AN OVERDETERMINED SYSTEM*

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In this paper we consider the overdetermined system of equations:

$$|\operatorname{grad} u| = g(u), \qquad \Delta u = h(u), \tag{1}$$

for a real function u defined on an open connected set Ω in $\mathfrak{R}^3$. Under the assumption that $\operatorname{grad} u$ never vanishes, we show that the solution u must have an extremely restricted behaviour. In particular, its level surfaces are necessarily either all concentric spheres, or all concentric circular cylinders, or all parallel planes (or pieces of these).

The system (1) arises in the theory of phase transitions, as shown in the paper in this journal by James Serrin. His suggestion that the above result would be useful in Korteweg's theory was the motivation for this note.

We now formally state the result and its proof.

THEOREM. Let $u\colon \Omega \to \mathfrak{R}$ be of class C^2 on a connected open set Ω in $\mathfrak{R}^3$, and suppose that

$$|\operatorname{grad} u| = g(u), \qquad \Delta u = h(u) \quad \text{in } \Omega,$$

where $g\colon u(\Omega) \to \mathfrak{R}^+$, $h\colon u(\Omega) \to \mathfrak{R}$ are of class C^1 and class C^0 respectively.

Then the level surfaces of u in Ω are all of one of the following three types:

(i) pieces of concentric spheres;
(ii) pieces of concentric circular cylinders;
(iii) pieces of parallel planes.

Proof. We first show that it is possible to take $g(u) \equiv 1$, without loss of generality.

Indeed, consider the function $v\colon \Omega \to \mathfrak{R}$ defined by

$$v(x) = \int_a^{u(x)} dt/g(t),$$

where a is some fixed number in $u(\Omega)$. Since g is of class C^1 and $g(t) \neq 0$ for $t \in u(\Omega)$, it is evident that v is of class C^2 in Ω, and moreover that

$$|\operatorname{grad} v| \equiv 1, \qquad \Delta v = [h(u)/g(u)] - g'(u) \quad \text{in } \Omega.$$

Because of the identity

$$u(x) = a + \int_0^{v(x)} g(t)\, dt \quad \text{in } \Omega$$

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clearly the function $[h(u)/g(u)] - g'(u)$ can be written in the form $\tilde{h}(v)$, with $\tilde{h}\colon v(\Omega) \to \mathfrak{R}$ of class C^0.

If we can show that the level surfaces of v are either concentric spheres, or concentric circular cylinders, or parallel planes, then the same would of course hold true for u, and the proof would be complete. Consequently in what follows we can assume $g(u) \equiv 1$.

Consider now any orthogonal trajectory Γ of the level surfaces of u. This is defined as a solution $x\colon I \to \mathfrak{R}^3$ of the differential equation

$$\frac{d}{ds}x(s) = (\operatorname{grad} u)(x(s)),$$

where $s \in I$ and I is an interval of $\mathfrak{R}$. Naturally the solution can be continued until it leaves Ω, because the function $(\operatorname{grad} u)(x)$ is of class C^1 in Ω. By differentiation, for each $i = 1, 2, 3$ we obtain

$$\begin{aligned}\frac{d^2}{ds^2}(x_i(s)) &= \frac{d}{ds}\left(\frac{\partial u}{\partial x_i}(x(s))\right) = \sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{d}{ds}x_j(x)\\ &= \sum_{j=1}^{3} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{1}{2}\sum_{j=1}^{3}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j}\right)\\ &= \frac{\partial}{\partial x_i}\left(\frac{1}{2}|g(u)|^2\right);\end{aligned}$$

where the expressions on the right hand side are evaluated at the position $x(s)$.

Since $g(u) \equiv 1$, it follows that $d^2x/ds^2 \equiv 0$; that is, Γ is straight. Even more, since

$$\left|\frac{dx}{ds}\right| = |\operatorname{grad} u| = 1,$$

we see that s is the arc-length along Γ; and because

$$\frac{d}{ds}u(x(s)) = \sum_{j=1}^{3}\frac{\partial u}{\partial x_j}\frac{dx_j}{ds} = \sum_{j=1}^{3}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} = |g(u)|^2 = 1,$$

the value of u varies linearly with s along Γ.

We now evaluate the mean curvature $H\colon \Sigma_c \to \mathfrak{R}$ of any level surface

$$\Sigma_c = \{x \in \Omega\colon u(x) = c\}$$

of u, using the well-known, and easily proved, formula (cf. [3] pages 306–307)

$$\Delta u = u_{\nu\nu} + 2Hu_\nu \quad \text{on } \Sigma_c,$$

where u_ν denotes the first directional derivative, and $u_{\nu\nu}$ the second directional derivative of u along the normal to Σ_c at x.

Observing that

$$u_\nu = \frac{du}{ds} = 1, \quad u_{\nu\nu} = \frac{d^2u}{ds^2} = 0 \quad \text{on } \Sigma_c,$$

the above formula yields

$$H(x) = \tfrac{1}{2}h(u(x)), \quad \text{for } x \in \Sigma_c.$$

It follows that the mean curvature of Σ_c everywhere has the same constant value $\frac{1}{2}h(c)$.

Let Σ_0 denote the fixed level surface Σ_{c_0} of u. The level surfaces Σ_c near Σ_0 are obviously parallel to Σ_0, since by the condition $du/ds = 1$ on Γ they can be obtained as the locus of points on the normals to Σ_0 which have distance $c - c_0$ from Σ_0. A well-known formula for the mean curvature of parallel surfaces (see [1, p. 209]) states that

$$H(x) = \frac{H(y) - sK(y)}{1 - 2sH(y) + s^2K(y)}, \qquad x \in \Sigma_c, y \in \Sigma_0,$$

where s is the distance between the surfaces Σ_c and Σ_0 along any normal Γ and x and y lie on the same normal.

In our case $H(x) = h(c)/2$, $H(y) = h(c_0)/2$ and $s = c - c_0$. Thus for $y \in \Sigma_0$ and c near c_0 we have

$$h(c) = \frac{h(c_0) - 2(c - c_0)K(y)}{1 - (c - c_0)h(c_0) + (c - c_0)^2K(y)}.$$

Since the right hand side, for fixed y, is a rational function of c, it follows that h is also. Hence we can differentiate both sides with respect to c and then put $c = c_0$ to obtain

$$h'(c_0) = -2K(y) + \left(h(c_0)\right)^2,$$

or, in other words,

$$K(y) = \left[\left(h(c_0)\right)^2 - h'(c_0)\right]/2.$$

Thus Σ_0, and in consequence any level surface Σ of u, has constant Gaussian curvature as well as constant mean curvature.

Now any surface having both constant Gaussian curvature and mean curvature is a piece of either a sphere, or a circular cylinder, or a plane (see [1, p. 263] and use Theorem 2.7, p. 255, and formulas on pp. 86–87 and 91). Thus each level surface Σ of u is a piece of either a sphere, or a circular cylinder, or a plane. But also these level surfaces are parallel. Thus, if Σ_0 were a piece of a sphere, for example, then all the level surfaces Σ of u would be pieces of concentric spheres. Similarly, if Σ_0 were a piece of either a circular cylinder, or a plane, then the other level surfaces would be pieces of either concentric circular cylinders, or parallel planes.

The proof is now complete.

References

[1] B. O'Neill, *Elementary differential geometry*, Academic Press, New York, 1966

[2] J. Serrin, *Recent methods in nonlinear analysis and applications*, Proc. IV Intern. Meeting of S.A.F.A. 1980, eds. Canfora, Rionero, Sbordone & Trombetti, Liquori ed., Neaples, Italy, 169–175

[3] G. Talenti, ibid. 301–322