An orthogonal-collocation integral formulation for transient radiative transport

J.I. Frankel

Mechanical and Aerospace Engineering Department, University of Tennessee, Knoxville, Tennessee, 37996-2210, USA

ABSTRACT

A new formulation is offered for transient radiative transport which promotes the use of orthogonal collocation. An intermediate variable is introduced which permits the efficient and rapid development of accurate numerical results. Chebyshev polynomials of the first kind are used as the basis functions for the spatial variable while the temporal variable is resolved by an initial value method. Some a posteriori error estimates are presented illustrating the effectiveness of the approach. This new formulation has potential impact to the boundary element community with regard to nonlinear problems.

INTRODUCTION

The accurate numerical simulation of nonlinear, weakly-singular integro-partial differential equations of mathematical physics often represents a formidible challenge to researchers. Both algebraic nonlinearities in the temperature variable and the appearance of kernels containing logarithmic singularities arise in applications involving transient heat transfer [1] in a participating medium and multidimensional heat transfer in a participating medium.

Recently, Kumar and Sloan [2] proposed a new formulation of one-dimensional Hammerstein integral equations which permits efficient computation by a collocation method. Frankel [3] illustrated that the approach of Kumar and Sloan can be extended to multidimensional and transient studies.

ALTERNATIVE INTEGRO-DIFFERENTIAL FORMULATION

In the present context, we consider [1]

$$\frac{\partial \theta}{\partial t}(\eta, t) = g(\eta) - \theta^4(\eta, t) + \lambda\alpha \int_{\xi=-1}^{1} \theta^4(\xi, t) E_1(\alpha|\xi - \eta|) d\xi, \qquad (1a)$$

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$$\eta \epsilon [-1,1], \quad t > 0,$$

where

$$g(\eta) = \lambda \Big[E_2[\alpha(1+\eta)] + \theta_1^4 E_2[\alpha(1-\eta)] \Big], \tag{1b}$$

$$\theta(\eta, 0) = \theta_1, \quad \eta \epsilon [-1,1], \tag{1c}$$

with $\lambda = 1/2$ and $\alpha > 0$.

Here, $\theta(\eta, t)$ is the unknown dependent variable requiring resolution and η, t are the spatial and temporal independent variables, respectively. The n^{th} exponential integral function [4] is denoted by $E_n(z)$ where $E_1(z)$ contains a logarithmic singularity as $z \to 0$.

Let

$$\varphi(\eta, t) = \theta^4(\eta, t), \tag{2}$$

thus Equation (1a) can be written as

$$\frac{\partial \theta}{\partial t}(\eta, t) = g(\eta) - \varphi(\eta, t) + \lambda\alpha \int_{\xi=-1}^{1} \varphi(\xi, t) E_1(\alpha|\xi - \eta|) d\xi, \tag{3}$$

$$\eta \epsilon [-1,1], \quad t > 0.$$

Next, we integrate Equation (3) with respect to t, to get

$$\theta(\eta, t) = h(\eta, t) - \Psi(\eta, t) + \lambda\alpha \int_{\xi=-1}^{1} \Psi(\xi, t) E_1(\alpha|\xi - \eta|) d\xi, \tag{4a}$$

$$\eta \epsilon [-1,1], \quad t \geq 0,$$

where

$$\Psi(\eta, t) = \int_{t_o=0}^{t} \varphi(\eta, t_o) dt_o, \quad t \geq 0, \tag{4b}$$

and

$$h(\eta, t) = g(\eta) t + \theta_1, \quad t \geq 0. \tag{4c}$$

Clearly,

$$\frac{\partial \Psi}{\partial t}(\eta, t) = \varphi(\eta, t) = \theta^4(\eta, t), \tag{5a}$$

and

$$\Psi(\eta, 0) = 0, \quad \eta\epsilon[-1, 1]. \tag{5b}$$

Substituting Equation (4a) into Equation (5a), we obtain the new nonlinear, weakly-singular integro-partial differential equation

$$\frac{\partial \Psi}{\partial t}(\eta, t) = \left[h(\eta, t) - \Psi(\eta, t) + \lambda\alpha \int_{\xi=-1}^{1} \Psi(\xi, t) E_1(\alpha|\xi - \eta|) d\xi \right]^4, \tag{6}$$

$$\eta\epsilon[-1, 1], \quad t > 0,$$

subject to the initial condition displayed in Equation (5b). Note that the algebraic nonlinearity has been peeled away from within the integral operator shown in Equation (1) to a new position outside the integral operator. This new form permits the implementation of a collocation method in a highly efficient manner in the new variable, $\Psi(\eta, t)$. Once we resolve $\Psi(\eta, t)$, we can reconstruct $\theta(\eta, t)$ through the integral transform shown in Equation (4a).

SOLUTION BY ORTHOGONAL COLLOCATION

Let the unknown function $\Psi(\eta, t)$ be represented by the series expansion

$$\Psi(\eta, t) = \sum_{m=0}^{\infty} c_m^*(t) T_m(\eta), \quad \eta\epsilon[-1, 1], \quad t \geq 0, \tag{7}$$

where the basis functions $\{T_m(\eta)\}_{m=0}^{\infty}$ are chosen as the Chebyshev polynomials of the first kind [4]. The unknown time varying expansion coefficients requiring resolution are denoted as $\{c_m^*(t)\}_{m=0}^{\infty}$. In practice, we must truncate this series representation at a finite number of terms, say N. Thus, we express an approximation to $\Psi(\eta, t)$ as $\Psi_N(\eta, t)$, namely

$$\Psi(\eta, t) \approx \Psi_N(\eta, t) = \sum_{m=0}^{N} c_m^N(t) T_m(\eta), \quad \eta\epsilon[-1, 1], \quad t \geq 0, \tag{8}$$

where $c_m^N(t)$ is an approximation to $c_m^*(t)$.

Upon substituting Equation (8) into Equation (6), we arrive at

$$R_N(\eta, t) + \sum_{m=0}^{N} \frac{dc_m^N}{dt}(t) T_m(\eta) = \tag{9a}$$

$$\left[h(\eta, t) - \sum_{m=0}^{N} c_m^N(t) \{T_m(\eta) - \lambda\alpha A_m(\eta)\} \right]^4, \quad \eta\epsilon[-1, 1], \quad t > 0,$$

where

$$A_m(\eta) = \int_{\xi=-1}^{1} T_m(\xi)E_1(\alpha|\eta-\xi|)d\xi, \quad m = 0,1,...,N, \quad \eta\epsilon[-1,1], \quad (9b)$$

which is analytically expressible [5]. The residual function $R_N(\eta,t)$ is introduced in order to maintain the equal sign displayed in Equation (9a). Correspondingly, we find, from Equation (5b), the initial conditions for $c_m^N(t)$, $m = 0,1,...,N$, at $t = 0$, namely

$$R_N^{IC}(\eta) + \sum_{m=0}^{N} c_m^N(0)T_m(\eta) = 0, \quad \eta\epsilon[-1,1]. \quad (9c)$$

Unless the exact solution to $\Psi(\eta,t)$, at any instant in time, $t \geq 0$, is a linear combination of $\{T_m(\eta)\}_{m=0}^N$, we cannot obtain $\{c_m^N(t)\}_{m=0}^N$ which makes both $R_N(\eta,t)$ for $t > 0$ and $R_N^{IC}(\eta)$ at $t = 0$ vanish for all $\eta\epsilon[-1,1]$. However, we can obtain suitable time varying expansion coefficients by making the residuals indicated in Equation (9a) and Equation (9c) small in some sense. Using the definition of the inner product of two functions shown in Frankel [5], we define the weighted residual method through

$$\langle R_N(\eta,t), \Omega_k(\eta)\rangle_{w_k} = 0, \quad t > 0, \quad (10a)$$

and

$$\langle R_N^{IC}(\eta), \Omega_k(\eta)\rangle_{w_k} = 0, \quad t = 0. \quad (10b)$$

For the collocation method, we have $\Omega_k(\eta) = 1, w_k = \delta(\eta-\eta_k)$, $k = 0,1,...,N$. Here the Dirac delta function is denoted by δ while the $N+1$ collocation points are indicated by η_k, $k = 0,1,...,N$ and are defined by the closed rule [5]

$$\eta_k = cos\left(\frac{\pi k}{N}\right), \quad k = 0,1,...,N. \quad (11)$$

By choosing this set of N+1 collocation points, we ensure that both $R_N(\pm 1,t) = 0$ for $t > 0$ and $R_N^{IC}(\pm 1) = 0$ at $t = 0$.

Applying Equation (10a) on Equation (9a), and Equation (10b) on Equation (9c) formally produces

$$\sum_{m=0}^{N} \frac{dc_m^N}{dt}(t)T_m(\eta_k) = \left[h(\eta_k,t) - \sum_{m=0}^{N} c_m^N(t)\{T_m(\eta_k) - \lambda\alpha A_m(\eta_k)\}\right]^4, \quad (12a)$$

$$k = 0, 1, ..., N, \quad t > 0,$$

and

$$\sum_{m=0}^{N} c_m^N(0) T_m(\eta_k) = 0, \quad k = 0, 1, ..., N, \quad t = 0, \tag{12b}$$

respectively. Since $\{T_m(\eta)\}_{m=0}^N$ forms a set of linearly independent basis functions, we see that Equation (12b) reduces to $c_m^N(0) = 0, \quad m = 0, 1, ..., N$, which now represent the initial conditions necessary for resolving $c_m^N(t)$, $m = 0, 1, ..., N$.

Clearly, once $c_m^N(t)$, $m = 0, 1, ..., N$, $t \geq 0$ are resolved, $\Psi_N(\eta, t)$ is reconstructed through Equation (8). Finally, the approximate solution to $\theta(\eta, t)$, namely $\theta_N(\eta, t)$ is arrived at through Equation (4a) i.e.,

$$\theta_N(\eta, t) = h(\eta, t) - \sum_{m=0}^{N} c_m^N(t)[T_m(\eta) - \lambda\alpha A_m(\eta)], \quad \eta\epsilon[-1, 1], \quad t > 0. \tag{13}$$

STEADY-STATE ANALYSIS

At steady-state conditions, Equation (1) reduces to the linear (in $\hat{\theta}^4(\eta)$) Fredholm integral equation of the second kind

$$\hat{\theta}^4(\eta) = g(\eta) + \lambda\alpha \int_{\xi=-1}^{1} \hat{\theta}^4(\xi) E_1(\alpha|\xi - \eta|) d\xi, \quad \eta\epsilon[-1, 1], \tag{14}$$

where $\hat{\theta}(\eta) = \lim_{t\to\infty} \theta(\eta, t)$.

As before, we can develop a series representation for $\hat{\theta}^4(\eta)$, namely

$$\hat{\theta}^4(\eta) = \sum_{m=0}^{\infty} b_m^* T_m(\eta), \quad \eta\epsilon[-1, 1], \tag{15a}$$

while the N^{th} order approximation is given by

$$\hat{\theta}^4(\eta) \approx \hat{\theta}_N^4(\eta) = \sum_{m=0}^{N} b_m^N T_m(\eta), \quad \eta\epsilon[-1, 1], \tag{15b}$$

where $T_m(\eta)$ was previously defined. Following a similar procedure as described previously, we can obtain the expansion coefficients $\{b_m^N\}_{m=0}^N$ by solving a closed system of linear algebraic equations using conventional means.

TRANSIENT ANALYSIS

From viewing the findings at steady-state conditions, we choose to express the approximate solution for $\theta_N^4(\eta, t_i)$ at discrete time t_i as

$$\theta_N^4(\eta, t_i) = \sum_{m=0}^{N} d_m^N(t_i) T_m(\eta), \quad \eta \epsilon [-1, 1]. \tag{16a}$$

Evaluating Equation (16a) at $\eta = \eta_k$, $k = 0, 1, ..., N$ and substituting Equation (13) into the left-hand side of Equation (16a), we obtain $\{d_m^N(t_i)\}_{m=0}^N$ through matrix inversion at the indicated discrete time, $t = t_i$. These discrete times t_i correspond to the times used in obtaining the numerical results mandated by a initial value method for finding $\{c_m^N(t)\}_{m=0}^N$ as indicated by Equation (12). Doing so produces the linear system of equations for $\{d_m^N(t_i)\}_{m=0}^N$

$$\theta_N^4(\eta_k, t_i) = \sum_{m=0}^{N} d_m^N(t_i) T_m(\eta_k), \quad k = 0, 1, ..., N. \tag{16b}$$

By comparing Equation (15b) to Equation (16b), it is clear that in the limit as $t \to \infty$ (assuming no numerical errors in time)

$$\lim_{t \to \infty} d_m^N(t) = b_m^N, \quad m = 0, 1, ..., N. \tag{16c}$$

RESULTS AND CONCLUSIONS

Let us define the dimensionless dependent variables [1]

$$f^4(\eta, t) = \frac{\theta^4(\eta, t) - \theta_1^4}{1 - \theta_1^4}, \quad \hat{f}^4(\eta) = \frac{\hat{\theta}^4(\eta) - \theta_1^4}{1 - \theta_1^4}, \tag{17a, b}$$

which will be used for transient and steady-state analyses purposes, respectively. At this juncture, we can readily establish a posteriori error bounds for $\hat{f}_N^4(\eta)$. Let the local error of $\hat{f}_N^4(\eta)$ be denoted by

$$\hat{\gamma}_N(\eta) = \hat{f}^4(\eta) - \hat{f}_N^4(\eta), \quad \eta \epsilon [-1, 1], \tag{18}$$

and its size may be measured by means of some functional norm. In general, the error is typically as inaccessible as the exact solution. However, the residual $\hat{R}_N(\eta)$ is a computable measure of how close $\hat{f}_N^4(\eta)$ is to $\hat{f}^4(\eta)$. Following Frankel [5], we arrive at

$$\frac{\|\hat{R}_N\|_\infty/(1 - \theta_1^4)}{1 + |\lambda\alpha| \|\kappa\|_\infty} \le \|\hat{\gamma}_N\|_\infty \le \frac{\|\hat{R}_N\|_\infty/(1 - \theta_1^4)}{1 - |\lambda\alpha| \|\kappa\|_\infty}, \tag{19}$$

when $1 - |\lambda\alpha| \|\kappa\|_\infty > 0$. Here, $\|\kappa\|_\infty$ is the infinity norm of the integral operator [5] indicated in Equation (14).

A program was developed using $Mathematica^{TM}$, Version 2.2 on a NeXT TurboStation having 16 MBytes of memory. Table 1 presents a comparision of steady-state results using $\hat{f}_N^4(\eta)$ as defined by Equation (17b) between two previous investigations and the current study when $\alpha = 0.5$ $(L = 1)$ for various N. From viewing the upper- and lower-error estimates, the results shown when N=12 appear to be accurate to ± 0.001.

Table 2 presents $f_N^4(\eta, t)$, where $\theta_N^4(\eta, t)$ is arrived at through Equation (16a), when $\alpha = 0.5$ $(L = 1)$ and N=12. In this case, the functions $\{c_m^N(t)\}_{m=0}^N$ defined by the differential equations in Equation (12) are resolved numerically using a conventional fully explicit, fifth-order, six-stage Runge-Kutta then the function $\Psi_N(\eta, t)$ is reconstructed through Equation (8). The time step used in presenting this table is $\Delta t = 0.2$, which represents a relatively large time step. It was found that the numbers presented here when compared to smaller time steps ($\Delta t = 0.1$, 0.05) converged to six places of accuracy with exception of a few (rare) occasions. In contrast, Prasad and Hering [1] often required time steps of up to 20 times smaller then used in the present study.

The alternative formulation described in this communication illustrates that an effective and accurate orthogonal collocation method can be conceived and applied to transient radiative transport.

	Heaslet-Warming	Prasad-Hering	Present Investigation, Equation (14).			
η			N = 6	N = 8	N = 10	N = 12
-1	0.756	0.760	0.758186	0.758159	0.758152	0.758149
-0.8	0.698	0.692	0.693707	0.694002	0.694616	0.694771
-0.6	0.646	0.642	0.642139	0.643344	0.642915	0.642706
-0.4	0.590	0.594	0.594817	0.594241	0.593962	0.594316
-0.2	0.551	0.545	0.547736	0.54636	0.547025	0.546715
0	0.500	0.499	0.5	0.5	0.5	0.5
0.2	0.449	0.452	0.452264	0.45364	0.452975	0.453285
0.4	0.410	0.405	0.405183	0.405759	0.406038	0.405684
0.6	0.354	0.355	0.357861	0.356656	0.357085	0.357294
0.8	0.302	0.305	0.306293	0.305998	0.305384	0.305229
1	0.244	0.240	0.241814	0.241841	0.241848	0.241851
Error Bounds for the present study, Equation (19):						
Upper-Error Estimate:			0.004533	0.002587	0.001667	0.001161
Lower-Error Estimate:			0.0008849	0.0005050	0.0003255	0.0002267

Table 1. Steady-state solution $\hat{f}_N^4(\eta)$ when $\alpha = 0.5$ and compared with previous investigations [1]. The error estimates make use of Equation (19).

η	t = 0.2	t = 1	t = 5	t = 15
-1	0.0622082	0.423464	0.749361	0.758148
-0.8	0.0425642	0.297264	0.682926	0.694769
-0.6	0.0326082	0.226019	0.628382	0.642704
-0.4	0.0261886	0.178075	0.577727	0.594314
-0.2	0.021283	0.141842	0.528047	0.546712
0	0.0176481	0.115012	0.479479	0.499997
0.2	0.014785	0.0942173	0.43123	0.453282
0.4	0.0124121	0.0774039	0.382579	0.40568
0.6	0.010571	0.0643619	0.333788	0.35729
0.8	0.00896476	0.0531695	0.282468	0.305225
1	0.00765155	0.0435793	0.222282	0.241848

Table 2. Transient distribution for $f_N^4(\eta, t)$ at equally spaced locations when $N = 12$ and $\alpha = 0.5$.

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