banach spaces $C^*_\lambda(G)_h=\{x\in C^*_\lambda(G): x=x^*\}$, $L^1(G)_h=L^1(G)\cap C^*_\lambda(G)_h$, $L^\infty(G)_h=\{g\in L^\infty(G): g=\tilde{g}\}$ and $B_\lambda(G)_h:=B_\lambda(G)\cap L^\infty(G)_h$. Then $L^1(G)'_h=L^\infty(G)_h$ and $C^*_\lambda(G)'_h=B_\lambda(G)_h$. Moreover, $L^1(G)_{\lambda p}\subset L^1(G)_h$ and $L^1(G)'_{\lambda p}[\subset L^1(G)'_p=P(G)]\subset L^\infty(G)_h$. It follows from (2.5) and (1.2) that $C^*_\lambda(G)_p$

$=\{x\in C^*_\lambda(G):\langle x,u\rangle\geqq 0$ for all $u\in P_\lambda(G)\}$. Thus $L^1(G)_{\lambda p}=C^*_\lambda(G)_p\cap L^1(G)$ is the predual cone of $P_\lambda(G)$. It follows from (1.4) that $L^1(G)'_{\lambda p}$ is the $\sigma(L^\infty(G), L^1(G))$-closure of $P_\lambda(G)$. But $P_\lambda(G)$ is $\sigma(L^\infty(G)_h, L^1(G)_h)$-closed by [5, 18.3.5]. □

Since G is amenable if and only if $P_\lambda(G)=P(G)$ ([7, p. 61], see also [5, 18.3.6]), we obtain as a consequence of (2.6):

$$G \text{ is amenable if and only if } L^1(G)_{\lambda p}=L^1(G)_p. \tag{2.7}$$

It follows from [5, 13.4.4 and 13.7.4] that for $\varphi\in L^1(G)\cap C(G)$

$$\varphi\in L^1(G)_{\lambda p} \text{ if and only if } \Delta^{1/2}\varphi\in P(G). \tag{2.8}$$

We will need the following properties of $P(G)$.

$$\text{The topology of compact convergence coincides with } \sigma(L^\infty(G), L^1(G)) \text{ on uniformly bounded subsets of } P(G) \text{ [5, 13.5.2]}. \tag{2.9}$$

Moreover, $P_\lambda(G)$ can be described as follows. Let $u\in P(G)$. Then

$$u\in P_\lambda(G) \text{ if and only if there exists a net } (k_i)\subset C_c(G) \text{ such that } u=c-\lim k_i * \tilde{k}_i, \tag{2.10}$$

where $c-\lim$ is the limit for the topology of compact convergence ([5, 18.3.5] or [6, 1.25]). In particular,

$$\text{For } u\in P_\lambda(G), \text{ there exists a net } (u_i) \text{ in } A(G)\cap P_\lambda(G) \text{ such that } \sup_i \|u_i\|_\infty=\sup_i u_i(e)<\infty \text{ and } c-\lim_i u_i=\sigma(L^\infty(G), L^1(G))-\lim_i u_i=u. \tag{2.11}$$

[In fact, let $u_i=k_i*\tilde{k}_i$. Then $u_i\in A(G)\cap P_\lambda(G)$ and $u=c-\lim_i u_i$. In particular, $u(e)=\lim u_i(e)$. So there exists i_0 such that $\sup_{i\geqq i_0}\|u_i\|_\infty=\sup_{i\geqq i_0} u_i(e)<\infty$. Thus the net $(u_i)_{i\geqq i_0}$ satisfies (2.11).]

For the proof of Theorem 2.1 we need to characterize the evaluation functionals $\delta_t\in B_\lambda(G)'$ ($t\in G$) defined by $\langle u,\delta_t\rangle=u(t)$ ($u\in B_\lambda(G)$). The following definitions will be convenient.

Definition 2.2. Let $\varphi\in B_\lambda(G)'$. We call φ *p-continuous* if for every net (u_i) in $P_\lambda(G)$ such that $\sup_i\|u_i\|_\infty<\infty$ and $u\in P_\lambda(G)$ $\sigma(L^\infty(G), L^1(G))-\lim_i u_i=u$ implies that $\lim_i\langle u_i,\varphi\rangle=\langle u,\varphi\rangle$.

Since $\operatorname{span}P_\lambda(G)=B_\lambda(G)$ [6, (2.6)], it follows from (2.11) that

$$\text{if } \varphi\in B_\lambda(G)' \text{ is } p\text{-continuous, then } \varphi|_{A(G)}=0 \text{ implies } \varphi=0. \tag{2.12}$$

Definition 2.3. Let $\varphi\in B_\lambda(G)'_+$. Then φ is called a *p-atom* if φ is p-continuous and for every p-continuous $\psi\in B_\lambda(G)'_+$, $\psi\leqq\varphi$ implies that $\psi=c\cdot\varphi$ for some constant $c\geqq 0$.

Here we let $B_\lambda(G)_+=B(G)_+\cap B_\lambda(G)$. Moreover, $\psi\leqq\varphi$ means that $\varphi-\psi\in B_\lambda(G)'_+$.

Lemma 2.4. *Let $\varphi \in B_\lambda(G)'_+$. Then φ is a p-atom if and only if $\varphi = c \cdot \delta_t$ for some $t \in G$ and some constant $c \geqq 0$.*

Proof. Let $\varphi = \delta_t$. It follows from (2.9) that φ is p-continuous. Let $\psi \in B_\lambda(G)'_+$ be p-continuous such that $\psi \leqq \varphi$. Consider $\psi_0 = \psi|_{A(G)}$. Then it follows from [1, 4.1] that $\operatorname{supp}\psi_0 \subset \{t\}$ (see [6, (4.5)] for the definition of the support). By [6, (4.9)], this implies that $\psi_0 = c \cdot \delta_t|_{A(G)}$. Hence by (2.12), $\psi = c \cdot \delta_t$. We have shown that δ_t is a p-atom. Conversely, assume that $\varphi \in B_\lambda(G)'_+$ is a p-atom. We show that $\operatorname{supp}\varphi_0$ contains at most one point (where $\varphi_0 = \varphi|_{A(G)}$). Indeed, if this is not the case, then there exist $t_1, t_2 \in \operatorname{supp}\varphi_0$ such that $t_1 \neq t_2$. Let U be an open neighborhood of t_2 such that $t_1 \notin \bar{U}$. There exists $u \in A(G)_+$, which is a linear combination of functions in $P_\lambda(G)$, such that $u(t_1) = 1$, $u(t) = 0$ for all $t \in U$ and $u(t) \leqq 1$ for all $t \in G$ [6, (3.2)]. Consider $\psi \in B_\lambda(G)'$ defined by $\langle v, \psi \rangle = \langle u \cdot v, \psi \rangle$ $(v \in B_\lambda(G))$. Since $u \in \operatorname{span} P_\lambda(G)$, it follows that ψ is p-continuous [use (2.9)]. Moreover, $0 \leqq \psi \leqq \varphi$. Since φ is a p-atom, there exists $c \geqq 0$ such that $\varphi = c \cdot \psi$. Since $t_2 \in \operatorname{supp}\varphi_0$, there exists $u_2 \in A(G)$ such that $u_2(t) = 0$ for all $t \notin U$ and $\langle u_2, \varphi \rangle \neq 0$ [6, (4.4)]. Then $\langle u_2, \psi \rangle = \langle u \cdot u_2, \varphi \rangle = \langle 0, \varphi \rangle = 0$. Consequently, $c = 0$. But since $t_1 \in \operatorname{supp}\varphi_0$ and $u(t_1) = 1$, it follows that $\psi \neq 0$ [6, (4.4) (ii)], which is absurd since $\psi = c \cdot \varphi = 0$.

We have proved that $\operatorname{supp}\varphi_0$ contains at most one point. This implies that $\varphi_0 = c \cdot \delta_t$ for some $c \geqq 0$, $t \in G$ by [6, (4.9) and (4.6)]. From (2.12) it follows, that $\varphi = c \cdot \delta_t$. □

Proof of Theorem 2.1. Let $T: L^1(G_1) \to L^1(G_2)$ be bijective and linear such that $TL^1(G_1)_+ = L^1(G_2)_+$ and $TL^1(G_1)_{\lambda p} = L^1(G_2)_{\lambda p}$. Then T is continuous [12, II, 5.3]. It follows from (2.6) that $T'P_\lambda(G_2) = P_\lambda(G_1)$, hence $T'B_\lambda(G_2) = B_\lambda(G_1)$. Let $S: B_\lambda(G_2) \to B_\lambda(G_1)$ be the restriction of T'. Then S is continuous for the uniform topology and satisfies $SP_\lambda(G_2) = P_\lambda(G_1)$. This implies that S' maps p-continuous functionals onto p-continuous functionals. Moreover, since $TL^1(G_1)_+ = L^1(G_2)_+$, it follows that $SB_\lambda(G_2)_+ = B_\lambda(G_1)_+$ and consequently, S' is an order isomorphism for the pointwise ordering (i.e. $S'B_\lambda(G_1)'_+ = B_\lambda(G_2)'_+$). It follows that S' maps p-atoms onto p-atoms. Consequently, by Lemma 2.4, for every $t \in G_1$ there exist $\alpha(t) \in G_2$ and $c(t) > 0$ such that $S'\delta_t = c(t)\delta_{\alpha(t)}$. Thus $S: B_\lambda(G_2) \to B_\lambda(G_1)$ is given by $(Su)(t) = c(t)u(\alpha(t))$ for all $t \in G_1$, $u \in B_\lambda(G_2)$. We show that α is continuous. If this is not the case, there exist $t_0 \in G_1$, a neighborhood U of $\alpha(t_0)$ and a net (t_i) in G_1 converging to t_0 such that $\alpha(t_i) \notin U$ for all i. Choose $u \in B_\lambda(G_2)$ such that $u(\alpha(t_0)) = 1$ and $u(\alpha(t_i)) = 0$ for all i [6, (3.2)]. Then $(Su)(t_i) = 0$ for all i, but $(Su)(t_0) = 1$. This is a contradiction, since Su is continuous. By the same arguments, S^{-1} is given by $(S^{-1}v)(s) = k(s)v(\beta(s))$ $(v \in B_\lambda(G),\ s \in G_2)$ for some $k: G_2 \to (0, \infty)$ and some continuous function $\beta: G_2 \to G_1$. One sees easily that β is the inverse of α, so that α is actually a homeomorphism. This implies that functions with compact support are mapped by S onto functions with compact support. Moreover, since $SP_\lambda(G_2) = P_\lambda(G_1)$, it follows that S is continuous (see e.g. [1, 3.1]). Hence, $SA(G_2) = S(\overline{B_\lambda(G_2) \cap C_c(G_2)}) \subset \overline{S(B_\lambda(G_2) \cap C_c(G_2))} \subset \overline{B_\lambda(G_1) \cap C_c(G_1)} = A(G_1)$. Applying the same argument to S^{-1}, we see that S restricted to $A(G_2)$ is a biorder isomorphism from $A(G_1)$ onto $A(G_2)$. We conclude from [1, 4.3] that $c(t) \equiv \text{const}$ and α is a topological group isomorphism or anti-isomorphism. Using the fact that S is the restriction of T', one obtains as in [2, 6.2] that T has the desired form. □

Remark 2.5. We proved Theorem 2.1 by a reduction to the corresponding result for $A(G)$ [1, 4.3], whereas in the case where the positive-definite ordering is defined by the cone $L^1(G)_p$ [2, 6.2] the proof was a simple deduction from the analogous result for $B(G)$ [2, 5.3]. Here we cannot take this path since we do not know whether $(B_\lambda(G), B_\lambda(G)_+, P_\lambda(G))$ is a complete invariant. The techniques used in [2, Sects. 4 and 5] for $B(G)$ cannot be applied to $B_\lambda(G)$, it seems.

3. A Characterization of the Fourier Transform

In this section we characterize the Fourier transform for a locally compact abelian group G as an order anti-isomorphism from $L^1(G)$ onto $A(\hat{G})$.

Proposition 3.1. *Let G be an abelian locally compact group with dual group $\hat{G}$. Denote by $\mathscr{F}: L^1(G) \to A(\hat{G})$ the Fourier transform. Then*

$$\mathscr{F}L^1(G)_+ = A(\hat{G}) \cap P(\hat{G}), \tag{3.1}$$

$$\mathscr{F}L^1(G)_p = A(\hat{G})_+ . \tag{3.2}$$

Proof. (3.1) is an immediate consequence of Bochner's theorem. In order to prove (3.2), consider the real Banach spaces $L^1(G)_h = \{f \in L^1(G): f = f^*\}$ and $A(\hat{G})_{\mathbb{R}} = \{u \in A(\hat{G}): u(t) \in \mathbb{R} \text{ for all } t \in G\}$. Then $\mathscr{F}L^1(G)_h = A(\hat{G})_{\mathbb{R}}$ (since $\mathscr{F}(f^*) = (\mathscr{F}f)^-$ for all $f \in L^1(G)$). The set $\mathscr{F}L^1(G)_p$ is a closed cone in $A(\hat{G})_{\mathbb{R}}$. Since $\mathscr{F}(f * f^*) = |\mathscr{F}f|^2$ for all $f \in L^1(G)$, it follows that $\mathscr{F}L^1(G)_p = \{|u|^2 : u \in A(\hat{G})\} \subset A(\hat{G})_+$. In order to prove the converse inclusion, by (1.3) it is enough to show that $(\mathscr{F}L^1(G)_p)' \subset A(\hat{G})'_+$. Let $\varphi \in (\mathscr{F}L^1(G)_p)'$, i.e. $\varphi \in A(\hat{G})'$ and $\varphi \circ \mathscr{F} \in L^1(G)'_p$. Then by (1.8), there exists $q \in P(G)$ such that $\langle \mathscr{F}f, \varphi \rangle = \langle f, \varphi \circ \mathscr{F} \rangle = \langle f, \varphi \rangle$ for all $f \in L^1(G)$. By Bochner's theorem there exists $\mu \in M(\hat{G})_+$ such that $q = \hat{\mu}$. Let $u \in A(\hat{G})_+$, $f = \mathscr{F}^{-1}u$. Then $\langle u, \varphi \rangle = \langle \mathscr{F}f, \varphi \rangle = \langle f, \hat{\mu} \rangle = \int\int f(t)\overline{(\gamma, t)}\, d\mu(\gamma)\, dt = \int \hat{f}(\gamma)\, d\mu(\gamma) = \int u(\gamma)\, d\mu(\gamma) \geqq 0$. Thus $\varphi \in A(\hat{G})'_+$. □

Next we characterize the commutativity of G by a property of the positive-definite ordering on $A(G)$.

Proposition 3.2. *Let G be a locally compact group. The following are equivalent.*
(i) *G is abelian.*
(ii) *$(A(G), A(G) \cap P(G))$ is a vector lattice.*

Proof. If G is abelian, then by (3.1) $\mathscr{F}$ is an order isomorphism from $(L^1(\hat{G}), L^1(\hat{G})_+)$ onto $(A(G), A(G) \cap P(G))$. Since $(L^1(\hat{G}), L^1(\hat{G})_+)$ is a vector lattice, (ii) follows.

We show the converse. Consider the real Banach space $A(G)_h = \{u \in A(G): u = \tilde{u}\}$. Since $u \to \tilde{u}$ is an involution on $A(G)$ [6, (3.8) and (2.6)], the dual space of $A(G)_h$ is $VN(G)_h := \{T \in VN(G): \overline{\langle \tilde{u}, T \rangle} = \langle u, T \rangle \text{ for all } u \in A(G)\} = \{T \in VN(G): T^* = T\}$ (use (1.18) for the last equality). It follows from [6, (3.15)] that $A(G)_h = (A(G) \cap P(G)) - (A(G) \cap P(G))$. So $(A(G)_h, A(G) \cap P(G))$ is a vector lattice. Since the dual cone of $A(G) \cap P(G)$ is $VN(G)_p$ (1.18), it follows from [12, II, 4.2 and II, 4.2, Corollary 2] that $(VN(G)_h, VN(G)_p)$ is a vector lattice. This implies that $VN(G)$ is a commutative C^*-algebra (see [13] or [3, Example 4.2.6]). In particular, $\lambda_{st} = \lambda_s \cdot \lambda_t = \lambda_t \cdot \lambda_s = \lambda_{ts}$ and hence $st = ts$ for all $s, t \in G$. Thus G is abelian. □

Theorem 3.3. *Let G_1, G_2 be locally compact groups and $F: L^1(G_1) \to A(G_2)$ a bijective, linear mapping such that*

$$FL^1(G_1)_+ = A(G_2) \cap P(G_2) \tag{3.3}$$

holds. Suppose in addition that one of the following two conditions

$$FL^1(G_1)_p = A(G_2)_+ \tag{3.4}$$

$$FL^1(G_1)_{\lambda p} = A(G_2)_+ \tag{3.5}$$

is satisfied.

Then G_1 and G_2 are abelian and there exists a topological group isomorphism $\alpha: G_2 \to \hat{G}_1$ and a constant $c > 0$ such that

$$(Ff)(t) = c \cdot \hat{f}(\alpha(t)) \qquad (t \in G_2) \tag{3.6}$$

for all $f \in L^1(G_1)$, where $\hat{f}$ denotes the Fourier transformation of f.

Proof. Since $L^1(G_1)$ is a vector lattice for the pointwise ordering, it follows from (3.3) that $A(G_2)$ is a vector lattice for the positive-definite ordering. Hence G_2 is abelian by Proposition 3.2. Denote by $\mathcal{F}_2: L^1(\hat{G}_2) \to A(G_2)$ the Fourier transform. It follows from Proposition 3.1 and the hypotheses that $\mathcal{F}_2^{-1} \circ F$ is a biorder isomorphism from $L^1(G_1)$ onto $L^1(\hat{G}_2)$, where the positive-definite ordering is defined by the cone $L^1(G)_p$ when (3.4) holds, and by the cone $L^1(G)_{\lambda p}$ if (3.5) holds. Thus it follows from [2, 6.2] in the first case and from Theorem 2.1 in the second that G_1 is isomorphic to $\hat{G}_2$. Hence G_1 is abelian as well. Let $\mathcal{F}_1: L^1(G_1) \to A(\hat{G}_1)$ denote the Fourier transform. Then $F \circ \mathcal{F}_1^{-1}$ is a biorder isomorphism from $A(\hat{G}_1)$ onto $A(G_2)$. Thus by [1, 4.3] there exist a topological group isomorphism $\alpha: G_2 \to \hat{G}_1$ and a constant $c > 0$ such that $(F \circ \mathcal{F}_1^{-1})u = c \cdot u \circ \alpha$ for all $u \in A(\hat{G}_1)$. Hence F is given by (3.6). □

4. Order Isomorphisms of Measure Algebras

The purpose of this section is to show that also $M(G)$ can be considered as a biordered vector space which is a complete isomorphism invariant. The proof will be given by reduction to the corresponding results ([2, 6.2] and Theorem 2.1) for $L^1(G)$.

There are two canonical positive-definite orderings on $M(G)$. The corresponding cones are given by

$$M(G)_p = \overline{\text{co}}\{\mu^* * \mu : \mu \in M(G)\} \tag{4.1}$$

and

$$M(G)_{\lambda p} = \{\mu \in M(G) : \lambda(\mu) \in \mathcal{L}(L^2(G))_p\}. \tag{4.2}$$

Here $\lambda(\mu) \in \mathcal{L}(L^2(G))$ is the operator defined by $\lambda(\mu) f = \mu * f$ $(f \in L^2(G))$. Hence $\mu \in M(G)_{\lambda p}$ if and only if $\lambda(\mu)$ is a positive definite operator (that is $(\lambda(\mu) f | f) \geqq 0$ for all $f \in L^2(G)$; this again means that $\langle \tilde{f} * f, \mu \rangle \geqq 0$ for all $f \in C_c(G)$ [5, 13.7.4]).

Both cones are proper and closed, and clearly

$$M(G)_p \subseteqq M_{\lambda p}(G). \tag{4.3}$$

The induced cones on $L^1(G)$ are

$$M(G)_p \cap L^1(G) = L^1(G)_p, \tag{4.4}$$

$$M(G)_{\lambda p} \cap L^1(G) = L^1(G)_{\lambda p}. \tag{4.5}$$

[(4.5) follows immediately from the definitions. To see (4.4), observe that $C := M(G)_p \cap L^1(G)$ is a closed cone in $L^1(G)$. Clearly, $L^1(G)_p \subset C$. To see the converse, let $\varphi \in L^1(G)'_p = P(G)$. Then by [5, 13.4.4 (ii)], $\varphi \in C'$. Hence $L^1(G)'_p \subset C'$. Applying (1.3) to the real Banach space $L^1(G)_h$ with dual space $L^\infty(G)_h = \{f \in L^1(G) : f = \tilde{f}\}$ one obtains $C \subset L^1(G)_p$.]

The cones $M(G)_p$ and $M(G)_{\lambda p}$ are different if G is non-amenable (since $L^1(G)_p \neq L^1(G)_{\lambda p}$ in that case [by (2.7).)] But in contrast to the situation for $L^1(G)$, one also has $M(G)_p \neq M(G)_{\lambda p}$ if G is abelian and non-discrete (see: Remark 6.2).

Now we formulate the main result of this section. Let $\alpha : G_1 \to G_2$ be a topological group isomorphism or anti-isomorphism. Denote by $V_\alpha : C_0(G_2) \to C_0(G_1)$ the mapping $f \to f \circ \alpha$, and by $(V_\alpha)' : M(G_1) \to M(G_2)$ its adjoint. Then it is easy to see that $V'_\alpha M(G_1)_+ = M(G_2)_+$, $V'_\alpha M(G_1)_p = M(G_2)_p$ and $V'_\alpha M(G_1)_{\lambda p} = M(G_2)_{\lambda p}$. Consequently, the biordered vector spaces $(M(G), M(G)_+, M(G)_p)$ as well as $(M(G), M(G)_+, M(G)_{\lambda p})$ are isomorphism invariants. Our theorem says that they both are complete.

Theorem 4.1. *Let $T : M(G_1) \to M(G_2)$ be a bijective linear mapping such that*

$$TM(G_1)_+ = M(G_2)_+. \tag{4.6}$$

Moreover, assume that one of the following two conditions

$$TM(G_1)_p = M(G_2)_p \tag{4.7}$$

$$TM(G_1)_{\lambda p} = M(G_2)_{\lambda p} \tag{4.8}$$

holds.

Then there exist a topological group isomorphism or anti-isomorphism $\alpha : G_1 \to G_2$ and a constant $c > 0$ such that $T = cV'_\alpha$.

For the proof of this theorem we need several intermediate results. First we show how $L^1(G)$ can be abstractly defined in the ordered Banach algebra $(M(G), M(G)_+, *)$. Recall that a subspace J of $M(G)$ is called a *lattice ideal*, if for $\mu \in J$ and $\nu \in M(G)$, $|\nu| \leq |\mu|$ implies that $\nu \in J$. The space $L^1(G)$ is a closed lattice and algebra ideal in $M(G)$.

Proposition 4.2. *$L^1(G)$ is the smallest non-zero closed lattice and algebra ideal in $M(G)$.*

Proof. Let J be a non-zero closed lattice and algebra ideal in $M(G)$. We have to show that $L^1(G) \subset J$.

a) Let $t_0 \in G$. Then there exists $f \in J \cap C^b(G)$ such that $f(t_0) \neq 0$. [Indeed, if this is not true, then $f(t_0) = 0$ for all $f \in C^b(G) \cap J$. Since J is an algebra ideal, it follows that

$$f(e) = (\delta_{t_0} * f)(t_0) = 0 \quad \text{for all} \quad f \in C^b(G) \cap J. \tag{4.9}$$

Let $\mu \in J$ be non-zero. Given $q \in C_c(G)$, let $f = \mu * q$. Then $f \in J \cap C^b(G)$. Hence $\langle q, \mu \rangle = \mu * \check{q}(e) = f(e) = 0$ [by (4.9)]. Since $q \in C_c(G)$ is arbitrary, this implies that $\mu = 0$, contradiction.]

b) Since span $J_+ = J$, it follows from a) that for all $t \in G$ there exists a pointwise positive $f \in J \cap C^b(G)$ such that $f(t) > 0$. Consequently, for every compact set $K \subset G$, there exists $f \in C^b(G) \cap J_+$ such that $\inf_{t \in K} f(t) > 0$. Since J is a lattice ideal in $M(G)$, this implies that $C_c(G) \subset J$. It follows that $L^1(G) \subset J$, since J is closed. □

If A_1, A_2 are algebras, a bijective, linear mapping $T: A_1 \to A_2$ is called Jordan-isomorphism if $T\{x, y\} = \{Tx, Ty\}$ for all $x, y \in A_1$, where $\{x, y\} = xy + yx$.

Corollary 4.3. *Let $T: M(G_1) \to M(G_2)$ be a Jordan isomorphism such that $TM(G_1)_+ = M(G_2)_+$. Then $TL^1(G_1) = L^1(G_2)$.*

Proof. Let J_1 be a non-zero closed lattice and algebra ideal in $M(G_1)$. Since T is an order isomorphism for the pointwise ordering, $J_2 := TJ_1$ is a lattice ideal in $M(G_2)$. By [12, II, 5.3] T^{-1} is continuous, and so J_2 is closed. Since T is a Jordan isomorphism, it follows that J_2 is a Jordan ideal, i.e. $\{\mu, \nu\} \in J_2$, whenever $\mu \in M(G_2)$, $\nu \in J_2$. Now let $\mu \in M(G_2)_+$, $\nu \in (J_2)_+$. Then $0 \leqq \mu * \nu \leqq \{\mu, \nu\} \in J_2$ and $0 \leqq \nu * \mu \leqq \{\mu, \nu\} \in J_2$. Since J_2 is a lattice ideal, it follows that $\nu * \mu$, $\mu * \nu \in J_2$. Since span $M(G_2)_+ = M(G_2)$ and span$(J_2)_+ = J_2$, it follows that J_2 is actually an algebra ideal. We have shown that T maps non-zero closed lattice and algebra ideals onto non-zero closed lattice and algebra ideals. The same is true for T^{-1}. Thus it follows from Proposition 4.2 that $TL^1(G_1) = L^1(G_2)$. □

Proposition 4.4. *Let $T_j: M(G_1) \to M(G_2)$ $(j = 1, 2)$ be Jordan isomorphisms such that $T_j L^1(G_1) = L^1(G_2)$. If $T_1 f = T_2 f$ for all $f \in L^1(G_1)$, then $T_1 = T_2$.*

Proof. Let $\mu \in M(G_1)$, $\nu_j = T_j \mu$ $(j = 1, 2)$. Let $g \in L^1(G_2)$. Then

$$\{g, \nu_1\} = \{g, \nu_2\} \tag{4.10}$$

[for let $f = T_1^{-1} g = T_2^{-1} g \in L^1(G_1)$; then $\{g, \nu_1\} = T_1\{f, \mu\} = T_2\{f, \mu\} = \{g, \nu_2\}$ since $\{f, \mu\} \in L^1(G_1)$]. But for $\nu \in M(G_2)$, $g \in C_c(G_2)$, $\nu * g$ and $g * \nu$ are continuous functions and $\nu * \check{g}(e) = \langle g, \nu \rangle$ and $(\check{g} * \nu)(e) = \langle g, \Delta^{-1} \nu \rangle$, where Δ denotes the modular function of G_2. Hence (4.10) implies that $\langle g, (1 + \Delta^{-1}) \nu_1 \rangle = \langle g, (1 + \Delta^{-1}) \nu_2 \rangle$ for all $g \in C_c(G_2)$. This implies that $\nu_1 = \nu_2$. □

An order isomorphism between unital C^*-algebras which maps the identity onto the identity is a Jordan isomorphism [3, 3.2.3]. We will extend this result to certain ordered algebras. Let A be an involutive algebra and A_p a cone in A. We say that (A, A_p) is *C^*-ordered* if A is a dense involutive subalgebra of a C^*-algebra $\mathfrak{A}$ such that $A_p = \mathfrak{A}_p \cap A$, where $\mathfrak{A}_p$ denotes the usual positive cone in $\mathfrak{A}$ (see 1 B). For example,

$$(M(G), M(G)_p) \quad \text{and} \quad (M(G), M(G)_{\lambda p}) \text{ are } C^*\text{-ordered algebras}. \tag{4.11}$$

[By (1.6), one can take for $\mathfrak{A}$ the enveloping C^*-algebra of $M(G)$ in the first case, and by definition $\mathfrak{A} = \mathcal{L}(L^2(G))$ in the second.]

Proposition 4.5. *Let (A_j, A_{jp}) be C^*-ordered algebras with identity e_j $(j = 1, 2)$. If $T: A_1 \to A_2$ is an order isomorphism such that $Te_1 = e_2$, then T is a Jordan isomorphism.*

For the proof we need the following result, which is a special case of [4, I, Sect. 6, Theorem 1].

Lemma 4.6. *Let E be a real ordered vector space with positive cone E_+ and F a subspace of E. If for every $x \in E$ there exists $y \in F$ such that $x \leqq y$ then every positive linear form on F has a positive extension on E.*

Proof of Proposition 4.5. There exist C^*-algebras $\mathfrak{A}_j$ $(j=1,2)$ such that A_j is a dense involutive subalgebra of $\mathfrak{A}_j$ and $A_{jp}=\mathfrak{A}_{jp} \cap A_j$ $(j=1,2)$. Since $e_1 \in A_{1+}$, the hypotheses of Lemma 4.6 are satisfied for $E=\mathfrak{A}_{1h}$, $E_+=\mathfrak{A}_{1p}$ and $F=A_{1h}$. Let $f \in \mathfrak{A}'_{2p}$. Then $g_0 := f|_{A_2} \circ T$ is a positive functional on A_1. By 4.6, g_0 has a unique positive extension $g \in \mathfrak{A}'_{1p}$. The mapping $f \to g$ from $\mathfrak{A}'_{2p}$ into $\mathfrak{A}'_{1p}$ is clearly positive homogeneous and additive. So its linear extension defines a positive linear mapping $S: \mathfrak{A}'_2 \to \mathfrak{A}'_1$. In particular, S is continuous (see e.g. [1, 3.1]). Applying the same argument to S^{-1}, we see that S is an order isomorphism. Hence $S': \mathfrak{A}''_1 \to \mathfrak{A}''_2$ an order isomorphism as well. It is easy to see that

$$Tx = S'x \quad \text{for} \quad x \in A_1 . \tag{4.12}$$

In particular, $S'e_2 = e_1$. By [3, 3.2.3], S' is a Jordan isomorphism. So it follows from (4.12) that T is a Jordan isomorphism as well. □

Let us call a positive measure $\mu \in M(G)$ an *atom*, if $\mu = c\delta_t$ for some $c \geqq 0$, $t \in G$ (where δ_t denotes the Dirac measure in t). Then for $\mu \in M(G)_+$, it is easy to see that

$$\begin{gathered}\mu \text{ is an atom if and only if } 0 \leqq \nu \leqq \mu \text{ implies } \nu = c\mu \\ \text{for some } c \geqq 0 \text{ for all } \nu \in M(G) .\end{gathered} \tag{4.13}$$

Proof of Theorem 4.1. T is an order isomorphism for the pointwise ordering. So it follows from (4.13) that T maps atoms onto atoms. In particular, $T\delta_e = c \cdot \delta_t$ for some $t \in G_2$, $c > 0$. Since $\delta_e \in M(G_1)_p \subset M(G_1)_{\lambda p}$, it follows from assumption (4.7), resp. (4.8), that $c \cdot \delta_t \in M(G_2)_{\lambda p}$. Since $\lambda_t = \lambda(\delta_t)$ is unitary, this implies that $t = e$. Considering $c^{-1}T$ instead of T, we can assume that $c=1$. Then it follows from (4.11) and Proposition 4.5 that T is a Jordan isomorphism. From Corollary 4.3 we obtain that $TL^1(G_2) = L^1(G_2)$. We can now apply [2, 6.2] [in the case of (4.7)], resp., Theorem 2.1 [in the case of (4.8)] to the restriction T_0 of T to $L^1(G_1)$ and conclude that T_0 is given by (2.2) or (2.3). Let $\alpha: G_1 \to G_2$ be the inverse of β (in the notation of Theorem 2.1). Then it is easy to see that $T_0 = c \cdot V'_{\alpha|L^1(G_1)}$ for some constant $c > 0$. Thus it follows from Proposition 4.4 that $T = c \cdot V'_\alpha$. □

5. A Second Look at $B(G)$

In considering the Fourier-Stieltjes transform in the next section we will be concerned with a second "pointwise" cone in $B(G)$, namely

$$B(G)_\oplus = \overline{\mathrm{co}}\{|u|^2 : u \in B(G)\} .$$

This is the positive cone defined by the involution $u \to \bar{u}$ on $B(G)$. Clearly,

$$B(G)_\oplus \subset B(G)_+ . \tag{5.1}$$

But we will see that the inclusion is proper in general. The question arises whether the results of [2, Sect. 5] remain true if the cone $B(G)_+$ is replaced by $B(G)_\oplus$, in particular if the biordered vector space $(B(G), B(G)_\oplus, P(G))$ is a complete invariant. And indeed, we will prove the following.

Theorem 5.1. *Let* $T: B(G_1) \to B(G_2)$ *be a bijective linear mapping such that* $TB(G_1)_\oplus = B(G_2)_\oplus$. *Then* $TB(G_1)_+ = B(G_2)_+$.

As a consequence, one obtains the following result with the help of [2, 5.3]:

Corollary 5.2. *Let* $T: B(G_1) \to B(G_2)$ *be bijective, linear such that* $TB(G_1)_\oplus = B(G_2)_\oplus$ *and* $TP(G_1) = P(G_2)$. *Then there exists a topological group isomorphism or anti-isomorphism* $\alpha: G_2 \to G_1$ *and a constant* $c > 0$ *such that*

$$Tu = c \cdot u \circ \alpha \quad \textit{for all} \quad u \in B(G_1).$$

Let us now show that the two "pointwise" cones are different in general.

Proposition 5.3. *If* G *is abelian and non-compact, then* $B(G)_\otimes \neq B(G)_+$.

Proof. Let Δ denote the space of all multiplicative continuous linear forms on $B(G)$ and $\Delta_h = \{\varphi \in \Delta : \varphi(\bar{u}) = \overline{\varphi(u)}\}$. Then we have

$$\Delta_h = \Delta \cap B(G)'_\oplus \qquad \text{(cf. 1 B)}. \tag{5.2}$$

By evaluation, one can identify G with a subset of Δ_h. We show that

$$\bar{G} = \Delta_h \cap B(G)'_+ \qquad (= \Delta \cap B(G)'_+). \tag{5.3}$$

It is obvious that $\bar{G} \subset \Delta_h \cap B(G)'_+$. To see the reverse inclusion, denote by $\bar{B}$ the uniform closure of $B(G)$ in $C^b(G)$. $\bar{B}$ is a commutative C^*-algebra. The space of all continuous multiplicative linear functionals on $\bar{B}$ can be identified with $\Delta_\infty = \{\varphi \in \Delta : |\langle u, \varphi \rangle| \leqq \|u\|_\infty \ (u \in B(G))\}$. Δ_∞ is obviously a closed subset of Δ and $\bar{G} \subset \Delta_\infty$. We can identify $\bar{B}$ with $C(\Delta_\infty)$. Since $B(G)$ is dense in $\bar{B}$, it follows that

$$\bar{G} = \Delta_\infty. \tag{5.4}$$

Moreover,

$$B(G)'_+ \cap \Delta \subset \Delta_\infty. \tag{5.5}$$

In fact, let $\varphi \in B(G)'_+ \cap \Delta$. If $u \in B(G)$ is real valued, then

$$-\|u\|_\infty 1 \leqq u \leqq \|u\|_\infty 1.$$

Consequently,

$$-\|u\|_\infty \leqq \langle u, \varphi \rangle \leqq \|u\|_\infty;$$

hence

$$|\langle u, \varphi \rangle| \leqq \|u\|_\infty.$$

For arbitrary $u \in B(G)$ we obtain,

$$|\langle u, \varphi \rangle| = |\langle \mathrm{Re}\, u, \varphi \rangle + i \langle \mathrm{Im}\, u, \varphi \rangle| \leqq 2\|u\|_\infty.$$

Thus φ is uniformly continuous. So it can be extended to a continuous multiplicative linear form on $\bar{B}$. Hence the extension has norm smaller or equal 1 [5, B3], which implies $\varphi \in \Delta_\infty$.

From (5.4) and (5.5) we obtain (5.3).

We now prove the proposition.

By [9, Theorem 3] there exists $\varphi_0 \in \Delta_h$ such that $\varphi_0 \notin \bar{G}$. So we conclude from (5.2) and (5.3) that $B(G)'_{\oplus} \neq B(G)'_+$. This implies by (1.3) (applied to the real Banach space $B(G)_{\mathbb{R}} = \{u | u = \bar{u}\}$) that $B(G)_+ \neq B(G)_{\oplus}$. □

Remark 5.4. For the Fourier algebra the situation is different. In fact, one can show that $A(G)_+ = \overline{\text{co}}\{|u|^2 : u \in A(G)\} = A(G) \cap B(G)_{\oplus}$ for every locally compact group. We omit the proof, since this will not be needed here.

Concerning the proof of Theorem 5.1, the same technique as in [2, Sects. 4, 5] can be used. So we will not give all the details. However, a more general approach seems appropriate and of independent interest.

We consider a commutative Banach algebra A with identity e. Let $\Delta_h = \{\varphi \in A' : \varphi(x \cdot y) = \varphi(x)\varphi(y) \text{ and } \varphi(x) = \overline{\varphi(x^*)} \text{ for all } x, y \in A\}$. *We assume that* Δ_h *separates points*, i.e.

$$\varphi(x) = 0 \quad \text{for all} \quad \varphi \in \Delta_h \quad \text{implies} \quad x = 0. \tag{5.6}$$

We define the cone $A_p = \overline{\text{co}}\{x^* \cdot x : x \in A\}$ (cf. 1 B). The ordered vector space (A, A_p) has been investigated by Kelley and Vaught [10], and we shall use their results. It follows from (5.6) that the cone A_p is proper. For $x \in A_h$ we have

$$\|x - e\| \leqq 1 \quad \text{implies} \quad x \in A_p \tag{5.7}$$

(see [10, Sect. 2]), and consequently

$$(\|x\| e - x) \in A_p \qquad (x \in A_n). \tag{5.8}$$

(This follows from [10, 1.3a)] and (1.2), since for all $f \in A'_p$, $\langle \|x\| e - x, f \rangle = \|x\| \, \|f\| - \langle x, f \rangle \geqq 0$.)

We conclude that A_p is generating, i.e.,

$$A_p - A_p = A_h. \tag{5.9}$$

[In fact, let $x \in A_h$. Then by (5.8), $(\|x\| e \pm x) \in A_p$. Consequently, $x = \frac{1}{2}(x + \|x\| e) - \frac{1}{2}(\|x\| e - x) \in A_p - A_p$.]

For $a \in A$ denote by $M_a \in \mathscr{L}(A)$ the *multiplier* given by $M_a x = ax$ $(x \in A)$. The *multiplier algebra* $\mathscr{M}(A) := \{M_a : a \in A\}$ is a closed subalgebra of $\mathscr{L}(A)$ and is isometrically isomorphic to A.

Lemma 5.5. *Let $M : A \to A$ be a linear mapping. Then $M \in \mathscr{M}(A)$ if and only if for all $x \in A$ and all $\varphi \in \Delta_h$*

$$\varphi(x) = 0 \quad \textit{implies} \quad \varphi(Mx) = 0. \tag{5.10}$$

Lemma 5.6. *Let $M : A \to A$ be a positive linear mapping. Then M is a multiplier if and only if there exists a constant $c > 0$ such that*

$$Mx \leqq cx \qquad (x \in A_p). \tag{5.11}$$

The proofs of these two lemmas are exactly the same as of [2, 4.1]; resp., [2, 4.2] if the elements of Δ_h take the place of the point evaluations.

Now let A_1 and A_2 be two commutative involutive Banach algebras with identity which satisfy the hypothesis (5.6).

Theorem 5.7. *Let $T: A_1 \to A_2$. Then T is an order isomorphism if and only if there exist a *-algebra isomorphism $V: A_1 \to A_2$ and an invertible element $a \in A_{2p}$ such that*

$$T = M_a V. \tag{5.12}$$

Proof. Assume that T is given by (5.12). Then $\varphi(a^{-1}) = \varphi(a)^{-1} > 0$ for all $\varphi \in \Delta_h$. Let $f \in A'_{2p}$, $\|f\| = 1$. Then $f \in \overline{\mathrm{co}}\, \Delta_h$. (This follows from [10, 2.1] by the Krein-Milman theorem.) Hence $f(a^{-1}) \geqq 0$ for all $f \in A'_{2p}$. This implies that $a^{-1} \in A_{2p}$ [by (1.2)]. Hence M_a is an order isomorphism. It follows immediately from the definition of the ordering that V is an order isomorphism. So $T = MV$ is an order isomorphism as well. The converse can be proved with help of Lemmas 5.5 and 5.6 as in [2, Theorem 5.2]. □

Remark. One might compare Theorem 5.7 with Proposition 4.5. Even though they have a common special case (namely if $Te = e$ in Theorem 5.7) none of the proofs can be applied to the other case, it seems.

It is clear that $B(G)$ satisfies the assumptions made above. In fact, $\{\delta_t : t \in G\} \subset \Delta_h$ (where $\delta_t(u) = u(t)$ $(u \in B(G))$ so that (5.6) holds. Hence Theorem 5.1 follows from Theorem 5.7 and [2, 5.1].

6. A Characterization of the Fourier-Stieltjes Transform

Now we give the characterization of the Fourier-Stieltjes transform which is analogous to Proposition 2.1 and Theorem 2.3 for the Fourier transform. The proofs are based on the results of Sects. 4 and 5.

Proposition 6.1. *Let G be an locally compact abelian group with dual $\hat{G}$. Denote by $\mathscr{F}: M(G) \to B(\hat{G})$ the Fourier-Stieltjes transform. Then $\mathscr{F}$ is bijective, linear and satisfies*

$$\mathscr{F} M(G)_+ = P(\hat{G}), \tag{6.1}$$

$$\mathscr{F} M(G)_p = B(\hat{G})_\oplus, \tag{6.2}$$

$$\mathscr{F} M(G)_{\lambda p} = B(\hat{G})_+ . \tag{6.3}$$

Proof of Proposition 6.1. (6.1) is Bochner's theorem. Since $\mathscr{F}(\mu^* * \mu) = |\mathscr{F}\mu|^2$ $(\mu \in M(G))$, it follows immediately from the definitions of the cones that (6.2) holds. Finally, denote by $\mathfrak{U}: L^2(G) \to L^2(\hat{G})$ the Plancherel transform. Then for $\mu \in M(G)$, the operator $T_\mu = \mathfrak{U}\lambda(\mu)\mathfrak{U}^{-1}$ on $L^2(\hat{G})$ is given by $T_\mu f = \mathscr{F}(\mu) \cdot f$ $(f \in L^2(\hat{G}))$. This implies (6.3). □

Remark 6.2. It follows from (6.2), (6.3) and Proposition 5.3 that $M(G)_p \neq M(G)_{\lambda p}$ for every locally compact non-discrete abelian group G.

Proposition 6.1 shows that the Fourier-Stieltjes transform is a biorder anti-isomorphism from $M(G)$ onto $B(\hat{G})$, where one has two choices for the positive-definite ordering in $M(G)$ and the corresponding pointwise ordering in $B(\hat{G})$. The next theorem shows that the Fourier transform is characterized by these properties. Before that we need a description of commutativity which is analogous to Proposition 2.2.

Proposition 6.3. *Let G be a locally compact group. The following are equivalent:*
(i) *G is abelian*
(ii) *$(B(G), P(G))$ is a vector lattice.*

Proof. Since the Fourier-Stieltjes transform is an order isomorphism from the vector lattice $(M(\hat{G}), M(\hat{G})_+)$ onto $(B(G), P(G))$, (i) implies (ii). Assume now that (ii) holds. Then $B(G)' = C^*(G)''$ is a lattice for the usual ordering of the C^*-algebra $C^*(G)''$. It follows from [3, Example 4.2.6] that $C^*(G)''$ is commutative. Hence $C^*(G)$ and consequently $L^1(G)$ are commutative as well. This implies that G is abelian by [8, (20.24)]. □

Theorem 6.4. *Let G_1, G_2 be locally compact groups. Let $F: M(G_1) \to B(G_2)$ be a bijective linear mapping such that*

$$FM(G_1)_+ = P(G_2) \tag{6.4}$$

holds. Suppose, moreover, that one of the two conditions

$$FM(G_1)_p = B(G_2)_\oplus \tag{6.5}$$

$$FM(G_1)_{\lambda p} = B(G_2)_+ \tag{6.6}$$

is satisfied.

Then G_1 and G_2 are abelian and there exists a topological group isomorphism $\alpha: G_2 \to \hat{G}_1$ and a constant $c > 0$ such that

$$F\mu = c \cdot \hat{\mu} \circ \alpha \tag{6.7}$$

for all $\mu \in M(G_1)$, where $\hat{\mu}$ is the Fourier-Stieltjes transformation of μ.

Proof. Since $(M(G_1), M(G_1)_+)$ is a lattice, it follows from (6.4) that $(B(G_2), P(G_2))$ is a lattice. Hence G_2 is abelian by Proposition 6.3. Let $\mathscr{F}_2 : M(\hat{G}_2) \to B(G_2)$ denote the Fourier-Stieltjes transform. Then $F^{-1} \circ \mathscr{F}_2$ is a biorder isomorphism from $M(\hat{G}_2)$ onto $M(G_1)$ where the positive-definite ordering is given by the cone $M(G)_p$ if (6.5) holds and by the cone $M(G)_{\lambda p}$ if (6.6) holds. In both cases it follows from Theorem 4.1 that G_1 and $\hat{G}_2$ are isomorphic. Hence G_1 is abelian as well. Finally, consider the operator $F \circ \mathscr{F}_1^{-1} : B(\hat{G}_1) \to B(G_2)$. Then $F \circ \mathscr{F}_1^{-1}$ is a biorder isomorphism where the pointwise ordering is defined by the cone $B(G)_\oplus$ if (6.5) holds and by $B(G)_+$ if (6.6) holds. It follows from Corollary 5.2 in the first case and from [2, 5.3] in the second that there exist a topological group isomorphism $\alpha: G_2 \to \hat{G}_1$ and a constant $c > 0$ such that $(F \circ \mathscr{F}_1^{-1})u = c \cdot u \circ \alpha$ $(u \in B(\hat{G}_1))$. Thus (6.7) is satisfied. □

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