An Over-the-Counter Approach to the FOREX Market

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ABSTRACT

The FOREX market is an over-the-counter market (in fact, the largest in the world) characterized by bilateral trade, intermediation, and significant bid-ask spreads. The existing international macroeconomics literature has failed to account for these stylized facts largely due to the fact that it models the FOREX as a standard Walrasian market, therefore overlooking some important institutional details of this market. In this paper, we build on recent developments in monetary theory and finance to construct a dynamic general equilibrium model of intermediation in the FOREX market. A key concept in our approach is that immediate trade between ultimate buyers and sellers of foreign currencies is obstructed by search frictions (e.g., due to geographic dispersion). We use our framework to compute standard measures of FOREX market liquidity, such as bid-ask spreads and trade volume, and to study how these measures are affected both by macroeconomic fundamentals and the FOREX market microstructure. We also show that the FOREX market microstructure critically affects the volume of international trade and, consequently, welfare. Hence, our paper highlights that modeling the FOREX as a frictionless Walrasian market is *not* without loss of generality.

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1 Introduction

The foreign exchange (FOREX) market is an institution of paramount importance since it constitutes the channel through which international liquidity is allocated, thus assisting international trade and investment. Moreover, the FOREX market is an over-the-counter (OTC) market, in fact, the largest in the world, characterized by bilateral trade, intermediation, and bid-ask spreads (see Lyons (2001) and Burnside, Eichenbaum, and Rebelo (2009)). The traditional international macroeconomics literature has failed to account for these characteristics largely due to the fact that it models the FOREX as a standard, frictionless Walrasian market, therefore overlooking some important institutional details of this market. To remedy this deficiency we build on recent developments in monetary theory and finance to construct a dynamic general equilibrium model of intermediation in the FOREX market. We use this framework to compute explicitly standard measures of FOREX market liquidity, such as bid-ask spreads and trade volume, and to study how these measures are affected both by monetary policy and the FOREX market microstructure. Our model offers new and important insights in comparison to the conventional approach, since it allows us to examine how the FOREX trading frictions affect the volume of international trade and, consequently, welfare.

To motivate our story, consider an agent who resides in country A and wishes to acquire currency of country B (e.g., in order to purchase some goods or services from a firm in that country, buy some assets in that country, or go on vacation to that country). At the same time, an agent who resides in country B might wish to acquire currency of country A for similar reasons. If these two agents could contact each other, they might be able to carry out a mutually beneficial currency trade.[1] However, an immediate contact between the two agents might be difficult: the environment described here is clearly characterized by *search frictions*. If there exists a third party who can bypass these frictions, then intermediation services will arise naturally. This idea, which can be traced back to Rubinstein and Wolinsky (1987), seems especially relevant within the context of the FOREX market, given the difficulty of immediate trade among (ultimate) buyers and sellers of foreign currencies (e.g., due to geographic dispersion).

To formalize this idea, we develop a two-country, two-currency monetary-search model based on Lagos and Wright (2005) (henceforth, LW). Due to frictions, such as anonymity and limited commitment, trade of home goods in each country necessitates the use of local currency.[2] Agents work in their home country and receive local currency which they use to purchase home goods, but they can also exchange for foreign currency in the FOREX market,

[1] This argument implicitly assumes the absence of a global centralized market for currencies, which is the case in practice. In this paper, we remain agnostic as to why such a market does not exist, and choose to model the FOREX market in an empirically relevant way, i.e., as a decentralized OTC market.

[2] Strictly speaking, these frictions make the use of a medium of exchange necessary, and agents are free to carry out transactions using either currency. The fact that only local currency is accepted in home goods trade is a result of the model rather than an assumption. For more details, see Section 2.1.

should an opportunity to consume abroad arise. We model the FOREX market as an OTC market following the influential work of Duffie, Gârleanu, and Pedersen (2005) (henceforth, DGP). In this market, agents who wish to acquire foreign currency meet with FOREX intermediators or dealers in a bilateral fashion and privately negotiate over the terms of trade. A key feature of the model is that FOREX dealers can participate in a well-networked interdealer market, which guarantees access to a large pool of agents (i.e., traders represented by other dealers) who offer what their clients are searching for, namely, foreign currency. The unique ability of dealers to access this frictionless market is precisely what allows them to bypass the search frictions that obstruct direct trade among agents, and, hence, charge positive bid-ask spreads.

The model delivers closed form solutions for the dealers' bid and ask prices, and allows us to study how the spread is affected both by market microstructure (e.g., dealer availability and agents' bargaining positions), and by macroeconomic fundamentals (e.g., inflation). We find that the spread is negatively related to dealer availability. A lower ex ante likelihood of contacting a dealer discourages agents from carrying large amounts of real home money balances (which is costly), thus, making them more liquidity constrained and increasing their marginal benefit from consuming foreign goods. As a result, these agents are willing to give up more units of home currency in exchange for one unit of foreign currency, which allows dealers to extract higher fees. Since the ease with which agents can contact a dealer is typically interpreted as a measure of market liquidity, this result implies that bid-ask spreads will be tighter in more liquid markets, a finding which is well-established both in the theoretical and the empirical finance literature (e.g., see DGP and the references therein). However, the channel through which this result emerges in our framework is quite different. For instance, in DGP, a higher probability of contacting a dealer effectively increases the agent's bargaining power (and tightens the spread) by making access to alternative trading partners easier. In our paper, a higher probability of contacting a dealer effectively increases the agent's bargaining power by making her less liquidity constraint, and, thus, less eager to acquire foreign currency in the FOREX market.

We also find that the bid-ask spread is increasing in the dealers' bargaining power. An increase in the dealers' bargaining power induces agents to carry a larger amount of real home money balances into the FOREX market, because they realize that they will now need to give up more units of home currency to acquire one unit of foreign currency. As we have already seen, this tends to make agents less liquidity constrained, and effectively improve their bargaining position. However, while the typical agent carries more (real) home currency, a disproportionately large fraction of this currency is collected by the dealer as a fee, and, ultimately, the net effect on the bid-ask spread is positive.

In addition, we show that an increase in anticipated inflation in the home country leads to a wider bid-ask spread. Given the earlier discussion, this result is quite intuitive. Since inflation in the home currency captures the cost of carrying real home money balances, a higher inflation makes home agents more liquidity constrained and increases the marginal benefit of consuming foreign goods. Put simply, a high rate of inflation in the home country makes agents more desperate for the foreign good and, hence, the foreign currency, and effectively worsens their bargaining position allowing dealers to extract higher fees.

Our model also has interesting implications for the FOREX trade volume. We characterize trade volume at both layers of the FOREX market, i.e., agent-dealer and interdealer trade, and we find a positive correlation between dealer availability and trade volume at both levels. Like before, an increase in the ex ante likelihood of contacting a dealer raises the amount of real home money balances that agents bring into the FOREX market. As a result, in any given agent-dealer meeting, a larger volume of currencies change hands, but, moreover, dealers (who represent agents who now have a higher demand for foreign currency) place larger orders for foreign currencies in the interdealer market. Except from this *indirect* positive effect on the intensive margin, an increase in dealer availability also *directly* increases the extensive margin of agent-dealer trade volume, since it implies a higher number of agent-dealer matches.

Interestingly, changes in the dealers' bargaining position affect the volume at the two layers of FOREX trade differently. For instance, consider an increase in the dealers' bargaining power, which, as we saw, induces agents to carry more real home money balances into the FOREX market. Since an even larger fraction of the (higher) real balances is now reaped by the dealers, the effect of such change on the agent-dealer trade volume is undoubtedly positive. However, since a large fraction of real balances ends up directly in the dealers' pockets as a fee, the amount of currencies that get re-shuffled through the interdealer market, i.e., the interdealer trade volume, decreases. Finally, we show that a higher inflation in either country lowers the trade volume at both layers of FOREX trade through the usual negative effect on real balances.

To the best of our knowledge, this is the first paper that models the FOREX market as a decentralized OTC market within a dynamic general equilibrium framework. However, modeling the FOREX market in an empirically relevant way is not an end in itself. We show that the FOREX market microstructure critically affects the volume of international trade and, consequently, welfare. Hence, our paper highlights that modeling the FOREX as a frictionless Walrasian market is *not* without loss of generality. We find that as dealer availability improves, the equilibrium real home money balances increase, and agents can afford to purchase more foreign currency, and, hence, more foreign goods. Thus, a more liquid FOREX market boosts the volume of international trade and improves welfare. An increase in the dealers' bargaining power has even more interesting effects. On the one hand, it hurts agents who obtain a foreign consumption opportunity, because these agents now have to pay higher intermediation fees. On the other hand, it may benefit agents who do not obtain such an opportunity (ex post), because it induces them to carry a larger amount of real home money balances in anticipation of the higher fees (ex ante). We show that if inflation is relatively high, and foreign consumption opportunities and/or dealer availability are relatively low, the second (positive) effect dominates, and an increase in the dealers' bargaining power can actually improve social welfare.

1.1 Related Literature

After the collapse of the Bretton Woods System in the early 70s, advanced economies started adopting a floating exchange rate regime, which spurred a large literature on the FOREX rate determination. Some early works include Dornbusch (1976), Lucas (1982), and Meese and Rogoff (1983). These seminal papers, and the ones inspired by them, are useful to study the effect of macroeconomic fundamentals (e.g., inflation and productivity in each country) on the determination of the FOREX rate. However, the international macroeconomics literature typically models the FOREX market as a perfectly competitive market, therefore overlooking some important institutional details of this market, such as intermediation, spreads, etc.

As a response, a new approach, often referred to as the FOREX microstructure literature, has emerged over the last two decades. Influential works in this dimension of research include Admati and Pfleiderer (1988), Ito, Lyons, and Melvin (1998), Evans and Lyons (2002), and Evans and Lyons (2005). In this literature, the role of intermediation in the FOREX market is explicitly studied, and it arises due to the presence of frictions, such as adverse selection and inventory costs.[3] Although the microstructure literature has given us fruitful insights on many aspects of the FOREX rate determination which had been overlooked by the international macroeconomics literature, it has itself neglected the role of macroeconomic fundamentals, which is arguably very important for the determination of exchange rates in the long-run.

Our paper can be viewed as an attempt to bridge the gap between the two strands of the literature.[4] Modeling the FOREX market as an OTC market within a dynamic general equilibrium framework allows us to study questions that neither of the two existing approaches can study in isolation. For instance, our model can be used to examine the effect of monetary policy on standard measures of FOREX market liquidity, such as bid-ask spreads. This would not be possible within the international macroeconomics literature, because, as is well-known, in a Walrasian market there is no room for intermediation and spreads. Also, studying this question would not be possible within the microstructure literature, because the majority of these papers adopt a partial equilibrium approach, where foreign currency is not explicitly modeled as money whose holding cost is controlled by monetary policy. Similarly, our paper offers a framework for studying how the FOREX market microstructure can affect international trade and welfare, which would not be possible within either of the existing strands of the literature.

In addition, our search-based approach to modeling intermediation in the FOREX market sets this paper aside from the microstructure literature, where intermediation typically arises

[3] The first is based on the idea that some information relevant to exchange rates is not publicly available. In the presence of asymmetric information, intermediaries can arise and charge bid-ask spreads due to their ability to buffer against adverse selection (for example, see Copeland and Galai (1983), Glosten and Milgrom (1985) and Kyle (1985)). The inventory cost based models revolve around the idea that intermediaries can provide immediacy (i.e., guarantee of fast service) in an environment where holding positive inventories is costly (for example, see Amihud and Mendelson (1980) and Ho and Stoll (1981)).

[4] Lyons (2001) estimates that “the uneasy dichotomy between macro and micro approaches is destined to fade”. Nevertheless, even today the gap between the macroeconomic and the microstructure approach remains large.

due to the existence of adverse selection or inventory costs.[5] Although these frictions seem relevant within the context of the FOREX market, we believe that the search frictions approach is also extremely relevant, given the inherent difficulty of immediate trade among buyers and sellers of foreign currencies, mainly, but not exclusively, due to the geographic dispersion of these agents (see Section 2.1 for a more detailed discussion). Hence, it is somewhat surprising that this simple idea has not been formally described in the FOREX market literature before.

Methodologically our paper is closely related to Lagos and Zhang (2014), who also develop a monetary-search model augmented to include OTC financial trade, and use it to study the effect of monetary policy on asset prices and the OTC market liquidity. Our model extends this framework to an open-economy setting in order to specifically study the performance of the FOREX market, and how it affects international trade and welfare. Geromichalos and Herrenbrueck (2012) consider a model where agents can allocate their wealth between money and an illiquid asset, and, following an idiosyncratic consumption shock, they can acquire additional liquidity in an OTC financial market. The present paper has similar structure since agents who get an opportunity to consume abroad can acquire foreign currency in the OTC FOREX market. Our paper is also related to Trejos and Wright (2012), who develop a framework that nests the DGP model into a "second-generation" monetary-search model (e.g. Shi (1995) and Trejos and Wright (1995)) and discuss similarities and differences between the two literatures.

The present paper is closely related to a number of works that employ monetary-search models to address long-standing questions in international macroeconomics. For instance, Matsuyama, Kiyotaki, and Matsui (1993) develop a two-country search model and study the conditions under which the two currencies arise as media of exchange in different countries. Wright and Trejos (2001) study the same question, but they employ a second generation monetary-search model where prices are endogenized using bargaining theory. Head and Shi (2003) develop a two-country model and show that the nominal exchange rate depends on the stocks and growth rates of the two monies. More recently, Geromichalos and Simonovska (2014) build a two-country model where assets can help agents facilitate international transactions, and use it to rationalize the well-known asset home bias puzzle. Zhang (2014) develops an information-based theory of international currency, and shows that the threat of losing international status imposes an inflation discipline on the issuing country. Finally, Bignon, Breton, and Rojas Breu (2013) build a two-country model of currency and endogenous default to study whether impediments to credit market integration can affect the desirability of a currency union.[6]

The remainder of the paper proceeds as follows. Section 2 describes the physical environment and the salient features of the FOREX market that our model aims to capture. Section 3 studies the agents' optimal behavior. Section 4 defines a stationary equilibrium in the two-

[5] In that respect, our theory of intermediation is closer to Shevchenko (2004) and Wright and Wong (2014).

[6] Jung and Pyun (2015) and Jung and Lee (2015) also study how liquidity premia of assets can rationalize other international asset pricing puzzle such as excess international reserves hoarding and violation of the uncovered interest parity puzzle respectively.

country model, and describes how the key variables are affected by changes in macroeconomic fundamentals and the FOREX market microstructure. Section 5 concludes.

2 Physical Environment

Time is infinite and discrete. There are two countries, A and B. Each country has a unit measure of buyers, and sellers with a measure equal to $1+\delta$, $\delta \in [0,1]$. The identity of buyers and sellers is fixed over time. We will use the terms "buyer (seller) from country i" and "buyer (seller) i" interchangeably. There exists a third type of agents called *dealers* with a measure of v. Dealers have no national identity. All agents are infinitely lived and discount future at rate $\beta \in (0,1)$. Three divisible and non-storable consumption goods are produced and traded: a general good produced by all agents and a special good i produced only by sellers in each country $i \in \{A, B\}$. Each country's monetary authority issues a perfectly divisible and storable fiat currency, which we will refer to as $money_i$, $i \in \{A, B\}$. Let $A_{m_i,t}$ denote the stock of $money_i$ at time t. The money stock is initially given by $A_{m_i,0} \in \mathbb{R}_{++}$, and thereafter it grows at a constant rate γ_i (i.e., $A_{m_i,t+1} = \gamma_i A_{m_i,t}$), where $\gamma_i \geq \beta$ is chosen by the monetary authority in country i. New $money_i$ is introduced (if $\gamma_i > 1$) or withdrawn (if $\gamma_i < 1$) via lump-sum transfers to buyers of country i at the end of every period.

Each period is divided into three subperiods characterized by different economic activities. We begin with an intuitive description of the environment. A formal decription of each subperiod will follow. In the third subperiod, agents trade in perfectly competitive or Walrasian markets. This subperiod can be thought of as the settlement stage, where agents from each country work and choose a portfolio of (local) money holdings to bring with them in the following period. In the second subperiod, trade takes place in decentralized markets characterized by anonymity and imperfect credit. Due to these frictions, trade in this subperiod necessitates the use of a medium of exchange (MOE). Agents who wish to acquire foreign currency, in order to purchase foreign goods during the round of decentralized trade, can do so in the FOREX market which opens in the first subperiod of each period. Hence, the FOREX market is strategically placed before the decentralized goods markets open, but after agents have found out whether they have an opportunity to consume the foreign (special) good in the current period.

We now proceed to a formal description of the subperiods, starting with the third one and moving backwards. In the third subperiod, all agents have access to a technology that allows them to transform a unit of labor into a unit of general good. Buyers and sellers from country i trade the general good with $money_i$ within country i's spot Walrasian or centralized market (henceforth, CM_i). The two CMs are distinct from each other: agents from country i cannot participate in CM_{-i}, and $money_i$ is not traded in CM_{-i}. However, dealers can access both CMs, which plays a key role in their ability to serve as intermediators in the FOREX market. At the end of the third subperiod, a fraction $\delta \in [0, 1]$ of buyers obtain an opportunity to consume the foreign special good in the forthcoming period. These buyers are referred to as the C-types, and the rest are referred to as the N-types. *All buyers* get to consume the local special good.

In the second subperiod, a distinct decentralized market opens in each country (henceforth, DM_i). In DM_i, local sellers and buyers, who might be locals or foreigners, trade special good i. Within any DM, trade is bilateral and anonymous, and buyers cannot commit to repaying their debt. Thus, all trade has to be *quid pro quo*. When a seller meets a foreign buyer, the buyer can, in principle, pay the seller with a combination of local and foreign currency. However, as we have seen, seller i cannot visit CM_{-i}, and, therefore, she will not accept foreign currency as payment.[7] Hence, although we do not make assumptions that explicitly preclude $money_{-i}$ from serving as a MOE in DM_i, it turns out that only local currency will serve as a MOE in each DM. This implies that C-type buyers $-i$ who did not acquire $money_i$ in the FOREX market (see next paragraph), will not participate in DM_i. Thus, in any DM, the measure of sellers $(1 + \delta)$ is weakly greater than the measure of buyers, and we assume that all buyers (i.e., agents on the short side of the market) match with a seller. The probability with which a seller matches with a local or foreign buyer only depends on the relative measures of these two groups. Finally, within any given match, buyers make a take-it-or-leave-it (TIOLI) offer to the seller.

Given the discussion so far, it follows that C-type buyers want to acquire foreign currency before DM trade begins. Interestingly, buyers from country i hold precisely what (C-type) buyers from country $-i$ need: $money_i$. However, to make things interesting *and* realistic, we assume that immediate trade between these agents is impossible. Buyers who wish to acquire foreign currency have to visit the FOREX market which operates in the first subperiod. Following DGP, we model this market as an OTC market characterized by search and bilateral trade between dealers and buyers. Let $\alpha_D \in [0, 1]$ denote the probability with which the typical dealer contacts a buyer in the FOREX, so, by symmetry, $\alpha_D/2$ is the probability that this buyer is a citizen of country i, $i \in \{A, B\}$. Similarly, $\alpha_i \in [0, 1]$ represents the probability with which buyer i contacts a dealer. Within any given buyer-dealer pair, the terms of trade are determined through proportional bargaining (Kalai (1977)), with $\theta \in [0, 1]$ denoting the dealer's bargaining power.

When a dealer meets with a C-type buyer $-i$ she can provide that agent with $money_i$ that comes from two potential sources. First, the dealer may carry some $money_i$ that she acquired in the preceding CM_i (recall that a dealer can visit both CMs). Second, the dealer has immediate access to a perfectly competitive *interdealer market*, where she can acquire $money_i$, at the ongoing market price, from dealers who either contacted buyers i or carry $money_i$ on their own account.

[7] Given that currencies are fiat, an agent will hold money only for two reasons: i) to use it as a MOE to purchase special goods in the DM round of trade; or ii) to sell it for the general good in the CM round of trade. Since the identity of agents is fixed, a seller will never hold money in order to use it as a MOE. A seller might be willing to accept money as a MOE in the DM, if she hopes to trade it for some general good in the forthcoming CM. But, by assumption, seller i never gets to visit CM_{-i}, i.e., the market where $money_{-i}$ is traded for general good. Hence, a seller's valuation for the foreign currency is zero, and, she will never accept it as a MOE.

The ability of dealers to access this frictionless market is precisely what allows them to bypass the search frictions that obstruct direct currency trade between buyers.[8] This assumption is also consistent with the fact that, in practice, FOREX dealers have access to a well-networked inter-dealer market. We have now finished describing all three subperiods. The timing of events is summarized in Figure 1.

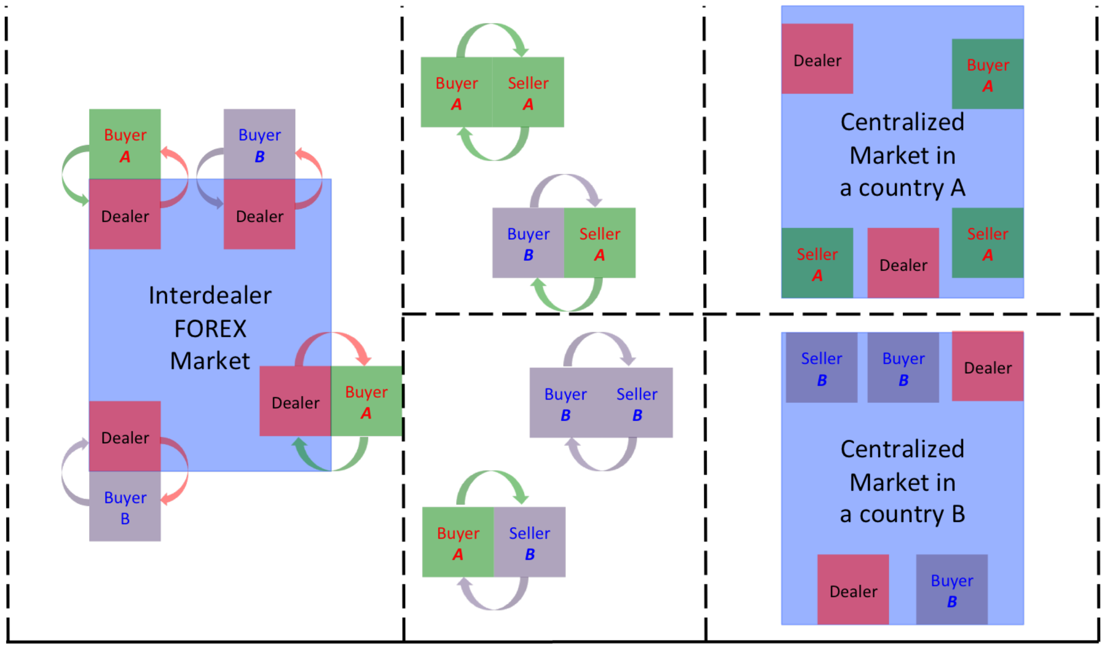


Figure 1: Timing of trading process

Finally, consider agents' preferences. The utility of the typical buyer i is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(q_t) + u(\tilde{q}_t) + X_t - H_t \right\},$$

where q_t ($\tilde{q}_t$) denotes consumption of the local (foreign) special good in the second subperiod of period t. The terms X_t and H_t stand for the consumption of general good and the effort to produce that good in the third subperiod of period t, respectively.[9] We assume that $u(\cdot)$ is twice continuously differentiable, with $u(0) = 0$, $u'(\cdot) > 0$, $u'(0) = \infty$, $u'(\infty) = 0$, and $u''(\cdot) < 0$. The term $\mathbb{E}_0$ denotes the expectation with respect to the probability measure induced by the random trading process in the DMs. The utility of the typical seller i is given by

[8] Assuming that dealers have access to a competitive interdealer market is standard in the recent literature on OTC financial trade; e.g., see Weill (2011) and Lagos, Rocheteau, and Weill (2011).

[9] All the results will remain unaltered if one assumes quasi-linear (instead of linear) preferences, as in LW. The linear utility specification is consistent with the interpretation of the CM as a pure liquidity or settlement market; e.g., see Chiu and Molico (2010).

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ -q_t + X_t - H_t \right\},$$

where $X_t, H_t, \mathbb{E}_0$ are as above, and $-q_t$ is the disutility of producing q_t units special good in the second subperiod of t (i.e., without loss of generality, we assume that the disutility is linear). Finally, the utility of the typical dealer (who does not participate in the DM) is given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ X_t - H_t \right\},$$

where all the symbols have already been explained.

2.1 Discussion of the Physical Environment

Since this is (to the best of our knowledge) the first open-economy monetary model that specifies the FOREX market explicitly as a decentralized OTC market, some of the model's assumptions might appear to be non-standard, and, hence, deserve some discussion.

For instance, we assume that agents from country i cannot visit CM_{-i}, where $money_{-i}$ is traded in a competitive environment.[10] This assumption aims to capture the simple and empirically relevant idea that citizens of country i live and work in their home country, get paid in local currency, and, whenever they have a need to purchase foreign goods, they exchange local for foreign currency within a special institution known as the FOREX market. If agents from country i were allowed to acquire $money_{-i}$ in CM_{-i} (which, recall, is a perfectly competitive market), this would defeat the very purpose of this paper, which is to explicitly model the FOREX as an OTC market characterized by bilateral trade and intermediation.

We have also seen that only $money_i$ serves as a MOE in DM_i. Strictly speaking, this is a result rather than an assumption of the model, and it follows directly from the assumption that sellers i cannot visit CM_{-i} (see footnote 7), which, in turn, is adopted for the reasons described in the previous paragraph. However, we do not claim that our paper offers a "deep" theory of which assets serve as MOE in various types of international meetings, because this is not the question that we are after. Zhang (2014) asks precisely this question (among others) and shows that, if sellers must pay a relatively high cost to verify the genuineness of foreign currency (an assumption which is quite reasonable), then a "local currency dominance" equilibrium, where $money_i$ serves exclusively as a MOE in DM_i, will arise *endogenously*.

In our model, intermediation in the FOREX market arises because direct currency trade between buyers from the two countries is difficult. One can think of buyer i as a Swiss importer

[10] This assumption is new even within the class of models that employ two-country versions of LW. For instance, Geromichalos and Simonovska (2014) interpret CM_i as an analogue of *E-trade*, where agents from all over the world can purchase assets of country i. In Zhang (2014), there is only one CM, in which agents from both countries trade the two currencies. In other words, in Zhang (2014), the CM *is* the FOREX market.

of US computers, and buyer $-i$ as an American importer of Swiss chocolate. Since the former agent wishes to transform Swiss francs into dollars and the latter wishes to purchase Swiss francs with her dollars, it seems like the two agents could carry out a mutually beneficial currency trade. However, contacting each other and carrying out this trade is clearly difficult for the two buyers in our example.[11] In this environment, the buyers are happy to purchase foreign currency from FOREX dealers, and the dealers can charge an intermediation fee which reflects their ability to access a well-networked (and practically competitive) interdealer market, and, hence, bypass the frictions that prevent direct trade between currency buyers.[12]

This discussion clarifies that the search frictions, which play a central role in our analysis, refer to the difficulty of agent i to contact agent $-i$ and purchase foreign currency directly from her, and not (necessarily) to the difficulty of agent i to contact a dealer. The latter is simply captured by the parameter $\alpha_i \in [0, 1]$. Thus, if a buyer of currency can find a dealer fairly quickly, in the context of our model, this would imply a large value of α_i.[13] It is also important to clarify that trade among dealers is not characterized by any frictions whatsoever (dealers trade with each other in a perfectly competitive interdealer market). This is precisely why our model predicts the existence of positive bid-ask spreads in dealer-customer transactions, but no spreads in interdealer transactions, which is consistent with the empirical observation (for instance, see Lyons (2001)).

To conclude this section, we briefly discuss how two of the most important parameters of the model, namely α_i and θ, map into the real-world FOREX market. As we have already described, α_i captures the degree of dealer availability in that market. Therefore, one can interpret the recent emergence of many online FOREX dealers and multilateral trading facilities (MTFs) as an increase in α_i.[14] The term θ stands for the bargaining power of the dealers in a typical buyer-dealer match. Hence, generally speaking, this parameter reflects the negotiating strength of the dealers, but, more specifically, it can be thought of as a shortcut for capturing certain market conditions which we have not explicitly modeled. For instance, in our model the

[11] Even if the search friction *per se* is not immense enough to preclude contact between the two parties, there are other reasons, such as the asymmetric demand for foreign currencies, that can render such trade difficult. For instance, even if it is relatively easy for the two buyers to contact each other, say, through the internet, it might still be the case that the Swiss importer of computers seeks to purchase 10 million US dollars, while the American importer of chocolate only wants to convert 100,000 USD into Swiss francs. In this case, the Swiss importer would have to search for multiple trading partners, which would make the overall transaction complicated.

[12] Here, for the sake of simplicity, we have assumed that currency trade between buyers from the two countries is *impossible*. However, all one needs to assume in order to generate a role for intermediation in the FOREX market, is that direct trade between these agents is (just) *difficult*. For instance, one could assume that the C-type buyer i first attempts to trade directly with buyer $-i$, and then, if that attempt proves unsuccessful, she resorts to dealers' services. This alternative specification would make the analysis more cumbersome, but it would not add any interesting insights to the model.

[13] In fact, most of the results of the paper would remain valid even in the extreme case with $\alpha_i = 1$. This is true because setting $\alpha_i = 1$ does not imply that we are in a "frictionless" environment, since, as we just explained, the friction here concerns the difficulty to buy currency directly from foreign agents and not the availability of dealers.

[14] Examples of such online dealers include FXall, Atriax, FXchange, FXconnect, FXtrade, Gain.com, MatchbookFX, FOREX, LMAX.com, and others.

measure of dealers is fixed, however, θ may capture the degree of competition for customers among dealers (i.e., a low θ can be interpreted as a market where many dealers compete for few currency customers). Similarly, in our model we have excluded direct currency trade between buyers from different countries (see footnote 12), however, θ may capture the degree to which it is possible for buyer i to contact (and directly trade with) buyer $-i$ (i.e., a high θ can be interpreted as a market where direct trade between ultimate buyers and sellers of currency is extremely difficult).

3 Value Functions and Optimal Behavior

3.1 Value Functions

Consider first the typical buyer i, who enters CM_i with m_i units of home currency. For this agent, the Bellman's equation satisfies[15]

$$\begin{gathered} W_i^B(m_i) = \max_{X,H,\widehat{m}_i} \left\{ X - H + \beta \mathbb{E}_k \{ \Omega_i^k(\widehat{m}_i) \} \right\} \\ s.t. \quad X + \varphi_i \widehat{m}_i = H + \varphi_i m_i + T_i, \end{gathered}$$

where variables with a hat indicate next period's choices. The term φ_i denotes the price of $money_i$ in terms of the general good, and T_i is the real value of the lump-sum monetary transfer by the monetary authority of country i. The function Ω_i^k represents the value function in the FOREX market for a buyer of type $k = C, N$. Eliminating H from the budget constraint yields

$$\begin{aligned} W_i^B(m_i) &= W_i^B(0) + \varphi_i m_i, \\ W_i^B(0) &\equiv T_i + \max_{\widehat{m}_i} \left\{ -\varphi_i \widehat{m}_i + \beta \mathbb{E}_k \{ \Omega_i^k(\widehat{m}_i) \} \right\}, \quad k \in \{C, N\}. \end{aligned} \tag{1}$$

As is standard in models that build on LW, the buyer's value function is linear in the money holdings, implying that there are no wealth effects on the choice of $\widehat{m}_i$.

Next, consider the CM value function for the typical seller i. This agent will never want

[15] A buyer who acquired some foreign currency in the FOREX market of period t, may fear that she might not be able to trade again in that market in $t + 1$ (given the search frictions). Hence, she might choose to not spend all of her foreign currency in the DM of period t, but instead keep some to trade in the DM of $t + 1$. The problem with this type of strategy is that, if buyers (in period t) can keep some foreign currency to trade in the DM of $t + 1$, they can do the same for the DM of $t + 2, t + 3$, and so on. Allowing this type of strategy would make the model intractable. Hence, we require that buyers spend all the foreign currency acquired in the FOREX market of period t in the DM of that period. This implies that as these agents enter their home CM they can only hold home currency. Although here, for the sake of simplicity, we assume that buyers *must* spend all their foreign currency in the current DM, as long as the probability of meeting a dealer, α_i, is large enough, buyers will actually *choose* to do so. In other words, under certain (and quite reasonable) parameter values, the assumption in question would arise endogenously as a result.

to leave the CM with any money holdings, since she does not want to consume in the DM round of trade (see Rocheteau and Wright (2005) for a rigorous proof). The seller will typically hold some home currency, which she received as payment (either from a local or from a foreign buyer) in the DM. For this agent, the Bellman's equation satisfies

$$\begin{aligned} W_i^S(m_i) &= \max_{X,H} \left\{ X - H + \beta V_i^S(0) \right\} \\ s.t. \quad X &= H + \varphi_i m_i, \end{aligned}$$

where $V_i^S(0)$ denotes the seller's value function in DM_i. Replacing H from the budget constraint into W_i^S yields

$$W_i^S(m_i) = \beta V_i^S(0) + \varphi_i m_i.$$

A dealer is the only type of agent who might enter the third subperiod with a portfolio that contains both currencies, $\mathbf{m} \equiv (m_A, m_B)$. For this agent, the Bellman's equation satisfies

$$\begin{aligned} W^D(\mathbf{m}) &= \max_{X,H,\widehat{\mathbf{m}}} \left\{ X - H + \beta \Omega_D(\widehat{\mathbf{m}}) \right\} \\ s.t. \quad X + \boldsymbol{\varphi}\widehat{\mathbf{m}} &= H + \boldsymbol{\varphi}\mathbf{m}, \end{aligned}$$

where $\Omega_D(\widehat{\mathbf{m}})$ denotes value function of a dealer who enters the FOREX market with portfolio $\widehat{\mathbf{m}}$. Also, we define $\boldsymbol{\varphi} \equiv (\varphi_A, \varphi_B)$, and we let $\boldsymbol{\varphi}\mathbf{m}$ denote the dot product of $\boldsymbol{\varphi}$ and $\mathbf{m}$. Eliminating H from the budget constraint implies that

$$\begin{aligned} W^D(\mathbf{m}) &= W^D(\mathbf{0}) + \boldsymbol{\varphi}\mathbf{m}, \\ W^D(\mathbf{0}) &\equiv \max_{\widehat{\mathbf{m}}} \left\{ -\boldsymbol{\varphi}\widehat{\mathbf{m}} + \beta \Omega_D(\widehat{\mathbf{m}}) \right\}. \end{aligned} \tag{2}$$

Before we proceed to the description of the FOREX market value functions, we introduce two useful definitions. First, let $\widetilde{W}^D(\mathbf{m})$ denote the continuation value of a dealer who just (optimally) rebalanced her portfolio, $\mathbf{m}$, in the interdealer FOREX market. This function satisfies

$$\begin{aligned} \widetilde{W}^D(\mathbf{m}) &= \max_{\widetilde{\mathbf{m}}} W^D(\widetilde{\mathbf{m}}) \\ s.t. \quad \widetilde{m}_A + \varepsilon \widetilde{m}_B &\leq m_A + \varepsilon m_B, \end{aligned} \tag{3}$$

where $\widetilde{\mathbf{m}} \equiv (\widetilde{m}_A, \widetilde{m}_B)$ denotes the post-interdealer market portfolio of the dealer, and the constraint requires that the value of the post-interdealer market portfolio (measured in terms of $money_A$) cannot exceed the value of the pre-interdealer market portfolio. Note that ε is defined as the price of $money_B$ in terms of $money_A$ in the interdealer FOREX market. Hence, an increase (decrease) in ε is equivalent to a depreciation of $money_A$ ($money_B$).

Second, consider a meeting between a dealer who enters the FOREX market with portfolio $\mathbf{m}^d$ and a buyer from country i who carries m_i units of home currency. Then, the terms

$$\left[\bar{m}_A^i\left(m_i,\varepsilon,\boldsymbol{\varphi}\right),\bar{m}_B^i\left(m_i,\varepsilon,\boldsymbol{\varphi}\right)\right],$$
$$\left[\bar{m}_A^d\left(\mathbf{m}^d,m_i,\varepsilon,\boldsymbol{\varphi}\right),\bar{m}_B^d\left(\mathbf{m}^d,m_i,\varepsilon,\boldsymbol{\varphi}\right)\right]$$

denote the portfolios that the buyer and the dealer (respectively) hold after all FOREX trading has concluded (i.e., it includes the buyer-dealer trade and the interdealer market trade).[16]

Now consider the expected FOREX value function for the tyipcal buyer i who enters the first subperiod with m_i units of home currency. This functions satisfies

$$\mathbb{E}_k\{\Omega_i^k(m_i)\}=\delta\Omega_i^C(m_i)+(1-\delta)V_i^n(m_i), \tag{4}$$

where $\Omega_i^C(m_i)$ denotes the FOREX value function for a buyer i who gets to consume the foreign special good in the forthcoming DM_{-i}, and $V_i^n(m_i)$ denotes the value function for a buyer i who proceeds to DM_i with her original (home) money holdings. Furthermore, we have

$$\Omega_i^C(m_i)=\alpha_i V_i^y\left(\bar{m}_A^i,\bar{m}_B^i\right)+(1-\alpha_i)V_i^n(m_i), \tag{5}$$

where V_i^n has been explained earlier, and $V_i^y(\bar{m}_A^i,\bar{m}_B^i)$ denotes the DM value function for buyer i, conditional on having met with a dealer in the preceding FOREX market.

The value fuunction for a dealer who enters the FOREX market with portfolio $\mathbf{m}^d$ satisfies[17]

$$\begin{aligned}\Omega_D(\mathbf{m}^d)=&(1-\alpha_D)\widetilde{W}^D(\mathbf{m}^d)\\&+\frac{\alpha_D}{2}\int\widetilde{W}^D\left(\bar{m}_A^d\left(\mathbf{m}^d,m_A,\varepsilon,\boldsymbol{\varphi}\right),\bar{m}_B^d\left(\mathbf{m}^d,m_A,\varepsilon,\boldsymbol{\varphi}\right)\right)dF^A\left(m_A\right)\\&+\frac{\alpha_D}{2}\int\widetilde{W}^D\left(\bar{m}_A^d\left(\mathbf{m}^d,m_B,\varepsilon,\boldsymbol{\varphi}\right),\bar{m}_B^d\left(\mathbf{m}^d,m_B,\varepsilon,\boldsymbol{\varphi}\right)\right)dF^B\left(m_B\right),\end{aligned} \tag{6}$$

where F^i is the cumulative distribution function over the $money_i$ holdings of the random buyer i whom the dealer may contact in the FOREX market.

Finally, consider the value functions in the DM round. For a C-type buyer i who matched

[16] The exact solutions for the post-trade portfolios will be rigorously analyzed in Section 3.2. For now, we use the terms $\bar{m}_A^i$ and $\bar{m}_A^i\left(m_i,\varepsilon,\boldsymbol{\varphi}\right)$, or $\bar{m}_A^d$ and $\bar{m}_A^d\left(\mathbf{m}^d,m_i,\varepsilon,\boldsymbol{\varphi}\right)$, etc., interchangeably.

[17] As we have already explained, the various $\widetilde{W}^D$ terms denote the continuation value for a dealer who rebalances her portfolio in the interdealer FOREX market. However, the composition of that portfolio critically depends on whether the dealer matched with a buyer, and, if yes, on the buyer's citizenship and her money holdings. So, for example, the first line on the RHS of eq.(6) describes the event in which the dealer does not contact any buyer and proceeds to the interdealer market with her original portfolio, $\mathbf{m}^d$. The second line describes the event in which the dealer contacts a buyer from country A, in which case she proceeds to the interdealer market with portfolio $\left(\bar{m}_A^d\left(\mathbf{m}^d,m_A,\varepsilon,\boldsymbol{\varphi}\right),\bar{m}_B^d\left(\mathbf{m}^d,m_A,\varepsilon,\boldsymbol{\varphi}\right)\right)$ (the third line admits a similar interpretation).

with a dealer in the preceding FOREX market and carries a portfolio (m_A, m_B), we have[18]

$$V_i^y(m_A, m_B) = u(q) + u(\tilde{q}) + W_i^B\left(m_i - p\right), \tag{7}$$

where q ($\tilde{q}$) denotes the consumption of local (foreign) special good, and p the units of $money_i$ that the buyer transfers to seller i in DM_i. These terms will be determined in Section 3.2. Furthermore, for the typical buyer i who only participates in her home DM (either because she is an N-type or because she did not contact a dealer in the FOREX market), we have

$$V_i^n(m_i) = u(q) + W_i^B\left(m_i - p\right). \tag{8}$$

The DM value function for seller i, who enters DM_i with no money, is given by

$$V_i^S = \frac{1}{1+\delta}\left[-q + W_i^S\left(p\right)\right] + \frac{\delta\alpha_{-i}}{1+\delta}\left[-\tilde{q} + W_i^S\left(\tilde{p}\right)\right] + \frac{\delta(1-\alpha_{-i})}{1+\delta}W_i^S\left(0\right), \tag{9}$$

where q, p denote the production of special good, and the units of $money_i$ exchanged in a meeting with a local buyer, and $\tilde{q}, \tilde{p}$ are the analogue expressions for a meeting with a foreign buyer. The expression $\delta(1-\alpha_{-i})/(1+\delta)$ in the third term on the RHS of eq.(9) is the probability with which the seller does not match with a buyer, in which case she proceeds to CM_i with no money.

3.2 Terms of Trade

In this section, we study the determination of the terms of trade in the various markets. First, consider a meeting in DM_i between a seller i and a buyer i (a local) who carries m_i units of local currency. The two parties negotiate over a quantity of special good, q, to be produced, and an amount of $money_i$, p, to be delivered to the seller. Given that the buyer makes a TIOLI offer to the seller, the bargaining problem can be expressed as[19]

$$\max_{p,q}\{u(q) + W_i^{\mathcal{B}}(m_i - p) - W_i^{\mathcal{B}}(m_i)\}$$
$$s.t. \quad q = W_i^{\mathcal{S}}(p) - W_i^{\mathcal{S}}(0),$$

[18] As we have already argued, the buyer will spend all her foreign currency in DM_{-i}, hence, the only argument inside W_i^B is home money holdings.

[19] At this stage the buyer has already made her portfolio decisions (i.e., how much home currency to keep and how much to exchange for foreign currency). Hence, all that matters for the bargaining problem is the buyer's local money holdings, and not whether she is a C or an N-type and/or whether she traded in the FOREX market.

and the feasibility constraint $p \leq m_i$. Given the linearity of $W_i^{\mathcal{B}}, W_i^{\mathcal{S}}$, the problem simplifies to

$$\max_{p,q}\{u(q) - \varphi_i p\}$$
$$s.t.\ \ q = \varphi_i p,$$

and $p \leq m_i$. The next lemma describes the solution to this bargaining problem.

Lemma 1. *Define* $q^* = \{q : u'(q) = 1\}$ *and* $m_i^* = q^*/\varphi_i$. *Then, in a* DM_i *meeting between a seller* i *and a local buyer, the bargaining solution is given by* $q(m_i) = \min\{\varphi_i m_i, q^*\}$ *and* $p(m_i) = \min\{m_i, m_i^*\}$.

Proof. The proof is obvious, and it is, therefore, omitted. □

The interpretation of Lemma 1 is standard. The terms of trade, (q, p), depend only on the buyer's $money_i$ holdings. When m_i exceeds a certain level, m_i^*, the buyer purchases the first-best quantity, q^*, and gives up exactly m_i^* units of $money_i$. On the other hand, if m_i falls short of m_i^*, the buyer is liquidity constrained. In this case, she gives up all her $money_i$, and receives the amount of good that the seller is willing to produce for that money, i.e., $q = \varphi_i m_i$.

Next, consider a DM_i meeting between a seller i and a buyer $-i$ (a foreigner) who carries m_i units of $money_i$ (which to her is foreign currency acquired in the preceding FOREX market). The two parties negotiate over a quantity of special good, $\tilde{q}$, to be produced, and an amount of $money_i$, $\tilde{p}$, to be delivered to the seller. In this type of meeting, the determination of the terms of trade is even more straightforward than before, because the buyer spends all her $money_i$ by assumption (see footnote 15). The solution to this bargaining problem follows trivially.

Lemma 2. *In a* DM_i *meeting between a seller* i *and a foreign buyer, the bargaining solution is given by* $\tilde{q}(m_i) = \varphi_i m_i$ *and* $\tilde{p}(m_i) = m_i$.

We now proceed to the characterization of the terms of trade in the FOREX market. Consider first a dealer who enters the FOREX market with a portfolio $\mathbf{m}^d$ and does not match with a buyer. This dealer can still participate in the interdealer market and potentially sell her money to dealers who want to acquire it (e.g., because they matched with C-type buyers). The next lemma describes the continuation value of this agent.

Lemma 3. *A dealer who enters the first sub-period with portfolio* $\mathbf{m}^d$, *and does not contact any buyer, enters the third sub-period with portfolio* $\widetilde{\mathbf{m}}^d \equiv \left(\widetilde{m}_A^d\left(\mathbf{m}^d, \varepsilon, \boldsymbol{\varphi}\right), \widetilde{m}_B^d\left(\mathbf{m}^d, \varepsilon, \boldsymbol{\varphi}\right)\right)$, *given by*

$$\widetilde{m}_A^d = \begin{cases} 0, & \text{if } \varepsilon\varphi_A < \varphi_B, \\ \in [0, m_A^d + \varepsilon m_B^d], & \text{if } \varepsilon\varphi_A = \varphi_B, \\ m_A^d + \varepsilon m_B^d, & \text{if } \varepsilon\varphi_A > \varphi_B. \end{cases} \quad \widetilde{m}_B^d = \begin{cases} m_B^d + m_A^d/\varepsilon, & \text{if } \varepsilon\varphi_A < \varphi_B, \\ m_B^d + (m_B^d - \widetilde{m}_A^d)/\varepsilon, & \text{if } \varepsilon\varphi_A = \varphi_B, \\ 0, & \text{if } \varepsilon\varphi_A > \varphi_B. \end{cases}$$

Moreover, the dealer's maximum expected discounted payoff is

$$\widetilde{W}^D(\mathbf{m}^d) = \bar{\varphi}\left(m_A^d + \varepsilon m_B^d\right) + W^D(\mathbf{0}), \tag{10}$$

where $\bar{\varphi} \equiv \max\{\varphi_A, \varphi_B/\varepsilon\}$.

Proof. See the appendix. □

Lemma 3 admits an intuitive interpretation. If $\varepsilon\varphi_A < \varphi_B$, then a dealer who holds any $money_A$ in the interdealer FOREX market can use a unit of $money_A$ to buy $1/\varepsilon$ units of $money_B$. The net return of this trading strategy is $\varphi_B/\varepsilon - \varphi_A$, which is strictly positive. Therefore, under these prices, the typical dealer will sell all her $money_A$ for $money_B$ in the interdealer market. The same intuition applies to the complementary case (i.e., when $\varepsilon\varphi_A > \varphi_B$). When $\varepsilon\varphi_A = \varphi_B$, the dealer is indifferent with respect to the composition of her portfolio (between $money_A$ and $money_B$). As we shall see later, this case is the only one that can arise in equilibrium.

We now study the bargaining outcome in a meeting between a C-type buyer i and a dealer in the FOREX market. Since the buyer has an opportunity to consume special good in DM_{-i}, she may want to exchange some of her $money_i$ for $money_{-i}$. In turn, dealers, have the unique ability to access the frictionless interdealer market, where they can acquire $money_{-i}$ from other dealers, who either contacted buyers from country $-i$ or carry $money_{-i}$ on their own account. Hence, in this type of bilateral meeting there are clear gains from trade, and the two parties will split the surplus according to Kalai's proportional bargaining solution (with $\theta \in (0,1)$ being the dealer's bargaining power).[20] The bargaining problem is given by

$$\max_{\bar{m}_A^d, \bar{m}_B^d, \bar{m}_A^i, \bar{m}_B^i \geq 0} \left\{\widetilde{W}^D(\bar{\mathbf{m}}^d) - \widetilde{W}^D(\mathbf{m}^d)\right\}$$

$$s.t. \;\; 1. \;\; \frac{\theta}{1-\theta} = \frac{\widetilde{W}^D(\bar{\mathbf{m}}^d) - \widetilde{W}^D(\mathbf{m}^d)}{V_i^y(\bar{\mathbf{m}}^i) - V_i^n(m_i^i)},$$

$$2. \;\; \bar{m}_A^d + \bar{m}_A^i + \varepsilon[\bar{m}_B^d + \bar{m}_B^i] \leq m_A^d + m_A^i \mathbb{I}_{\{i=A\}} + \varepsilon\left[m_B^d + m_B^i \mathbb{I}_{\{i=B\}}\right],$$

where $\bar{\mathbf{m}}^d \equiv \left(\bar{m}_A^d, \bar{m}_B^d\right)$, $\bar{\mathbf{m}}^i \equiv (\bar{m}_A^i, \bar{m}_B^i)$ denote the post-FOREX trade portfolio for the dealer and the buyer, respectively, and $\mathbb{I}_{\{i=n\}}, n \in \{A, B\}$, is an indicator function that equals 1 if $i = n$.

As is standard with proportional bargaining, the problem above maximizes the dealer's surplus (i.e., her post-negotiation continuation value net of her threat point), subject to two constraints. The first is the so-called *Kalai* constraint, which requires the dealer-to-buyer surplus ratio to equal the ratio of the two players' bargaining powers (i.e., $\theta/(1-\theta)$). The second is the feasibility constraint which requires that the combined value of the pre-trade portfolios is

[20] For a discussion on the benefits of using Kalai's (1977) over Nash Jr's (1950) bargaining solution in monetary theory, see Aruoba, Rocheteau, and Waller (2007).

enough to finance the combined value of the post-trade portfolios in the interdealer market. Exploiting the linearity of the value functions the problem simplifies to

$$\max_{\bar{m}_A^d,\bar{m}_B^d,\bar{m}_A^i,\bar{m}_B^i\geq 0}\left\{\bar{\varphi}\left[\bar{m}_A^d+\varepsilon\bar{m}_B^d-\left(m_A^d+\varepsilon m_B^d\right)\right]\right\}$$

$$st.\ 1.\ \frac{\theta}{1-\theta}=\frac{\bar{\varphi}\left[\bar{m}_A^d+\varepsilon\bar{m}_B^d-\left(m_A^d+\varepsilon m_B^d\right)\right]}{u\left(q\left(\bar{m}_i^i\right)\right)+u\left(\tilde{q}\left(\bar{m}_{-i}^i\right)\right)-u\left(q\left(m_i^i\right)\right)+\varphi_i\left[p(m_i^i)-p(\bar{m}_i^i)-(m_i^i-\bar{m}_i^i)\right]},$$
$$2.\ \bar{m}_A^d+\bar{m}_A^i+\varepsilon[\bar{m}_B^d+\bar{m}_B^i]\leq m_A^d+m_A^i\mathbb{I}_{\{i=A\}}+\varepsilon\left[m_B^d+m_B^i\mathbb{I}_{\{i=B\}}\right].$$

The solution to this bargaining problem is described in the following lemma.

Lemma 4. *Consider the bargaining problem between a buyer i and a dealer, who enter the first subperiod with portfolios m_i^i and $\mathbf{m}^d$, respectively. Moreover, define the following objects:*

$$\chi_{-i}^*\equiv\left\{\chi_{-i}:u'\left(\varphi_{-i}\chi_{-i}\right)=\frac{\left(\varphi_A\varepsilon\right)\mathbb{I}_{\{i=A\}}+\varphi_B\mathbb{I}_{\{i=B\}}}{\varphi_B\mathbb{I}_{\{i=A\}}+\left(\varphi_A\varepsilon\right)\mathbb{I}_{\{i=B\}}}\right\},\tag{11}$$

$$G(\chi_{-i})\equiv\left\{G(\chi_{-i}):\frac{\varphi_{-i}u'\left(\varphi_{-i}\chi_{-i}\right)}{\varphi_i u'\left(\varphi_i G(\chi_{-i})\right)}=\frac{\bar{\varphi}\mathbb{I}_{\{i=B\}}+\bar{\varphi}\varepsilon\mathbb{I}_{\{i=A\}}}{\bar{\varphi}\varepsilon\mathbb{I}_{\{i=B\}}+\bar{\varphi}\mathbb{I}_{\{i=A\}}}\right\},\tag{12}$$

$$\tau_i(\chi_{-i})\equiv\frac{\theta u\left(\varphi_{-i}\chi_{-i}\right)+(1-\theta)\bar{\varphi}\chi_{-i}[\varepsilon\mathbb{I}_{\{i=A\}}+\mathbb{I}_{\{i=B\}}]}{\theta\varphi_i+(1-\theta)\bar{\varphi}[\varepsilon\mathbb{I}_{\{i=B\}}+\mathbb{I}_{\{i=A\}}]}.\tag{13}$$

We have the following results:

a) The buyer's post-FOREX trade portfolio, $\bar{\mathbf{m}}^i$, is given by

$$\bar{m}_{-i}^i\left(m_i^i\right)=\begin{cases}\chi_{-i}^*, & \text{if } m_i^i\geq m_i^*+\tau_i(\chi_{-i}^*),\\ \left\{\chi_{-i}:m_i^i=\Gamma\left(\chi_{-i}\right)\right\}, & \text{if } m_i^*\leq m_i^i\leq m_i^*+\tau_i(\chi_{-i}^*),\\ \left\{\chi_{-i}:m_i^i=\Lambda\left(\chi_{-i}\right)\right\}, & \text{if } m_i^i\leq m_i^*,\end{cases}$$

$$\bar{m}_i^i\left(m_i^i\right)=\begin{cases}m_i^i-\tau_i(\chi_{-i}^*), & \text{if } m_i^i\geq m_i^*+\tau_i(\chi_{-i}^*),\\ G\left(\bar{m}_{-i}^i\left(m_i^i\right)\right), & \text{otherwise},\end{cases}$$

where we have defined

$$\Gamma\left(\chi_{-i}\right)\equiv G(\chi_{-i})+\tau_i(\chi_{-i})+\frac{\theta\left\{u\left(\varphi_i G(\chi_{-i})\right)-\varphi_i G(\chi_{-i})\right\}-\theta\left\{u\left(q_i^*\right)-q_i^*\right\}}{\theta\varphi_i+(1-\theta)\bar{\varphi}[\varepsilon\mathbb{I}_{\{i=B\}}+\mathbb{I}_{\{i=A\}}]},$$

$$\Lambda\left(\chi_{-i}\right)\equiv G(\chi_{-i})+\frac{\theta\left\{u\left(\varphi_{-i}\chi_{-i}\right)+u\left(\varphi_i G(\chi_{-i})\right)-u\left(\varphi_i m_i^i\right)\right\}+(1-\theta)\bar{\varphi}\chi_{-i}[\varepsilon\mathbb{I}_{\{i=A\}}+\mathbb{I}_{\{i=B\}}]}{(1-\theta)\bar{\varphi}[\varepsilon\mathbb{I}_{\{i=B\}}+\mathbb{I}_{\{i=A\}}]}.$$

Also, we have $\Gamma'\left(\chi_{-i}\right)>0$, $\Lambda'\left(\chi_{-i}\right)>0$.

b) Define the "ask" and the "bid" price of $money_B$ as follows:

✓ $\varepsilon^a\equiv$ *the price of $money_B$ in terms of $money_A$ that a dealer asks buyer A to pay for $money_B$.*

✓ $\varepsilon^b \equiv$ *the price of* $money_B$ *in terms of* $money_A$ *that a dealer bids to buyer* B *to buy* $money_B$*.*

Then, we have $\varepsilon^a = \frac{m_A^A - \bar{m}_A^A}{\bar{m}_B^A}$, $\varepsilon^b = \frac{\bar{m}_A^B}{m_B^B - \bar{m}_B^B}$*, and* $\varepsilon^b \leq \varepsilon \leq \varepsilon^a$*.*

c) The dealer's post-FOREX trade portfolio, $\left(\bar{m}_A^d, \bar{m}_B^d\right)$*, is given by*

$$\bar{m}_A^d \in \left(0, m_A^d + \varepsilon m_B^d + \left(\left(\varepsilon^a - \varepsilon\right)\mathbb{I}_{\{i=A\}} + \left(\frac{\varepsilon - \varepsilon^b}{\varepsilon^b}\right)\mathbb{I}_{\{i=B\}}\right)\bar{m}_{-i}^i\right),$$

$$\bar{m}_B^d = \frac{1}{\varepsilon}\left(m_A^d + \varepsilon m_B^d + \left(\left(\varepsilon^a - \varepsilon\right)\mathbb{I}_{\{i=A\}} + \left(\frac{\varepsilon - \varepsilon^b}{\varepsilon^b}\right)\mathbb{I}_{\{i=B\}}\right)\bar{m}_{-i}^i - \bar{m}_A^d\right).$$

Proof. See the appendix. □

The formal proof of the lemma has been relegated to the appendix. Here, we provide an intuitive description of the solution. When buyer i gives up one unit of $money_i$, this typically reduces the amount of good, say q, that she can purchase in DM_i, but also allows her to acquire $money_{-i}$, which she can use to purchase good, say $\tilde{q}$, in DM_{-i}. This transaction will undoubtedly create a benefit, or surplus, because the buyer's preferences in the DM round are given by $u(q) + u(\tilde{q})$, with $u' > 0, u'' < 0$, and $u'(0) = \infty$.[21] The proportional bargaining solution first makes sure that the surplus generated by the transaction (i.e., the transfer of $money_{-i}$ to buyer i) is maximized, and then determines the terms of trade so that the so-called Kalai constraint is satisfied (this last part simply means that the dealer obtains a fraction θ of the surplus).

A key observation from Lemma 4 is that the solution to the bargaining problem depends only on the buyer's pre-trade $money_i$ holdings, m_i^i. Hence, it turns out that the key contribution of the dealer in this match is her ability to access the interdealer market, and *not* the fact that she may be carrying some money on her own account. Technically speaking, this is attributed to the fact that the marginal rate of substitution (MRS) between $money_A$ and $money_B$ for dealers is exogenously pinned down by the currency prices in the (perfectly competitive) interdealer FOREX market (i.e., the terms $\varepsilon, \boldsymbol{\varphi}$ on the right-hand side of eq.(12)).

Before we explain the solution to the bargaining problem we provide an intuitive interpretation of the various terms that appear in Lemma 4. The term χ_{-i} stands for the units of foreign currency that buyer i holds after the FOREX meeting, and χ_{-i}^* is the amount of foreign currency that allows buyer i to purchase the first best quantity in DM_{-i}. The term $G(\chi_{-i})$ stands for the post-FOREX amount of $money_i$ held by buyer i, and it is defined in eq.(12), which states that surplus maximization requires that the MRS between $money_i$ and $money_{-i}$ for the buyer (left-hand side of the equation in the curly bracket) should be equal to the MRS between $money_i$ and $money_{-i}$ for the dealer (right-hand side of the equation in the curly bracket). The term $\tau_i(\chi_{-i}^*)$

[21] Hence, if $\tilde{q} = 0$ and $q > 0$, there is always a benefit from reducing q and increasing $\tilde{q}$. In fact, for $\tilde{q} \approx 0$ the marginal benefit of increasing $\tilde{q}$ is infinite, as it follows from the Inada condition.

stands for the units of $money_i$ that the buyer needs to carry in addition to m_i^* in order to purchase the first best quantity in *both* DMs. Naturally, this term is increasing both in χ_{-i}^* and θ.[22] Finally, the terms $\Gamma(\chi_{-i}), \Lambda(\chi_{-i})$ have been defined such that the equations $m_i^i = \Gamma(\chi_{-i})$ and $m_i^i = \Lambda(\chi_{-i})$ represent the Kalai constraint (for different levels of m_i^i).[23]

Given this discussion, the interpretation of Lemma 4 becomes quite intuitive. Given her money holdings, m_i^i, the buyer can find herself in three possible regions.

1. $m_i^i \geq m_i^* + \tau_i\left(\chi_{-i}^*\right)$.
   In this region, the buyer's $money_i$ holdings are so plentiful that she ends up purchasing the first best quantity in both $DM's$. This requires that the buyer's post-FOREX $money_{-i}$ holdings equal χ_{-i}^*, and her post-FOREX $money_i$ holdings equal her original holdings, m_i^i, net of the term $\tau_i(\chi_{-i}^*)$ described in the previous paragraph.

2. $m_i^* \leq m_i^i \leq m_i^* + \tau_i(\chi_{-i}^*)$.
   In this intermediate region, the buyer can afford to purchase q^* in her home DM but not in both DMs. Of course, the buyer could choose to keep m_i^* units of $money_i$, which would allow her to purchase q^* in DM_i, but optimality requires that the MRS between $money_i$ and $money_{-i}$ for the buyer should be equal to the MRS between $money_i$ and $money_{-i}$ for the dealer. Put simply, the post-FOREX $money_i$ and $money_{-i}$ holdings of the buyer are pinned down by eq.(12) and the equation $m_i^i = \Gamma(\chi_{-i})$ (i.e., the Kalai constraint).

3. $m_i^i \leq m_i^*$.
   In this region, the buyer's pre-trade money holdings are so scarce that she cannot even afford to purchase q^* in the home DM. The interpretation of the bargaining solution is similar to the one in Region 2: the buyer's post-FOREX $money_i$ and $money_{-i}$ holdings are pinned down by eq.(12) and the Kalai constraint. However, given that we are in the case where $m_i^i \leq m_i^*$, the relevant Kalai constraint is now given by the equation $m_i^i = \Lambda(\chi_{-i})$.

Part (b) of the lemma simply determines the bid and ask price of the dealer (in terms of $money_B$)

[22] The more $money_{-i}$ the buyer wishes to acquire, the more $money_i$ she should bring. Moreover, when θ is higher the transaction fee charged by the dealer is higher, thus, to acquire a given amount of $money_{-i}$ the buyer should, on average, bring more $money_i$.

[23] To see this point, focus on the equation $m_i^i = \Gamma(\chi_{-i})$ (an analogous argument applies to the second equation). Also, to simplify the illustration here, set $\varepsilon\varphi_A = \varphi_B$ (which, of course, will hold in equilibrium, and) which implies that the term $\bar{\varphi}[\varepsilon\mathbb{I}_{\{i=B\}} + \mathbb{I}_{\{i=A\}}]$, which appears in $\Gamma(\chi_{-i})$, simplifies to φ_i, $\forall i$. Multiply both sides by φ_i and solve with respect to $\varphi_i m_i^i - \varphi_i G(\chi_{-i})$, to obtain

$$\begin{aligned}\varphi_i m_i^i - \varphi_i G(\chi_{-i}) =& \varphi_{-i}\chi_{-i} \\ &+\theta\left\{[u(\varphi_i G(\chi_{-i})) - \varphi_i G(\chi_i)] + [u(\varphi_{-i}\chi_{-i}) - \varphi_{-i}\chi_{-i}] - [u(q^*) - q^*]\right\}.\end{aligned}$$

This equation states that the dealer should leave the meeting with an amount of $money_i$ (the LHS of the equation) whose value equals the value of the $money_{-i}$ that she brings into the match either through intermediation or on her own account (the term $\varphi_{-i}\chi_{-i}$), plus a fraction θ of the surplus generated when $\varphi_{-i}\chi_{-i}$ (real) units of foreign currency are transferred to the buyer (the second line in the equation).

as functions of the post and pre-trade money holdings of buyer i. Part (c) describes the dealer's post-trade portfolio, which turns out to be indeterminate. This follows from the fact that the dealer can visit both CMs (and sell the currencies for general good), so there is a continuum of portfolio allocations that give her the same payoff. Nevertheless, as the following corollary shows, the combined value of the dealer's post-trade portfolio is uniquely pinned down.

Corollary 1. *The $money_A$ value of the dealer's post-trade portfolio is uniquely given by*

$$\bar{m}_A^d + \varepsilon \bar{m}_B^d = m_A^d + \varepsilon m_B^d + \left((\varepsilon^a - \varepsilon)\, \mathbb{I}_{\{i=A\}} + \left(\frac{\varepsilon - \varepsilon^b}{\varepsilon^b} \right) \mathbb{I}_{\{i=B\}} \right) \bar{m}_{-i}^i.$$

Corollary 1 also clarifies that the dealer extracts a transaction fee from the buyer. For example, when the dealer encounters a buyer A who purchases $\bar{m}_B^A$ units of foreign currency, she extracts a total fee equal to $(\varepsilon^a - \varepsilon)\, \bar{m}_B^A$. Similarly, when the dealer encounters a buyer B who purchases $\bar{m}_A^B$ of foreign currency, she extracts a total fee equal to $\left[(\varepsilon - \varepsilon^b)/\varepsilon^b\right] \bar{m}_A^B$.

The following lemma highlights some interesting properties of certain key variables that can be calculated directly from the bargaining solution (Lemma 4), such as the *ask* and *bid* price of $money_B$, and the amount of currency that changes hands. These terms will be crucial for the determination of the *equilibrium* bid-ask spread and FOREX trade volume later on.

Lemma 5. *The Ask and Bid Price satisfy*

$$\frac{\partial \varepsilon^a}{\partial \theta} > 0, \quad \frac{\partial \varepsilon^b}{\partial \theta} < 0, \quad \frac{\partial \varepsilon^a}{\partial m_A^A} \begin{cases} = 0, & \text{in Region 1}, \\ < 0, & \text{in Region 2,3}, \end{cases} \quad \frac{\partial \varepsilon^b}{\partial m_B^B} \begin{cases} = 0, & \text{in Region 1}, \\ > 0, & \text{in Region 2,3}. \end{cases}$$

The Volume of $money_{-i}$ the dealer hands over satisfies

$$\frac{\partial \bar{m}_{-i}^i}{\partial \theta} \begin{cases} = 0, & \text{in Region 1}, \\ < 0, & \text{in Region 2,3}, \end{cases} \qquad \frac{\partial \bar{m}_{-i}^i}{\partial m_i^i} \begin{cases} = 0, & \text{in Region 1}, \\ > 0, & \text{in Region 2,3}. \end{cases}$$

The Volume of $money_i$ the buyer i hands over satisfies

$$\frac{\partial \left(m_i^i - \bar{m}_i^i\right)}{\partial \theta} \begin{cases} > 0, & \text{in Region 1}, \\ > 0, & \text{in Region 2,3}, \end{cases} \qquad \frac{\partial \left(m_i^i - \bar{m}_i^i\right)}{\partial m_i^i} \begin{cases} = 0, & \text{in Region 1}, \\ > 0, & \text{in Region 2,3}. \end{cases}$$

Proof. See the appendix. □

These results admit an intuitive interpretation. When the dealer's bargaining power θ increases, other things equal, the dealer extracts a larger fraction of the surplus, so that the bid price decreases and the ask price increases (and so does the spread between the two). Moreover, when θ is higher, the dealer hands over less $money_{-i}$ to buyer i for any given $money_i^i$, with the exception of Region 1, where the buyer is not constrained by her home money holdings.[24] Naturally, the volume of $money_i$ that buyer i hands over to the dealer is increasing in θ. The effects of an increase in m_i^i work in the opposite way than an increase in θ. An increase in m_i^i typically allows the buyer to purchase more (home and) foreign good, but since u is concave, her marginal benefit from acquiring one additional unit of foreign currency, or, equivalently, from consuming one additional unit of foreign good, is diminishing. In a sense, an increase in m_i^i makes buyer i less eager to acquire foreign currency, and effectively allows her to obtain better terms of trade in the FOREX market. Finally, with the exception of Region 1, the more $money_i$ the buyer carries, the more she hands over to the dealer, due to the fact that the increase in $\bar{m}_{-i}^i$ more than offsets the improvement in the terms of trade, i.e., the decrease in ε^a.

3.3 Optimal Behavior

In this section, we describe the optimal portfolio choices of buyers and dealers. The first step is to characterize the *objective functions* for these agents. First, consider the typical dealer. Substitute eq.(10) into eq.(6), and lead the emerging expression by one period to obtain

$$\begin{aligned}\Omega_D\left(\widehat{\mathbf{m}}^d\right) &= \widehat{\varphi}(\widehat{m}_A^d + \widehat{\varepsilon}\widehat{m}_B^d) + W^D(\widehat{\mathbf{0}}) \\ &+ \widehat{\varphi}\int \frac{\alpha_D}{2}\left[\varepsilon^a\left(\widehat{m}_A\right) - \widehat{\varepsilon}\right]\bar{m}_B^A\left(\widehat{m}_A\right)dF^A\left(\widehat{m}_A\right) \\ &+ \widehat{\varphi}\int \frac{\alpha_D}{2}\left[\frac{\widehat{\varepsilon} - \varepsilon^b\left(\widehat{m}_B\right)}{\varepsilon^b\left(\widehat{m}_B\right)}\right]\bar{m}_A^B\left(\widehat{m}_B\right)dF^B\left(\widehat{m}_B\right).\end{aligned}$$

Next, substitute Ω_D from the last expression into eq.(2), and define the term inside the $\max$ operator (i.e., the objective function) as J^D. Then, we have

$$\begin{aligned}J^D\left(\widehat{m}_A^d, \widehat{m}_B^d\right) &\equiv \left(-\varphi_A + \beta\widehat{\varphi}\right)\widehat{m}_A^d + \left(-\varphi_B + \beta\widehat{\varphi}\widehat{\varepsilon}\right)\widehat{m}_B^d \\ &+ \beta\widehat{\varphi}\int \frac{\alpha_D}{2}\left[\varepsilon^a\left(\widehat{m}_A\right) - \widehat{\varepsilon}\right]\bar{m}_B^A\left(\widehat{m}_A\right)dF^A\left(\widehat{m}_A\right) \\ &+ \beta\widehat{\varphi}\int \frac{\alpha_D}{2}\left[\frac{\widehat{\varepsilon} - \varepsilon^b\left(\widehat{m}_B\right)}{\varepsilon^b\left(\widehat{m}_B\right)}\right]\bar{m}_A^B\left(\widehat{m}_B\right)dF^B\left(\widehat{m}_B\right).\end{aligned}$$

In the last expression, the fist line represents the cost of carrying $money_A$ and $money_B$, the second line captures the expected discounted benefit from executing buyer A's order (i.e., purchasing $money_B$) in the interdealer market, and the last line admits a similar interpretation (when the dealer meets a buyer B). As we already know from Lemma 4, these expected intermediation fees are independent of the dealer's own portfolio. Also, it is understood that the expressions

[24] However, for a given m_i^i, increasing θ makes it more likely that the bargaining solution will "fall" in Region 2. Hence, this statement implicitly assumes that the change in θ is such that we are still in the interior of Region 1.

$\bar{m}_{-i}^{i}, \varepsilon^{a}$, and ε^{b} are described by Lemma 4.

Next, consider the typical buyer i. Substitute equations (7), (8) into (5), and then plug the emerging expression into eq.(4) and lead it by one period to obtain

$$\begin{aligned}
\mathbb{E}_k\{\Omega_i^k(\widehat{m}_i)\} =& \widehat{\varphi}_i\widehat{m}_i + W_i^{\mathcal{B}}(\widehat{0}) \\
&+ \delta\alpha_i \left\{u\left(\tilde{q}_i\left(\bar{m}_{-i}^{i}\left(\widehat{m}_i, \widehat{\varepsilon}, \widehat{\boldsymbol{\varphi}}\right)\right)\right) + u\left(q_i\left(\widehat{m}_i - \vartheta\left(\widehat{m}_i\right)\right)\right)\right\} \\
&- \delta\alpha_i \left[\widehat{\varphi}_i\vartheta\left(\widehat{m}_i\right) + \widehat{\varphi}_i p_i\left(\widehat{m}_i - \vartheta\left(\widehat{m}_i\right)\right)\right] \\
&+ \left(1 - \delta\alpha_i\right)\left\{u\left(q_i\left(\widehat{m}_i\right)\right) + \widehat{\varphi}_i\left[\widehat{m}_i - p_i\left(\widehat{m}_i\right)\right]\right\},
\end{aligned}$$

where $\vartheta\left(\widehat{m}_i\right)$ represents the units of $money_i$ that buyer i hands over to the dealer.[25] To obtain buyer i's objective function, substitute $\mathbb{E}_k\{\Omega_i^k(\widehat{m}_i)\}$ from the last expression into eq.(1). We have

$$\begin{aligned}
J^i\left(\widehat{m}_i\right) \equiv& \left(-\varphi_i + \beta\widehat{\varphi}_i\right)\widehat{m}_i \\
&+ \beta\delta\alpha_i \left\{u\left(\tilde{q}_i\left(\bar{m}_{-i}^{i}\left(\widehat{m}_i, \widehat{\varepsilon}, \widehat{\boldsymbol{\varphi}}\right)\right)\right) + u\left(q_i\left(\widehat{m}_i - \vartheta\left(\widehat{m}_i\right)\right)\right)\right\} \\
&- \beta\delta\alpha_i \left[\widehat{\varphi}_i\vartheta\left(\widehat{m}_i\right) + \widehat{\varphi}_i p_i\left(\widehat{m}_i - \vartheta\left(\widehat{m}_i\right)\right)\right] \\
&+ \beta\left(1 - \delta\alpha_i\right)\left\{u\left(q_i\left(\widehat{m}_i\right)\right) + \widehat{\varphi}_i\left[\widehat{m}_i - p_i\left(\widehat{m}_i\right)\right]\right\}.
\end{aligned} \tag{14}$$

In the last expression, the first line represents the buyer's net benefit from holding $\widehat{m}_i$ units of local money until the end of the next period, and the remaining terms represent the expected net gain from (potential) trade. More precisely, the second line stands for the expected utility gain in the forthcoming DMs, if the buyer is a C-type and matches with a dealer. The third line represents the cost from giving up some of her $money_i$ to a dealer (the first term inside the square bracket) and to a seller i (the second term inside the square bracket), in the same event. Finally, the fourth line represents the buyer's net benefit in the event that she does not trade in the FOREX market (either because she is an N-type or because she did not match with a dealer). The terms $q_i(\cdot)$, $\tilde{q}_{-i}(\cdot)$, and $p_i(\cdot)$ are described by the solutions to the DM bargaining problems.

Before we proceed to the description of agents' optimal behavior, the following auxiliary lemma describes some important properties of the buyer's objective function.

Lemma 6. *Define $J_r^i\left(\widehat{m}_i\right)$ as buyer i's objective function, when her money holdings, $\widehat{m}_i$, are such that the relevant region in the buyer-dealer FOREX bargaining solution is $r = \{1, 2, 3\}$. Then, we have:*

$$\frac{\partial J_1^i\left(\widehat{m}_i\right)}{\partial \widehat{m}_i} = -\,\varphi_i + \beta\widehat{\varphi}_i, \tag{15}$$

[25] In particular, $\vartheta\left(\widehat{m}_i\right) = \widehat{\varepsilon}^a\left(\widehat{m}_i\right)\bar{m}_{-i}^{i}\left(\widehat{m}_i, \widehat{\varepsilon}, \widehat{\boldsymbol{\varphi}}\right)\mathbb{I}_{\{i=A\}} + \bar{m}_{-i}^{i}\left(\widehat{m}_i, \widehat{\varepsilon}, \widehat{\boldsymbol{\varphi}}\right)/\widehat{\varepsilon}^b\left(\widehat{m}_i\right)\mathbb{I}_{\{i=B\}}$.

$$\frac{\partial J_2^i(\widehat{m}_i)}{\partial \widehat{m}_i} = -\varphi_i + \beta\widehat{\varphi}_i + \beta\delta\alpha_i\widehat{\varphi}_{-i}u'\left(\widehat{\varphi}_{-i}\bar{m}_{-i}^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)\frac{\partial\left(\bar{m}_{-i}^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)}{\partial\widehat{m}_i} + \beta\delta\alpha_i\widehat{\varphi}_i\left\{u'\left(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)\frac{\partial\left(\bar{m}_i^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)}{\partial\widehat{m}_i} - 1\right\}, \tag{16}$$

$$\frac{\partial J_3^i(\widehat{m}_i)}{\partial \widehat{m}_i} = -\varphi_i + \beta\widehat{\varphi}_i + \beta\delta\alpha_i\widehat{\varphi}_{-i}u'\left(\widehat{\varphi}_{-i}\bar{m}_{-i}^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)\frac{\partial\left(\bar{m}_{-i}^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)}{\partial\widehat{m}_i} + \beta\delta\alpha_i\widehat{\varphi}_i\left\{u'\left(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)\frac{\partial\left(\bar{m}_i^i(\widehat{m}_i,\widehat{\varepsilon},\widehat{\boldsymbol{\varphi}})\right)}{\partial\widehat{m}_i} - 1\right\} + \beta(1-\delta\alpha_i)\widehat{\varphi}_i\left\{u'(\widehat{\varphi}_i\widehat{m}_i) - 1\right\}. \tag{17}$$

Proof. Replacing q_i, $\tilde{q}_i$, and p_i from Lemmas 1, 2, and obtaining the derivative with respect to $\widehat{m}_i$ yields the desired result. ☐

We are now ready to describe agents' optimal behavior, starting with the typical dealer. It is important to keep in mind that, in any equilibrium, the following two conditions must hold:[26]

$$\varphi_A \geq \beta\max(\widehat{\varphi}_A, \widehat{\varphi}_B/\widehat{\varepsilon}) \quad \textit{and} \quad \varphi_B \geq \beta\max(\widehat{\varphi}_A\widehat{\varepsilon}, \widehat{\varphi}_B). \tag{18}$$

Lemma 7. *Taking prices $\boldsymbol{\Psi} \equiv (\boldsymbol{\varphi}, \widehat{\boldsymbol{\varphi}}, \widehat{\overline{\varphi}}, \varepsilon, \widehat{\varepsilon})$ as given, the optimal portfolio choice of the typical dealer, $\widehat{\mathbf{m}}^d$, is as follows:*

1. *If $\varphi_i = \beta\widehat{\overline{\varphi}}\left\{\widehat{\varepsilon}\mathbb{I}_{\{i=B\}} + \mathbb{I}_{\{i=A\}}\right\}$, then $\widehat{m}_i^d \in \mathbb{R}_+$.*
2. *If $\varphi_i > \beta\widehat{\overline{\varphi}}\left\{\widehat{\varepsilon}\mathbb{I}_{\{i=B\}} + \mathbb{I}_{\{i=A\}}\right\}$, then $\widehat{m}_i^d = 0$.*

Proof. The proof is trivial, it is, therefore, omitted. ☐

The dealer's money demand is simple. The dealer does not have a benefit from carrying $money_i$, since, as we have already established, the terms of trade in the FOREX meetings are not affected by her money holdings. Thus, the dealer will typically choose to leave the CM round of trade without any money, unless the cost of holding money is zero.

Next, consider the optimal portfolio choice of the typical buyer i.

Lemma 8. *Taking prices $\boldsymbol{\Psi} \equiv (\boldsymbol{\varphi}, \widehat{\boldsymbol{\varphi}}, \widehat{\overline{\varphi}}, \varepsilon, \widehat{\varepsilon})$ as given, the optimal portfolio choice of the typical buyer i, $\widehat{m}_i$, is as follows:*

[26] The proof of this claim is standard in monetary theory. If it was $\varphi_A < \beta\max(\widehat{\varphi}_A, \widehat{\varphi}_B/\widehat{\varepsilon})$, then dealers would have an infinite demand for $money_A$, which is clearly inconsistent with the existence of equilibrium.

1. *If* $\varphi_i/\beta\widehat{\varphi}_i = 1$, *then* $\widehat{m}_i = m_i^* + \tau_i\left(\chi_{-i}^*\right)$.

2. *If* $1 < \varphi_i/\beta\widehat{\varphi}_i \leq \bar{\mu}_i$, *then there exits a unique optimal* $\widehat{m}_i \in \left[m_i^*, m_i^* + \tau_i\left(\chi_{-i}^*\right)\right)$, *which satisfies* $\partial J_2^i\left(\widehat{m}_i\right)/\partial\widehat{m}_i = 0$.

3. *If* $\varphi_i/\beta\widehat{\varphi}_i \geq \bar{\mu}_i$, *then there exits a unique optimal* $\widehat{m}_i \in (0, m_i^*]$, *which satisfies* $\partial J_3^i\left(\widehat{m}_i\right)/\partial\widehat{m}_i = 0$.

Proof. The proof and the definition of the term $\bar{\mu}_i$ are relegated to the appendix. □

The typical buyer i's money demand, $D_{\widehat{\varepsilon}}^i$, is plotted in Figure 2 (for some given $\widehat{\varepsilon}$) against the ratio $\varphi_i/\left(\beta\widehat{\varphi}_i\right)$, which captures the cost of holding home money . The money demand curve has a standard negative slope, but it also kinks at the point $\left(m_i^*, \bar{\mu}_i\right)$. Intuitively, the term $\bar{\mu}_i$ (which is shown to be greater than 1 in the appendix) captures the level of inflation that induces the buyer to carry enough money in order to purchase q^* in the home DM.[27] When the cost of holding money drops below $\bar{\mu}_i$, the buyer carries an even greater amount of home currency, or, in terms of the language introduced in Lemma 4, the relevant region of the FOREX bargaining protocol switches from Region 3 to Region 2. The change in the slope of the demand function around $\bar{\mu}_i$ captures the fact that the marginal benefit from carrying one more unit of money differs between Regions 2 and 3. In both Regions 2 and 3, an additional unit of $money_i$ allows buyer i to purchase more special good in DM_{-i} if she trades in the FOREX market, a benefit which is represented by the second and third lines in equations (16), (17). However, in Region 3, an additional unit of $money_i$ also allows the buyer to purchase more special good in DM_i, if she does not trade in the FOREX market. This benefit is represented by the fourth line in eq.(17), and it does not have a counterpart in eq.(16), since in Region 2 the buyer is already able to purchase the first-best quantity.

4 Equilibrium in the Two-Country Model

4.1 Definition of Equilibrium

In this section we characterize the steady state equilibrium of the model. Before stating the formal definition, we introduce some additional notation. Let $A_{m_i}^d$ and $A_{m_i}^i$ denote the amount of $money_i$ held by all dealers and all buyers i, respectively, at the beginning of the current period. Also, let $\widetilde{A}_{m_i}^d$ and $\widetilde{A}_{m_i}^i$ denote the amount of $money_i$ held by all dealers and all buyers i, respectively, at the end of the preceding period. Finally, let $\bar{A}_{m_i}^d$, $\bar{A}_{m_i}^i$, and $\bar{A}_{m_i}^{-i}$ denote the

[27] In the standard one-country model, $\bar{\mu}_i$ would be equal to 1, i.e., the buyer would carry enough money to purchase q^* only if the cost of holding money is zero. However, here the buyer realizes that she might have an opportunity to purchase the foreign good as well. Hence, if the cost of holding money is not too high ($1 < \varphi_i/\beta\widehat{\varphi}_i \leq \bar{\mu}_i$), she will carry an amount of money that exceeds m_i^*.

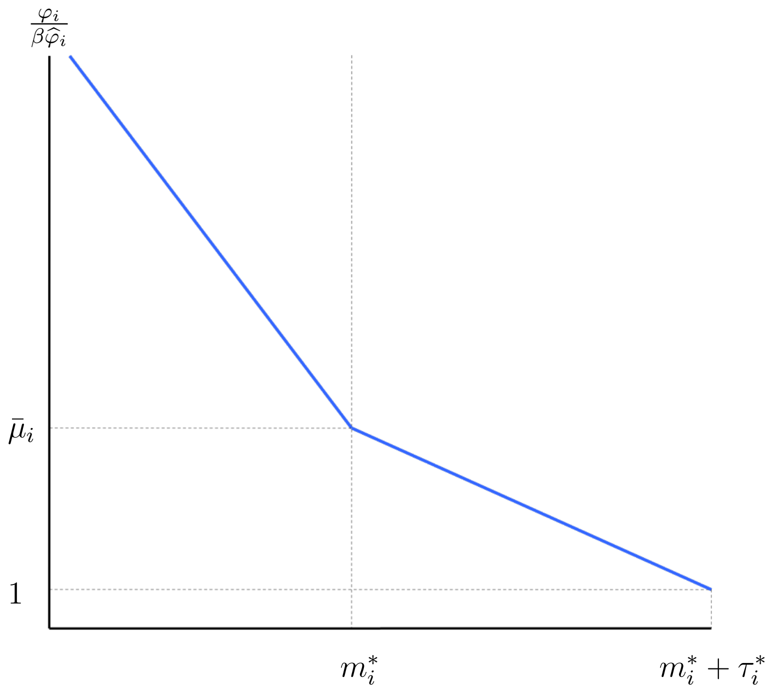


Figure 2: $money_i$ demand by a buyer from country i

amount of $money_i$ held by all dealers, all buyers i, and all buyers $-i$ who traded in the FOREX market, respectively, after the current period's FOREX trading has concluded. We have,

$$\begin{aligned}
\bar{A}^d_{m_i} =& (1-\alpha_D)v \int \widetilde{m}^d_i \left(\mathbf{m}^d, \varepsilon, \boldsymbol{\varphi}\right) dK^d \left(\mathbf{m}^d\right) \\
& + \frac{\alpha_D}{2} v \int \widetilde{m}^d_i \left(\bar{\mathbf{m}}^d, \varepsilon, \boldsymbol{\varphi}\right) dK^d \left(\mathbf{m}^d\right) dF^i \left(m^i_i\right) \\
& + \frac{\alpha_D}{2} v \int \widetilde{m}^d_i \left(\bar{\mathbf{m}}^d, \varepsilon, \boldsymbol{\varphi}\right) dK^d \left(\mathbf{m}^d\right) dF^{-i} \left(m^{-i}_{-i}\right), \\
\bar{A}^i_{m_i} =& \alpha_i \delta \int \bar{m}^i_i \left(m^i_i, \varepsilon, \boldsymbol{\varphi}\right) dF^i \left(m^i_i\right), \\
\bar{A}^{-i}_{m_i} =& \alpha_{-i} \delta \int \bar{m}^{-i}_i \left(m^{-i}_{-i}, \varepsilon, \boldsymbol{\varphi}\right) dF^{-i} \left(m^{-i}_{-i}\right),
\end{aligned}$$

where K^d is the cumulative distribution function over the portfolio held by the typical dealer in the beginning of the current period.[28] We are now ready to define a stationary equilibrium.

Definition 1. A steady state equilibrium is a list of terms of trade in the DMs, the interdealer

[28] Given the uniqueness of agents' optimal choice (Lemmas 7, 8), the *cdf*'s K^d, F^i, F^{-i} will be degenerate. Here, we describe the variables $\bar{A}^d_{m_i}, \bar{A}^i_{m_i}, \bar{A}^{-i}_{m_i}$ explicitly as functions of these *cdf*'s only for the sake of generality.

market, and the typical buyer-dealer FOREX meeting,

$$\left\{(p_i, q_i), (\tilde{p}_i, \tilde{q}_i), \widetilde{\mathbf{m}}^d, \bar{\mathbf{m}}^d, \bar{\mathbf{m}}^i, \varepsilon^a, \varepsilon^b\right\}, i = \{A, B\},$$

given by Lemmas 1, 2, 3, and 4, together with a list of money holdings

$$\left\{\left(\widehat{m}_A^d, \widehat{m}_B^d\right), \widehat{m}_i^i\right\}, i = \{A, B\},$$

and prices, $\mathbf{\Psi}$, such that:

- The money holdings, $\left\{\left(\widehat{m}_A^d, \widehat{m}_B^d\right), \widehat{m}_i^i\right\}$, solve the individual optimization problems (1) and (2), taking prices as given.
- Prices are such that all Walrasian markets (i.e., CM and interdealer FOREX market) clear:
  - $\widetilde{A}_{m_i}^d + \widetilde{A}_{m_i}^i = A_{m_i}^d + A_{m_i}^i, i = \{A, B\}$, ($CM_i$),
  - $\bar{A}_{m_i}^d + \bar{A}_{m_i}^i + \bar{A}_{m_i}^{-i} = A_{m_i}^d + \delta\alpha_i A_{m_i}^i, i = \{A, B\}$, (interdealer FOREX market for $money_i$).
- The law of one price holds, i.e., no arbitrage conditions exist in the interdealer FOREX market: $\varphi_B = \varepsilon\varphi_A$.
- Real $money_i$ balances for all i remain constant over time:
  - $\varphi_i/\widehat{\varphi}_i = (\widehat{A}_{m_i}^d + \widehat{A}_{m_i}^i)/(A_{m_i}^d + A_{m_i}^i) = \gamma_i, , i = \{A, B\}$,
  - $\widehat{\varepsilon}/\varepsilon = \gamma_A/\gamma_B$.

Definition 1 reveals an important property of equilibrium. Due to the competitive nature of the interdealer market, no arbitrage conditions can arise in equilibrium, i.e., we must have $\varphi_B = \varepsilon\varphi_A$. Imposing this condition in eq.(12) (for i equal to A or B), implies that the buyer's post-FOREX trade $(money_i, money_{-i})$ holdings are such that she will purchase exactly the same quantity of special good in the local and the foreign DM, i.e., in equilibrium, $q_i = \tilde{q}_{-i}$.[29]

In the next section, we will examine how the various equilibrium variables depend on the policy parameters γ_i, γ_{-i}, and the structural parameters θ, α_i, α_D, and δ. Before we proceed, notice that the three regions introduced in the discussion following Lemma 4 have their "general equilibrium" counterparts, i.e., they can be expressed in terms of equilibrium real balances held by buyer i, Z^i. More precisely, we will say that equilibrium is in Region 1, when $Z^i = q^* + \tilde{q}^*$, where $\tilde{q}^* \equiv \theta u(q^*) + (1-\theta)q^*$.[30] In the intermediate Region 2, we will have $Z^i \in [q^*, q^* + \tilde{q}^*)$. Finally, $Z^i < q^*$ means that equilibrium lies in Region 3.

[29] For instance, letting $i = A$ and substituting for $\varepsilon = \varphi_B/\varphi_A$ in eq.(12) implies that $\varphi_B\chi_B = \varphi_A G(\chi_B)$.

[30] Since money is fiat there is never a point in carrying $Z^i > q^* + \tilde{q}^*$. Hence, Region 1 is a singleton rather than an interval.

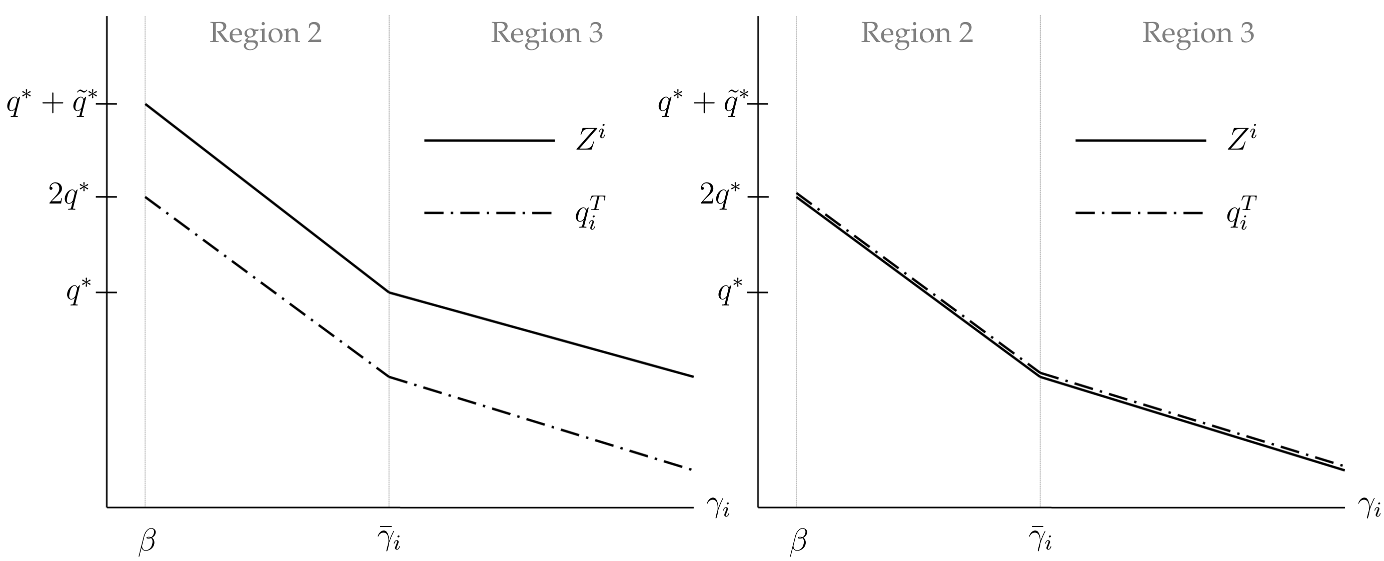
4.2 Characterization of Equilibrium

4.2.1 International Trade

We are now ready to state the main results of the paper. Proposition 1 describes the effect of changes in home inflation, γ_i, on the aggregate DM good consumption of country i. Proposition 2 describes the effect of changes in foreign inflation, γ_{-i}, and the bargaining power of dealers, θ, on the equilibrium exports and imports of country i. To simplify the presentation, throughout Section 4.2.1 we assume, without loss of generality, that all agents get to consume the foreign good and match with a dealer, i.e., we set $\delta\alpha_i = 1$, $\forall i \in \{A, B\}$.

Proposition 1. *Define q_i^T as country i's total consumption of special goods both in the local and the foreign DM. We have the following results:*
i) If $\theta > 0$ ($\theta = 0$), then $Z^i > q_i^T$ ($Z^i = q_i^T$), $\forall \gamma_i$.
ii) Constrained efficiency, i.e., $q_i^T = 2q^*$, requires that $\gamma_i = \beta$. Otherwise, $q_i^T < 2q^*$ and $\partial q_i^T / \partial \gamma_i < 0$.

Proof. See the appendix. □



Case 1: $\theta > 0$ **Case 2:** $\theta = 0$

Figure 3: Effects of γ_i on the country i's DM goods consumption (q_i^T) with $\delta\alpha_i = 1$

The intuition behind Proposition 1 is straightforward. The amount of special goods that buyers i can purchase in both DMs depends on the amount of real home money balances that they carry. If $\theta > 0$, Z^i will exceed q_i^T, because some of the real balances that buyers carry will end up in the pockets of dealers as intermediation fees. The first best will be achieved only if the local monetary authority follows the Friedman rule, i.e., if $\gamma_i = \beta$. In any other case, DM

consumption will fall short of the first best, i.e., $q_i^T < 2q^*$. Finally, since inflation acts as a tax on holding home real balances, a higher γ_i reduces Z^i and leads to a lower consumption of both special goods. These results are depicted in Figure 3. The kink that both Z^i and q_i^T exhibit at $\bar{\gamma}_i$ follows directly from the fact that the buyer's money demand function also has a kink as we move from Region 2 to Region 3 (see Figure 2 and the discussion following Lemma 8).

Proposition 2. *Define $\tilde{q}_{-i}^T$ as country i's total exports, i.e., the total amount of special good that sellers i sell to foreign buyers. Likewise, define $\tilde{q}_i^T$ as country i's total imports, i.e., the total amount of special good that buyers i purchase from foreign sellers. Finally, let Z^{-i} denote the real $money_{-i}$ balances held by buyers $-i$. We have the following results:*
i) If $\theta > 0$ ($\theta = 0$), then $Z^{-i} > 2\tilde{q}_{-i}^T$ ($Z^{-i} = 2\tilde{q}_{-i}^T$), $\forall \gamma_{-i}$.
ii) The effects of foreign inflation, γ_{-i}, on the imports and exports of country i satisfy:

$$\frac{\partial \tilde{q}_i^T}{\partial \gamma_{-i}} = 0 \quad \text{and} \quad \frac{\partial \tilde{q}_{-i}^T}{\partial \gamma_{-i}} < 0.$$

iii) An increase in γ_{-i} reduces country i's net exports.
iv) An increase in θ reduces the volume of international trade, i.e., $\partial \tilde{q}_i^T / \partial \theta < 0, \forall i$.

Proof. See the appendix. □

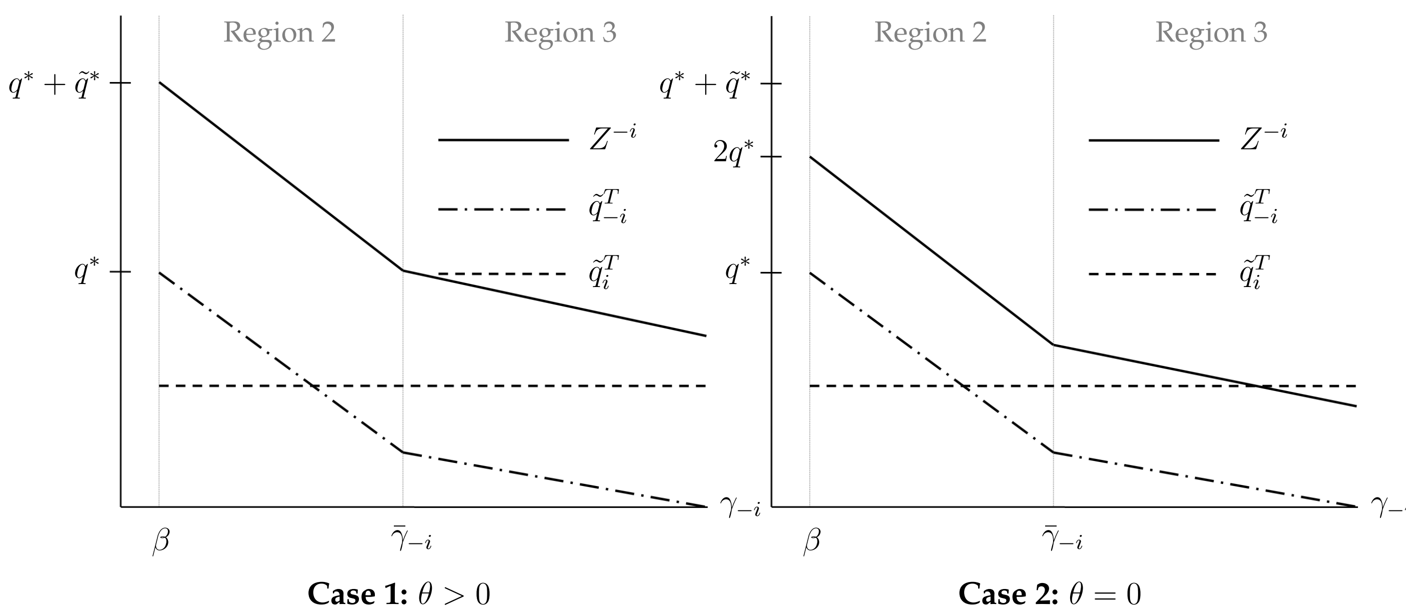


Figure 4: Effects of γ_{-i} on the country i's DM goods consumption (q_i^T) with $\delta\alpha_i = 1$, $\beta < \gamma_i$

The negative effect of foreign inflation on a country's exports is intuitive. A rising foreign inflation lowers the real money balances held by foreign buyers and, consequently, reduces the amount of special good that these agents can afford to purchase in the home DM. However, a higher foreign inflation does not affect the level of imports of the home country. The intuition is as follows. A change in foreign inflation does not affect the demand for real home money balances. Hence, buyers from the home country carry exactly the same amount of home currency into the FOREX market. Since the higher foreign inflation leads to an appreciation of the home currency in the FOREX market, buyers from the home country end up entering the foreign DM with more foreign currency. However, due to the competitive nature of the FOREX interdealer market (and the consequent no-arbitrage condition in that market), this increase exactly offsets the decrease in the value of foreign currency, which was caused by the initial rise in γ_{-i}. As a result, the amount of foreign DM good purchased by home buyers is unaffected by changes in foreign inflation. Given this discussion, part (iii) of the proposition follows immediately. These results are repeated in Figure 4.

The last part of Proposition 2 describes the effect of a change in the dealers' bargaining power on the volume of international trade. As θ increases, buyers from both countries anticipate a less liquid FOREX market in the form of higher transaction fees. As we show in the appendix, this induces buyers to carry a higher amount of real home money balances into the FOREX market. However, the increase in the transaction fee overweighs the one in real money balances, so that buyers leave the FOREX market with less real money available for purchases in the foreign DM. Therefore, other things equal, an increase in θ decreases the level of imports for both countries, and, hence, the total volume of international trade.

4.2.2 FOREX Market Liquidity

In this section, we examine the determinants of standard measures of FOREX market liquidity. More specifically, we describe the trade volume in the FOREX market, both at the interdealer and the buyer-dealer level, and the bid-ask spread, and we study how these variables are affected by monetary policy and the FOREX market microstructure. The relevant results are stated in Propositions 3, 4, and 5, respectively.

Proposition 3. *The trade volume in the interdealer FOREX market, $\mathcal{V}_{ID}$, is defined as the post-buyer-dealer FOREX trade real $money_A$ ($money_B$) balances held by country B (A). That is,*

$$\mathcal{V}_{ID} \equiv \varphi_A \bar{A}^B_{m_A} + \varphi_B \bar{A}^A_{m_B}.$$

We have the following results:
i) $\mathcal{V}_{ID}$ is decreasing in either country's inflation rate, i.e., $\partial \mathcal{V}_{ID}/\partial \gamma_i < 0, \forall i \in \{A, B\}$.
ii) For given γ_i, γ_{-i}, $\mathcal{V}_{ID}$ is decreasing in the dealer's bargaining power, i.e., $\partial \mathcal{V}_{ID}/\partial \theta < 0$.
iii) For given γ_i, γ_{-i}, $\mathcal{V}_{ID}$ is increasing in the probability with which the typical buyer matches with a dealer, i.e., $\partial \mathcal{V}_{ID}/\partial(\delta\alpha_i) > 0, \forall i \in \{A, B\}$.

Proof. See the appendix. □

To understand the definition of $\mathcal{V}_{ID}$ recall the dealer's optimal portfolio decision discussed in Lemma 7. In the stationary equilibrium, where $\gamma_i > \beta$, for all i, dealers never hold any of the two currencies on their own account. In other words, dealers trade in the interdealer FOREX market only with the currencies of the customers that they represent. Moreover, from the two previous propositions we know that a rising inflation in country i leads to a fall in the real value of buyer i's post-FOREX portfolio. Hence, part (i) of Proposition 3 follows naturally. Part (ii) of the proposition states that as θ increases, the amount of liquidity provided by dealers in the typical buyer-dealer meeting decreases, and so does the volume of trade in the interdealer market. This result follows closely from the properties of the buyer-dealer FOREX bargaining protocol (i.e., Lemma 5) and is quite intuitive: when θ is high a large fraction of real balances ends up directly in the dealers' pockets in the form of fees. Thus, the amount of currencies that get re-shuffled through the interdealer market decreases.

Part (iii) of the proposition is also intuitive.[31] An increase in $\delta\alpha_i$ leads to a greater number of matches between dealers and buyers, which, in turn, raises the volume of interdealer market trade because now a greater number of buyers is represented in that market (recall that dealers trade in the interdealer market only with the currencies of customers that they represent). Also, an increase in $\delta\alpha_i$ has another less obvious effect: when this term is high buyers realize that they are more likely to obtain a fruitful consumption opportunity abroad, which induces them to carry more real home money balances (formally, $\partial Z^i/\partial(\delta\alpha_i) > 0, \forall i$). As a result, when $\delta\alpha_i$ is high, not only more buyers are represented in the interdealer market, but also each one of these buyers wants to trade a larger amount of home currency, thus increasing $\mathcal{V}_{ID}$ even further.

Even though part (iii) of Proposition 3 describes the effect on $\mathcal{V}_{ID}$ of changes in $\delta\alpha_i$, rather than δ, α_i individually (see footnote 31), it should be noted that each one of these terms has an important economic meaning on its own: α_i captures the dealer availability in the FOREX market, which is often interpreted as a measure of market liquidity in the finance literature. Also, δ, the probability of obtaining a foreign consumption opportunity, can be viewed as the degree of economic integration between the two countries.

Proposition 4. *The trade volume in the buyer-dealer FOREX market, $\mathcal{V}_{BD}$, is defined as the sum of real $money_A$ and $money_B$ balances collected by dealers during the round of bilateral trade. That is,*

$$\mathcal{V}_{BD} \equiv \varphi_A \left[\varepsilon^a \bar{m}_B^A \left(\delta\alpha_A A_{m_A}^A \right) \right] + \varphi_B \left[\frac{\bar{m}_A^B \left(\delta\alpha_B A_{m_B}^B \right)}{\varepsilon^b} \right].$$

We have the following results:

i) $\mathcal{V}_{BD}$ is decreasing in either country's inflation rate, i.e., $\partial\mathcal{V}_{BD}/\partial\gamma_i < 0, \forall i \in \{A, B\}$.

ii) For given γ_i, γ_{-i}, $\mathcal{V}_{BD}$ is increasing in the dealers' bargaining power, i.e., $\partial\mathcal{V}_{BD}/\partial\theta > 0$.

[31] The term $\mathcal{V}_{ID}$ is shown to be an implicit function only of the effective matching probability $\delta\alpha_i$, i.e., the probability with which the buyer is a C-type *and* matches in the FOREX market. Hence, here we study the effect on $\mathcal{V}_{ID}$ of changes in $\delta\alpha_i$, rather than δ, α_i individually.

iii) For given γ_i, γ_{-i}, $\mathcal{V}_{BD}$ is increasing in the probability with which the typical buyer of either country matches with a FOREX dealer, i.e., $\partial \mathcal{V}_{BD}/\partial(\delta\alpha_i) > 0$, $\forall i \in \{A, B\}$.

Proof. See the appendix. □

Part (i) of Proposition 4 has a similar interpretation as part (i) of Proposition 3. A rising inflation in country i depresses the equilibrium real balances held by the typical buyer i. Moreover, we know from Lemma 5 that the amount of $money_i$ that buyer i hands over to the dealer is positively related to the amount that she brought into the FOREX market originally. Hence, the negative relationship between $\mathcal{V}_{BD}$ and γ_i follows immediately. While changes in either country's inflation have the same effect on the volume of trade at both levels of the FOREX market, this is not true for changes in the dealers' bargaining position. When θ is large buyers carry a lot of real balances (i.e., $\partial Z^i/\partial\theta > 0$, $\forall i$) which, however, never make it to the interdealer market simply because they end up in the dealers pockets in the form of fees. Hence, an increase in θ reduces $\mathcal{V}_{ID}$ but it raises $\mathcal{V}_{BD}$. Part (iii) of Proposition 4 admits a similar interpretation as its analogue in Proposition 3 (also, a comment similar to the one in footnote 31 applies here). An increase in $\delta\alpha_i$ raises the volume of trade at the buyer-dealer level both through the extensive margin (more matches between buyers and dealers) and through the intensive margin (within each match a larger amount of real balances changes hands, because the increase in $\delta\alpha_i$ induces buyers to carry more real balances).

Proposition 5. *Recall the definition of $\varepsilon^a, \varepsilon^b$ from Lemma 4, and define the spread in the buyer-dealer FOREX market as the percentage difference between the 'Ask' and the 'Bid' price of $money_B$ and the competitive interdealer price of currency, ε. That is,*

$$\mathcal{S}^A \equiv \frac{\varepsilon^a - \varepsilon}{\varepsilon} \quad \text{and} \quad \mathcal{S}^B \equiv \frac{\varepsilon - \varepsilon^b}{\varepsilon^b}.$$

We have the following results:
i) $\mathcal{S}^i$ is increasing in country i's inflation rate, i.e., $\partial \mathcal{S}^i/\partial\gamma_i > 0$, $\forall i \in \{A, B\}$.
ii) $\mathcal{S}^i$ is increasing in the dealers' bargaining power, i.e., $\partial \mathcal{S}^i/\partial\theta > 0$, $\forall i \in \{A, B\}$.
iii) $\mathcal{S}^i$ is decreasing in the probability with which the typical buyer of either country matches with a FOREX dealer, i.e., $\partial \mathcal{S}^i/\partial(\delta\alpha_i) < 0$, $\forall i \in \{A, B\}$.

Proof. See the appendix. □

From the earlier definition, it follows that $\mathcal{S}^i$ is the per unit profit of a dealer who matched with buyer i.[32] Hence, an increase in $\mathcal{S}^i$ can be viewed as a higher intermediation fee. Part (i)

[32] For instance, consider a dealer who matched with buyer A who wants to acquire $money_B$. The dealer visits the interdealer market where she can purchase $money_B$ at price ε (in terms of $money_A$), but she sells the acquired money to the buyer at the negotiated price ε^a, where $\varepsilon^a > \varepsilon$.

of the proposition states that an increase in country i's inflation increases the equilibrium profit of dealers that meet buyers from country i.[33] This result is quite intuitive. Inflation in the home currency decreases the amount of real balances held by buyers i, and makes them more liquidity constrained. More precisely, a high γ_i implies that buyers can only purchase a small amount of good in the foreign DM, and, since u is strictly concave, these buyers will have a high valuation for an additional unit of foreign currency (which allows them to purchase a little more foreign good). Simply put, a high γ_i makes buyers i more desperate for foreign currency and effectively worsens their bargaining position. In turn, dealers take advantage of the buyers' willingness to purchase foreign currency at high prices and charge high intermediation fees.

As we have seen, an increase in the dealers' bargaining power induces agents to carry a larger amount of real home money balances into the FOREX market, which effectively improves their bargaining position. However, while the typical buyer carries more real home money, a disproportionately large fraction of this money is collected by the dealer, so that, ultimately, the net effect on the spread is positive (part (ii) of the proposition). Finally, an increase in the matching probability $\delta\alpha_i$ decreases the dealers' profits through the aforementioned real balances channel: an increase in $\delta\alpha_i$ induces buyers to carry more (real) local money into the FOREX market, which effectively allows her to achieve more favorable terms of trade in the negotiations with the dealer (part (iii) of the proposition).

Given that a high dealer availability (i.e., a high α_i) is often interpreted as an index of high market liquidity, part (iii) of Proposition 5 indicates that spreads will be tighter in a more liquid market, a finding which is well-established both in the theoretical and the empirical finance literature. However, it should be noted that the channel through which this result emerges in our model is different than the one highlighted in the existing finance literature. For instance, in DGP, a higher probability of contacting a dealer effectively improves the buyer's (or, in the DGP language, the investor's) bargaining position by improving her search alternatives and, hence, forcing the dealer to offer better prices. In our *monetary* model, a higher probability of contacting a dealer effectively increases the agent's bargaining power by making her less liquidity constraint, and, thus, less eager to acquire foreign currency in the FOREX market.

4.2.3 Social Welfare and FOREX Market Microstructure

In this section, we study the effect of the FOREX market microstructure on social welfare. Following Rocheteau and Wright (2005), we define the social welfare function, $\mathcal{W}$, as the sum (with equal weights) of all agents' steady state net utilities. Also, in what follows, for simplicity, we focus on the case with symmetric dealer availability for both countries, i.e., $\alpha_A = \alpha_B = \bar{\alpha}$. As

[33] Notice that a change in γ_{-i} has no effect on $\mathcal{S}^i$. The reason is the same as the one discussed in part (ii) of Proposition 2: an increase in γ_{-i} allows buyers i to purchase more $money_{-i}$ in the FOREX market, but, at the same time, the extra $money_{-i}$ that they acquire is not so valuable any more. Moreover, due to the competitive nature of the interdealer market, these two opposing forces exactly offset one-another.

we show in the appendix, the welfare function is given by

$$\mathcal{W} = \delta\bar{\alpha}2\left\{u(\tilde{q}_A^T) - \tilde{q}_A^T\right\} + \delta\bar{\alpha}2\left\{u(\tilde{q}_B^T) - \tilde{q}_B^T\right\} \\ + (1-\delta\bar{\alpha})\left\{u(\min\{Z^A, q^*\}) - \min\{Z^A, q^*\}\right\} + (1-\delta\bar{\alpha})\left\{u(\min\{Z^B, q^*\}) - \min\{Z^B, q^*\}\right\}. \tag{19}$$

As is standard in models that build on LW, the welfare function depends only on net DM utilities, i.e., the expressions $u(q) - q$, evaluated at $\tilde{q}_i^T$ or Z^i (or q^*), which, are equilibrium objects that have been described in detail in the previous propositions. To see why the CM utilities of agents do not show up in $\mathcal{W}$, notice that, in this model, with linear preferences in the CM, agents work in that market only in order to acquire money. This has two implications. First, agents who do not wish to hold money (i.e., sellers and dealers) never work in the CM. Second, buyers work in order to purchase real balances which are later enjoyed (i.e., sold for general good) either by a seller or a dealer.[34] Hence, in the steady state, all the negative entries in $\mathcal{W}$ (see eq.(a.32) in the appendix), which represent the buyers' disutility from work, cancel out with some positive entries, which represent the CM consumption of agents who sold the money that they received from these buyers during the earlier rounds of trade.[35]

Proposition 6. *i) For any $\gamma_A, \gamma_B > \beta$, we have $\partial\mathcal{W}/\partial(\bar{\alpha}\delta) > 0$.*
ii) If $\gamma_i \leq \bar{\mu}_i$, $\forall i$, then $\partial\mathcal{W}/\partial\theta < 0$. On the other hand, if $\gamma_i > \bar{\mu}_i$, $\forall i$, then the effect of changes in θ on $\mathcal{W}$ differs depending on the value of $\delta\bar{\alpha}$. In particular, there exists a unique $\widetilde{\delta\alpha} \in (0,1)$, such that

$$\frac{\partial\mathcal{W}}{\partial\theta} > 0 \quad \textit{iff} \quad \delta\bar{\alpha} < \widetilde{\delta\alpha}.$$

Proof. See the appendix. □

Part (i) of the proposition is quite intuitive. An increase in the effective probability of matching with a dealer improves welfare for two reasons. First, it implies that a greater number of buyers is able to purchase the foreign DM good. This undoubtedly increases welfare, because the buyers' utility in the DM round is given by $u(q) + u(\tilde{q})$, with $u' > 0, u'' < 0$, and $u'(0) = \infty$.[36] Second, an increase in $\delta\bar{\alpha}$ raises equilibrium real home money balances. Hence, other things equal, within any given match the buyer is able to purchase an amount of good which is closer to the first best, q^* (and which, clearly, maximizes $\mathcal{W}$).

[34] Or even by the buyers themselves, if they carry $Z^i > q^*$, and they do not purchase any foreign DM good.

[35] Given this discussion, the interpretation of eq.(19) is quite intuitive. The first (second) term on the RHS of this equation represents the net surplus generated in DM meetings between sellers and matched C-type buyers from country A (B). The relevant weight on this term, i.e., the measure of matched C-types from country A, $\delta\bar{\alpha}$, is multiplied by 2 because these agents choose to consume the same amount of good ($\tilde{q}_A^T$) in both DMs. Similarly, the third (fourth) term on the RHS of eq.(19) represents the net surplus generated in DM meetings between sellers and buyers from country A (B) who only traded in their local DM.

[36] Hence, a buyer can always increase her benefit by decreasing the local DM consumption q and increasing the foreign DM consumption $\tilde{q}$. For more details, see the discussion following Lemma 4.

The welfare effect of changes in θ, described in part (ii) of the proposition, is richer and more interesting. Generally, an increase in θ lowers $\tilde{q}_i^T$, $\forall i$ (part (iv) of Proposition 2), which undoubtedly hurts welfare since it decreases the first two terms on the RHS of eq.(19). However, we have also seen that an increase in θ raises Z^i, $\forall i$, which *could* improve welfare since it may increase the third and fourth terms on the RHS of eq.(19). While the first (negative) effect is relevant for any level of inflation, the second (potentially positive) effect is relevant only when buyers who trade exclusively at the home DM carry $Z^i < q^*$, i.e., only if equilibrium lies in Region 3. Hence, if $\gamma_i \leq \bar{\mu}_i$, $\forall i$, so that equilibrium lies in Region 2, an increase in θ only has a negative effect on $\mathcal{W}$ by reducing the welfare generated in DM meetings between sellers and matched C-type buyers (i.e., $\tilde{q}_i^T$ falls in the first two terms in eq.(19)). If, on the other hand, $\gamma_i > \bar{\mu}_i$, $\forall i$, so that equilibrium lies in Region 3, an increase in θ still causes the aforementioned negative effect on $\mathcal{W}$, but now it also creates a benefit by increasing the welfare generated in DM meetings between sellers and buyers who only trade at home (i.e., Z^i rises in the last two terms in eq.(19)). Which of these forces prevails depends on the relative value of $\delta\bar{\alpha}$. If $\delta\bar{\alpha}$ is relatively low, in the precise sense that $\delta\bar{\alpha} < \widetilde{\delta\alpha}$, the number of matches for which a high θ is beneficial is large, and, hence an increase in the dealers' bargaining power can improve welfare.

These results highlight that the FOREX market microstructure critically affects welfare. Hence, our model delivers important insights which have been overlooked by the existing literature, where the FOREX market is typically modeled as a frictionless Walrasian market.

5 Conclusion

In this paper, we develop a two-country dynamic general equilibrium framework, where the FOREX market is modeled in an empirically relevant way, namely, as a decentralized over-the-counter (OTC) market characterized by bilateral trade and intermediation. Our paper can be viewed as an attempt to bridge the gap between the two existing literatures on FOREX rate determination: the traditional international macroeconomics approach and the recent FOREX microstructure approach. Our model allows us to study questions that cannot be studied within either of the two existing literatures in isolation. For instance, we are able to explicitly compute standard measures of FOREX market liquidity, such as bid-ask spreads and trade volume, and to analyze how these measures are affected both by monetary policy and the FOREX market microstructure. Among other results, we find that: a) an increase in either country's inflation raises the bid-ask spreads; b) an increase in either country's inflation decreases the total trade volume in the FOREX market; and c) an increase in the FOREX dealer's bargaining power raises the investor-dealer trade volume, but lowers the interdealer trade volume.

Our theory also sheds new light on how the FOREX market microstructure can affect international trade and welfare. We find that an increase in the FOREX market liquidity (measured by dealer availability) will undoubtedly increase international trade and welfare, but an increase in the dealers' bargaining power could increase or decrease welfare depending on the FOREX market liquidity and the degree of economic integration between the two countries. These results highlight that modeling the FOREX market as a frictionless Walrasian market is *not* without loss of generality, and our model makes a first step towards incorporating the institutional details of the FOREX market within a dynamic general equilibrium framework.

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A Appendix

Proof of Lemma 3.

Plugging eq.(2) into (3) yields

$$\widetilde{W}_D(\mathbf{m}^d) = \max_{\{\widetilde{m}_A^d, \widetilde{m}_B^d\}} \left\{\varphi_A \widetilde{m}_A^d + \varphi_B \widetilde{m}_B^d + \mu\left(m_A^d + \varepsilon m_B^d - \widetilde{m}_A^d - \varepsilon \widetilde{m}_B^d\right) + \lambda_A \widetilde{m}_A^d + \lambda_B \widetilde{m}_B^d\right\} + W_D(\mathbf{0}),$$

where μ represents the Lagrangian multiplier on the feasibility constraint, and λ_A, λ_B denote the multipliers on the nonnegativity constraints $\widetilde{m}_A^d, \widetilde{m}_B^d \in \mathbb{R}_+$, respectively. The FOCs for are given by

$$\varphi_A - \mu + \lambda_A = 0, \tag{a.1}$$

$$\varphi_B - \mu\varepsilon + \lambda_B = 0, \tag{a.2}$$

$$\mu \left(m_A^d + \varepsilon m_B^d - \widetilde{m}_A^d - \varepsilon \widetilde{m}_B^d \right) = 0. \tag{a.3}$$

Combining eq.(a.1) and (a.2), one can verify that $\mu > 0$ and $\varepsilon\varphi_A + \varepsilon\lambda_A = \varphi_B + \lambda_B$. Then, three are the possible scenarios: (1): $\lambda_A > 0 = \lambda_B$, (2): $\lambda_A = \lambda_B = 0$, and (3): $\lambda_A = 0 < \lambda_B$. In case (1), eq.(a.3) imply $\widetilde{m}_A^d = 0$, $\widetilde{m}_B^d = m_B^d + m_A^d/\varepsilon$, and $\varepsilon\varphi_A < \varphi_B$. In case (2), the FOCs imply $\widetilde{m}_A^d \in [0, m_A^d + \varepsilon m_B^d]$, $\widetilde{m}_B^d = m_B^d + (m_B^d - \widetilde{m}_A^d)/\varepsilon$, and $\varepsilon\varphi_A = \varphi_B$. In case (3), eq.(a.3) implies $\widetilde{m}_A^d = m_A^d + \varepsilon m_B^d$, $\widetilde{m}_B^d = 0$, and $\varepsilon\varphi_A > \varphi_B$. Eq.(10) can be derived by plugging the post-interdealer FOREX portfolio $\left(\widetilde{m}_A^d, \widetilde{m}_B^d\right)$ into eq.(3). *Q.E.D.*

Proof of Lemma 4.

Two possible pairs can be formed: (i): a meeting between a buyer A and a dealer, (ii): a meeting between a buyer B and a dealer. We only show the solution for the former in detail (the solution for the latter follows identical steps).

<u>Proof of part (a):</u>

In the bargaining game between a buyer A and a dealer, the Lagrangian function becomes

$$\begin{aligned}
\mathcal{L} &= \bar{\varphi}\left[\bar{m}_A^d + \varepsilon \bar{m}_B^d - (m_A^d + \varepsilon m_B^d)\right] \\
&\quad + \lambda_1 \left\{ \frac{\theta}{1-\theta} \left[u\left(q_A\left(\bar{m}_A^A\right)\right) + u\left(\widetilde{q}_A\left(\bar{m}_B^A\right)\right) - u\left(q_A\left(m_A^A\right)\right) + \varphi_A \left[p_A(m_A^A) - p_A(\bar{m}_A^A) - \left(m_A^A - \bar{m}_A^A\right)\right]\right] \right\} \\
&\quad - \lambda_1 \left\{ \bar{\varphi}\left[\bar{m}_A^d + \varepsilon \bar{m}_B^d - (m_A^d + \varepsilon m_B^d)\right] \right\} \\
&\quad + \lambda_2 \left\{ m_A^d + m_A^A + \varepsilon m_B^d - \left(\bar{m}_A^d + \bar{m}_A^A + \varepsilon[\bar{m}_B^d + \bar{m}_B^A]\right) \right\} \\
&\quad + \lambda_3 \bar{m}_A^d + \lambda_4 \bar{m}_B^d + \lambda_5 \bar{m}_A^A + \lambda_6 \bar{m}_B^A,
\end{aligned}$$

where λ_1 denotes the Lagrangian multiplier on the Kalai constraint, and λ_2 denotes the Lagrangian multiplier on the feasibility constraint. The terms λ_3 to λ_6 represent the multipliers on the nonnegativity constraints. The corresponding FOCs are given by

$$\bar{\varphi} - \lambda_1 \bar{\varphi} - \lambda_2 + \lambda_3 = 0, \tag{a.4}$$

$$\lambda_1 \frac{\theta}{1-\theta} \left\{ u'\left(q(\bar{m}_A^A)\right) \left.\frac{\partial q}{\partial m_A}\right|_{m_A = \bar{m}_A^A} - \varphi_A \left(\left.\frac{\partial p}{\partial m_A}\right|_{m_A = \bar{m}_A^A} - 1 \right) \right\} - \lambda_2 + \lambda_5 = 0, \tag{a.5}$$

$$\bar{\varphi}\varepsilon - \lambda_1 \bar{\varphi}\varepsilon - \lambda_2 \varepsilon + \lambda_4 = 0, \tag{a.6}$$

$$\lambda_1 \frac{\theta}{1-\theta} u'\left(\widetilde{q}(\bar{m}_B^A)\right) \left.\frac{\partial \widetilde{q}}{\partial m_B}\right|_{m_B = \bar{m}_B^A} - \lambda_2 \varepsilon + \lambda_6 = 0. \tag{a.7}$$

Suppose $\lambda_2 = 0$ then, from eq.(a.7) and Lemma 2 the following holds:

$$\underbrace{\lambda_1 \frac{\theta}{1-\theta} u'\left(\varphi_B \bar{m}_B^A\right) \varphi_B}_{>0} + \lambda_6 = 0.$$

For the equation above to hold, it must be the case that $\lambda_1 = \lambda_6 = 0$. Then, from eq.(a.6), we have $\bar{\varphi}\varepsilon + \lambda_4 = 0$, which is a contradiction. Thus, $\lambda_2 > 0$ always. Since $\lambda_1 > 0$ as well, six possible cases remain: (1) $\lambda_3 = \lambda_4 = \lambda_5 = 0 < \lambda_6$, (2) $\lambda_3 = \lambda_4 = \lambda_6 = 0 < \lambda_5$, (3) $\lambda_4 = \lambda_5 = \lambda_6 = 0 < \lambda_3$, (4) $\lambda_3 = \lambda_5 = \lambda_6 = 0 < \lambda_4$, (5) $\lambda_5 = \lambda_6 = 0$ and $\lambda_3 > 0$ and $\lambda_4 > 0$, (6) $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$.[37] Moreover, $\lambda_1 > 0$ and $\lambda_2 > 0$ imply

$$\bar{\varphi}\left[\bar{m}_A^d + \varepsilon \bar{m}_B^d - (m_A^d + \varepsilon m_B^d)\right] = \tag{a.8}$$
$$\frac{\theta}{1-\theta}\left[u\left(q_A\left(\bar{m}_A^A\right)\right) + u\left(\widetilde{q}_A\left(\bar{m}_B^A\right)\right) - u\left(q_A\left(m_A^A\right)\right) + \varphi_A\left[p_A(m_A^A) - p_A(\bar{m}_A^A) - \left(m_A^A - \bar{m}_A^A\right)\right]\right],$$

$$m_A^d + m_A^A + \varepsilon m_B^d = \bar{m}_A^d + \bar{m}_A^A + \varepsilon[\bar{m}_B^d + \bar{m}_B^A]. \tag{a.9}$$

We analyze each case one by one.

Case 1: $\lambda_3 = \lambda_4 = \lambda_5 = 0 < \lambda_6 \Rightarrow \bar{m}_B^A = 0$.

From eq.(a.4) and (a.6), we get $\lambda_2 = \bar{\varphi}(1 - \lambda_1)$. Then eq.(a.5) implies that

$$\lambda_1 \frac{\theta}{1-\theta} \left\{ u'\left(q(\bar{m}_A^A)\right) \left.\frac{\partial q}{\partial m_A}\right|_{m_A = \bar{m}_A^A} - \varphi_A \left(\left.\frac{\partial p}{\partial m_A}\right|_{m_A = \bar{m}_A^A} - 1 \right) \right\} = \bar{\varphi}(1 - \lambda_1).$$

[37] Technically speaking, there are other possible cases to consider. In what follows, however, we explain why those cases are ruled out.

Suppose $\bar{m}_A^A \geq m_A^*$, then

$$\lambda_1 \frac{\theta}{1-\theta} \varphi_A = \bar{\varphi}(1-\lambda_1),$$
$$\Rightarrow \lambda_1 \left[\frac{\theta}{1-\theta} + \frac{\bar{\varphi}}{\varphi_A}\right] = \frac{\bar{\varphi}}{\varphi_A},$$
$$\Rightarrow \lambda_1 = \frac{\bar{\varphi}/\varphi_A}{(\bar{\varphi}/\varphi_A) + (\theta/(1-\theta))} < 1.$$

But from eq.(a.7), we have $u'(0) = \infty$, which is a contradiction. Suppose $\bar{m}_A^A < m_A^*$. Then, from eq.(a.5) we obtain

$$\lambda_1 = \frac{\bar{\varphi}/(\varphi_A u'(\bar{\varphi}\bar{m}_A^A))}{\bar{\varphi}/(\varphi_A u'(\bar{\varphi}\bar{m}_A^A)) + (\theta/(1-\theta))} < 1.$$

But again, from eq.(a.7), we have $u'(0) = \infty$, which is a contradiction. Thus, case (1) cannot be a solution.

Case 2: $\lambda_3 = \lambda_4 = \lambda_6 = 0 < \lambda_5 \Rightarrow \bar{m}_B^A > 0$ and $\bar{m}_A^A = 0$.

Again, the RHS of eq.(a.5) goes to infinity as $u'(q(0)) = \infty$. This restricts our attention to $\lambda_5 = \lambda_6 = 0$ (i.e., $\bar{m}_A^A > 0$ and $\bar{m}_B^A > 0$) for the rest of the cases considered.

Case 3: $\lambda_5 = \lambda_6 = \lambda_4 = 0 < \lambda_3 \Rightarrow \bar{m}_A^d = 0$ and $\bar{m}_B^d > 0$.

From eq.(a.6), we have $\lambda_2 = \bar{\varphi}(1-\lambda_1)$, but from eq.(a.4) we get

$$\bar{\varphi}(1-\lambda_1) + \underbrace{\lambda_3}_{>0} = \lambda_2,$$

which is a contradiction. This implies that case (3) cannot be a solution to the problem.

Case 4: $\lambda_5 = \lambda_6 = \lambda_3 = 0 < \lambda_4 \Rightarrow \bar{m}_A^d > 0$ and $\bar{m}_B^d = 0$.

From eq.(a.4), we have $\lambda_2 = \bar{\varphi}(1-\lambda_1)$. Also, eq.(a.6) becomes

$$\underbrace{\bar{\varphi}(1-\lambda_1)}_{\lambda_2} + \underbrace{(\lambda_4/\varepsilon)}_{>0} = \lambda_2.$$

This is also a contradiction.

Case 5: $\lambda_5 = \lambda_6 = 0$ and $\lambda_3 > 0$ & $\lambda_4 > 0 \Rightarrow \bar{m}_A^d = 0$ and $\bar{m}_B^d = 0$.

In this case, the dealer's surplus becomes non-positive (i.e., $\bar{\varphi}[\bar{m}_A^d + \varepsilon\bar{m}_B^d - (m_A^d + \varepsilon m_B^d)] \leq 0$). Thus, this case cannot be the solution either.

Case 6: $\lambda_3 = \lambda_4 = \lambda_4 = \lambda_5 = \lambda_6 = 0 \Rightarrow \bar{m}_A^d > 0$ and $\bar{m}_B^d > 0$.

From equations (a.4) and (a.6), we obtain $\lambda_2 = \bar{\varphi}(1-\lambda_1)$. Now, suppose $\bar{m}_A^A \geq m_A^*$. Then, from eq.(a.5), $\lambda_1(\theta/(1-\theta))\varphi_A = \lambda_2$. Combine this with $\lambda_2 = \bar{\varphi}(1-\lambda_1)$ to obtain

$$\lambda_1 = \frac{\bar{\varphi}}{\bar{\varphi} + \varphi_A\theta/(1-\theta)} < 1,$$
$$\lambda_2 = \bar{\varphi}\frac{\varphi_A\theta/(1-\theta)}{\bar{\varphi} + \varphi_A\theta/(1-\theta)}.$$

From eq.(a.7), $\lambda_1\theta/(1-\theta)u'(\varphi_B\bar{m}_B^A)\varphi_B = \lambda_2\varepsilon$. Replacing λ_1 and λ_2 from above, one can derive the following:

$$u'(\varphi_B\bar{m}_B^A) = \frac{\varphi_A\varepsilon}{\varphi_B}. \tag{a.10}$$

Therefore, the solution for $\bar{m}_B^A$ must satisfy eq.(a.10) which corresponds to the χ_B^* in eq.(11). Next, we solve for the $\bar{m}_A^A$. First, eq.(a.8) can be re-written as

$$\bar{\varphi}\left[\bar{m}_A^d + \varepsilon\bar{m}_B^d - (m_A^d + \varepsilon m_B^d)\right] = \frac{\theta}{1-\theta}\left\{u(q^*) + u(\varphi_B\bar{m}_B^A) - u(q^*) + \varphi_A\left[\bar{m}_A^A - m_A^A\right]\right\}.$$

After rearranging, the following equation can be derived:

$$\bar{m}_A^A = m_A^A - \frac{u(\varphi_B\bar{m}_B^A)}{\varphi_A} + \frac{1-\theta}{\theta}\frac{\bar{\varphi}}{\varphi_A}\left[\bar{m}_A^d + \varepsilon\bar{m}_B^d - (m_A^d + \varepsilon m_B^d)\right].$$

Lastly, using $m_B^d - \bar{m}_B^d = \chi_B^*$ and $\bar{m}_A^d - m_A^d = m_A^A - \bar{m}_A^A$, $\bar{m}_A^A$ is given by

$$\bar{m}_A^A = m_A^A - \frac{\theta u(\varphi_B\chi_B^*) + \bar{\varphi}(1-\theta)\varepsilon\chi_B^*}{\varphi_A\theta + \bar{\varphi}(1-\theta)}, \tag{a.11}$$

which is the solution provided in the lemma. Eq.(a.11) implies a couple of important things regarding the bargaining solution. First, it corresponds to $\tau_A(\chi_B^*)$ in eq.(13). Second, the assumption that $\bar{m}_A^A \geq m_A^*$ implies $m_A^A \geq m_A^* + \tau_A(\chi_B^*)$, which, in turn, verifies the first cutoff level for m_A^A in Lemma 4.

Now, let us consider the alternative case where $\bar{m}_A^A \leq m_A^*$. Then, from eq.(a.5), we have $\lambda_1\theta/(1-\theta)\varphi_A u'(\varphi\bar{m}_A^A) = \lambda_2$. Combine this with $\lambda_2 = \bar{\varphi}(1-\lambda_1)$ to get

$$\lambda_1 = \frac{\bar{\varphi}}{\bar{\varphi} + \varphi_A u'(\varphi\bar{m}_A^A)\theta/(1-\theta)} < 1,$$
$$\lambda_2 = \bar{\varphi}\frac{\varphi_A u'(\varphi\bar{m}_A^A)\theta/(1-\theta)}{\bar{\varphi} + \varphi_A u'(\varphi\bar{m}_A^A)\theta/(1-\theta)}.$$

From eq.(a.7), we have $\lambda_1\theta/(1-\theta)u'(\varphi_B\bar{m}_B^A)\varphi_B = \lambda_2\varepsilon$. Replacing λ_1 and λ_2 from above, one can derive the following necessary condition:

$$\frac{\varphi_B u'(\varphi_B \bar{m}_B^A)}{\varphi_A u'(\varphi_A \bar{m}_A^A)} = \frac{\bar{\varphi}\varepsilon}{\bar{\varphi}}, \tag{a.12}$$

where the LHS captures the MRS between the special good B and A for the buyer A, while the RHS represents the MRS of the dealer. Like before, we need to rearrange the Kalai constraint eq.(a.8). Here, we consider two different subcases.

Subcase 1: $m_A^* \leq m_A^A$.
Using Lemma 1, and after some algebra, eq.(a.8) can be rearranged to

$$\begin{aligned} m_A^A =& \bar{m}_A^A + \frac{\theta u(\varphi_B \bar{m}_B^A) + (1-\theta)\bar{\varphi}\varepsilon \bar{m}_B^A}{\theta\varphi_A + (1-\theta)\bar{\varphi}} - \frac{\theta\left(u(q^*) - q^*\right) - \theta\left(u(\varphi_A \bar{m}_A^A) - \varphi_A \bar{m}_A^A\right)}{\theta\varphi_A + (1-\theta)\bar{\varphi}} \\ =& \bar{m}_A^A + \tau_A(\bar{m}_B^A) - \xi(\bar{m}_A^A) = \Gamma(\bar{m}_B^A), \end{aligned} \tag{a.13}$$

where $\xi(\bar{m}_A^A)$ is a simpler notation for $\{\theta\left(u(q^*) - q^*\right) - \theta\left(u(\varphi_A \bar{m}_A^A) - \varphi_A \bar{m}_A^A\right)\}/\{\theta\varphi_A + (1-\theta)\bar{\varphi}\}$, and $\Gamma(\cdot)$ is defined in Lemma 4. As a result, $(\bar{m}_A^A, \bar{m}_B^A)$ must solve the system of equations (a.12),(a.13). Furthermore, since $\bar{m}_A^A \leq m_A^*$, $\max\{\tau_A(\bar{m}_B^A)\} = \tau_A(\chi_B^*)$, and $\min\{\xi(\bar{m}_A^A)\} = 0$, we must have $m_A^A \leq m_A^* + \tau_A(\chi_B^*)$, which verifies the second cutoff level for m_A^A in Lemma 4.

The existence and uniqueness of $(\bar{m}_A^A, \bar{m}_B^A)$ can be proven in the following way. First, given the form of eq.(a.12), it suffices to show the existence and uniqueness of $\bar{m}_B^A$ only. Let $G(\bar{m}_B^A)$ represent $\bar{m}_A^A$ such that eq.(a.12) holds. Then, if we define the function

$$H(\bar{m}_B^A) \equiv G(\bar{m}_B^A) - m_A^A + \frac{\theta u(\varphi_B \bar{m}_B^A) + (1-\theta)\bar{\varphi}\varepsilon \bar{m}_B^A - \theta\left(u(q^*) - q^*\right) + \theta\left(u(\varphi_A G(\bar{m}_B^A)) - \varphi_A G(\bar{m}_B^A)\right)}{\theta\varphi_A + (1-\theta)\bar{\varphi}}, \tag{a.14}$$

a solution (to our existence problem) will be a $\bar{m}_B^A$ that solves $H(\bar{m}_B^A) = 0$. It is easy to see that $H(0) < 0$. Also, $H'(\cdot) > 0$ in the whole domain, since $G' > 0$ from eq.(a.12), and $u'(\cdot) > 0$. Lastly, $\max\{\bar{m}_B^A\} = \chi_B^*$ from eq.(a.12). This leads to

$$H(\chi_B^*) = m_A^* + \tau_A(\chi_B^*) - m_A^A \geq 0.$$

This proves the existence and uniqueness of $(\bar{m}_A^A, \bar{m}_B^A)$ when $m_A^* \leq m_A^A \leq m_A^* + \tau_A(\chi^*)$.

Subcase 2: $m_A^A \leq m_A^*$.
Using Lemma 1, after some eq.(a.8) can be rearranged as follows:

$$m_A^A = \bar{m}_A^A + \frac{\theta u(\varphi_B \bar{m}_B^A) + (1-\theta)\bar{\varphi}\varepsilon \bar{m}_B^A}{(1-\theta)\bar{\varphi}} - \frac{\theta\left(u(\varphi_A m_A^A) - u(\varphi_A \bar{m}_A^A)\right)}{(1-\theta)\bar{\varphi}} = \Lambda(\bar{m}_B^A), \tag{a.15}$$

where $\Lambda(\cdot)$ is defined in Lemma 4. Similar to subcase 1, $(\bar{m}_A^A, \bar{m}_B^A)$ must solve the system of equations (a.12), (a.15). Define the function

$$R(\bar{m}_B^A) \equiv G(\bar{m}_B^A) - m_A^A + \frac{\theta u(\varphi_B \bar{m}_B^A) + (1-\theta)\bar{\varphi}\varepsilon \bar{m}_B^A + \theta\left(u(\varphi_A G(\bar{m}_B^A)) - u(\varphi_A m_A^A)\right)}{(1-\theta)\bar{\varphi}}. \quad \text{(a.16)}$$

It can be easily verified that $R(0) < 0$ and $R'(\bar{m}_B^A) > 0$, since $G' > 0$ and $u' > 0$. Also, $\bar{m}_B^A$ never exceeds χ_B^* from eq.(a.12), and therefore $R(\chi_B^*)$ can be re-written as

$$R(\chi_B^*) = m_A^* - m_A^A + \varepsilon\chi_B^* + \frac{\theta\left(u(\varphi_B \bar{m}_B^A) + u(\varphi_A G(\bar{m}_B^A)) - u(\varphi_A m_A^A)\right)}{(1-\theta)\bar{\varphi}},$$

where $u(\varphi_B \bar{m}_B^A) + u(\varphi_A G(\bar{m}_B^A)) - u(\varphi_A m_A^A)$ corresponds to buyer A's surplus which is non-negative. Since $m_A^A \leq m_A^*$, we also have $R(\chi_B^*) > 0$. This completes the proof for the existence and uniqueness.

Proof of parts (b) and (c):

When $m_A^A \geq m_A^* + \tau_A(\chi_B^*)$, the term ε^a is defined as $\varepsilon^a \bar{m}_B^A = m_A^A - \bar{m}_A^A$. Combining this definition with eq.(a.11) and rearranging yields[38]

$$\varepsilon^a = \varepsilon + \frac{\theta\left\{u(\varphi_B \chi_B^*) - \varphi_A \varepsilon \chi_B^*\right\}}{\left\{\varphi_A \theta + \bar{\varphi}(1-\theta)\right\}\chi_B^*}. \quad \text{(a.17)}$$

Combining equations (a.9),(a.11), and the fact that $\bar{m}_B^A = \chi_B^*$, we obtain

$$\bar{m}_A^d + \varepsilon \bar{m}_B^d = m_A^d + \varepsilon m_B^d + \tau_A(\chi_B^*) = m_A^d + \varepsilon m_B^d + (\varepsilon^a - \varepsilon)\chi_B^*,$$

where the second equality follows from eq.(a.17). This result verifies Corollary 1 in the case where $m_A^A \geq m_A^* + \tau_A(\chi_B^*)$. Then, the terms $\bar{m}_A^d$ and $\bar{m}_B^d$ in part (c) of the lemma follow immediately. When $m_A^* \leq m_A^A \leq m_A^* + \tau_A(\chi^*)$, one can obtain[39]

$$\varepsilon^a = \varepsilon + \frac{\theta\left\{u(\varphi_B \bar{m}_B^A) - \varphi_A \varepsilon \bar{m}_B^A\right\} - \theta\left\{u(q^*) - q^*\right\} + \theta\left\{u(\varphi_A G(\bar{m}_B^A)) - \varphi_A G(\bar{m}_B^A)\right\}}{\left\{\varphi_A \theta + \bar{\varphi}(1-\theta)\right\}\bar{m}_B^A}, \quad \text{(a.18)}$$

by combining the definition of ε^a with eq.(a.13) with some algebra. Using equations (a.18), (a.9),

[38] Note that the second pair counter part, namely ε^b, can be derived using similar steps:

$$1/\varepsilon^b = 1/\varepsilon + \frac{\theta\left\{u(\varphi_A \chi_A^*) - (\varphi_B/\varepsilon)\chi_A^*\right\}}{\left\{\varphi_B \theta + \bar{\varphi}\varepsilon(1-\theta)\right\}\chi_A^*}.$$

[39] Note that ε^b can be derived following similar steps:

$$1/\varepsilon^b = (1/\varepsilon) + \frac{\theta\left\{u(\varphi_A \bar{m}_A^B) - (\varphi_B/\varepsilon)\bar{m}_A^B\right\} - \theta\left\{u(q^*) - q^*\right\} + \theta\left\{u(\varphi_B G(\bar{m}_A^B)) - \varphi_B G(\bar{m}_A^B)\right\}}{\left\{\varphi_B \theta + \bar{\varphi}\varepsilon(1-\theta)\right\}\bar{m}_A^B}.$$

and (a.13), the following holds true

$$\bar{m}_A^d + \varepsilon \bar{m}_B^d = m_A^d + \varepsilon m_B^d + (\varepsilon^a - \varepsilon)\bar{m}_B^A.$$

This result again verifies Corollary 1 in the case of $m_A^* \leq m_A^A \leq m_A^* + \tau_A(\chi^*)$. Then, immediate access to the interdealer market after bargaining makes it straightforward to solve for the terms $\bar{m}_A^d$ and $\bar{m}_B^d$ that appear in part (c) of the lemma. When $m_A^A \leq m_A^*$, The term ε^a can be derived as in subcase 1:[40]

$$\varepsilon^a = \varepsilon + \frac{\theta \left\{ u(\varphi_B \bar{m}_B^A) + u(\varphi_A \bar{m}_A^A) - u(\varphi_A m_A^A) \right\}}{(1-\theta)\bar{\varphi}\bar{m}_B^A}. \tag{a.19}$$

Using equations (a.19), (a.9), and (a.15), one can show that

$$\bar{m}_A^d + \varepsilon \bar{m}_B^d = m_A^d + \varepsilon m_B^d + (\varepsilon^a - \varepsilon)\bar{m}_B^A,$$

which verifies Corollary 1 in the case of $m_A^A \leq m_A^*$. To derive the terms $\bar{m}_A^d$ and $\bar{m}_B^d$ in part (c) of the lemma one simply has to follow the same steps as in subcase 1. *Q.E.D.*

Proof of Lemma 5.

Again, we focus on the case where a dealer meets a buyer A.

Region 1: $m_A^A \geq m_A^* + \tau_A(\chi_B^*)$.

From eq.(a.17) it can be easily verified that $\partial \varepsilon^a / \partial \theta = 0$ and $\partial \bar{m}_B^A / \partial m_A^A = \partial \bar{m}_B^A / \partial \theta = 0$. Moreover, eq.(a.11) confirms that $\partial (m_A^A - \bar{m}_A^A)/\partial m_A^A = 0$. From eq.(a.17) it suffices to show that $u(\varphi_B \chi_B^*) > \varphi_A \varepsilon \chi_B^*$ in order to prove that $\partial \varepsilon^a / \partial \theta > 0$. The proof of this claim is straightforward. We have

$$\frac{u(\varphi_B \chi_B^*)}{\varphi_A \varepsilon \chi_B^*} > 1 \;\Rightarrow\; \frac{u(\varphi_B \chi_B^*)}{\varphi_B \chi_B^*} > \frac{\varphi_A \varepsilon}{\varphi_B},$$

where the second inequality follows from the concavity of u. This inequality also confirms that $\varepsilon^a > \varepsilon$ (following from eq.(a.17)), which, in turn, proves that $\partial (m_A^A - \bar{m}_A^A)/\partial \theta > 0$.

Region 2: $m_A^* \leq m_A^A \leq m_A^* + \tau_A(\chi_B^*)$.

From eq.(a.8), the surplus for the buyer A must be positive. Along with the fact that $q^* > \varphi_A \bar{m}_A^A$ in this region, the following must be true:

$$\begin{aligned} u(\varphi_B \bar{m}_B^A) \geq& \varphi_A (m_A^A - \bar{m}_A^A) + u(q^*) - q^* - (u(\varphi_A \bar{m}_A^A) - \varphi_A \bar{m}_A^A), \\ u(\varphi_B \bar{m}_B^A) >& \varphi_A \varepsilon \bar{m}_B^A + u(q^*) - q^* - (u(\varphi_A \bar{m}_A^A) - \varphi_A \bar{m}_A^A), \end{aligned} \tag{a.20}$$

[40] Again, the term ε^b takes the form

$$1/\varepsilon^b = (1/\varepsilon) + \frac{\theta \left\{ u(\varphi_A \bar{m}_A^B) + u(\varphi_B \bar{m}_B^B) - u(\varphi_B m_B^B) \right\}}{(1-\theta)\bar{\varphi}\varepsilon \bar{m}_A^B}.$$

where the second inequality follows from the fact that $(m_A^A - \bar{m}_A^A) = \varepsilon^a \bar{m}_B^A > \varepsilon \bar{m}_B^A$. Notice that $\varepsilon^a > \varepsilon$ follows from eq.(a.18) and the fact that buyer surplus (i.e., the numerator in the second part of eq.(a.18)) is positive.

In what follows, let $Q(\bar{m}_B^A) \equiv u(\varphi_B \bar{m}_B^A) - \varphi_A \varepsilon \bar{m}_B^A - u(q^*) + q^* + (u(\varphi_A \bar{m}_A^A) - \varphi_A \bar{m}_A^A)$. First, we prove that $\partial \bar{m}_B^A / \partial \theta < 0$. Using the Implicit Function Theorem (IFT) on eq.(a.14), we know that

$$\frac{\partial \bar{m}_B^A}{\partial \theta} = -\frac{\partial H / \partial \theta}{\partial H / \partial \bar{m}_B^A} < 0,$$

because

$$\frac{\partial H}{\partial \bar{m}_B^A} = \underbrace{G'(\bar{m}_B^A)}_{>0} + \varepsilon + \theta \underbrace{\left\{ \varphi_B u'(\varphi_B \bar{m}_B^A) - \varphi_A \varepsilon \right\}}_{>0} > 0. \tag{a.21}$$

That the second highlighted term in the expression above is positive follows directly from eq.(a.12). Moreover,

$$\frac{\partial H}{\partial \theta} = \frac{Q(\bar{m}_B^A)\left(\varphi_A \theta + (1-\theta)\bar{\varphi}\right) - \theta(\varphi_A - \bar{\varphi}) Q(\bar{m}_B^A)}{\left(\varphi_A \theta + (1-\theta)\bar{\varphi}\right)^2} > 0.$$

This inequality follows from that the fact that $Q(\bar{m}_B^A) > 0$ (see (a.20)).

Given the analysis so far, the claims that $\partial \varepsilon^a / \partial \theta > 0$, $\partial \bar{m}_B^A / \partial m_A^A > 0$, $\partial \varepsilon^a \bar{m}_B^A / \partial m_A^A > 0$, and $\partial \varepsilon^a \bar{m}_B^A / \partial \theta > 0$ can all be verified by combining $\partial \bar{m}_B^A / \partial \theta < 0$ and eq.(a.13).

That $\partial \varepsilon^a / \partial m_A^A < 0$ can be shown as follows: Substitute $Q(\bar{m}_B^A)$ into eq.(a.18), to obtain $\varepsilon^a = \varepsilon + \left\{ \theta Q(\bar{m}_B^A) \right\} / \left\{ \varphi_A \theta + \bar{\varphi}(1-\theta) \bar{m}_B^A \right\}$. Then, consider any two arbitrary values of $\bar{m}_B^A$, say x and y, where $x < y$. Since $\partial \bar{m}_B^A / \partial m_A^A > 0$, it suffices to show that $\varepsilon + \{\theta Q(x)\} / \{\varphi_A \theta + \bar{\varphi}(1-\theta)x\} > \varepsilon + \{\theta Q(y)\} / \{\varphi_A \theta + \bar{\varphi}(1-\theta)y\}$. The concavity of $Q(\cdot)$ implies that $Q(x)/x > Q(y)/y$, and, hence, the proof is complete.

Region 3: $m_A^A \leq m_A^*$

As in Region 2, eq.(a.19) and the fact that the buyer's surplus is positive confirm that $\partial \varepsilon^a / \partial \theta > 0$ and $\varepsilon^a > \varepsilon$. We now prove that $\partial \bar{m}_B^A / \partial \theta < 0$. Exploiting the IFT on eq.(a.16), we obtain

$$\frac{\partial \bar{m}_B^A}{\partial \theta} = -\frac{\partial R / \partial \theta}{\partial R / \partial \bar{m}_B^A} < 0,$$

where it is straightforward to see that $\partial R / \partial \bar{m}_B^A > 0$. Moreover,

$$\frac{\partial R}{\partial \theta} = \frac{\left(u(\varphi_B \bar{m}_B^A) + u(\varphi_A \bar{m}_A^A) - u(\varphi_A m_A^A)\right)\{(1-\theta)\bar{\varphi} + \theta \bar{\varphi}\}}{\left((1-\theta)\bar{\varphi}\right)^2} > 0,$$

which follows from that the fact that the buyer's surplus, $u(\varphi_B \bar{m}_B^A) + u(\varphi_A \bar{m}_A^A) - u(\varphi_A m_A^A)$, is positive. Then, the claims that $\partial\varepsilon^a/\partial\theta > 0$, $\partial\bar{m}_B^A/\partial m_A^A > 0$, $\partial\varepsilon^a\bar{m}_B^A/\partial m_A^A > 0$, and $\partial\varepsilon^a\bar{m}_B^A/\partial\theta > 0$ can all be verified from combining $\partial\bar{m}_B^A/\partial\theta < 0$ and eq.(a.15). The proof of the fact that $\partial\varepsilon^a/\partial m_A^A < 0$ is similar to the one in Region 2 above.*Q.E.D.*

Proof of Lemma 8.

We have the following cases: **Case 1:** $\varphi_i/\beta\widehat{\varphi}_i = 1$ rules out Regions 2 and 3. To see this point, note from eq.(16) and (17), that for any value of $\widehat{m}_i$ in these regions, J_r^i is greater than zero. Then, only $\widehat{m}_i = m_i^* + \tau_i\left(\chi_{-i}^*\right)$ guarantees that eq.(15) is satisfied.

Case 2: Suppose $1 < \varphi_i/\beta\widehat{\varphi}_i$ and the optimal $\widehat{m}_i < m_i^* + \tau_i(\chi_{-i}^*)$ (i.e., $\bar{m}_i < m_i^*$). Then, the FOC must correspond to eq.(16), which can be rewritten as

$$\frac{\varphi_i}{\beta\widehat{\varphi}_i} = 1 + \delta\alpha_i\frac{\widehat{\varphi}_{-i}}{\widehat{\varphi}_i}u'(\widehat{\varphi}_{-i}\bar{m}_{-i}^i)\frac{\partial\bar{m}_{-i}^i}{\partial\widehat{m}_i^i} + \delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\partial\bar{m}_i^i}{\partial\widehat{m}_i^i} - 1\right\}.$$

Using eq.(12) we can simplify this expression further and write

$$\frac{\varphi_i}{\beta\widehat{\varphi}_i} = 1 + \delta\alpha_i[\widehat{\varepsilon}\mathbb{I}_{\{i=A\}} + (1/\widehat{\varepsilon})\mathbb{I}_{\{i=B\}}]u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\partial\bar{m}_{-i}^i}{\partial\widehat{m}_i^i} + \delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\partial\bar{m}_i^i}{\partial\widehat{m}_i^i} - 1\right\}.$$

Next, solve for $\partial\bar{m}_i^i/\partial m_i^i$ and $\partial\bar{m}_{-i}^i/\partial m_i^i$, employing the IFT on eq.(a.14). We have

$$\begin{aligned}\frac{\partial\bar{m}_i^i}{\partial m_i^i} &= -\frac{-1}{\left(\theta\varphi_i + (1-\theta)\bar{\varphi}[\cdot] + \theta\left\{\varphi_i u'(\varphi_i\bar{m}_i^i) - \varphi_i\right\}\right)/\left(\theta\varphi_i + (1-\theta)\bar{\varphi}[\cdot]\right)} \\ &= \frac{\theta\varphi_i + (1-\theta)\bar{\varphi}[\cdot]}{\theta\varphi_i u'(\varphi_i\bar{m}_i^i) + (1-\theta)\bar{\varphi}[\cdot]},\end{aligned} \tag{a.22}$$

and

$$\begin{aligned}\frac{\partial\bar{m}_{-i}^i}{\partial m_i^i} &= -\frac{-1}{\left(\theta\varphi_{-i} + (1-\theta)\bar{\varphi}[\bar{\cdot}] + \theta\left\{\varphi_{-i}u'(\varphi_{-i}\bar{m}_{-i}^i) - \varphi_{-i}\right\}\right)/\left(\theta\varphi_i + (1-\theta)\bar{\varphi}[\cdot]\right)} \\ &= \frac{\theta\varphi_i + (1-\theta)\bar{\varphi}[\cdot]}{\theta\varphi_{-i}u'(\varphi_{-i}\bar{m}_{-i}^i) + (1-\theta)\bar{\varphi}[\bar{\cdot}]},\end{aligned} \tag{a.23}$$

where $[\bar{\cdot}] = \varepsilon\mathbb{I}_{\{i=A\}} + \mathbb{I}_{\{i=B\}}$ and $[\cdot] = \varepsilon\mathbb{I}_{\{i=B\}} + \mathbb{I}_{\{i=A\}}$. Then, plug eq.(a.22) and (a.23) into eq.(a.22) to obtain

$$\begin{aligned}\frac{\varphi_i}{\beta\widehat{\varphi}_i} =& 1 + \delta\alpha_i[\widehat{\varepsilon}\mathbb{I}_{\{i=A\}} + (1/\widehat{\varepsilon})\mathbb{I}_{\{i=B\}}]u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i + (1-\theta)\widehat{\bar{\varphi}}[\cdot]}{\theta\widehat{\varphi}_{-i}u'(\widehat{\varphi}_{-i}\bar{m}_{-i}^i) + (1-\theta)\widehat{\bar{\varphi}}[\bar{\cdot}]} \\ &+ \delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i + (1-\theta)\widehat{\bar{\varphi}}[\cdot]}{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\bar{m}_i^i) + (1-\theta)\widehat{\bar{\varphi}}[\cdot]} - 1\right\}.\end{aligned}$$

Exploiting eq.(12) once again leads to the following Euler equation:

$$\begin{aligned}\frac{\varphi_i}{\beta\widehat{\varphi}_i} =&1+\delta\alpha_i x u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i+(1-\theta)\widehat{\bar{\varphi}}[\cdot]}{\theta x\widehat{\varphi}_i u'(\widehat{\varphi}_i\bar{m}_i^i)+(1-\theta)\widehat{\bar{\varphi}}[\bar{\cdot}]}\\ &+\delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i+(1-\theta)\widehat{\bar{\varphi}}[\cdot]}{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\bar{m}_i^i)+(1-\theta)\widehat{\bar{\varphi}}[\cdot]}-1\right\},\end{aligned}\tag{a.24}$$

where $x=[\widehat{\varepsilon}\mathbb{I}_{\{i=A\}}+(1/\widehat{\varepsilon})\mathbb{I}_{\{i=B\}}]$ and $\bar{m}_i^i$ is a function of $\widehat{m}_i$ taking prices as given. The RHS of eq.(a.24) is strictly decreasing in $\bar{m}_i^i$ due to the concavity of u. This, combined with Lemma 4, confirms the uniqueness of the optimal $\widehat{m}_i$ in Region 2. Furthermore, it can be easily observed that the optimal $\widehat{m}_i$ is independent of the $money_{-i}$ holding cost, $\varphi_{-i}/(\beta\widehat{\varphi}_{-i})$.

Lastly, since so far we have assumed $\bar{m}_i < m_i^*$, $\varphi_i/\beta\widehat{\varphi}_i$ must be bounded from above by $\bar{\mu}$ which solves

$$\begin{aligned}\bar{\mu} =&1+\delta\alpha_i x u'(\widehat{\varphi}_i m_i^*)\frac{\theta\widehat{\varphi}_i+(1-\theta)\widehat{\bar{\varphi}}[\cdot]}{\theta x\widehat{\varphi}_i u'(\widehat{\varphi}_i m_i^*)+(1-\theta)\widehat{\bar{\varphi}}[\bar{\cdot}]}\\ &+\delta\alpha_i\left\{u'(\widehat{\varphi}_i m_i^*)\frac{\theta\widehat{\varphi}_i+(1-\theta)\widehat{\bar{\varphi}}[\cdot]}{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i m_i^*)+(1-\theta)\widehat{\bar{\varphi}}[\cdot]}-1\right\}>1.\end{aligned}$$

Case 3: Suppose $\bar{\mu}\leq\varphi_i/\beta\widehat{\varphi}_i$. The FOC corresponds to eq.(17), which can be re-written as

$$\frac{\varphi_i}{\beta\widehat{\varphi}_i}=1+\delta\alpha_i\frac{\widehat{\varphi}_{-i}}{\widehat{\varphi}_i}u'(\widehat{\varphi}_{-i}\bar{m}_{-i}^i)\frac{\partial\bar{m}_{-i}^i}{\partial\widehat{m}_i^i}+\delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\partial\bar{m}_i^i}{\partial\widehat{m}_i^i}-1\right\}+(1-\delta\alpha_i)\left\{u'(\widehat{\varphi}_i\widehat{m}_i)-1\right\}.$$

Using eq.(12), we can simplify this to

$$\begin{aligned}\frac{\varphi_i}{\beta\widehat{\varphi}_i} =&1+\delta\alpha_i[\widehat{\varepsilon}\mathbb{I}_{\{i=A\}}+(1/\widehat{\varepsilon})\mathbb{I}_{\{i=B\}}]u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\partial\bar{m}_{-i}^i}{\partial\widehat{m}_i^i}\\ &+\delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\partial\bar{m}_i^i}{\partial\widehat{m}_i^i}-1\right\}+(1-\delta\alpha_i)\left\{u'(\widehat{\varphi}_i\widehat{m}_i)-1\right\}.\end{aligned}\tag{a.25}$$

Next, solve for $\partial\bar{m}_i^i/\partial m_i^i$ and $\partial\bar{m}_{-i}^i/\partial m_i^i$ employing the IFT into eq.(a.16). We have

$$\frac{\partial\bar{m}_i^i}{\partial m_i^i}=\frac{\theta\varphi_i u'(\varphi_i m_i^i)+(1-\theta)\bar{\varphi}[\cdot]}{\theta\varphi_i u'(\varphi_i\bar{m}_i^i)+(1-\theta)\bar{\varphi}[\cdot]},\tag{a.26}$$

and

$$\frac{\partial\bar{m}_{-i}^i}{\partial m_i^i}=\frac{\theta\varphi_i u'(\varphi_A m_i^i)+(1-\theta)\bar{\varphi}[\cdot]}{\theta\varphi_{-i}u'(\varphi_{-i}\bar{m}_{-i}^i)+(1-\theta)\bar{\varphi}[\bar{\cdot}]},\tag{a.27}$$

where $[\bar{\cdot}]=\varepsilon\mathbb{I}_{\{i=A\}}+\mathbb{I}_{\{i=B\}}$ and $[\cdot]=\varepsilon\mathbb{I}_{\{i=B\}}+\mathbb{I}_{\{i=A\}}$. Substitute eq.(a.26) and (a.27) into eq.(a.25) to obtain

$$\frac{\varphi_i}{\beta\widehat{\varphi}_i} =1 + \delta\alpha_i[\widehat{\varepsilon}\mathbb{I}_{\{i=A\}} + (1/\widehat{\varepsilon})\mathbb{I}_{\{i=B\}}]u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\widehat{m}_i^i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]}{\theta\widehat{\varphi}_{-i}u'(\widehat{\varphi}_{-i}\bar{m}_{-i}^i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]}$$
$$+ \delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\widehat{m}_i^i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]}{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\bar{m}_i^i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]} - 1\right\}.$$

Exploiting eq.(12) one more time finally leads to the following Euler equation:

$$\frac{\varphi_i}{\beta\widehat{\varphi}_i} =1 + \delta\alpha_i x u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\widehat{m}_i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]}{\theta x\widehat{\varphi}_i u'(\widehat{\varphi}_i\bar{m}_i^i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]}$$
$$+ \delta\alpha_i\left\{u'(\widehat{\varphi}_i\bar{m}_i^i)\frac{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\widehat{m}_i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]}{\theta\widehat{\varphi}_i u'(\widehat{\varphi}_i\bar{m}_i^i) + (1-\theta)\widehat{\widehat{\varphi}}[\cdot]} - 1\right\}, \tag{a.28}$$

where $x = [\widehat{\varepsilon}\mathbb{I}_{\{i=A\}} + (1/\widehat{\varepsilon})\mathbb{I}_{\{i=B\}}]$ and $\bar{m}_i^i$ is a function of $\widehat{m}_i$ taking prices as given. As in the previous case, the RHS of eq.(a.28) is strictly decreasing in $\widehat{m}_i$. This completes the proof of uniqueness. *Q.E.D.*

Proof of Proposition 1.
We have shown that $\widehat{\varphi}_i\widehat{m}_i$ is strictly decreasing in $\varphi_i/(\beta\widehat{\varphi}_i)$. In an equilibrium with $\gamma_i > \beta$, Lemma 7 implies that $A_{m_i}^d = 0$. Using $\widehat{m}_i = A_{m_i}^i$ in equilibrium, one can calculate the real balances $Z^i = \varphi_i A_{m_i}^i = \widehat{\varphi}_i\widehat{A}_{m_i}^i$, and verify they are strictly decreasing in γ_i. When $\gamma_i = \beta$ (i.e., at the *Friedman Rule*), then $\widehat{m} = m_i^* + \tau_i(\chi_{-i}^*)$ from Lemma 8, which leads to $Z^i = q^* + [\theta u(q^*) + (1-\theta)q^*]$ in equilibrium. From Lemmas 1, 2 buyers from a country i get the first best in this case. Note that $Z^i = q_i^T$ only if $\theta = 0$. The difference between these two terms when $\theta > 0$ represents the transaction fees that dealers extract in terms of real $money_i$ balances. *Q.E.D.*

Proof of Proposition 2.
From eq.(12) and the fact that $\varphi_B = \varepsilon\varphi_A$ in equilibrium, $\widetilde{q}_i^T$ equals to country i's total consumption of local special good. This explains the fact that $q_{-i}^T = 2\widetilde{q}_{-i}^T$. Next, in the proof of Proposition 1 we showed that the amount of real home money balances are decreasing in a country's inflation rate. Thus, q_{-i}^T is decreasing in γ_{-i}. Also, since $q_{-i}^T = 2\widetilde{q}_{-i}^T$, we have $\partial\widetilde{q}_{-i}^T/\partial\gamma_{-i} < 0$.

In the proof of Lemma 8 earlier, we showed that the demand for $money_i$ is not affected by $\varphi_{-i}/\beta\widehat{\varphi}_{-i}$, which equals to γ_{-i} in equilibrium. This suffices to show that $\partial\widetilde{q}_i^T/\partial\gamma_{-i} = 0$. Lastly, $\theta > 0$ guarantees a positive transaction fee earned by dealers and therefore $Z^{-i} > 2\widetilde{q}_{-i}^T$, $\forall\gamma_{-i}$ when $\theta > 0$.

For part (iv), it suffices to show that FOCs of buyer i imply $\partial(\varphi_{-i}\bar{m}_{-i}^i)/\partial\theta < 0$ at equilibrium. By applying the equilibrium conditions, $u'(\hat{\varphi}_{-i}\bar{m}_{-i}^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi})) = u'(\hat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi}))$, and $\widehat{\varphi}_A\widehat{\varepsilon} = \widehat{\varphi}_B$ into eq.(a.24) and (a.28), one can get the following two FOCs for Regions 2 and 3 respectively:

$$\frac{\varphi_i}{\beta\widehat{\varphi}_i} = (1-\delta\alpha_i) + \frac{2\delta\alpha_i u'(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi})))}{\theta u'(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi}))) + (1-\theta)},$$

$$\frac{\varphi_i}{\beta\widehat{\varphi}_i} = (1-\delta\alpha_i) + \frac{2\delta\alpha_i u'(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi})))\left\{\theta u'(\widehat{\varphi}_i\widehat{m}_i) + (1-\theta)\right\}}{\theta u'(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi}))) + (1-\theta)}.$$

Given prices and the fact that $u'(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi}))) > 1$ for all equilibrium regions, and given that $u'(\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi}))) > u'(\widehat{\varphi}_i\widehat{m}_i) > 1$ in Region 3, it is straightforward to see that $\partial(\varphi_i\bar{m}_i^i)/\partial\theta = \partial(\varphi_{-i}\bar{m}_{-i}^i)/\partial\theta < 0$ for both regions. *Q.E.D.*

Proof of Proposition 3.
In the proof of Proposition 1, it was shown that $\widehat{\varphi}_i\widehat{m}_i$ is strictly decreasing in γ_i. Thus, on aggregate, the equilibrium $\varphi_i\bar{A}_{m_i}^i$ must also be decreasing in γ_i. Moreover, since eq.(12) indicates a positive correlation between $\widehat{\varphi}_i\bar{m}_i^i$ and $\widehat{\varphi}_{-i}\bar{m}_{-i}^i$, $\varphi_{-i}\bar{A}_{m_{-i}}^i$ in equilibrium must be decreasing in γ_i. Given the definition of $\mathcal{V}_{ID}$, it follows that $\partial\mathcal{V}_{ID}/\partial\gamma_i < 0$, $\forall i \in \{A,B\}$. Part (ii) of the proposition follows from part (iv) of Proposition 2. The proof for part (iii) follows from the the Euler equations for buyer i, i.e., eq.(a.24) and (a.28). A higher $\delta\alpha_i$ increases the expected benefit from carrying real local money balances, which, in turn, leads to a greater $\widehat{\varphi}_i\bar{m}_i^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi})$ and a greater $\widehat{\varphi}_{-i}\bar{m}_{-i}^i(\widehat{m}_i,\varepsilon,\boldsymbol{\varphi})$ in equilibrium. *Q.E.D.*

Proof of Proposition 4.
Lemma 5 confirms that $m_i^i - \bar{m}_i^i$ is increasing in the units of $money_i$ that buyer i brings to the match. Since in equilibrium γ_i depresses Z^i, the total volume of real $money_i$ balance transfer to dealers is negatively correlated to γ_i. This proves part (i). Part (ii) of the proposition follows from the fact that Z^i is increasing is increasing in θ. The proof of part (iii) of Proposition 4 follows identical steps as the one of part (ii) of Proposition 3. *Q.E.D.*

Proof of Proposition 5.
We focus on the case $i = A$ (the proof for $i = B$ follows identical steps). First, we impose the equilibrium condition for the interdealer exchange rate, $\varepsilon\varphi_B = \varphi_A$. Then, in Region 1, eq.(a.17) can be rearranged as to

$$(\varepsilon^a - \varepsilon)/\varepsilon = \frac{\theta\left\{u(q^*) - q^*\right\}}{q^*}. \tag{a.29}$$

The term $(\varepsilon^a - \varepsilon)/\varepsilon$ for Regions 2 and 3 is then implied by eq.(a.18) and (a.19), respectively, and it is given as follows:

$$(\varepsilon^a - \varepsilon)/\varepsilon = \frac{\theta\left\{u(\varphi_B \bar{m}_B^A) - \varphi_B \bar{m}_B^A + \left[u(\varphi_A G(\bar{m}_B^A)) - \varphi_A G(\bar{m}_B^A))\right] - (u(q^*) - q^*)\right\}}{\varphi_B \bar{m}_B^A}, \tag{a.30}$$

$$(\varepsilon^a - \varepsilon)/\varepsilon = \frac{\theta\left\{u(\varphi_B \bar{m}_B^A) + u(\varphi_A G(\bar{m}_B^A)) - u(Z^A)\right\}}{\varphi_B \bar{m}_B^A}, \tag{a.31}$$

where $\bar{m}_B^A = \bar{m}_B^A(\delta\alpha_A A_{m_A}^A)$. We have previously seen that Z^A and $\varphi_B \bar{m}_B^A(\delta\alpha_A A_{m_A}^A)$ are decreasing in γ_A. In addition, the expressions in the RHS of eq.(a.30) and (a.31) are concave in $\varphi_B \bar{m}_B^A(\delta\alpha_A A_{m_A}^A)$. Combining these pieces of information, one can verify that $(\varepsilon^a - \varepsilon)/\varepsilon$ is increasing in γ_A. Part (ii) of Proposition 5 can be easily verified by a simple inspection of equations (a.29)-(a.31). Part (iii) of the proposition follows directly from the fact that an increase in $\delta\alpha_i$ raises Z^A and $\varphi_B \bar{m}_B^A(\delta\alpha_A A_{m_A}^A)$ (i.e., it has the opposite effect from an increase in γ_i). *Q.E.D.*

Proof of Proposition 6.

Before we proceed with the proof of the two statements in the proposition, we describe the function $\mathcal{W}$ in its most general form, and we explain how it simplifies to the expression seen in eq.(19). Notice that the equilibrium CM consumption and work effort will differ among buyers with different trading histories. For instance, a buyer who traded in both DM's carries less money than a buyer who traded only in the home DM, so she will have to work harder to rebalance her portfolio. What is less obvious is that the equilibrium CM consumption and work hours will also differ among dealers and sellers, depending on the type of buyer with whom these agents traded earlier in the period. For instance, if γ_{-i} is much greater than γ_i, then a seller i who traded with a buyer $-i$ will enter the CM with fewer real balances than a seller i who traded with a local buyer, and this will negatively affect her CM consumption.

We have the follwoing possibilities. First, within country i, $i \in \{A, B\}$, there are two types of buyers: Buyers *B1* who traded only at the home DM, and buyers *B2* who traded in both DMs.[41] Let X_j^i (H_j^i) denote the equilibrium CM consumption (work effort) for the buyer i of type $j \in \{B1, B2\}$. Moreover, within each country i, there are four types of sellers: Sellers *S1* who traded with buyers i of type *B1*, sellers *S2* who traded with buyers i of type *B2*, sellers *S3* who traded with (foreign) buyers $-i$ of type *B2*, and sellers *S4* who did not trade with anyone. Let X_j^i (H_j^i) denote the equilibrium CM consumption (work effort) for the seller i of type $j \in \{S1, S2, S3, S4\}$. Finally, there are three types of dealers: Dealers *Di*, $i \in \{A, B\}$ who traded with buyers i (of type *B2*), and dealers *DN* who did not trade with anyone. Let X_j (H_j) denote the equilibrium CM consumption (work effort) for a dealer of type $j \in \{DA, DB, DN\}$.

Given this discussion, the welfare function is simply the sum of net utilities over all these

[41] Notice that, within country i, the *B2* buyers are the C-types who also matched in the FOREX market.

types of agents weighted by their appropriate measure. We have,[42]

$$\begin{aligned}\mathcal{W} = {} & v\frac{\alpha_D}{2}\left(X_{DA} - H_{DA}\right) + v\frac{\alpha_D}{2}\left(X_{DB} - H_{DB}\right) + v(1-\alpha_D)\left(X_{DN} - H_{DN}\right) \\ & + (1-\delta\alpha_A)\left(X^A_{S1} - H^A_{S1}\right) + \delta\alpha_A\left(X^A_{S2} - H^A_{S2}\right) + \delta\alpha_B\left(X^A_{S3} - H^A_{S3}\right) + \delta(1-\alpha_B)\left(X^A_{S4} - H^A_{S4}\right) \\ & + (1-\delta\alpha_B)\left(X^B_{S1} - H^B_{S1}\right) + \delta\alpha_B\left(X^B_{S2} - H^B_{S2}\right) + \delta\alpha_A\left(X^B_{S3} - H^B_{S3}\right) + \delta(1-\alpha_A)\left(X^B_{S4} - H^B_{S4}\right) \\ & + (1-\delta\alpha_A)\left(X^A_{B1} - H^A_{B1}\right) + \delta\alpha_A\left(X^A_{B2} - H^A_{B2}\right) + (1-\delta\alpha_B)\left(X^B_{B1} - H^B_{B1}\right) + \delta\alpha_B\left(X^B_{B2} - H^B_{B2}\right) \\ & + \delta\alpha_A\left\{u(q^{B2}_A) - q^{B2}_A + u(\tilde{q}^{B2}_A) - \tilde{q}^{B2}_A\right\} + \delta\alpha_B\left\{u(q^{B2}_B) - q^{B2}_B + u(\tilde{q}^{B2}_B) - \tilde{q}^{B2}_B\right\} \\ & + (1-\delta\alpha_A)\left\{u(q^{B1}_A) - q^{B1}_A\right\} + (1-\delta\alpha_B)\left\{u(q^{B1}_B) - q^{B1}_B\right\}.\end{aligned} \tag{a.32}$$

It is now straightforward to calculate the various equilibrium objects that appear in eq.(a.32). For instance, only buyers, the agents who wish to acquire money, work in the CM. Hence, $H^i_j = 0\ \forall i \in \{A, B\}$ and $j \in \{S1, S2, S3, S4\}$, and $H_j = 0\ \forall j \in \{DA, DB, DN\}$. Dealers and sellers will only consume in the CM, and the level of their consumption equals the amount of real balances that they received from buyers earlier. Hence, letting $\mathcal{C}_{S,D}$ denote the total net CM utilities of (all) dealers and sellers (i.e., the first three lines in eq.(a.32)), we obtain[43]

$$\mathcal{C}_{S,D} = \delta\bar{\alpha}\sum_i\{Z^i - 2\tilde{q}^T_i\} + (1-\delta\bar{\alpha})\min\{Z^A, q^*\} + 2\delta\bar{\alpha}\tilde{q}^T_A + (1-\delta\bar{\alpha})\min\{Z^B, q^*\} + 2\delta\bar{\alpha}\tilde{q}^T_B.$$

Furthermore, letting $\mathcal{C}_B$ denote the total net CM utilities of all buyer (i.e., the fourth line in eq.(a.32)), we obtain

$$\mathcal{C}_B = -(1-\delta\bar{\alpha})\min\{Z^A, q^*\} - \delta\bar{\alpha}Z^A - (1-\delta\alpha_B)\min\{Z^B, q^*\} - \delta\bar{\alpha}Z^B.$$

Substituting these results into eq.(a.32) and some algebra yields the expression in eq.(19).

Now consider part (i) of Proposition 6. First notice that one can re-write $\mathcal{W}$ as a function of $\bar{\alpha}\delta$ as follows

$$\begin{gathered}\mathcal{W}(\bar{\alpha}\delta) = 2\delta\bar{\alpha}\{m_i(\bar{\alpha}\delta) + m_{-i}(\bar{\alpha}\delta)\} + (1-\delta\bar{\alpha})\{n_i(\bar{\alpha}\delta) + n_{-i}(\bar{\alpha}\delta)\}, \\ m_i(\bar{\alpha}\delta) = u(\tilde{q}^T_i) - \tilde{q}^T_i,\ \forall i, \\ n_i(\bar{\alpha}\delta) = u(\min\{Z^i, q^*\}) - \min\{Z^i, q^*\},\ \forall i,\end{gathered}$$

where $m'_i > 0$ and $n'_i \geq 0\,\forall i$. Then, the sign of $\partial\mathcal{W}/\partial(\bar{\alpha}\delta)$ depends on that of $\{2m_i(\bar{\alpha}\delta) - n_i(\bar{\alpha}\delta)\} + \bar{\alpha}\delta\{2m'_i - n'_i\}$. Since $Z^i > \tilde{q}^T_i$ for all $\theta > 0$, and u is concave, we have $\bar{\alpha}\delta\{2m'_i - n'_i\} > 0$. In

[42] The amount of DM good produced within a match also depends on the buyer's type. For instance, the $B1$ buyer i will typically purchase more q_i in her own DM_i than the $B2$ type. Thus, we also differentiate q_i in eq.(a.32): q^j_i refers to the amount of special good i produced when the buyer i is of type j, and $\tilde{q}^{B2}_i$ is the amount of special good $-i$ produced when the buyer i is of type $B2$.

[43] Here, we also use the following equilibrium conditions: $v\frac{\alpha_D}{2}(X_{Di} - H_{Di}) = \delta\alpha_i\{Z^i - (q_i + \tilde{q}^T_i)\}$, $q^{B1}_i = \min\{Z^i, q^*\}$, and $\tilde{q}^{B2}_i = q^{B2}_i = \tilde{q}^T_i$.

addition, as shown in the proof of Lemma 5, the Kalai constraint with $\theta > 0$ always guarantees that $\{2m_i(\bar{\alpha}\delta) - n_i(\bar{\alpha}\delta)\} > 0$. Thus, $\partial\mathcal{W}/\partial(\bar{\alpha}\delta) > 0$.

Consider now part (ii). If $\gamma_i \leq \bar{\mu}_i,\ \forall i$, the proof is obvious. So let $\gamma_i > \bar{\mu}_i,\ \forall i$. In this case, we can rewrite $\mathcal{W}$ as a function of θ in the following way:

$$\mathcal{W}(\theta) = 2\delta\bar{\alpha}\{x_i(\theta) + x_{-i}(\theta)\} + (1-\delta\bar{\alpha})\{y_i(\theta) + y_{-i}(\theta)\},$$
$$x_i(\theta) = u(\tilde{q}_i^T) - \tilde{q}_i^T,\ \forall i,$$
$$y_i(\theta) = u(Z^i) - Z^i,\ \forall i.$$

Since $x_i' < 0$ and $y_i' > 0,\ \forall i$, the sign of $\mathcal{W}'$ depends on the following condition:

$$\frac{y_i'}{|x_i'|} \gtreqless \frac{2\delta\bar{\alpha}}{1-\delta\bar{\alpha}} \Longleftrightarrow \mathcal{W}'(\theta) \gtreqless 0.$$

Finally, the existence and uniqueness of $\widetilde{\delta\alpha}$ is easily derived, since $y_i'/|x_i'|$ is strictly decreasing in $\delta\bar{\alpha}$ (due to the concavity of u) while $2\delta\bar{\alpha}/(1-\delta\bar{\alpha})$ is strictly increasing in $\delta\bar{\alpha}$. *Q.E.D.*