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An overview of interval-valued intuitionistic fuzzy information aggregations and applications

Zeshui Xu[1] · Xunjie Gou[1]

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Abstract Interval-valued intuitionistic fuzzy set, generalized by Atanassov and Gargov, can be used to characterize the uncertain information more sufficiently and accurately when we face the fact that the values of the membership function and the non-membership function in an intuitionistic fuzzy set are difficult to be expressed as exact real numbers in many real-world decision-making problems. In this paper, we provide an overview of interval-valued intuitionistic fuzzy information aggregation techniques, and their applications in various fields such as decision-making, entropy measure, supplier selection and some practical decision-making problems. Meanwhile, we also review some important methods for decision-making with interval-valued intuitionistic fuzzy information, including the QUALIFLEX-based method, the TOPSIS method, the extended VIKOR method, the module partition schemes evaluation (MPSE) approach, the outranking choice method, the inclusion-based LINMAP method and the risk attitudinal ranking method, the evidential reasoning methodology, etc. Finally, we point out some possible directions for future research.

Keywords Interval-valued intuitionistic fuzzy set · Aggregation operators · Decision-making methods · Applications

✉ Zeshui Xu
xuzeshui@263.net

Xunjie Gou
gouxunjie@qq.com

[1] Business School, Sichuan University, Chengdu 610064, China

1 Introduction

Fuzzy set (FS), introduced by Zadeh (1965), has been applied in various fields of modern society as decision-making, pattern recognition, medical diagnosis, etc. Later, consider that the FS only contains one single membership degree, Atanassov (1986) introduced the concept of intuitionistic fuzzy set (IFS) for extending the single membership degree of FS into three reasonable forms, namely, the membership degree, the non-membership degree and the hesitancy degree. Since the IFS can describe the uncertainty of an object more comprehensively and reasonably than the FS. In recent decades, lots of research on the IFS and its extensions have been done. However, with the rapid development of society, the IFSs cannot characterize the uncertain information sufficiently when we face the situations where the values of the membership function and the non-membership function in an IFS cannot be expressed as exact real numbers but the value ranges can be provided. Atanassov and Gargov (1989) generalized the IFS to the interval-valued intuitionistic fuzzy set (IVIFS), which can effectively accommodate such situations. For simplicity, Xu (2007a) defined the interval-valued intuitionistic fuzzy number (IVIFN) as an ordered pair, which is the basic component of the IVIFS.

Additionally, Xu and Chen (2007a), and Xu and Cai (2015) defined some basic operational laws of IVIFNs, which include "intersection", "union", "supplement", and "power", etc. Then, amounts of aggregation operators (Xu 2007b, c, 2010; Xu and Chen 2007b, 2011; Shi and He 2013; Zhao et al. 2010; Qi et al. 2013; Xu and Yager 2009; Dong and Wan 2015; Liu et al. 2015; Meng et al. 2013a, b, 2014, 2015; Yu 2013, 2014; He et al. 2013; Xu and Xia 2011; Zhang and Qi 2012; Wei and Yi 2008; Li 2014; Wang and Liu 2013; Chen et al. 2012; Wan and

Dong 2014; Yager 2004; Zhou et al. 2014; Yang and Yuan 2014; Gu et al. 2014) have been proposed for dealing with the decision-making problems with interval-valued intuitionistic fuzzy information, including the interval-valued intuitionistic fuzzy weighted averaging (IVIFWA) operator (Xu 2007b), the interval-valued intuitionistic fuzzy weighted geometric (IVIFWG) operator (Xu 2007b), the interval-valued intuitionistic fuzzy ordered weighted averaging (IVIFOWA) operator (Xu and Chen 2007b), the generalized interval-valued intuitionistic fuzzy weighted averaging (GIVIFWA) operator (Zhao et al. 2010), the induced generalized interval-valued intuitionistic fuzzy Choquet ordered averaging (I-GIVIFCOA) operator (Xu and Xia 2011), and the interval-valued intuitionistic fuzzy interactive weighted average (IVIFIWA) operator (Yu 2014), etc. In fact, all of these aggregation operations can be considered as granules in granular computing (GrC) (Pedrycz and Chen 2015; Peters and Weber 2016; Livi and Sadeghian 2016; Xu and Wang 2016; Antonelli et al. 2016; Lingras et al. 2016; Skowron et al. 2016; Dubois and Prade 2016; Loia et al. 2016; Yao 2016; Ciucci 2016; Wilke and Portmann 2016; Song and Wang 2016; Liu et al. 2016). Based on these aggregation operators, many decision-making approaches have been put forward to deal with different kinds of decision-making problems with interval-valued intuitionistic fuzzy information (Gu et al. 2014; Zhao et al. 2013; Jin et al. 2014; Ye 2011; Wei and Zhang 2015; Gupta et al. 2015; Xu and Shen 2014; Chen 2014; Wu and Chiclana 2014; Cai and Han 2014; Yue and Jia 2013; Chen et al. 2011, 2012; Xiao and Wei 2008; Zhang et al. 2013; Chen and Li 2013; Chen and Chiou 2015), such as the extended VIKOR method (Zhao et al. 2013), the entropy measures (Jin et al. 2014; Ye 2011; Wei and Zhang 2015; Gupta et al. 2015), the outranking choice method (Xu and Shen 2014), the inclusion-based LINMAP method (Chen 2014) and the risk attitudinal ranking method (Wu and Chiclana 2014), the evidential reasoning methodology (Chen and Chiou 2015), etc. To understand and learn these aggregation operators and decision-making methods better and more conveniently, it is necessary to make an overview of interval-valued intuitionistic fuzzy information aggregation techniques and their applications. In order to do that, this paper is organized as follows: Sect. 2 introduces some basic concepts and operational laws about IVIFNs. In Sect. 3, we mainly discuss some different kinds of aggregation operators about interval-valued intuitionistic fuzzy information as well as make some classifications for them. Section 4 reviews some important methods for decision-making with interval-valued intuitionistic fuzzy information and their applications in solving various practical decision-making problems. Finally, we make some concluding remarks and discuss the possible directions for future research.

2 Basic concepts and operations

In this section, we mainly review some basic concepts and operational laws related to the interval-valued intuitionistic fuzzy information

Given a crisp set X. Atanassov (1986) introduced the intuitionistic fuzzy set (IFS) $A=\{<x,\mu_A(x),v_A(x)>|x\in X\}$, where the functions $\mu_A(x)$ and $v_A(x)$ define the membership degree and the non-membership degree of the element $x\in X$ to the set A, respectively. For every $x\in X$, A satisfies $\mu_A(x)\in[0,1]$, $v_A(x)\in[0,1]$ and $0\le\mu_A(x)+v_A(x)\le 1$. Later, in order to deal with the situations where both the membership degree and the non-membership degree cannot be expressed as exact numerical values but the value ranges can be provided, Atanassov and Gargov (1989) extended the IFS to interval-valued intuitionistic fuzzy set (IVIFS):

$$\begin{aligned}\tilde{A}&=\left\{<x,\tilde{\mu}_{\tilde{A}}(x),\tilde{v}_{\tilde{A}}(x)>|x\in X\right\}\\&=\left\{<x,\left[\inf\tilde{\mu}_{\tilde{A}}(x),\sup\tilde{\mu}_{\tilde{A}}(x)\right],\right.\\&\qquad\left.\left[\inf\tilde{v}_{\tilde{A}}(x),\sup\tilde{v}_{\tilde{A}}(x)\right]>|x\in X\right\},\end{aligned}\tag{1}$$

where $\tilde{\mu}_{\tilde{A}}(x)\subset[0,1]$ and $\tilde{v}_{\tilde{A}}(x)\subset[0,1]$, which satisfy $\sup\tilde{\mu}_{\tilde{A}}(x)+\sup\tilde{v}_{\tilde{A}}(x)\le 1$. The basic component of an IVIFS is an ordered pair, characterized by an interval-valued membership degree and an interval-valued non-membership degree of x in $\tilde{A}$. Such an ordered pair is called an IVIFN (Xu 2007a). For convenience, an IVIFN is generally simplified as $\tilde{\alpha}=([a,b],[c,d])$, where $[a,b]\subset[0,1]$, $[c,d]\subset[0,1]$ and $b+d\le 1$. For convenience of application, Xu (2007a) defined some basic operational laws for IVIFNs:

Let $\tilde{\alpha}=([a,b],[c,d])$ and $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])(i=1,2)$ be three IVIFNs. Then

1. $\tilde{\alpha}_1\oplus\tilde{\alpha}_2=([a_1+a_2-a_1a_2,b_1+b_2-b_1b_2],\ [c_1c_2,d_1d_2])$;
2. $\tilde{\alpha}_1\otimes\tilde{\alpha}_2=([a_1a_2,b_1b_2],[c_1+c_2-c_1c_2,d_1+d_2-d_1d_2])$;
3. $\lambda\tilde{\alpha}=\left(\left[1-(1-a)^{\lambda},1-(1-b)^{\lambda}\right],\left[c^{\lambda},d^{\lambda}\right]\right),\lambda>0$;
4. $\tilde{\alpha}^{\lambda}=\left(\left[a^{\lambda},b^{\lambda}\right],\left[1-(1-c)^{\lambda},1-(1-d)^{\lambda}\right]\right),\lambda>0$.

3 Interval-valued intuitionistic fuzzy aggregation operators

3.1 Some basic interval-valued intuitionistic fuzzy aggregation operators

Based on the operational laws introduced in Sect. 2, Xu (2007b) introduced two basic interval-valued intuitionistic fuzzy aggregation operators:

Definition 3.1 (Xu 2007b) Let $\tilde{\alpha}_j(j=1,2,\cdots,n)$ be a collection of IVIFNs, and let IVIFWA: $\tilde{\theta}^n\to\tilde{\theta}$. If

$$\mathrm{IVIFWA}_{\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\mathop{\oplus}_{j=1}^{n}\omega_j\tilde{\alpha}_j$$
$$=\left(\left[1-\prod_{j=1}^{n}(1-a_j)^{\omega_j},1-\prod_{j=1}^{n}(1-b_j)^{\omega_j}\right],\left[\prod_{j=1}^{n}c_j^{\omega_j},\prod_{j=1}^{n}d_j^{\omega_j}\right]\right), \quad (2)$$

then the function IVIFWA is called the interval-valued intuitionistic fuzzy weighted averaging (IVIFWA) operator. Let IVIFWG: $\tilde{\theta}^n \to \tilde{\theta}$. If

$$\mathrm{IVIFWG}_{\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\mathop{\otimes}_{j=1}^{n}\tilde{\alpha}_j^{\omega_j}$$
$$=\left(\left[\prod_{j=1}^{n}a_j^{\omega_j},\prod_{j=1}^{n}b_j^{\omega_j}\right],\left[1-\prod_{j=1}^{n}(1-c_j)^{\omega_j},1-\prod_{j=1}^{n}(1-d_j)^{\omega_j}\right]\right), \quad (3)$$

then the function IVIFWG is called the interval-valued intuitionistic fuzzy weighted geometric (IVIFWG) operator, where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $\tilde{\alpha}_j (j=1,2,\ldots,n)$ with $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$ $(j=1,2,\ldots,n)$.

In addition, Xu and Chen (2007b) introduced two ordered weighted aggregation operators for IVIFNs:

Definition 3.2 (Xu and Chen 2007b) Let IVIFOWA: $\tilde{\theta}^n \to \tilde{\theta}$. If

$$\mathrm{IVIFOWA}_{\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\mathop{\oplus}_{j=1}^{n}\omega_j\tilde{\alpha}_{\sigma(j)}$$
$$=\left(\left[1-\prod_{j=1}^{n}(1-a_{\sigma(j)})^{\omega_j},1-\prod_{j=1}^{n}(1-b_{\sigma(j)})^{\omega_j}\right],\left[\prod_{j=1}^{n}c_{\sigma(j)}^{\omega_j},\prod_{j=1}^{n}d_{\sigma(j)}^{\omega_j}\right]\right), \quad (4)$$

then the function IVIFOWA is called an interval-valued intuitionistic fuzzy ordered weighted averaging (IVIFOWA) operator. Let IVIFOWG: $\tilde{\theta}^n \to \tilde{\theta}$. If

$$\mathrm{IVIFOWG}_{\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\mathop{\otimes}_{j=1}^{n}\tilde{\alpha}_{\sigma(j)}^{\omega_j}=\left(\left[\prod_{j=1}^{n}a_{\sigma(j)}^{\omega_j},\prod_{j=1}^{n}b_{\sigma(j)}^{\omega_j}\right],\right.$$
$$\left.\left[1-\prod_{j=1}^{n}(1-c_{\sigma(j)})^{\omega_j},1-\prod_{j=1}^{n}(1-d_{\sigma(j)})^{\omega_j}\right]\right), \quad (5)$$

then the function IVIFOWG is called an interval-valued intuitionistic fuzzy ordered weighted geometric (IVIFOWG) operator, where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the weight vector associated with the function IVIFOWA and IVIFOWG, with $\omega_j\in[0,1]$ $(j=1,2,\ldots,n)$ and $\sum_{j=1}^{n}\omega_j=1$, $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is any permutation of $(1,2,\cdots,n)$, such that $\tilde{\alpha}_{\sigma(j-1)}>\tilde{\alpha}_{\sigma(j)}$, for any j.

For the IVIFWA, IVIFWG, IVIFOWA and IVIFOWG operators, we can see that the IVIFWA and IVIFWG operators only consider the importance of each given IVIFN, while the IVIFOWA and IVIFOWG operators only weight the ordered position of each IVIFN instead of the IVIFN itself. Thus, these operators consider only one of the two different aspects. In order to overcome this limitation, Xu and Chen (2007a, b) developed the hybrid aggregation operators for IVIFNs:

Definition 3.3 (Xu and Chen 2007b) Let IVIFHA: $\tilde{\theta}^n \to \tilde{\theta}$. Suppose

$$\mathrm{IVIFHA}_{w,\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\mathop{\oplus}_{j=1}^{n}\omega_j\dot{\tilde{\alpha}}_{\sigma(j)}$$
$$=\left(\left[1-\prod_{j=1}^{n}(1-\dot{a}_{\sigma(j)})^{\omega_j},1-\prod_{j=1}^{n}(1-\dot{b}_{\sigma(j)})^{\omega_j}\right],\right.$$
$$\left.\left[\prod_{j=1}^{n}\dot{c}_{\sigma(j)}^{\omega_j},\prod_{j=1}^{n}\dot{d}_{\sigma(j)}^{\omega_j}\right]\right), \quad (6)$$

then the function IVIFHA is called an interval-valued intuitionistic fuzzy hybrid averaging (IVIFHA) operator, where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the weight vector associated with the function IVIFHA, with $\omega_j\in[0,1]$ $(j=1,2,\ldots,n)$ and $\sum_{j=1}^{n}\omega_j=1$, $\dot{\tilde{\alpha}}_{\sigma(j)}$ is the j-th largest of the weighted IVIFNs $\dot{\tilde{\alpha}}_i$ $(1=1,2,\cdots,n)$, here $\dot{\tilde{\alpha}}_i=nw_i\tilde{\alpha}_i$ $(i=1,2,\ldots,n)$, $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ is the weight vector of the IVIFNs $\tilde{\alpha}_i$ $(1=1,2,\cdots,n)$, with $w_j\in[0,1]$ $(j=1,2,\ldots n)$ and $\sum_{j=1}^{n}w_j=1$, and n is the balancing coefficient.

If we replace (6) with the following form:

$$\mathrm{IVIFHG}_{w,\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots\tilde{\alpha}_n)=\mathop{\otimes}_{j=1}^{n}\ddot{\tilde{\alpha}}_{\sigma(j)}^{\omega_j}=\left(\left[\prod_{j=1}^{n}a_{\sigma(j)}^{\omega_j},\prod_{j=1}^{n}b_{\sigma(j)}^{\omega_j}\right],\right.$$
$$\left.\left[1-\prod_{j=1}^{n}(1-c_{\sigma(j)})^{\omega_j},1-\prod_{j=1}^{n}(1-d_{\sigma(j)})^{\omega_j}\right]\right), \quad (7)$$

then the function IVIFHG is called an interval-valued intuitionistic fuzzy hybrid geometric (IVIFHG) operator, where $\ddot{\tilde{\alpha}}_{\sigma(j)}$ is the j-th largest of the weighted IVIFNs $\dot{\tilde{\alpha}}_i$ $(1=1,2,\ldots,n)$, here $\ddot{\tilde{\alpha}}_i=\tilde{\alpha}_i^{nw_i}$ $(i=1,2,\ldots,n)$.

In particular, if $\omega=(1/n,1/n,\ldots,1/n)^{\mathrm{T}}$, then the IVIFWA and IVIFWG operators reduce to the interval-valued intuitionistic fuzzy averaging (IVIFA) operator and interval-valued intuitionistic fuzzy geometric (IVIFG) operator (Xu 2007c), respectively:

$$\mathrm{IVIFA}(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\ldots,\tilde{\alpha}(x_n))$$
$$=\left(\left[1-\prod_{i=1}^{n}(1-a(x_i))^{1/n},1-\prod_{i=1}^{n}(1-b(x_i))^{1/n}\right],\right.$$
$$\left.\left[\prod_{i=1}^{n}(c(x_i))^{1/n},\prod_{i=1}^{n}(d(x_i))^{1/n}\right]\right) \quad (8)$$

$$\mathrm{IVIFG}(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \ldots, \tilde{\alpha}(x_n)) = \left(\left[\left(\prod_{i=1}^{n} (a(x_i)) \right)^{1/n}, \left(\prod_{i=1}^{n} (b(x_i)) \right)^{1/n} \right], \left[1 - \left(\prod_{i=1}^{n} (1 - c(x_i)) \right)^{1/n}, 1 - \left(\prod_{i=1}^{n} (1 - c(x_i)) \right)^{1/n} \right] \right) \tag{9}$$

The IVIFOWA and IVIFOWG operators reduce to the IVIFA and IVIFG operators, respectively; The IVIFHA and IVIFHG operators reduce to the IVIFWA and IVIFWG operators, respectively. If $w = (1/n, 1/n, \ldots, 1/n)^{\mathrm{T}}$, then the IVIFHA and IVIFHG operators reduce to the IVIFOWA and IVIFOWG operators, respectively.

Clearly, the IVIFHA operator generalizes both the IVIFWA and IVIFOWA operators and the IVIFHG operator generalizes both the IVIFWG and IVIFOWG operators. They can consider not only the importance of each given IVIFN itself, but also the importance of the ordered position of the IVIFN. All the IVIFWA, IVIFWG, IVIFOWA, IVIFOWG, IVIFHA and IVIFHG operators satisfy the desirable properties such as idempotency, boundedness and commutativity.

3.2 Interval-valued intuitionistic fuzzy Bonferroni means

The most fundamental characteristic of the Bonferroni means (BM) is that it can capture the interrelationship of the input arguments, which makes BM very useful in various application fields. Motivated by intuitionistic fuzzy Bonferroni mean (Xu and Yager 2011), and considering that sometimes the membership function and the non-membership function whose values are also intervals rather than exact real numbers, Xu and Chen (2011) introduced the interval-valued intuitionistic fuzzy Bonferroni mean, which can take the importance of each argument into account:

Definition 3.4 (Xu and Chen 2011) Let $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFNs, and $p, q > 0$. If

$$\mathrm{IVIFB}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\frac{1}{n(n-1)} \left(\underset{\substack{i,j=1 \\ i \neq j}}{\overset{n}{\oplus}} \left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q \right) \right) \right)^{\frac{1}{p+q}}$$
$$= \left(\left[\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (a_i)^p (a_j)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (b_i)^p (b_j)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right], \left[1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1-c_i)^p (1-c_j)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1-d_i)^p (1-d_j)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right] \right), \tag{10}$$

then $\mathrm{IVIFB}^{p,q}$ is called the interval-valued intuitionistic fuzzy Bonferroni mean (IVIFBM).

Considering that the input data may have different important degrees, Xu and Chen (2011) introduced the concept of weighted interval-valued intuitionistic fuzzy Bonferroni mean:

Definition 3.5 (Xu and Chen 2011) If

$$\mathrm{WIVIFB}_{\omega}^{p,q}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\left(\frac{1}{n(n-1)}\left(\underset{\substack{i,j=1\\i\neq j}}{\overset{n}{\oplus}}\left((\omega_i\tilde{\alpha}_i)^p\otimes(\omega_j\tilde{\alpha}_j)^q\right)\right)\right)^{\frac{1}{p+q}}=\left(\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-(1-a_i)^{\omega_i})^p(1-(1-a_j)^{\omega_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\right.$$
$$\left.\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-(1-b_i)^{\omega_i})^p(1-(1-b_j)^{\omega_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-(c_i)^{\omega_i})^p(1-(c_j)^{\omega_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$
$$\left.\left.1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-(d_i)^{\omega_i})^p(1-(d_j)^{\omega_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right) \tag{11}$$

then $\mathrm{WIVIFB}_{\omega}^{p,q}$ is called the weighted interval-valued intuitionistic fuzzy Bonferroni mean (WIVIFBM), where $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the weight vector of the IVIFNs $\tilde{\alpha}_i$ $(1=1,2,\ldots,n)$, and ω_i indicates the importance degree of $\tilde{\alpha}_i$, satisfying $\omega_i\in[0,1]$ and $\sum_{i=1}^{n}\omega_i=1$ $(i=1,2,\ldots,n)$.

Both the IVIFBM and the WIVIFBM have some desirable properties, such as idempotency, monotonicity, commutativity and boundedness. Additionally, Shi and He (2013) defined the interval-valued intuitionistic fuzzy optimized weighted Bonferroni mean (IVIFOWBM) and the generalized interval-valued intuitionistic fuzzy optimized weighted Bonferroni mean (GIVIFOWBM):

Definition 3.6 (Shi and He 2013) Let $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])$ $(i=1,2,\ldots,n)$ be a collection of IVIFNs with the weight vector $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ such that $p,q\geq 0$, $\omega\geq 0$ and $\sum_{i=1}^{n}\omega_i=1$. Then

$$\mathrm{IVIFOWB}^{p,q}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\left(\underset{\substack{i,j,=1\\i\neq j}}{\overset{n}{\oplus}}\frac{\omega_i\omega_j}{1-\omega_i}\left(\tilde{\alpha}_i^p\otimes\tilde{\alpha}_j^q\right)\right)^{\frac{1}{p+q}}=\left(\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-a_i^pa_j^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}},\right.\right.$$
$$\left.\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-b_i^pb_j^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}\right],\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-c_i)^p(1-c_j)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}},\right.$$
$$\left.\left.1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-d_i)^p(1-d_j)^q\right)^{\frac{\omega_i\omega_j}{1-\omega_i}}\right)^{\frac{1}{p+q}}\right]\right) \tag{12}$$

is called the interval-valued intuitionistic fuzzy optimized weighted Bonferroni mean (IVIFOWBM).

Definition 3.7 (Shi and He 2013) Let $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFNs with the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ such that $p, q, r \geq 0$, $\omega \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Then

$$\mathrm{GIVIFOWB}^{p,q,r}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\bigoplus_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \frac{\omega_i \omega_j \omega_k}{(1-\omega_i)(1-\omega_i-\omega_j)} \left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q \otimes \tilde{\alpha}_k^r \right) \right)^{\frac{1}{p+q+r}} = \left(\left[\left(1 - \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(1 - a_i^p a_j^q a_k^r \right)^{\frac{\omega_i \omega_j \omega_k}{(1-\omega_i)(1-\omega_i-\omega_j)}} \right)^{\frac{1}{p+q+r}}, \left(1 - \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(1 - a_i^p a_j^q a_k^r \right)^{\frac{\omega_i \omega_j \omega_k}{(1-\omega_i)(1-\omega_i-\omega_j)}} \right)^{\frac{1}{p+q+r}} \right], \left[1 - \left(1 - \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(1 - (1-c_i)^p (1-c_j)^q (1-c_k)^r \right)^{\frac{\omega_i \omega_j \omega_k}{(1-\omega_i)(1-\omega_i-\omega_j)}} \right)^{\frac{1}{p+q+r}}, 1 - \left(1 - \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(1 - (1-d_i)^p (1-d_j)^q (1-d_k)^r \right)^{\frac{\omega_i \omega_j \omega_k}{(1-\omega_i)(1-\omega_i-\omega_j)}} \right)^{\frac{1}{p+q+r}} \right] \right) \quad (13)$$

is called the generalized interval-valued intuitionistic fuzzy optimized weighted Bonferroni mean (GIVIFOWBM).

Both the IVIFOWBM and the GIVIFOWBM satisfy the idempotency, monotonicity and transformation.

3.3 Generalized interval-valued intuitionistic fuzzy aggregation operators

The advantage of the generalized operators is that it can consider all situations by changing the value of the variable λ. Zhao et al. (2010) developed a series of generalized aggregation operators for IVIFNs:

Definition 3.8 (Zhao et al. 2010) Let GIVIFWA: $\tilde{\theta}^n \to \tilde{\theta}$. If

$$\mathrm{GIVIFWA}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\omega_1 \tilde{\alpha}_1^{\lambda} \oplus \omega_2 \tilde{\alpha}_2^{\lambda} \oplus \cdots \oplus \omega_n \tilde{\alpha}_n^{\lambda} \right)^{\frac{1}{\lambda}} = \left(\left[\left(1 - \prod_{j=1}^{n} \left(1 - a_j^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} \left(1 - b_j^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}} \right], \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1-c_j)^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1-d_j)^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}} \right] \right), \quad (14)$$

then the function GIVIFWA is called the generalized interval-valued intuitionistic fuzzy weighted averaging (GIVIFWA) operator, where $\lambda > 0$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_j = \left([a_j, b_j], [c_j, d_j] \right)$ $(j = 1, 2, \ldots, n)$ with $\omega_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

There are some special cases obtained by using different choices of the parameters ω and λ (Shi and He 2013):

1. If $\lambda = 1$, then the GIVIFWA operator reduces to be the IVIFWA operator.
2. If $\omega = (1/n, 1/n, \ldots, 1/n)^{\mathrm{T}}$ and $\lambda = 1$, then the GIVIFWA operator reduces to the IVIFA operator.

Definition 3.9 (Zhao et al. 2010) Let GIVIFOWA: $\tilde{\theta}^n \to \tilde{\theta}$. If

$$\mathrm{GIVIFOWA}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\omega_1 \left(\tilde{\alpha}_{\sigma(1)} \right)^{\lambda} \oplus \omega_2 \left(\tilde{\alpha}_{\sigma(2)} \right)^{\lambda} \oplus \cdots \oplus \omega_n \left(\tilde{\alpha}_{\sigma(n)} \right)^{\lambda} \right)^{\frac{1}{\lambda}} = \left(\left[\left(1 - \prod_{j=1}^{n} \left(1 - a_{\sigma(j)}^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} \left(1 - b_{\sigma(j)}^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}} \right], \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1-c_{\sigma(j)})^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1-d_{\sigma(j)})^{\lambda} \right)^{\omega_j} \right)^{\frac{1}{\lambda}} \right] \right), \quad (15)$$

then the function GIVIFOWA is called the generalized interval-valued intuitionistic fuzzy ordered weighted averaging (GIVIFOWA) operator, where $\lambda > 0$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is the weight vector associated with the GIVIFOWA operator, with $\omega_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$, $\tilde{\alpha}_{\sigma(j)}$ is the jth largest of $\tilde{\alpha}_j$ $(j = 1, 2, \ldots, n)$.

There are some special cases by using different choices of the parameters ω and λ, which can be shown as:

1. If $\lambda = 1$, then the GIVIFOWA operator reduces to be the IVIFOWA operator;
2. If $\omega = (1/n, 1/n, \ldots, 1/n)^{\mathrm{T}}$ and $\lambda = 1$, then the GIVIFOWA operator reduces to the IVIFA operator;
3. If $\omega = (1, 0, \ldots, 0)^{\mathrm{T}}$, then the GIVIFOWA operator reduces to the following form:

$$\mathrm{IVIFMAX}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \max_j (\tilde{\alpha}_j) \quad (16)$$

4. If $\omega = (0, 0, \ldots, 1)^{\mathrm{T}}$, then the GIVIFOWA operator reduces to the following form:

$$\mathrm{IVIFMIN}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \min_j (\tilde{\alpha}_j) \quad (17)$$

The GIVIFWA operator only weights the IVIFNs, while the GIVIFOWA operator only weights the ordered positions of the IVIFNs instead of the IVIFNs themselves. To overcome this limitation, Zhao et al. (2010) introduced a generalized interval-valued intuitionistic fuzzy hybrid aggregation (GIVIFHA) operator:

Definition 3.10 (Zhao et al. 2010) A GIVIFHA operator of dimension n is a mapping GIVIFHA: $\tilde{\theta}^n \to \tilde{\theta}$, which has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$, $\lambda > 0$, such that

$$\begin{aligned} &\mathrm{GIVIFHA}_{w,\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \\ &= \left(\omega_1 \left(\dot{\tilde{\alpha}}_{\sigma(1)}\right)^{\lambda} \oplus \omega_2 \left(\dot{\tilde{\alpha}}_{\sigma(2)}\right)^{\lambda} \oplus \cdots \oplus \omega_n \left(\dot{\tilde{\alpha}}_{\sigma(n)}\right)^{\lambda}\right)^{\frac{1}{\lambda}} \\ &= \left(\left[\left(1 - \prod_{j=1}^{n}\left(1 - \dot{a}^{\lambda}_{\sigma(j)}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n}\left(1 - \dot{b}^{\lambda}_{\sigma(j)}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right], \right. \\ &\left[1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \dot{c}_{\sigma(j)}\right)^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}, \right. \\ &\left.\left. 1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \dot{d}_{\sigma(j)}\right)^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right]\right), \end{aligned} \quad (18)$$

where $\dot{\tilde{\alpha}}_{\sigma(j)}$ is the jth largest of the weighted IVIFNs $\dot{\tilde{\alpha}}_j$ $(\dot{\tilde{\alpha}}_j = nw_j\tilde{\alpha}_j,\ j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient, which plays a role of balance.

Additionally, based on these aggregation operators discussed above, Qi et al. (2013) introduced some other generalized interval-valued intuitionistic fuzzy aggregation operators:

Definition 3.11 (Qi et al. 2013) Let $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$ be a collection of IVIFNs, where $\tilde{\alpha}_j = \left(\left[a_j, b_j\right], \left[c_j, d_j\right]\right)$. Let $\bar{\mu}$ be the mean value of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$, and $\bar{\mu} = \left(\left[a_{\bar{\mu}}, b_{\bar{\mu}}\right], \left[c_{\bar{\mu}}, d_{\bar{\mu}}\right]\right)$, then $\bar{\mu}$ can be obtained by the IVIFWA operator with $\omega = (1/n, 1/n, \ldots, 1/n)^{\mathrm{T}}$, where $a_{\bar{\mu}} = 1 - \prod_{j=1}^{n}\left(1 - a_j\right)^{1/n}$, $b_{\bar{\mu}} = 1 - \prod_{j=1}^{n}\left(1 - b_j\right)^{1/n}$, $c_{\bar{\mu}} = \prod_{j=1}^{n} c_j^{1/n}$, and $d_{\bar{\mu}} = \prod_{j=1}^{n} d_j^{1/n}$.

Definition 3.12 (Xu and Yager 2009) Let $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$ be a collection of IVIFNs, where $\tilde{\alpha}_j = \left(\left[a_j, b_j\right], \left[c_j, d_j\right]\right)$ $(j = 1, 2, \ldots, n)$, $\bar{\mu} = \left(\left[a_{\bar{\mu}}, b_{\bar{\mu}}\right], \left[c_{\bar{\mu}}, d_{\bar{\mu}}\right]\right)$ denotes the mean value of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$, then the variance of $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n$ can be computed by $\xi = \sqrt{\frac{1}{n}\sum_{j=1}^{n}\left(d\left(\tilde{\alpha}_j, \bar{\mu}\right)\right)^2}$.

Definition 3.13 (Qi et al. 2013) Let $\bar{\mu}$ be the mean value of given interval-valued intuitionistic fuzzy arguments, ξ be the variance of given IVIFNs, then the Gaussian weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ can be defined as: $\omega_j = \frac{1}{\sqrt{2\pi}\xi} e^{-d^2\left(\tilde{\beta}_j - \bar{\mu}\right)/2\xi^2}$, where $\left(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n\right)$ is a permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$, with $\tilde{\beta}_{j-1} \geq \tilde{\beta}_j$ for all $j = 1, 2, \ldots, n$. Consider that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$ are commonly required in the aggregation operators, then we can normalize the Gaussian weighting vector by $\omega_j = \frac{\left(1/\sqrt{2\pi}\xi\right)e^{-d^2\left(\tilde{\beta}_j - \bar{\mu}\right)/2\xi^2}}{\sum_{j=1}^{n}\left(1/\sqrt{2\pi}\xi\right)e^{-d^2\left(\tilde{\beta}_j - \bar{\mu}\right)/2\xi^2}}$, $j = 1, 2, \ldots, n$.

The Gaussian generalized interval-valued intuitionistic fuzzy ordered weighted averaging (Gaussian-GIVIFOWA) operator and the Gaussian generalized interval-valued intuitionistic fuzzy ordered weighted geometric (Gaussian-GIVIFOWG) operator can be defined as follows:

Definition 3.14 (Qi et al. 2013) A Gaussian-GIVIFOWA operator of dimension n is a mapping Gaussian-GIVIFOWA: $\Omega^n \to \Omega$, which has an associated Gaussian weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, then

$$\text{Gaussian-GIVIFOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\omega_{\tilde{\alpha}_{\sigma(1)}} \tilde{\alpha}^{\lambda}_{\sigma(1)} \oplus \omega_{\tilde{\alpha}_{\sigma(2)}} \tilde{\alpha}^{\lambda}_{\sigma(2)} \oplus \cdots \oplus \omega_{\tilde{\alpha}_{\sigma(n)}} \tilde{\alpha}^{\lambda}_{\sigma(n)}\right)$$
$$= \left(\frac{1}{\sqrt{2\pi}\xi} e^{-d^2\left(\tilde{\beta}_1 - \bar{\mu}\right)/2\xi^2} \tilde{\beta}^{\lambda}_1 \oplus \frac{1}{\sqrt{2\pi}\xi} e^{-d^2\left(\tilde{\beta}_2 - \bar{\mu}\right)/2\xi^2} \tilde{\beta}^{\lambda}_2 \oplus \cdots \oplus \frac{1}{\sqrt{2\pi}\xi} e^{-d^2\left(\tilde{\beta}_n - \bar{\mu}\right)/2\xi^2} \tilde{\beta}^{\lambda}_n\right) \left(\left(\sum_{j=1}^{n} \frac{1}{\sqrt{2\pi}\xi} e^{-d^2\left(\tilde{\beta}_j - \bar{\mu}\right)/2\xi^2}\right)^{\frac{1}{\lambda}}\right)^{-1} \tag{19}$$

Additionally, a Gaussian-GIVIFOWG operator of dimension n is a mapping Gaussian-GIVIFOWG: $\Omega^n \to \Omega$, then

$$\text{Gaussian-GIVIFOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \frac{1}{\lambda}\left(\left(\lambda\tilde{\beta}_1\right)^{\omega_{\tilde{\beta}(1)}} \otimes \left(\lambda\tilde{\beta}_2\right)^{\omega_{\tilde{\beta}(2)}} \otimes \cdots \otimes \left(\lambda\tilde{\beta}_n\right)^{\omega_{\tilde{\beta}(n)}}\right)$$
$$= \frac{1}{\lambda}\left(\left(\lambda\tilde{\beta}_1\right)^{\left(1/\sqrt{2\pi}\xi\right) e^{-d^2\left(\tilde{\beta}_1 - \bar{\mu}\right)/2\xi^2}} \otimes \left(\lambda\tilde{\beta}_2\right)^{\left(1/\sqrt{2\pi}\xi\right) e^{-d^2\left(\tilde{\beta}_2 - \bar{\mu}\right)/2\xi^2}} \otimes \cdots \otimes \left(\lambda\tilde{\beta}_n\right)^{\left(1/\sqrt{2\pi}\xi\right) e^{-d^2\left(\tilde{\beta}_n - \bar{\mu}\right)/2\xi^2}}\right)^{1\Big/\sum_{j=1}^{n} \left(1/\sqrt{2\pi}\xi\right) e^{-d^2\left(\tilde{\beta}_j - \bar{\mu}\right)/2\xi^2}} \tag{20}$$

where $\left(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n\right)$ is a permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$, with $\tilde{\beta}_{j-1} \geq \tilde{\beta}_j$ for all $j = 1, 2, \ldots, n$.

Obviously, the aggregated results of the Gaussian-GIVIFOWA and Gaussian-GIVIFOWG operators are independent of orderings. Thus, the Gaussian-GIVIFOWA and Gaussian-GIVIFOWG operators are neat and dependent operators. Afterwards, Qi et al. (2013) defined a power generalized interval-valued intuitionistic fuzzy ordered weighted averaging (P-GIVIFOWA) operator and a power generalized interval-valued intuitionistic fuzzy ordered weighted geometric (P-GIVIFOWG) operator. Firstly, a hybrid support function can be defined as:

Definition 3.15 (Qi et al. 2013) Let $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$ be a collection of IVIFNs, and let $\bar{\mu}$ denote the mean value, then the hybrid support function can be defined as:

$$\text{Sup}(\tilde{\alpha}_j) = \frac{1}{n-1} \sum_{k=1, j \neq k}^{n} \left(1 - d(\tilde{\alpha}_j, \tilde{\alpha}_k)\right) + \left(1 - d(\tilde{\alpha}_j, \bar{\mu})\right)$$
$$= \frac{1}{n-1} \sum_{k=1, j \neq k}^{n} \text{Sup}(\tilde{\alpha}_j, \tilde{\alpha}_k) + \text{Sup}(\tilde{\alpha}_j, \bar{\mu}) \tag{21}$$

Definition 3.16 (Qi et al. 2013) A P-GIVIFOWA operator of dimension n is a mapping P-GIVIFOWA: $\Omega^n \to \Omega$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is the associated power weighting vector, $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, then

$$\text{P-GIVIFOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$$
$$= \left(\frac{\text{Sup}\left(\tilde{\beta}_1\right)}{\sum_{j=1}^{n} \text{Sup}\left(\tilde{\beta}_j\right)} \tilde{\beta}^{\lambda}_1 \oplus \frac{\text{Sup}\left(\tilde{\beta}_2\right)}{\sum_{j=1}^{n} \text{Sup}\left(\tilde{\beta}_j\right)} \tilde{\beta}^{\lambda}_2 \oplus \cdots \oplus \frac{\text{Sup}\left(\tilde{\beta}_n\right)}{\sum_{j=1}^{n} \text{Sup}\left(\tilde{\beta}_j\right)} \tilde{\beta}^{\lambda}_n\right)^{1/\lambda}$$
$$= \left(\frac{\text{Sup}\left(\tilde{\beta}_1\right)\tilde{\beta}^{\lambda}_1 \oplus \text{Sup}\left(\tilde{\beta}_2\right)\tilde{\beta}^{\lambda}_2 \oplus \cdots \oplus \text{Sup}\left(\tilde{\beta}_n\right)\tilde{\beta}^{\lambda}_n}{\sum_{j=1}^{n} \text{Sup}\left(\tilde{\beta}_j\right)}\right)^{1/\lambda} \tag{22}$$

and a P-GIVIFOWG operator of dimension n is a mapping P-GIVIFOWG: $\Omega^n \rightarrow \Omega$, then

$$\text{P-GIVIFOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \frac{1}{\lambda}\left(\left(\lambda\tilde{\beta}_1\right)^{\mathrm{Sup}(\tilde{\beta}_1)\Big/\sum_{j=1}^{n}\mathrm{Sup}(\tilde{\beta}_j)} \otimes \left(\lambda\tilde{\beta}_2\right)^{\mathrm{Sup}(\tilde{\beta}_2)\Big/\sum_{j=1}^{n}\mathrm{Sup}(\tilde{\beta}_j)} \otimes \cdots \otimes \left(\lambda\tilde{\beta}_n\right)^{\mathrm{Sup}(\tilde{\beta}_n)\Big/\sum_{j=1}^{n}\mathrm{Sup}(\tilde{\beta}_j)}\right)$$
$$= \frac{1}{\lambda}\left(\left(\lambda\tilde{\beta}_1\right)^{\mathrm{Sup}(\tilde{\beta}_1)} \otimes \left(\lambda\tilde{\beta}_2\right)^{\mathrm{Sup}(\tilde{\beta}_2)} \otimes \cdots \otimes \left(\lambda\tilde{\beta}_n\right)^{\mathrm{Sup}(\tilde{\beta}_n)}\right)^{1\Big/\sum_{j=1}^{n}\mathrm{Sup}(\tilde{\beta}_j)} \quad (23)$$

where $\left(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n\right)$ is a permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$, with $\tilde{\beta}_{j-1} \geq \tilde{\beta}_j$ for all $j = 1, 2, \ldots, n$.

The GIVIFWA, GIVIFOWA, GIVIFHA, Gaussian-GIVIFOWA, Gaussian-GIVIFOWG, P-GIVIFOWA and P-GIVIFOWG operators satisfy the commutativity, idempotency and boundedness.

Additionally, Dong and Wan (2015) defined some generalized aggregation operators of interval-valued trapezoidal intuitionistic fuzzy numbers (IVTrIFNs), including the generalized ordered weighted averaging operator and the generalized hybrid weighted averaging operator. Liu et al. (2015) proposed the interval-valued intuitionistic fuzzy generalized Einstein weighted averaging (IVIFGEWA) operator, the interval-valued intuitionistic fuzzy generalized Einstein ordered weighted averaging (IVIFGEOWA) operator, and the interval-valued intuitionistic fuzzy generalized Einstein hybrid weighted averaging (IVIFGE-HWA) operator. Some properties of these operators such as idempotency, commutativity, monotonicity and boundedness were discussed. Meng et al. (2015a) proposed the generalized Banzhaf interval-valued intuitionistic fuzzy geometric Choquet (GBIVIFGC) operator. In order to comprehensively reflect the interactions among elements and reduce the complexity of solving a fuzzy measure, they further introduced the GBIVIFGC operator with respect to 2-additive measures. Yu (2013) developed an approach to deal with the decision-making problems in the context of IVIFSs by introducing the generalized interval-valued intuitionistic fuzzy weighted geometric (GIVIFWG) and the generalized interval-valued intuitionistic fuzzy ordered weighted geometric (GIVIFOWG) operators. He et al. (2013) generalized the power averaging operators to interval-valued intuitionistic fuzzy environments, and developed a series of generalized interval-valued intuitionistic fuzzy power aggregation operators.

3.4 Interval-valued intuitionistic fuzzy aggregation operators based on Choquet integral

The Choquet integral is a very useful way of measuring the expected utility of an uncertain event and can be used to propose some interval-valued intuitionistic fuzzy aggregation operators. Xu (2010) utilized Choquet integral to propose two operators for aggregating IVIFNs together with their correlative weights:

Definition 3.17 (Xu 2010) Let ζ be a fuzzy measure on $X = \{x_1, x_2, \ldots, x_n\}$, and

$$\tilde{\alpha}(x_i) = ([a(x_i), b(x_i)], [c(x_i), d(x_i)])$$

$(i = 1, 2, \ldots, n)$ be n IVIFNs, then we call

$$(C_3)\int \tilde{\alpha}\mathrm{d}\zeta = \text{IVIFCA}(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \ldots, \tilde{\alpha}(x_n)) = \left(\zeta(B_{\sigma(1)}) - \zeta(B_{\sigma(0)})\right)\tilde{\alpha}(x_{\sigma(1)}) \oplus \left(\zeta(B_{\sigma(2)}) - \zeta(B_{\sigma(1)})\right)\tilde{\alpha}(x_{\sigma(2)}) \oplus \cdots \oplus \left(\zeta(B_{\sigma(n)}) - \zeta(B_{\sigma(n-1)})\right)\tilde{\alpha}(x_{\sigma(n)}) = \left(\left[1 - \prod_{i=1}^{n}\left(1 - a(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})}, 1 - \prod_{i=1}^{n}\left(1 - b(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})}\right], \left[\prod_{i=1}^{n}\left(c(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})}, \prod_{i=1}^{n}\left(d(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})}\right]\right) \quad (24)$$

and

$$(C_4)\int \tilde{\alpha}\mathrm{d}\zeta = \mathrm{IVIFCG}(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\cdots,\tilde{\alpha}(x_n)) = \tilde{\alpha}\left(x_{\sigma(1)}\right)^{\zeta\left(B_{\sigma(1)}\right)-\zeta\left(B_{\sigma(0)}\right)} \oplus \tilde{\alpha}\left(x_{\sigma(2)}\right)^{\zeta\left(B_{\sigma(2)}\right)-\zeta\left(B_{\sigma(1)}\right)} \oplus \cdots \oplus \tilde{\alpha}\left(x_{\sigma(n)}\right)^{\zeta\left(B_{\sigma(n)}\right)-\zeta\left(B_{\sigma(n-1)}\right)}$$
$$= \left(\left[\prod_{i=1}^{n}\left(a\left(x_{\sigma(i)}\right)\right)^{\zeta\left(B_{\sigma(i)}\right)-\zeta\left(B_{\sigma(i-1)}\right)}, \prod_{i=1}^{n}\left(b\left(x_{\sigma(i)}\right)\right)^{\zeta\left(B_{\sigma(i)}\right)-\zeta\left(B_{\sigma(i-1)}\right)}\right], \left[1-\prod_{i=1}^{n}\left(1-c\left(x_{\sigma(i)}\right)\right)^{\zeta\left(B_{\sigma(i)}\right)-\zeta\left(B_{\sigma(i-1)}\right)}, 1-\prod_{i=1}^{n}\left(1-d\left(x_{\sigma(i)}\right)\right)^{\zeta\left(B_{\sigma(i)}\right)-\zeta\left(B_{\sigma(i-1)}\right)}\right]\right) \tag{25}$$

the interval-valued intuitionistic fuzzy correlated averaging (IVIFCA) operator and the interval-valued intuitionistic fuzzy correlated geometric (IVIFCG) operator, respectively, where $(C_3)\int \tilde{\alpha}\mathrm{d}\zeta$ and $(C_4)\int \tilde{\alpha}\mathrm{d}\zeta$ are the notation of Choquet integrals, $\tilde{\alpha}(x_i)$ indicates that the indices have been permuted so that $\tilde{\alpha}\left(x_{\sigma(1)}\right) \geq \tilde{\alpha}\left(x_{\sigma(2)}\right) \geq \cdots \geq \tilde{\alpha}\left(x_{\sigma(n)}\right)$, $B_{\sigma(k)} = \left\{x_{\sigma(j)} | j \leq k\right\}$, when $k \geq 1$ and $B_{\sigma(0)} = \emptyset$. The aggregated values of these two operators are IVIFNs. These operators not only consider the importance degrees of the elements or their ordered positions, but also can reflect the correlations among the elements or their ordered positions. It is worth pointing out that most of the existing intuitionistic fuzzy aggregation operators are the special cases of these two operators. Some special cases of the IVIFCA and IVIFCG operators can be shown as follows:

1. If $\zeta(B) = \sum_{x_i \in B}\zeta(\{x_i\})$, for all $B \subseteq X$, and $\zeta\left(\left\{x_{\sigma(i)}\right\}\right) = \zeta\left(B_{\sigma(i)}\right) - \zeta\left(B_{\sigma(i-1)}\right)$, then the IVIFCA and IVIFCG operators reduce to the IVIFWA and IVIFWG operators, respectively. Especially, if $\zeta(\{x_i\}) = \frac{1}{n}$, for all $i = 1, 2, \ldots, n$, then the IVIFCA and IVIFCG operators reduce to the IVIFA and IVIFG operators, respectively.

2. If $\zeta(B) = \sum_{i=1}^{|B|}\omega_i$, for all $B \subseteq X$, and $\omega_i = \zeta\left(B_{\sigma(i)}\right) - \zeta\left(B_{\sigma(i)}\right)$, $i = 1, 2, \cdots, n$, then the IVIFCA and IVIFCG operators reduce to the IVIFOWA and IVIFOWG operators, respectively. Especially, if $\zeta(B) = \frac{|B|}{n}$, for all $B \subseteq X$, then both the IVIFCA and IVIFOWA operators reduce to the IVIFA operator. Similarly, both the IVIFCG and IVIFOWG operators reduce to the IVIFG operator.

3. If $\zeta(B) = \psi\left(\sum_{x_i \in B}\zeta(\{x_i\})\right)$, for all $B \subseteq X$, and $\omega_i = \zeta\left(B_{\sigma(i)}\right) - \zeta\left(B_{\sigma(i-1)}\right) = \psi\left(\sum_{j \leq i}\zeta\left(\left\{x_{\sigma(i)}\right\}\right)\right) - \psi\left(\sum_{j < i}\zeta\left(\left\{x_{\sigma(i)}\right\}\right)\right)$, $i = 1, 2, \ldots, n$ hold, then the IVIFCA and IVIFCG operators reduce to the interval-valued intuitionistic fuzzy weighted ordered weighted averaging (IVIFWOWA) operator and the interval-valued intuitionistic fuzzy weighted ordered weighted geometric (IIFWOWG) operator, respectively:

$$\begin{aligned}
&\mathrm{IVIFWOWA}(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\ldots,\tilde{\alpha}(x_n)) \\
&= \omega_1\tilde{\alpha}\left(x_{\sigma(1)}\right) \oplus \omega_2\tilde{\alpha}\left(x_{\sigma(2)}\right) \oplus \cdots \oplus \omega_n\tilde{\alpha}\left(x_{\sigma(n)}\right) \\
&= \left(\left[1-\prod_{i=1}^{n}\left(1-a\left(x_{\sigma(i)}\right)\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-b\left(x_{\sigma(i)}\right)\right)^{\omega_i}\right],\right. \\
&\left.\left[\prod_{i=1}^{n}\left(c\left(x_{\sigma(i)}\right)\right)^{\omega_i}, \prod_{i=1}^{n}\left(d\left(x_{\sigma(i)}\right)\right)^{\omega_i}\right]\right)
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&\mathrm{IVIFWOWG}(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\ldots,\tilde{\alpha}(x_n)) \\
&= \tilde{\alpha}\left(x_{\sigma(1)}\right)^{\omega_1} \otimes \tilde{\alpha}\left(x_{\sigma(2)}\right)^{\omega_2} \otimes \cdots \otimes \tilde{\alpha}\left(x_{\sigma(n)}\right)^{\omega_n} \\
&= \left(\left[\prod_{i=1}^{n}\left(a\left(x_{\sigma(i)}\right)\right)^{\omega_i}, \prod_{i=1}^{n}\left(b\left(x_{\sigma(i)}\right)\right)^{\omega_i}\right],\right. \\
&\left.\left[1-\prod_{i=1}^{n}\left(1-c\left(x_{\sigma(i)}\right)\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-d\left(x_{\sigma(i)}\right)\right)^{\omega_i}\right]\right)
\end{aligned} \tag{27}$$

Especially, if $\zeta(\{x_i\}) = \frac{1}{n}$, for all $i = 1, 2, \ldots, n$, then the IVIFWOWA and IVIFWOWG operators reduce to the IVIFOWA and IVIFOWG operators, respectively.

On the basis of the IVIFCA and IVIFCG operators, Meng et al. (2013a) proposed the arithmetical interval-valued intuitionistic fuzzy generalized λ-Shapley Choquet ($\mathrm{AIVIFGSC}_{g_\lambda}$) operator and the geometric interval-valued intuitionistic fuzzy generalized λ-Shapley Choquet ($\mathrm{GIVIFGSC}_{g_\lambda}$) operator. These operators not only consider the importance of combinations or their ordered positions, but also reflect the correlations among combinations or their ordered positions.

Let $N = \{1, 2, \ldots, n\}$ be a finite index set, and $P(N)$ be the power set of N, i.e., the set of all subsets of N.We often omit braces for singletons, e.g., by writing N/i, T and N/S instead of $N/\{i\}$, $\{T\}$ and $N/\{S\}$. Moreover, the cardinality of any subset $S \in P(N)$ will be denoted by the corresponding lower case s. Then the generalized Shapley index with respect to the λ-fuzzy measure g_λ on N can be shown as:

$$\varphi_S^{\mathrm{Sh}}(g_\lambda, N) = \sum\nolimits_{T \subseteq N/S} \frac{(n-s-t)!t!}{(n-s+t)!}(g_\lambda(S \cup T) - g_\lambda(T)), \quad \forall S \subseteq N.$$

Based on which, we introduce the following generalized λ-Shapley Choquet operators:

Definition 3.18 (Meng et al. 2013a) Let $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFNs, and g_λ be the λ-fuzzy measure on N, then the AIVIFGSC$_{g_\lambda}$ and GIVIFGSC$_{g_\lambda}$ operators of $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$ are defined as:

$$\int \tilde{\alpha} d\varphi^{\mathrm{Sh}}(g_\lambda, N) = \mathrm{AIVIFGSC}_{g_\lambda}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \overset{n}{\underset{i=1}{\oplus}} \left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda, N) - \varphi^{\mathrm{Sh}}_{A_{(i+1)}}(g_\lambda, N)\right)\tilde{\alpha}_{(i)}$$
$$= \left(\left[1 - \prod_{i=1}^{n} \left(1 - a_{(i)}\right)^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)}, 1 - \prod_{i=1}^{n} \left(1 - b_{(i)}\right)^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)}\right],\right.$$
$$\left.\left[\prod_{i=1}^{n} c_{(i)}^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)}, \prod_{i=1}^{n} d_{(i)}^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)}\right]\right) \tag{28}$$

and

$$\int \tilde{\alpha} d\varphi^{\mathrm{Sh}}(g_\lambda, N) = \mathrm{GIVIFGSC}_{g_\lambda}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \overset{n}{\underset{i=1}{\otimes}} \tilde{\alpha}_{(i)}^{\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i+1)}}(g_\lambda,N)}$$
$$= \left(\left[\prod_{i=1}^{n} a_{(i)}^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)}, \prod_{i=1}^{n} b_{(i)}^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)}\right], \left[1 - \prod_{i=1}^{n} \left(1 - c_{(i)}\right)^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)},\right.\right.$$
$$\left.\left. 1 - \prod_{i=1}^{n} \left(1 - d_{(i)}\right)^{\left(\varphi^{\mathrm{Sh}}_{A_{(i)}}(g_\lambda,N) - \varphi^{\mathrm{Sh}}_{A_{(i-1)}}(g_\lambda,N)\right)}\right]\right) \tag{29}$$

respectively, $(\cdot)$ indicates a permutation on N such that $\tilde{\alpha}_{(1)} \leq \tilde{\alpha}_{(2)} \leq \cdots \leq \tilde{\alpha}_{(n)}$, and $A_{(i)} = \{i, \ldots, n\}$ with $A_{(n+1)} = \emptyset$.

The IVIFCA, IVIFCA, AIVIFGSC$_{g_\lambda}$ and GIVIFGSC$_{g_\lambda}$ satisfy the idempotency, comonotonicity and boundedness.

3.5 Induced generalized interval-valued intuitionistic fuzzy aggregation operators

Recently, some studies have been done about a type of induced aggregation operators, which take their arguments as pairs, in which the first components called order-inducing variables are used to induce an ordering over the second components which are the aggregated variables. Based on these operators and the Choquet integral discussed above, Xu and Xia (2011) introduced the induced generalized intuitionistic fuzzy Choquet integral operators and the induced generalized intuitionistic fuzzy Dempster–Shafer operators:

Definition 3.19 (Xu and Xia 2011) Let $X = \{x_1, x_2, \cdots, x_n\}$ be a fixed set, m be a fuzzy measure on X, and $\tilde{\alpha}_i = ([a_{\tilde{\alpha}_i}, b_{\tilde{\alpha}_i}], [c_{\tilde{\alpha}_i}, d_{\tilde{\alpha}_i}])(i = 1, 2, \ldots, n)$ be a collection of IVIFNs on X. An induced generalized interval-valued intuitionistic fuzzy Choquet ordered averaging (I-GIVIFCOA) operator of dimension n is a function I-GIVIFCOA:

$\tilde{\theta}^n \rightarrow \tilde{\theta}$, which is defined to aggregate the set of second arguments of a collection of 2-tuples $\{\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle\}$ according to the following expression:

$$\text{I-GIVIFCOA}_\lambda(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) = \begin{cases} \left(\overset{n}{\underset{i=1}{\oplus}} \left(\left(\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)}) \right) \tilde{\alpha}^{\lambda}_{\sigma(i)} \right) \right)^{1/\lambda}, & \lambda > 0, \\ \overset{n}{\underset{i=1}{\otimes}} \left(\tilde{\alpha}_{\sigma(i)}^{\left(\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})\right)} \right), & \lambda = 0, \end{cases}$$

$$= \begin{cases} \left(\left[\left(1 - \prod_{i=1}^{n} \left(1 - \left(a_{\tilde{\alpha}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})} \right)^{1/\lambda}, \left(1 - \prod_{i=1}^{n} \left(1 - \left(b_{\tilde{\alpha}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})} \right)^{1/\lambda} \right], \right. \\ \left. \left[1 - \left(1 - \prod_{i=1}^{n} \left(1 - \left(1 - c_{\tilde{\alpha}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})} \right)^{1/\lambda}, 1 - \left(1 - \prod_{i=1}^{n} \left(1 - \left(1 - d_{\tilde{\alpha}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})} \right)^{1/\lambda} \right] \right), \lambda > 0, \\ \left(\left[\prod_{i=1}^{n} \left(a_{\tilde{\alpha}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})}, \prod_{i=1}^{n} \left(b_{\tilde{\alpha}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})} \right], \right. \\ \left. \left[1 - \prod_{i=1}^{n} \left(1 - c_{\tilde{\alpha}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})}, 1 - \prod_{i=1}^{n} \left(1 - d_{\tilde{\alpha}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})} \right] \right), \quad \lambda = 0, \end{cases} \tag{30}$$

where $\lambda > 0$, u_i in 2-tuples $\langle u_i, \tilde{\alpha}_i \rangle$ is referred to as the order-inducing variable and $\tilde{\alpha}_i$ as the argument variable, $\sigma(i) : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ is a permutation such that $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \cdots \geq u_{\sigma(n)}$, $A_{\sigma(i)} = \{x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\}$ when $i \geq 1$ and $A_{\sigma(0)} \neq \emptyset$. In particular, if $u_i = u_j$ in two 2-tuples $\langle u_i, \tilde{\alpha}_i \rangle$ and $\langle u_j, \tilde{\alpha}_j \rangle$, then we replace $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ by their average, i.e., $(\tilde{\alpha}_i \oplus \tilde{\alpha}_j)/2$.

In the case where $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \cdots \geq u_{\sigma(n)}$ and $\tilde{\alpha}_{\sigma(1)} \geq \tilde{\alpha}_{\sigma(2)} \geq \cdots \geq \tilde{\alpha}_{\sigma(n)}$, the I-GIVIFCOA operator becomes

$$\text{GIVIFCOA}_\lambda(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle) = \overset{n}{\underset{j=1}{\oplus}} \left(\left(\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)}) \right) \tilde{\beta}_{\sigma(i)} \right)$$

$$= \begin{cases} \left(\left[\left(1 - \prod_{j=1}^{n} \left(1 - \left(a_{\tilde{\beta}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right)^{1/\lambda}, \left(1 - \prod_{j=1}^{n} \left(1 - \left(b_{\tilde{\beta}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right)^{1/\lambda} \right], \right. \\ \left. \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - c_{\tilde{\beta}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - d_{\tilde{\beta}_{\sigma(i)}} \right)^{\lambda} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right)^{1/\lambda} \right] \right), \quad \lambda > 0, \\ \left(\left[\prod_{j=1}^{n} \left(a_{\tilde{\beta}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})}, \prod_{j=1}^{n} \left(b_{\tilde{\beta}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right], \right. \\ \left. \left[1 - \prod_{j=1}^{n} \left(1 - c_{\tilde{\beta}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})}, 1 - \prod_{j=1}^{n} \left(1 - d_{\tilde{\beta}_{\sigma(i)}} \right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right] \right), \quad \lambda = 0, \end{cases} \tag{31}$$

which we call a generalized interval-valued intuitionistic fuzzy Choquet ordered averaging (GIVIFCOA) operator, where $\tilde{\beta}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$.

Similar to Sect. 3.1, we can get some special operators as the parameter k changes in the I-GIVIFCOA operator:

1. If $k = 1$, then the I-GIVIFCOA operator reduces to the induced interval-valued intuitionistic fuzzy Choquet ordered averaging (I-IVIFCOA) operator:

$$\begin{aligned} &\text{I-IVIFCOA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) \\ &= \mathop{\oplus}_{i=1}^{n}\left(\left(\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)\right)\tilde{\alpha}_{\sigma(i)}\right) \\ &= \left(\left[1 - \prod_{i=1}^{n}\left(1 - a_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)},\right.\right. \\ &\qquad \left. 1 - \prod_{i=1}^{n}\left(1 - b_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)}\right], \\ &\quad \left.\left[\prod_{i=1}^{n}\left(c_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)}, \prod_{i=1}^{n}\left(d_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)}\right]\right) \end{aligned} \quad (32)$$

Especially, if $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \cdots \geq u_{\sigma(n)}$ and $\tilde{\alpha}_{\sigma(1)} \geq \tilde{\alpha}_{\sigma(2)} \geq \cdots \geq \tilde{\alpha}_{\sigma(n)}$, then the I-IVIFCOA operator becomes the interval-valued intuitionistic fuzzy Choquet ordered averaging (IVIFCOA) operator:

$$\begin{aligned} &\text{IVIFCOA}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) \\ &= \mathop{\oplus}_{i=1}^{n}\left(\left(\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)\right)\tilde{\beta}_{\sigma(i)}\right) \\ &= \left(\left[1 - \prod_{j=1}^{n}\left(1 - a_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)},\right.\right. \\ &\qquad \left. 1 - \prod_{j=1}^{n}\left(1 - b_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)}\right], \\ &\quad \left.\left[\prod_{i=1}^{n}\left(c_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)}, \prod_{i=1}^{n}\left(d_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)}\right]\right) \end{aligned} \quad (33)$$

where $\tilde{\beta}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$.

2. If $\lambda = 0$, then the I-GIVIFCOA operator becomes

$$\begin{aligned} &\text{I-IVIFCOG}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) = \mathop{\oplus}_{i=1}^{n} \tilde{\alpha}_{\sigma(i)}^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)} \\ &= \left(\left[\prod_{i=1}^{n}\left(a_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)}, \prod_{i=1}^{n}\left(b_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)}\right],\right. \\ &\left.\left[1 - \prod_{i=1}^{n}\left(1 - c_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)}, 1 - \prod_{i=1}^{n}\left(1 - d_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)}\right]\right) \end{aligned} \quad (34)$$

which we call an induced interval-valued intuitionistic fuzzy Choquet ordered geometric (I-IVIFCOG) operator. Especially, if $u_{\sigma(1)} \geq u_{\sigma(2)} \geq \cdots \geq u_{\sigma(n)}$ and $\tilde{\alpha}_{\sigma(1)} \geq \tilde{\alpha}_{\sigma(2)} \geq \cdots \geq \tilde{\alpha}_{\sigma(n)}$, then the I-IVIFCOG operator reduces to the interval-valued intuitionistic fuzzy Choquet ordered geometric (IVIFCOG) operator:

$$\begin{aligned} &\text{IVIFCOG}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) = \mathop{\oplus}_{i=1}^{n} \tilde{\beta}_{\sigma(i)}^{\zeta\left(A_{\sigma(i)}\right) - \zeta\left(A_{\sigma(i-1)}\right)} \\ &= \left(\left[\prod_{j=1}^{n}\left(a_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)}, \prod_{j=1}^{n}\left(b_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)}\right],\right. \\ &\left.\left[1 - \prod_{j=1}^{n}\left(1 - c_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)}, 1 - \prod_{j=1}^{n}\left(1 - d_{\tilde{\beta}_{\sigma(j)}}\right)^{\zeta\left(A_{\sigma(j)}\right) - \zeta\left(A_{\sigma(j-1)}\right)}\right]\right) \end{aligned} \quad (35)$$

where $\tilde{\beta}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$.

Besides, Xu and Xia (2011) also introduced an interval-valued intuitionistic Dempster–Shafer operator:

Definition 3.20 (Xu and Xia 2011) Let $X = \{x_1, x_2, \ldots, x_n\}$ be a fixed set, $(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle)$ be a collection of 2-tuples on X, where u_i $(i = 1, 2, \ldots, n)$ are the inducing variables and $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$ are the aggregated arguments in the form of IVIFNs. Then $\tilde{M} = (\tilde{M}_k | \tilde{M}_k = \{\langle u_i, \tilde{\alpha}_i t\rangle | x_i \in D_k, i = 1, 2, \ldots, n\}, k = 1, 2, \ \ldots, r.) = (\{\langle u_{1k}, \tilde{\gamma}_{1k}\rangle, \langle u_{2k}, \tilde{\gamma}_{2k}\rangle, \ldots, \langle u_{q_k k}, \tilde{\gamma}_{q_k k}\rangle\} | k = 1, 2, \ldots, r)$ is a collection of 2-tuple arguments with r focal elements D_k $(k = 1, 2, \ldots, r)$. A BSI-GIVIFOA operator of dimension r is a function BSI-GIVIFOA: $\tilde{V}^r \to V$ defined by

$$\text{BSI-GIVIFOA}_{\lambda_1, \lambda_2}(\tilde{M}) = \begin{cases} \left(\mathop{\oplus}_{k=1}^{r}\left(p(D_k)\left(\mathop{\oplus}_{j=1}^{q_k}\left(\omega_{ik}\tilde{\beta}_{ik}^{\lambda_1}\right)\right)^{\lambda_2/\lambda_1}\right)\right)^{1/\lambda_2}, \lambda_1 > 0, \lambda_2 > 0, \\ \left(\mathop{\oplus}_{k=1}^{r}\left(p(D_k)\left(\mathop{\otimes}_{j=1}^{q_k}\left(\tilde{\beta}_{ik}^{\omega_{ik}}\right)\right)^{\lambda_2}\right)\right)^{1/\lambda_2}, \lambda_1 = 0, \lambda_2 > 0, \\ \mathop{\otimes}_{k=1}^{r}\left(\left(\mathop{\oplus}_{j=1}^{q_k}\left(\omega_{jk}\tilde{\beta}_{jk}^{\lambda_1}\right)\right)^{1/\lambda_1}\right)^{p(D_k)}, \lambda_1 > 0, \lambda_2 = 0, \\ \mathop{\oplus}_{k=1}^{r}\left(\left(\mathop{\otimes}_{j=1}^{q_k}\left(\tilde{\beta}_{jk}^{\omega_{jk}}\right)\right)^{p(D_k)}\right), \lambda_1 = 0, \lambda_2 = 0, \end{cases} \quad (36)$$

where $\omega_k = (\omega_{1k}, \omega_{2k}, \ldots, \omega_{q_k k})^{\mathrm{T}}$ is the weight vector of the k-th focal element D_k such that $\omega_{ik} \in [0, 1]$ and $\sum_{j=1}^{q_k} \omega_{ik} = 1$, q_k is the number of elements in D_k, $\tilde{\beta}_{jk}$ is the $\tilde{\gamma}_{jk}$ value of the pair $\langle u_{ik}, \tilde{\gamma}_{ik}\rangle$ having the jth largest u_{ik} $(i = 1, 2, \ldots, q_k)$, u_{ik} is the order-inducing variable, $\tilde{\gamma}_{ik}$ is the argument variable, and $p(D_k)$ is the basic probability assignment.

By the operational laws of IVIFNs and mathematical induction on n, some special cases of the BSI-GIVIFOA operator can be given as follows:

1. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then the BSI-GIVIFOA operator reduces to

$$\text{BSI - GIVIFOA}_{\lambda_1,\lambda_2}(\tilde{M}) = \left(\mathop{\oplus}_{k=1}^{r} \left(p(D_k) \left(\mathop{\oplus}_{j=1}^{q_k} \left(\omega_{jk} \tilde{\beta}_{jk}^{\lambda_1} \right) \right)^{\lambda_2/\lambda_1} \right) \right)^{1/\lambda_2}$$
$$= \left(\left[\left(1 - \prod_{k=1}^{r} \left(1 - \left(1 - \prod_{j=1}^{q_k} \left(1 - \left(a_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{\lambda_2/\lambda_1} \right)^{p(D_k)} \right)^{1/\lambda_2}, \left(1 - \prod_{k=1}^{r} \left(1 - \left(1 - \prod_{j=1}^{q_k} \left(1 - \left(b_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{\lambda_2/\lambda_1} \right)^{p(D_k)} \right)^{1/\lambda_2} \right], \right.$$
$$\left. \left[1 - \left(1 - \prod_{k=1}^{r} \left(1 - \left(1 - \prod_{j=1}^{q_k} \left(1 - \left(1 - c_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{\lambda_2/\lambda_1} \right)^{p(D_k)} \right)^{1/\lambda_2}, 1 - \left(1 - \prod_{k=1}^{r} \left(1 - \left(1 - \prod_{j=1}^{q_k} \left(1 - \left(1 - d_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{\lambda_2/\lambda_1} \right)^{p(D_k)} \right)^{1/\lambda_2} \right] \right) \tag{37}$$

2. If $\lambda_1 = \lambda_1 = \lambda$, then the BSI-GIVIFOA operator reduces to

$$\text{BSI - GIVIFOA}_{\lambda_1,\lambda_2}(\tilde{M}) = \left(\mathop{\oplus}_{k=1}^{r} \left(p(D_k) \mathop{\oplus}_{j=1}^{q_k} \left(\omega_{jk} \tilde{\beta}_{jk}^{\lambda_1} \right) \right) \right)^{1/\lambda_2}$$
$$= \left(\left[\left(1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - \left(a_{\beta_{jk}} \right)^{\lambda} \right)^{\omega_{jk} p(D_k)} \right)^{1/\lambda}, \left(1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - \left(b_{\beta_{jk}} \right)^{\lambda} \right)^{\omega_{jk} p(D_k)} \right)^{1/\lambda} \right], \right.$$
$$\left. \left[1 - \left(1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - \left(c_{\beta_{jk}} \right)^{\lambda} \right)^{\omega_{jk} p(D_k)} \right)^{1/\lambda}, 1 - \left(1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - \left(c_{\beta_{jk}} \right)^{\lambda} \right)^{\omega_{jk} p(D_k)} \right)^{1/\lambda} \right] \right) \tag{38}$$

3. If $\lambda_1 = \lambda_2 = 1$, then the BSI-GIVIFOA operator becomes:

$$\text{BSI - GIVIFOA}_{\lambda_1,\lambda_2}(\tilde{M}) = \mathop{\oplus}_{k=1}^{r} \left(p(D_k) \mathop{\oplus}_{j=1}^{q_k} \left(\omega_{jk} \tilde{\beta}_{jk}^{\lambda_1} \right) \right)$$
$$= \left(\left[1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - a_{\beta_{jk}} \right)^{\omega_{jk} p(D_k)}, 1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - b_{\beta_{jk}} \right)^{\omega_{jk} p(D_k)} \right], \right.$$
$$\left. \left[\prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - c_{\beta_{jk}} \right)^{\omega_{jk} p(D_k)}, \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - d_{\beta_{jk}} \right)^{\omega_{jk} p(D_k)} \right] \right) \tag{39}$$

4. If $\lambda_1 = 0$ and $\lambda_2 > 0$, then the BSI-GIVIFOA operator reduces to:

$$\text{BSI - GIVIFOA}_{\lambda_1,\lambda_2}(\tilde{M}) = \left(\mathop{\oplus}_{k=1}^{r} \left(p(D_k) \left(\mathop{\oplus}_{j=1}^{q_k} \left(\omega_{jk} \tilde{\beta}_{jk}^{\lambda_1} \right) \right)^{\lambda_2/\lambda_1} \right) \right)^{1/\lambda_2}$$
$$= \left(\left[\left(1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(a_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right)^{1/\lambda_2}, \left(1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(b_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right)^{1/\lambda_2} \right], \right.$$
$$\left. \left[1 - \left(1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(c_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right)^{1/\lambda_2}, 1 - \left(1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(d_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right)^{1/\lambda_2} \right] \right) \tag{40}$$

5. If $\lambda_1 = 0$ and $\lambda_2 > 0$, then the BSI-GIVIFOA operator reduces to the BSI-IVIFOAG operator:

$$\text{BSI - IVIFOAG}(\tilde{M}) = \mathop{\oplus}_{k=1}^{r} \left(p(D_k) \mathop{\otimes}_{j=1}^{q_k} \left(\tilde{\beta}_{jk}^{\omega_{jkj}} \right) \right)$$
$$= \left(\left[1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(a_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)}, 1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(b_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right], \right.$$
$$\left. \left[\prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(1 - c_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)}, \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(1 - d_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right] \right) \tag{41}$$

6. If $\lambda_1 > 0$ and $\lambda_2 = 0$, then the BSI-GIVIFOA operator reduces to:

$$\text{BSI - GIVIFOA}_{\lambda_1}(\tilde{M}) = \underset{k=1}{\overset{r}{\otimes}} \left(\left(\underset{j=1}{\overset{q_k}{\oplus}} \left(\omega_{jk} \tilde{\beta}_{jk}^{\lambda_1} \right) \right)^{1/\lambda_1} \right)^{p(D_k)}$$
$$= \left(\left[\prod_{j=1}^{q_k} \left(\left(1 - \prod_{k=1}^{r} \left(1 - \left(a_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{1/\lambda_1} \right)^{p(D_k)}, \prod_{j=1}^{q_k} \left(\left(1 - \prod_{k=1}^{r} \left(1 - \left(b_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{1/\lambda_1} \right)^{p(D_k)} \right], \right.$$
$$\left. \left[1 - \prod_{j=1}^{q_k} \left(\left(1 - \prod_{k=1}^{r} \left(1 - \left(c_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{1/\lambda_1} \right)^{p(D_k)}, 1 - \prod_{j=1}^{q_k} \left(\left(1 - \prod_{k=1}^{r} \left(1 - \left(d_{\tilde{\beta}_{jk}} \right)^{\lambda_1} \right)^{\omega_{jk}} \right)^{1/\lambda_1} \right)^{p(D_k)} \right] \right) \quad (42)$$

Definition 3.21 (Zhang and Qi 2012) An I-IVIFHA operator of dimension n is a mapping I-IVIFHA: $R^n \to R$ defined by an associated weight vector ω of dimension n such that the sum of the weights is 1 and $\omega_j \in [0,1]$ $(j = 1,2,\ldots,n)$, and a set of order-inducing variables $u_j (j = 1,2,\ldots,n)$, according to the following formula:

$$\text{I - IVIFHA}_{\omega,w}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle)$$
$$= \omega_1 \dot{\tilde{\alpha}}_{\sigma(1)} \oplus \omega_2 \dot{\tilde{\alpha}}_{\sigma(2)} \oplus \cdots \oplus \omega_n \dot{\tilde{\alpha}}_{\sigma(n)}$$
$$= \left(\left[1 - \prod_{j=1}^{n} \left(1 - \dot{\tilde{a}}_{\sigma(j)} \right)^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \dot{\tilde{b}}_{\sigma(j)} \right)^{\omega_j} \right], \left[\prod_{j=1}^{n} \dot{\tilde{c}}_{\sigma(j)}^{\omega_j}, \prod_{j=1}^{n} \dot{\tilde{d}}_{\sigma(j)}^{\omega_j} \right] \right) \quad (45)$$

7. If $\lambda_1 = 1$ and $\lambda_2 = 0$, then from the BSI-GIVIFOA operator, we obtain the BSI-IVIFOGA operator:

$$\text{BSI - IVIFOGA}(\tilde{M}) = \underset{k=1}{\overset{r}{\otimes}} \left(\underset{j=1}{\overset{q_k}{\oplus}} \left(\omega_{jk} \tilde{\beta}_{jk} \right) \right)^{p(D_k)}$$
$$= \left(\left[\prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(1 - a_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)}, \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(1 - b_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right], \right.$$
$$\left. \left[1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(1 - c_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)}, 1 - \prod_{k=1}^{r} \left(1 - \prod_{j=1}^{q_k} \left(1 - d_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}} \right)^{p(D_k)} \right] \right) \quad (43)$$

8. If $\lambda_1 = 0$ and $\lambda_2 = 0$, then the BSI-GIVIFOA operator becomes:

$$\text{BSI - IVIFOGG}(\tilde{M}) = \underset{k=1}{\overset{r}{\otimes}} \left(\left(\underset{j=1}{\overset{q_k}{\oplus}} \left(\tilde{\beta}_{jk}^{\omega_{jk}} \right) \right)^{p(D_k)} \right)$$
$$= \left(\left[\prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(a_{\tilde{\beta}_{jk}} \right)^{\omega_{jk} p(D_k)}, \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(b_{\tilde{\beta}_{jk}} \right)^{\omega_{jk} p(D_k)} \right], \right.$$
$$\left. \left[1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - c_{\tilde{\beta}_{jk}} \right)^{\omega_{jk} p(D_k)}, 1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(d_{\tilde{\beta}_{jk}} \right)^{\omega_{jk} p(D_k)} \right] \right) \quad (44)$$

which we call the BSI-IVIFOGG operator.

Both the I-GIVIFCOA and BSI-GIVIFOA operators satisfy commutativity, monotonicity, boundedness and idempotency.

Furthermore, Zhang and Qi (2012) defined the induced interval-valued intuitionistic fuzzy hybrid averaging (I-IVIFHA) operator and the induced interval-valued intuitionistic fuzzy hybrid geometric (I-IVIFHG) operator, which can consider more preference information during the aggregation process by combination of order-inducing variables independent of arguments:

Similarly, the I-IVIFHG operator is

$$\text{I - IVIFHG}_{\omega,w}(\langle u_1, \tilde{\alpha}_1 \rangle, \langle u_2, \tilde{\alpha}_2 \rangle, \ldots, \langle u_n, \tilde{\alpha}_n \rangle)$$
$$= \dot{\tilde{\alpha}}_{\sigma(1)}^{\omega_1} \otimes \dot{\tilde{\alpha}}_{\sigma(2)}^{\omega_2} \otimes \cdots \otimes \dot{\tilde{\alpha}}_{\sigma(n)}^{\omega_n} = \left(\left[\prod_{j=1}^{n} \dot{\tilde{a}}_{\sigma(j)}^{\omega_j}, \prod_{j=1}^{n} \dot{\tilde{b}}_{\sigma(j)}^{\omega_j} \right], \right.$$
$$\left. \left[1 - \prod_{j=1}^{n} \left(1 - \dot{\tilde{c}}_{\sigma(j)} \right)^{\omega_j}, 1 - \prod_{j=1}^{n} \left(1 - \dot{\tilde{d}}_{\sigma(j)} \right)^{\omega_j} \right] \right), \quad (46)$$

where $\dot{\tilde{\alpha}}_i = \left(\left[\dot{\tilde{a}}_i, \dot{\tilde{b}}_i \right], \left[\dot{\tilde{c}}_i, \dot{\tilde{d}}_i \right] \right) = n w_i \tilde{\alpha}_i = \left(\left[\tilde{a}_i^{n w_i}, \tilde{b}_i^{n w_i} \right], \left[1 - (1 - \tilde{c}_i)^{n w_i}, 1 - \left(1 - \tilde{d}_i \right)^{n w_i} \right] \right)$, $\left(\dot{\tilde{\alpha}}_{\sigma(1)}, \dot{\tilde{\alpha}}_{\sigma(2)}, \ldots, \dot{\tilde{\alpha}}_{\sigma(n)} \right)$ is $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$ reordered in the decreasing order of the order-inducing variables values of $u_j (j = 1,2,\ldots,n)$, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector $\tilde{\alpha}_j$ $(j = 1,2,\ldots,n)$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0,1]$, and n is the balancing coefficient.

Both the I-IVIFHA and I-IVIFHG operators satisfy commutativity, boundedness and idempotency. Meng et al. (2013b) proposed the induced generalized interval-valued intuitionistic fuzzy hybrid Shapley averaging (IG-IVIFHSA) operator:

Definition 3.22 (Meng et al. 2013b) An IG-IVIFHSA operator of dimension n is a mapping IG-IVIFHSA: $\Omega^n \to \Omega$ defined on the set of second arguments of two tuples $\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle$ with a set of order-inducing variables u_i $(i = 1, 2, \ldots, n)$ and a parameter γ such that $\gamma \in (0, +\infty)$, denoted by

$$\text{IG - IVIFHSA}_{\mu,v}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) = \left(\frac{\overset{n}{\underset{j=1}{\oplus}} \varphi_j(\mu, N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v, A)\tilde{\alpha}_j^{\gamma}}{\sum_{j=1}^{n} \varphi_j(\mu, N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v, A)}\right)^{1/\gamma}$$

$$= \left(\left[\left(1 - \prod_{j=1}^{n}\left(1 - a_j^{\gamma}\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}, \left(1 - \prod_{j=1}^{n}\left(1 - b_j^{\gamma}\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}\right],\right.$$
$$\left.\left[1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - c_j^{\gamma}\right)\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}, 1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - d_j^{\gamma}\right)\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}\right]\right), \quad (47)$$

where u_j is the jth largest of u_j $(i = 1, 2, \ldots, n)$, $\varphi_j(\mu, N)$ is the Shapley value with respect to the associated fuzzy measure μ on $N = \{1, 2, \ldots, n\}$ for the j-th index, and $\varphi_{\tilde{\alpha}_j}(v, A)$ is the Shapley value with respect to the fuzzy measure v on $\tilde{\alpha} = \{\tilde{\alpha}_j\}_{j=1,2,\ldots,n}$ for $\tilde{\alpha}_j$ $(j = 1, 2, \ldots, n)$.

Moreover, Meng et al. (2013b) introduced the induced generalized interval-valued intuitionistic fuzzy hybrid Shapley averaging (IG-IVIFHSA) operator and the induced generalized quasi interval-valued intuitionistic fuzzy hybrid Shapley averaging (IGQ-IVIFHSA) operator, respectively:

Definition 3.23 (Meng et al. 2013b) An IG-IVIFHSA operator of dimension n is a mapping IG-IVIFHSA: $\Omega^n \to \Omega$ defined on the set of second arguments of two tuples $\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle$ with a set of order-inducing variables u_i $(i = 1, 2, \ldots, n)$ and a parameter γ such that $\gamma \in (0, +\infty)$, denoted by

$$\text{IG - IVIFHSA}_{\mu,v}(\langle u_1, \tilde{\alpha}_1\rangle, \langle u_2, \tilde{\alpha}_2\rangle, \ldots, \langle u_n, \tilde{\alpha}_n\rangle) = \left(\frac{\overset{n}{\underset{j=1}{\oplus}} \varphi_j(\mu, N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v, \tilde{\alpha})\tilde{\alpha}_j^{\gamma}}{\sum_{j=1}^{n} \varphi_j(\mu, N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v, \tilde{\alpha})}\right)^{1/\gamma}$$

$$= \left(\left[\left(1 - \prod_{j=1}^{n}\left(1 - a_j^{\gamma}\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}, \left(1 - \prod_{j=1}^{n}\left(1 - b_j^{\gamma}\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}\right],\right.$$
$$\left.\left[1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - c_j\right)^{\gamma}\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}, 1 - \left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - d_j\right)^{\gamma}\right)^{\frac{\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}}\right)^{1/\gamma}\right]\right) \quad (48)$$

Similarly, the IG-IVIFHSA operator can be shown as:

$$\text{IGQ - IVIFHSA}_{\mu,\nu}(\langle u_1,\tilde{\alpha}_1\rangle,\langle u_2,\tilde{\alpha}_2\rangle,\ldots,\langle u_n,\tilde{\alpha}_n\rangle) = g^{-1}\left(\frac{\overset{n}{\underset{j=1}{\oplus}}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})g\left(\tilde{\alpha}^{\gamma}_j\right)}{\sum_{j=1}^{n}\varphi_j(\mu,N)\varphi^{\gamma}_{\tilde{\alpha}_j}(v,\tilde{\alpha})}\right)^{1/\gamma} \quad (49)$$

where u_j is the jth largest of u_j $(i=1,2,\ldots,n)$, $\varphi_j(\mu,N)$ is the Shapley value with respect to the associated fuzzy measure μ on $N=\{1,2,\ldots,n\}$ for the j-th index, and $\varphi_{\tilde{\alpha}_j}(v,A)$ is the Shapley value with respect to the fuzzy measure v on $\tilde{\alpha}=\{\tilde{\alpha}_j\}_{j=1,2,\ldots,n}$ for $\tilde{\alpha}_j$ $(j=1,2,\ldots,n)$, and g is a strictly continuous monotonic function.

The IG-IVIFHSA operator satisfies commutativity, monotonicity, idempotency and boundedness. In addition, based on the Choquet integral and the generalized Shapley function, Meng et al. (2014) introduced two induced interval-valued intuitionistic fuzzy hybrid aggregation operators, which are called the induced generalized Shapley interval-valued intuitionistic fuzzy hybrid Choquet arithmetical averaging (IGS-IVIFHCAA) operator and the induced generalized Shapley interval-valued intuitionistic fuzzy hybrid Choquet geometric mean (IGS-IVIFHCGM) operator. These operators not only consider the importance degrees of elements and their ordered positions, but also reflect the correlations among them and their ordered positions. Wei and Yi (2008) defined a new aggregation operator called induced interval-valued intuitionistic fuzzy ordered weighted geometric (I-IVIFOWG) operator, and some desirable properties of the I-IVIFOWG operators were studied, such as commutativity, idempotency and monotonicity. Yang and Yuan (2014) developed the induced interval-valued intuitionistic fuzzy Einstein ordered weighted geometric (I-IVIFEOWG) operator and the induced interval-valued intuitionistic fuzzy Einstein hybrid geometric (I-IVIFEHG) operator, and also established some desirable properties of these two operators, such as commutativity, idempotency and monotonicity.

3.6 Interval-valued intuitionistic fuzzy interactive aggregation operators

Though the operational laws of IVIFNs defined by Xu (2007) have been used widely, they still have some shortcomings and can be further improved. So Yu (2014) proposed some new operational laws for IVIFNs, the main advantage of these new operational laws is that it can handle some extreme cases such as the non-membership degree range or the membership degree range is reduced to [0,0].

Definition 3.24 (Yu 2014) Suppose that $\tilde{\alpha}=([a,b],[c,d])$, $\tilde{\alpha}_2=([a_1,b_1],[c_1,d_1])$ and $\tilde{\alpha}_2=([a_2,b_2],[c_2,d_2])$ are three IVIFNs, some new operational laws can be defined as follows:

1. $\tilde{\alpha}_1\oplus\tilde{\alpha}_2=([1-(1-a_1)(1-a_2),1-(1-b_1)\ (1-b_2)],[(1-a_1)(1-a_2)-(1-(a_1+\ c_1))(1-(a_2+\ c_2)),(1-b_1)(1-b_2)-(1-(b_1+d_1))(1-(b_2+d_2))]])$
2. $\tilde{\alpha}_1\otimes\tilde{\alpha}_2=([(1-c_1)(1-c_2)-(1-(a_1+c_1))(1-\ (a_2+c_2)),\ (1-d_1)(1-d_2)\ -\ (1-(b_1+d_1))(1-(b_2+d_2))],\ [1-(1-c_1)\ (1-c_2),1-(1-d_1)(1-d_2)])$
3. $\lambda\tilde{\alpha}=\left(\left[1-(1-a)^{\lambda},1-(1-b)^{\lambda}\right],\ \left[(1-a)^{\lambda}-(1-\ (a+c))^{\lambda},(1-b)^{\lambda}-(1-(b+d))^{\lambda}\right]\right)$
4. $(\tilde{\alpha})^{\lambda}=\left(\left[(1-c)^{\lambda}-(1-(a+c))^{\lambda},(1-d)^{\lambda}-(1-(b+d))^{\lambda}\right],\left[1-(1-c)^{\lambda},1-(1-d)^{\lambda}\right]\right)$

Based on these operational laws, two interval-valued intuitionistic fuzzy interactive weighted operators can be defined below:

Definition 3.25 (Yu 2014) Suppose that $(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)$ is a group of IVIFNs and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^{\mathrm{T}}$ is the weight vector of them, such that $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$. Then

$$\text{IVIFIWA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\overset{n}{\underset{j=1}{\oplus}}\omega_j\tilde{\alpha}_j = \left(\left[1-\prod_{j=1}^{n}(1-a_j)^{\omega_j},1-\prod_{j=1}^{n}(1-b_j)^{\omega_j}\right],\left[\prod_{j=1}^{n}(1-a_j)^{\omega_j}-\prod_{j=1}^{n}(1-(a_j+c_j))^{\omega_j},\prod_{j=1}^{n}(1-b_j)^{\omega_j}-\prod_{j=1}^{n}(1-(b_j+d_j))^{\omega_j}\right]\right) \quad (50)$$

and

$$\text{IVIFIWG}(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n)=\overset{n}{\underset{j=1}{\otimes}}\tilde{\alpha}_j^{\omega_j} = \left(\left[\prod_{j=1}^{n}(1-c_j)^{\omega_j}-\prod_{j=1}^{n}(1-(a_j+c_j))^{\omega_j},\prod_{j=1}^{n}(1-d_j)^{\omega_j}-\prod_{j=1}^{n}(1-(b_j+d_j))^{\omega_j}\right],\left[1-\prod_{j=1}^{n}(1-c_j)^{\omega_j},1-\prod_{j=1}^{n}(1-d_j)^{\omega_j}\right]\right) \quad (51)$$

are called the interval-valued intuitionistic fuzzy interactive weighted average (IVIFIWA) operator and the interval-valued intuitionistic fuzzy interactive weighted geometric (IVIFIWG) operator, respectively.

Additionally, Yu (2014) defined the interval-valued intuitionistic fuzzy interactive ordered weighted average (IVIFIOWA) operator and the interval-valued intuitionistic fuzzy interactive ordered weighted geometric (IVIFIOWG) operator:

Definition 3.26 (Yu 2014). Suppose that a group of IVIFNs is expressed as $(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)$, the IVIFIOWA and IVIFIOWG operators can be defined, respectively, as follows:

$$\begin{aligned}\mathrm{IVIFIOWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \mathop{\oplus}_{j=1}^{n} \omega_j \tilde{\alpha}_{\sigma(j)} \\ &= \left(\left[1 - \prod_{j=1}^{n}\left(1 - a_{\sigma(j)}\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - b_{\sigma(j)}\right)^{\omega_j}\right],\right. \\ &\quad \left[\prod_{j=1}^{n}\left(1 - a_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(a_{\sigma(j)} + c_{\sigma(j)}\right)\right)^{\omega_j},\right. \\ &\quad \left.\left.\prod_{j=1}^{n}\left(1 - b_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(b_{\sigma(j)} + d_{\sigma(j)}\right)\right)^{\omega_j}\right]\right) \end{aligned} \tag{52}$$

$$\begin{aligned}\mathrm{IVIFIOWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \mathop{\otimes}_{j=1}^{n} \tilde{\alpha}_{\sigma(j)}^{\omega_j} \\ &= \left(\left[\prod_{j=1}^{n}\left(1 - c_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(a_{\sigma(j)} + c_{\sigma(j)}\right)\right)^{\omega_j},\right.\right. \\ &\quad \left.\prod_{j=1}^{n}\left(1 - d_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(b_{\sigma(j)} + d_{\sigma(j)}\right)\right)^{\omega_j}\right], \\ &\quad \left.\left[1 - \prod_{j=1}^{n}\left(1 - c_{\sigma(j)}\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - d_{\sigma(j)}\right)^{\omega_j}\right]\right), \end{aligned} \tag{53}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is the associated weight vector such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. σ: $(1, 2, \ldots, n) \to (1, 2, \ldots, n)$, $\tilde{\alpha}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_j$.

In addition, based on these four operators and the IVIFHA and IVIFHG operators, here we introduce two interval-valued intuitionistic fuzzy interactive hybrid operators:

Definition 3.27 Let IVIFIHA: $\tilde{\theta}^n \to \tilde{\theta}$. Suppose

$$\begin{aligned}\mathrm{IVIFIHA}_{w,\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) &= \omega_1 \dot{\tilde{\alpha}}_{\sigma(1)} \oplus \omega_2 \dot{\tilde{\alpha}}_{\sigma(2)} \oplus \cdots \oplus \omega_1 \dot{\tilde{\alpha}}_{\sigma(n)} \\ &= \left(\left[1 - \prod_{j=1}^{n}\left(1 - \dot{a}_{\sigma(j)}\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \dot{b}_{\sigma(j)}\right)^{\omega_j}\right],\right. \\ &\quad \left[\prod_{j=1}^{n}\left(1 - \dot{a}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(\dot{a}_{\sigma(j)} + \dot{c}_{\sigma(j)}\right)\right)^{\omega_j},\right. \\ &\quad \left.\left.\prod_{j=1}^{n}\left(1 - \dot{b}_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(\dot{b}_{\sigma(j)} + \dot{d}_{\sigma(j)}\right)\right)^{\omega_j}\right]\right), \end{aligned} \tag{54}$$

then the function IVIFIHA is called the interval-valued intuitionistic fuzzy interactive hybrid averaging (IVIFIHA) operator, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is the weight vector associated with the function IVIFHA, with $\omega_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$, $\dot{\tilde{\alpha}}_{\sigma(j)}$ is the j-th largest of the weighted IVIFNs $\dot{\tilde{\alpha}}_i$ $(1 = 1, 2, \cdots, n)$, here $\dot{\tilde{\alpha}}_i = nw_i\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots w_n)^{\mathrm{T}}$ is the weight vector of the IVIFNs $\tilde{\alpha}_i$ $(1 = 1, 2, \cdots, n)$, with $w_j \in [0, 1]$ $(j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient.

If we replace (54) with the following form:

$$\begin{aligned}\mathrm{IVIFIHG}_{w,\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \ddot{\tilde{\alpha}}_{\sigma(1)}^{\omega_1} \oplus \ddot{\tilde{\alpha}}_{\sigma(2)}^{\omega_2} \oplus \cdots \oplus \ddot{\tilde{\alpha}}_{\sigma(n)}^{\omega_n} \\ &= \left(\left[\prod_{j=1}^{n}\left(1 - c_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(a_{\sigma(j)} + c_{\sigma(j)}\right)\right)^{\omega_j},\right.\right. \\ &\quad \left.\prod_{j=1}^{n}\left(1 - d_{\sigma(j)}\right)^{\omega_j} - \prod_{j=1}^{n}\left(1 - \left(b_{\sigma(j)} + d_{\sigma(j)}\right)\right)^{\omega_j}\right], \\ &\quad \left.\left[1 - \prod_{j=1}^{n}\left(1 - c_{\sigma(j)}\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - d_{\sigma(j)}\right)^{\omega_j}\right]\right), \end{aligned} \tag{55}$$

then the function IVIFIHG is called the interval-valued intuitionistic fuzzy interactive hybrid geometric (IVIFIHG) operator, where $\ddot{\tilde{\alpha}}_{\sigma(j)}$ is the jth largest of the weighted IVIFNs $\dot{\tilde{\alpha}}_i$ $(1 = 1, 2, \ldots n)$, here $\ddot{\tilde{\alpha}}_i = \tilde{\alpha}_i^{nw_i}$ $(i = 1, 2, \ldots, n)$.

All the IVIFIWA, IVIFIWG, IVIFIOWA, IVIFIOWG, IVIFIHA and IVIFIHA operators satisfy commutativity, monotonicity, idempotency and boundedness. Furthermore, Li (2014) introduced some operations on the IVIFSs, such as Hamacher sum, Hamacher product, etc., and further developed the interval-valued intuitionistic fuzzy Hamacher correlated averaging (IVIFHCA) operator. Wang and Liu (2013) defined some Einstein operations on IVIFSs and developed three arithmetic averaging operators, i.e., the interval-valued intuitionistic fuzzy Einstein weighted averaging (IVIFWA ε) operator, the interval-valued intuitionistic fuzzy Einstein ordered weighted averaging (IVIFOWA ε) operator, and the interval-valued intuitionistic fuzzy Einstein hybrid weighted averaging (IVIFHWA ε) operator. Gu et al. (2014) developed the interval-valued intuitionistic fuzzy Einstein correlated averaging (IVIFECA) operator. The prominent characteristic of this operator is that it can not only consider the importance of the elements or their ordered positions, but also reflect the interaction phenomena among the decision-making attributes or their ordered positions.

3.7 Some operators for IVIFNs based on Karnik–Mendel algorithm

In most of the existing aggregation operators for IVIFNs, the weight vector of the aggregated arguments usually takes the form of real numbers. However, with the increasing complexity in many real decision-making situations, there are often some challenges for the decision-maker to provide precise weight information due to time pressure, lack of knowledge (or data) and the decision-maker's limited expertise about the problem domain. In other words, the weights are not necessarily real numbers and may be intervals, IFNs or IVIFNs. Chen et al. (2012) proposed the interval-valued intuitionistic fuzzy weighted average operator based on the traditional weighted average method and the Karnik–Mendel algorithms (2001). As we all know, both the arguments themselves and the ordered positions of the arguments are very important in the process of information aggregation. Wan and Dong (2014) proposed a new ranking method of IVIFNs based on the possibility degree from the probability viewpoint, and then developed the ordered weighted average operator and the hybrid weighted average operator for IVIFNs, respectively:

Definition 3.28 (Chen et al. 2012) Let $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$ be a collection of the IVIFNs, where $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i])$. If

$$Y_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \frac{\sum_{j=1}^n \omega_j \tilde{\alpha}_j}{\sum_{j=1}^n \omega_j}, \tag{56}$$

then we call Y the weighted average operator for the IVIFNs, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is the weight vector.

Definition 3.29 (Wan and Dong 2014) Let $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$ be a collection of the IVIFNs, where $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i]) = [[a_i, b_i], [1 - d_i, 1 - c_i]]$. If

$$Z_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \frac{\sum_{j=1}^n \omega_j \tilde{\alpha}_{\sigma(j)}}{\sum_{j=1}^n \omega_j}, \tag{57}$$

then we call Z the ordered weighted average operator for IVIFNs, where ω_j $(j = 1, 2, \ldots, n)$ are the weights which correlate with Z. $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{\alpha}_{\sigma(j-1)} \geq \tilde{\alpha}_{\sigma(j)}$ for any j.

Definition 3.30 (Wan and Dong 2014) Let $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$ be a collection of the IVIFNs, where $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i]) = [[a_i, b_i], [1 - d_i, 1 - c_i]]$. If

$$F_{w,\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \frac{\sum_{j=1}^n \omega_j \tilde{\alpha}'_{\sigma(j)}}{\sum_{j=1}^n \omega_j}, \tag{58}$$

then we call F the hybrid weighted average operator for IVIFNs, where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ is the weight vector which correlates with F, $\tilde{\alpha}'_{\sigma(j)}$ is the j-th largest number of IVIFNs $\tilde{\alpha}'_k$ $(k = 1, 2, \ldots, n)$, here $\tilde{\alpha}'_k = n w_k \tilde{\alpha}_k$, $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_1$ $(i = 1, 2, \ldots, n)$, satisfying that $0 \leq w_j \leq 1$ and $\sum_{j=1}^n w_j = 1$ $(j = 1, 2, \ldots, n)$, n is the balancing coefficient.

By changing the forms of weights ω_j $(j = 1, 2, \ldots, n)$, the weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ may take different forms, so these three operators can also be transformed into different forms as follows:

Case 1 If ω_j $(j = 1, 2, \ldots, n)$ are crisp values, then these three operators Y, Z and F are calculated as follows:

$$\begin{aligned} Y = &\left(\left[\frac{\sum_{j=1}^n \omega_j a_j}{\sum_{j=1}^n \omega_j}, \frac{\sum_{j=1}^n \omega_j b_j}{\sum_{j=1}^n \omega_j} \right], \right. \\ &\left. \left[\frac{\sum_{j=1}^n \omega_j (1 - d_j)}{\sum_{j=1}^n \omega_j}, \frac{\sum_{j=1}^n \omega_j (1 - c_j)}{\sum_{j=1}^n \omega_j} \right] \right) \\ &= \left[\left[y_1^L, y_1^U \right], \left[y_2^L, y_2^U \right] \right] = \left(\left[y_1^L, y_1^U \right], \left[1 - y_2^U, 1 - y_2^L \right] \right) \end{aligned} \tag{59}$$

$$\begin{aligned} Z = &\left(\left[\frac{\sum_{j=1}^n \omega_j a_{\sigma(j)}}{\sum_{j=1}^n \omega_j}, \frac{\sum_{j=1}^n \omega_j b_{\sigma(j)}}{\sum_{j=1}^n \omega_j} \right], \right. \\ &\left. \left[\frac{\sum_{j=1}^n \omega_j (1 - d_{\sigma(j)})}{\sum_{j=1}^n \omega_j}, \frac{\sum_{j=1}^n \omega_j (1 - c_{\sigma(j)})}{\sum_{j=1}^n \omega_j} \right] \right) \\ &= \left[\left[z_1^L, z_1^U \right], \left[z_2^L, z_2^U \right] \right] = \left(\left[z_1^L, z_1^U \right], \left[1 - z_2^U, 1 - z_2^L \right] \right) \end{aligned} \tag{60}$$

$$\begin{aligned} F = &\left(\left[\frac{\sum_{j=1}^n \omega_j a'_{\sigma(j)}}{\sum_{j=1}^n \omega_j}, \frac{\sum_{j=1}^n \omega_j b'_{\sigma(j)}}{\sum_{j=1}^n \omega_j} \right], \right. \\ &\left. \left[\frac{\sum_{j=1}^n \omega_j \left(1 - d'_{\sigma(j)}\right)}{\sum_{j=1}^n \omega_j}, \frac{\sum_{j=1}^n \omega_j \left(1 - c'_{\sigma(j)}\right)}{\sum_{j=1}^n \omega_j} \right] \right) \\ &= \left[\left[f_1^L, f_1^U \right], \left[f_2^L, f_2^U \right] \right] = \left(\left[f_1^L, f_1^U \right], \left[1 - f_2^U, 1 - f_2^L \right] \right) \end{aligned} \tag{61}$$

Case 2 If ω_j $(j = 1, 2, \ldots, n)$ are intervals, where $\omega_j = \left[\omega_j^L, \omega_j^U \right] (j = 1, 2, \ldots, n)$, then these three operators Y, Z and F are, respectively, expressed as:

$$\begin{aligned} Y = &\frac{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right] \tilde{\alpha}_j}{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right]} \\ = &\left[\frac{\sum_{j=1}^n [a_j, b_j] \left[\omega_j^L, \omega_j^U \right]}{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right]}, \frac{\sum_{j=1}^n [1 - d_j, 1 - c_j] \left[\omega_j^L, \omega_j^U \right]}{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right]} \right] \\ = &\left[\left[y_1^L, y_1^U \right], \left[y_2^L, y_2^U \right] \right] = \left(\left[y_1^L, y_1^U \right], \left[1 - y_2^U, 1 - y_2^L \right] \right) \end{aligned} \tag{62}$$

$$\begin{aligned} Z = &\frac{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right] \tilde{\alpha}_{\sigma(j)}}{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right]} = \left[\frac{\sum_{j=1}^n \left[a_{\sigma(j)}, b_{\sigma(j)} \right] \left[\omega_j^L, \omega_j^U \right]}{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right]}, \right. \\ &\left. \frac{\sum_{j=1}^n \left[1 - d_{\sigma(j)}, 1 - c_{\sigma(j)} \right] \left[\omega_j^L, \omega_j^U \right]}{\sum_{j=1}^n \left[\omega_j^L, \omega_j^U \right]} \right] = \left[\left[z_1^L, z_1^U \right], \left[z_2^L, z_2^U \right] \right] \\ &= \left(\left[z_1^L, z_1^U \right], \left[1 - z_2^U, 1 - z_2^L \right] \right) \end{aligned} \tag{63}$$

$$F = \frac{\sum_{j=1}^{n}\left[\omega_j^L,\omega_j^U\right]\tilde{\alpha}'_{\sigma(j)}}{\sum_{j=1}^{n}\left[\omega_j^L,\omega_j^U\right]} = \left[\frac{\sum_{j=1}^{n}\left[a'_{\sigma(j)},b'_{\sigma(j)}\right]\left[\omega_j^L,\omega_j^U\right]}{\sum_{j=1}^{n}\left[\omega_j^L,\omega_j^U\right]}, \frac{\sum_{j=1}^{n}\left[1-d'_{\sigma(j)},1-c'_{\sigma(j)}\right]\left[\omega_j^L,\omega_j^U\right]}{\sum_{j=1}^{n}\left[\omega_j^L,\omega_j^U\right]}\right] = \left[\left[f_1^L,f_1^U\right],\left[f_2^L,f_2^U\right]\right] = \left(\left[f_1^L,f_1^U\right],\left[1-f_2^U,1-f_2^L\right]\right) \tag{64}$$

Case 3 If $\omega_j\ (j=1,2,\ldots,n)$ are IFNs, where $\omega_j=\left(\varsigma_j,\tau_j\right)=\left[\varsigma_j,1-\tau_j\right]$, then these three operators Y, Z and F are, respectively, expressed as:

$$Y = \frac{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]\tilde{\alpha}_j}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]} = \left[\frac{\sum_{j=1}^{n}\left[a_j,b_j\right]\left[\varsigma_j,1-\tau_j\right]}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]}, \frac{\sum_{j=1}^{n}\left[1-d_j,1-c_j\right]\left[\varsigma_j,1-\tau_j\right]}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]}\right] = \left[\left[y_1^L,y_1^U\right],\left[y_2^L,y_2^U\right]\right] = \left(\left[y_1^L,y_1^U\right],\left[1-y_2^U,1-y_2^L\right]\right) \tag{65}$$

$$Z = \frac{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]\tilde{\alpha}_{\sigma(j)}}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]} = \left[\frac{\sum_{j=1}^{n}\left[a_{\sigma(j)},b_{\sigma(j)}\right]\left[\varsigma_j,1-\tau_j\right]}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]}, \frac{\sum_{j=1}^{n}\left[1-d_{\sigma(j)},1-c_{\sigma(j)}\right]\left[\varsigma_j,1-\tau_j\right]}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]}\right] = \left[\left[z_1^L,z_1^U\right],\left[z_2^L,z_2^U\right]\right] = \left(\left[z_1^L,z_1^U\right],\left[1-z_2^U,1-z_2^L\right]\right) \tag{66}$$

$$F = \frac{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]\tilde{\alpha}'_{\sigma(j)}}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]} = \left[\frac{\sum_{j=1}^{n}\left[a'_{\sigma(j)},b'_{\sigma(j)}\right]\left[\varsigma_j,1-\tau_j\right]}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]}, \frac{\sum_{j=1}^{n}\left[1-d'_{\sigma(j)},1-c'_{\sigma(j)}\right]\left[\varsigma_j,1-\tau_j\right]}{\sum_{j=1}^{n}\left[\varsigma_j,1-\tau_j\right]}\right] = \left[\left[f_1^L,f_1^U\right],\left[f_2^L,f_2^U\right]\right] = \left(\left[f_1^L,f_1^U\right],\left[1-f_2^U,1-f_2^L\right]\right) \tag{67}$$

Case 4 If $\omega_j\ (j=1,2,\ldots,n)$ are IVIFNs, where $\omega_j=\left(\left[\varsigma_j^L,\varsigma_j^U\right],\left[\tau_j^L,\tau_j^U\right]\right)=\left[\left[\varsigma_j^L,\varsigma_j^U\right],\left[1-\tau_j^U,1-\tau_j^L\right]\right]$, then these three operators Y, Z and F are, respectively, expressed as:

$$Y = \frac{\sum_{j=1}^{n}\left[\left[\varsigma_j^L,\varsigma_j^U\right],\left[1-\tau_j^U,1-\tau_j^L\right]\right]\tilde{\alpha}_j}{\sum_{j=1}^{n}\left[\left[\varsigma_j^L,\varsigma_j^U\right],\left[1-\tau_j^U,1-\tau_j^L\right]\right]} = \left[\frac{\sum_{j=1}^{n}\left[a_j,b_j\right]\left[\varsigma_j^L,\varsigma_j^U\right]}{\sum_{j=1}^{n}\left[\varsigma_j^L,\varsigma_j^U\right]}, \frac{\sum_{j=1}^{n}\left[1-d_j,1-c_j\right]\left[1-\tau_j^U,1-\tau_j^L\right]}{\sum_{j=1}^{n}\left[1-\tau_j^U,1-\tau_j^L\right]}\right] = \left[\left[y_1^L,y_1^U\right],\left[y_2^L,y_2^U\right]\right] = \left(\left[y_1^L,y_1^U\right],\left[1-y_2^U,1-y_2^L\right]\right) \tag{68}$$

$$Z = \frac{\sum_{j=1}^{n}\left[\left[\varsigma_j^L,\varsigma_j^U\right],\left[1-\tau_j^U,1-\tau_j^L\right]\right]\tilde{\alpha}_{\sigma(j)}}{\sum_{j=1}^{n}\left[\left[\varsigma_j^L,\varsigma_j^U\right],\left[1-\tau_j^U,1-\tau_j^L\right]\right]} = \left[\frac{\sum_{j=1}^{n}\left[a_{\sigma(j)},b_{\sigma(j)}\right]\left[\varsigma_j^L,\varsigma_j^U\right]}{\sum_{j=1}^{n}\left[\varsigma_j^L,\varsigma_j^U\right]}, \frac{\sum_{j=1}^{n}\left[1-d_{\sigma(j)},1-c_{\sigma(j)}\right]\left[1-\tau_j^U,1-\tau_j^L\right]}{\sum_{j=1}^{n}\left[1-\tau_j^U,1-\tau_j^L\right]}\right] = \left[\left[z_1^L,z_1^U\right],\left[z_2^L,z_2^U\right]\right] = \left(\left[z_1^L,z_1^U\right],\left[1-z_2^U,1-z_2^L\right]\right) \tag{69}$$

$$F = \frac{\sum_{j=1}^{n}\left[\left[\varsigma_j^L,\varsigma_j^U\right],\left[1-\tau_j^U,1-\tau_j^L\right]\right]\tilde{\alpha}_{\sigma(j)}}{\sum_{j=1}^{n}\left[\left[\varsigma_j^L,\varsigma_j^U\right],\left[1-\tau_j^U,1-\tau_j^L\right]\right]} = \left[\frac{\sum_{j=1}^{n}\left[a_{\sigma(j)},b_{\sigma(j)}\right]\left[\varsigma_j^L,\varsigma_j^U\right]}{\sum_{j=1}^{n}\left[\varsigma_j^L,\varsigma_j^U\right]}, \frac{\sum_{j=1}^{n}\left[1-d_{\sigma(j)},1-c_{\sigma(j)}\right]\left[1-\tau_j^U,1-\tau_j^L\right]}{\sum_{j=1}^{n}\left[1-\tau_j^U,1-\tau_j^L\right]}\right] = \left[\left[f_1^L,f_1^U\right],\left[f_2^L,f_2^U\right]\right] = \left(\left[f_1^L,f_1^U\right],\left[1-f_2^U,1-f_2^L\right]\right), \tag{70}$$

where y_1^L, y_1^U, y_2^L, y_2^U, z_1^L, z_1^U, z_2^L, z_2^U, f_1^L, f_1^U, f_2^L and f_2^U are calculated by the Karnik–Mendel algorithms (2001). All of these three operators satisfy commutativity, idempotency and boundedness.

3.8 Continuous interval-valued intuitionistic fuzzy aggregation operators

In order to aggregate the continuous interval values, Yager (2004) developed the OWA operator into the continuous OWA (C-OWA) operator, which can be defined as follows:

Definition 3.31 (Yager 2004) A C-OWA operator is a mapping $f: M \to R^+$ associated with a basic unit interval monotonic (BUM) function Q such that

$$f_Q(\tilde{a}) = f_Q\left(\left[\tilde{a}^L,\tilde{a}^U\right]\right) = \int_0^1 \frac{\mathrm{d}Q(y)}{\mathrm{d}y}\left(\tilde{a}-y\left(\tilde{a}^U-\tilde{a}^L\right)\right)\mathrm{d}y$$

, where $\tilde{a}=\left[\tilde{a}^L,\tilde{a}^U\right]\in M$, M is the set of all nonnegative interval numbers.

If $\lambda=\int_0^1 Q(y)\mathrm{d}y$ is the attitudinal character of Q, then a general formulation of $f_Q(\tilde{a})$ can be obtained as: $f_Q(\tilde{a})=f_Q(\left[\tilde{a}^L,\tilde{a}^U\right])=\lambda\tilde{a}^U+(1-\lambda)\tilde{a}^U$.

In order to deal with the continuous interval-valued intuitionistic fuzzy information, Zhou et al. (2014)

introduced some continuous interval-valued intuitionistic fuzzy aggregation operators, including the continuous interval-valued intuitionistic fuzzy ordered weighted averaging (C-IVIFOWA) operator, the weighted continuous interval-valued intuitionistic fuzzy ordered weighted averaging (WC-IVIFOWA) operator, the ordered weighted continuous interval-valued intuitionistic fuzzy ordered weighted averaging (OWC-IVIFOWA) operator, and the combined continuous interval-valued intuitionistic fuzzy ordered weighted averaging (CC-IVIFOWA) operator. Firstly, we introduce the C-IVIFOWA operator:

Definition 3.32 (Zhou et al. 2014) A C-IVIFOWA operator is a mapping g: $\sum^n \to \Omega$ associated with the BUM function Q, such that

$$g_Q(\tilde{\alpha}) = \left(\mu_{g_Q(\tilde{\alpha})}, v_{g_Q(\tilde{\alpha})}\right) = (f_\lambda([a_{\tilde{\alpha}}, b_{\tilde{\alpha}}]), f_\lambda([c_{\tilde{\alpha}}, d_{\tilde{\alpha}}]))$$

, where $\tilde{\alpha} = (\tilde{\mu}_{\tilde{\alpha}}, \tilde{v}_{\tilde{\alpha}}) = ([a_{\tilde{\alpha}}, b_{\tilde{\alpha}}], [c_{\tilde{\alpha}}, d_{\tilde{\alpha}}]) \in \Sigma$, f_Q is the C-OWA operator. Q is a BUM function Q: $[0,1] \to [0,1]$, which is monotonic with $Q(0) = 0$ and $Q(1) = 1$. If $\lambda = \int_0^1 Q(y)\mathrm{d}y$ is the attitudinal character of Q, then $g_Q(\tilde{\alpha}) = (\lambda b_{\tilde{\alpha}} + (1-\lambda)a_{\tilde{\alpha}}, \lambda d_{\tilde{\alpha}} + (1-\lambda)c_{\tilde{\alpha}})$, and $g_Q(\tilde{\alpha})$ is an IFN.

It is obvious that the C-IVIFOWA operator can aggregate the IVIFNs, but this operator is applied only to one IVIFN rather than a series of IVIFNs. Therefore, it is necessary for us to extend the C-IVIFOWA operator to accommodate the aggregation of a sequence of IVIFNs. Then, Zhou et al. (2014) developed some extended continuous interval-valued intuitionistic fuzzy aggregation operators, including the WC-IVIFOWA operator, the OWC-IVIFOWA, and the CC-IVIFOWA operator:

Definition 3.33 (Zhou et al. 2014) A WC-IVIFOWA operator of dimension n is a mapping g': $\sum^n \to \Omega$ which is defined by an associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^{\mathrm{T}}$ satisfying $\omega_i \in [0,1]$ and $\sum_{i=1}^n \omega_i$, such that

$$g'_\lambda(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus}_{i=1}^{n} \omega_i g_\lambda(\tilde{\alpha}_i)$$
$$= \left(1 - \prod_{i=1}^n (1 - f_\lambda(\tilde{\mu}_{\tilde{\alpha}_i}))^{\omega_i}, \prod_{i=1}^n (f_\lambda(\tilde{v}_{\tilde{\alpha}_i}))^{\omega_i}\right), \quad (71)$$

where λ is the attitudinal character of the BUM function Q, and

$$\tilde{\alpha}_i = \left(\tilde{\mu}_{\tilde{\alpha}_i}, \tilde{v}_{\tilde{\alpha}_i}\right) = ([a_{\tilde{\alpha}_i}, b_{\tilde{\alpha}_i}], [c_{\tilde{\alpha}_i}, d_{\tilde{\alpha}_i}]) \in \Sigma$$

.

Based on Definition 3.33, the OWC-IVIFOWA operator can be shown as follows:

Definition 3.34 (Zhou et al. 2014) A OWC-IVIFOWA operator of dimension n is a mapping g'': $\sum^n \to \Omega$, such that

$$g''_\lambda(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus}_{i=1}^{n} \omega_i g_\lambda\left(\tilde{\alpha}_{\sigma(i)}\right)$$
$$= \left(1 - \prod_{i=1}^n \left(1 - f_\lambda\left(\tilde{\mu}_{\tilde{\alpha}_{\sigma(i)}}\right)\right)^{\omega_i}, \prod_{i=1}^n \left(f_\lambda\left(\tilde{v}_{\tilde{\alpha}_{\sigma(i)}}\right)\right)^{\omega_i}\right), \quad (72)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$.

Finally, the CC-IVIFOWA operator is an extension of the WC-IVIFOWA and OWC-IVIFOWA operators:

Definition 3.35 (Zhou et al. 2014) A CC-IVIFOWA operator of dimension n is a mapping g''' : $\sum^n \to \Omega$, such that

$$g'''_\lambda(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \mathop{\oplus}_{i=1}^{n} \omega_i G_\lambda\left(\tilde{\alpha}_{\sigma(i)}\right) \quad (73)$$

where $(\sigma(1), \sigma(2), , \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $G_\lambda\left(\tilde{\alpha}_{\sigma(i-1)}\right) \geq G_\lambda\left(\tilde{\alpha}_{\sigma(i)}\right)$ for all i. $G_\lambda\left(\tilde{\alpha}_{\sigma(i)}\right)$ is the i-th largest value of $(nw_1 g_\lambda(\tilde{\alpha}_1), nw_2 g_\lambda(\tilde{\alpha}_2), \ldots, nw_n g_\lambda(\tilde{\alpha}_n))$ and $g_\lambda(\tilde{\alpha}_i)$ $(i = 1, 2, \ldots, n)$ are confirmed by Eqs. (71) or (72), where n is the balanced factor.

The C-IVIFOWA, WC-IVIFOWA, OWC-IVIFOWA and CC-IVIFOWA operators satisfy commutativity, idempotency and boundedness. Here we utilize a figure (Fig. 1) to show the relations among the above operators:

4 The applications about the aggregation operators of IVIFNs

In this section, we provide a survey of some applications of the interval-valued intuitionistic fuzzy aggregation techniques and methods in the fields of different kinds of decision-making [such as multi-criteria decision-making (MCDM), multi-attribute decision-making (MADM) and group decision-making (GDM), etc.], entropy measures, supplier selection and some practical decision-making problems as emergency risk management (ERM), software quality evaluation, material selection, investment company selection, and supply chain management, etc.

4.1 Decision-making

Considering that these aggregation operators discussed above are classified by different forms, such as generalized forms, induced forms, Choquet integral forms, continuous

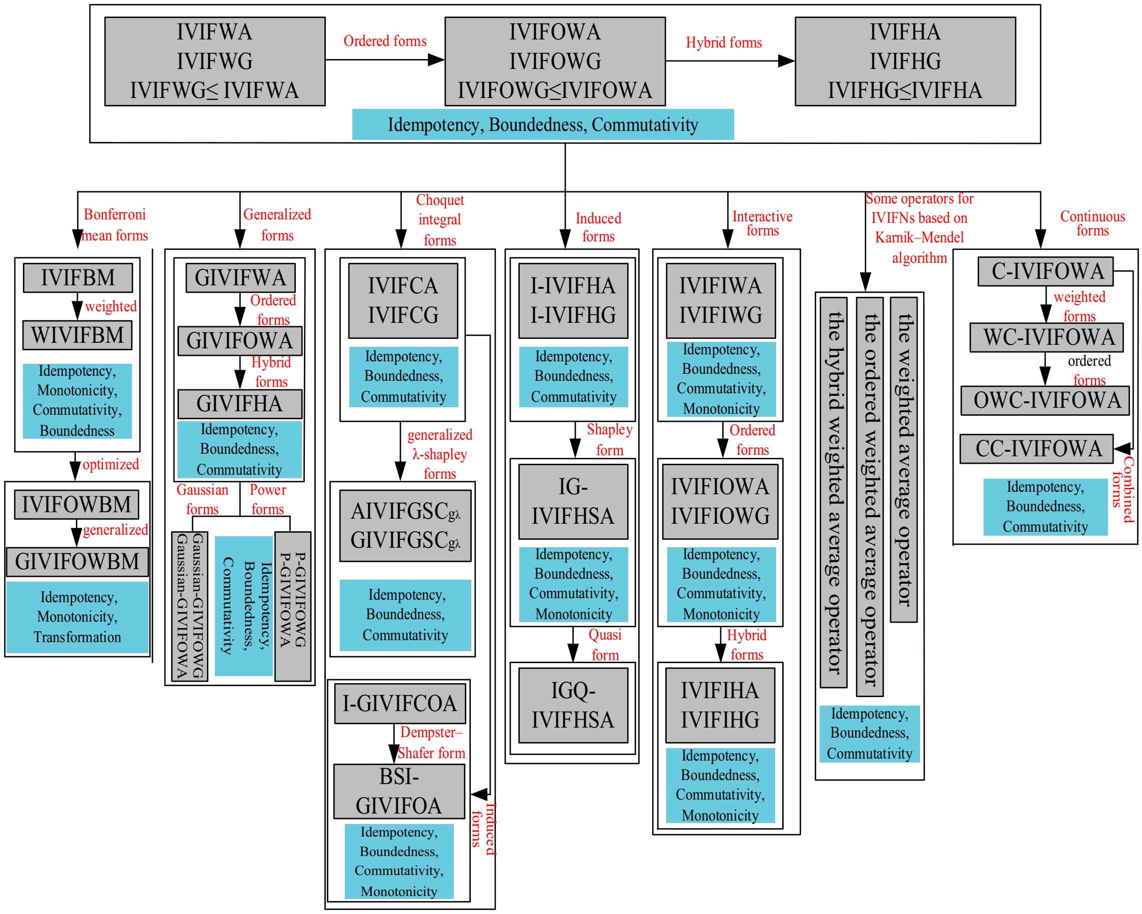

Fig. 1 The relations among main operators

forms, and some other operational laws, etc. Here we also summarize the decision-making applications based on these forms.

Xu and Chen (2007a, b) and Xu (2007b) developed some basic interval-valued intuitionistic fuzzy operators, such as the IVIFWA, IVIFWG, IVIFOWA, IVIFOWG, IVIFHA and IVIFHG operators, and then applied these operators to solve some the MADM and GDM problems involving the prioritization of a set of information technology improvement projects, etc. Based on the IVIFHG and IVIFWG operators, Liu et al. (2014) proposed the interval-valued intuitionistic fuzzy principal component analysis (IVIF-PCA) model to deal with the complex multi-attribute large-group decision-making (CMALGDM) problems. Moreover, Ye (2009) introduced the interval-valued intuitionistic fuzzy weighted arithmetic average operator and the interval-valued intuitionistic fuzzy weighted geometric average operator. To identify the best alternatives in the MCDM problems, a multi-criteria fuzzy decision-making method was established in which the decision values are IVIFSs. Later, Chen et al. (2012) proposed a new method for MCDM, which can overcome the drawback of Ye's method. Motivated by these basic operators and generalized forms, lots of operators have been developed and applied to different situations: Zhao et al. (2010) developed the GIVIFWA and GIVIFHA operators and applied them to MADM with interval-valued intuitionistic fuzzy information. Moreover, Qi et al. (2013) proposed the Gaussian-GIVIFOWA, the Gaussian-GIVIFOWG, the P-GIVIFOWA and the P-GIVIFOWG operators. They applied these proposed operators to exploitation investment evaluation of tourist spots. Additionally, Liu et al. (2015) proposed some generalized Einstein aggregation operators based on the IVIFNs, such as the IVIFGEWA, IVIFGEOWA and IVIFGE-HWA operators. Furthermore, the method for the multi-attribute group decision-making (MAGDM) problems based on these operators was developed. Meng et al. (2013a, 2015a) introduced the GBIVIFGC, AIVIFGSC$_{g\lambda}$ and GIVIFGSC$_{g\lambda}$ operators. Then they developed some approaches to multi-criteria group decision-making (MCGDM) under interval-valued intuitionistic fuzzy environment. He et al. (2013) developed a series of generalized interval-valued intuitionistic fuzzy power aggregation operators. Based on these operators, some MAGDM approaches were given. Chen (2014) presented an interval-valued intuitionistic fuzzy prioritized aggregation operator to aggregate the interval-valued intuitionistic fuzzy ratings of the alternatives with respect to the prioritized criteria and developed a prioritized aggregation operator-based approach to handle the MCDM problems. Li and Deng (2014) pointed out a shortcoming in the interval-valued intuitionistic fuzzy prioritized weighted average (IVIFPWA) operator when dealing with GDM in some extreme situations and provided an improvement of making slight adjustment on initial evaluations to overcome the shortcoming.

Similarly, considering these basic operators and the induced forms, some scholars have extended a series of the induced operators and given their applications to decision-making: Zhang and Qi (2012) proposed the I-IVIFHA and I-IVIFHG operators, and extended these operators to form a novel GDM method for selecting the most desirable alternative in multi-attribute multi-interest group decision-making problems with the attribute values and the decision-makers' preference values taking the form of IVIFNs. Moreover, Meng et al. (2013b) used the Shapley function to propose an IG-IVIFHSA operator, and developed an approach to MADM under interval-valued intuitionistic fuzzy environment. Based on the Choquet integral and the generalized Shapley function, Meng et al. (2014) introduced the IGS-IVIFHCAA and IGS-IVIFHCGM operators, and developed an approach to MADM under interval-valued intuitionistic fuzzy environment. Additionally, Wei and Yi (2008) proposed the I-IVIFOWG, IVIFOWGA and IVIFHGA operators. Based on these operators, some approaches have been developed to solve the MAGDM problems under the interval-valued intuitionistic fuzzy environment. Meng et al. (2015b) defined the induced generalized Shapley interval-valued intuitionistic uncertain linguistic hybrid Choquet averaging (IGS-IVIULHCA) operator and investigated interval-valued intuitionistic uncertain linguistic MAGDM with incomplete weight information and interactive conditions.

Besides, by defining some operations and considering some other situations, some operators and their applications have been put forward. For example, Wang and Liu (2013) defined some Einstein operations and developed the IVIFWA, IVIFOWA and IVIFHWA operators, and then applied the IVIFHWA operator into MADM. Based on these Einstein operations, Gu et al. (2014) developed the IVIFECA operator and applied it to the MADM problems for selecting the construction project with interval-valued intuitionistic fuzzy information. Based on the Karnik–Mendel algorithms, Wan and Dong (2014) defined the ordered weighted average operator and the hybrid weighted average operator for IVIFNs, and applied these two operators to solve the MAGDM problems with IVIFNs. Considering the continuous situation, Zhou et al. (2014) introduced the C-IVIFOWA, WC-IVIFOWA, OWC-IVIFOWA and CC-IVIFOWA operators, and applied these operators to GDM which is to study a human resource management problem where a university wants to introduce the oversea outstanding teachers. Wang et al. (2015) proposed some operations and a possibility degree of trapezium clouds, as well as several new aggregation operators. Moreover, a method based on trapezium clouds and IVIFNs was presented, which can provide solutions to MCGDM. Wu and Su (2015) developed an IVIF-PHWA operator-based MADM method to solve the decision-making problems in interval-valued intuitionistic fuzzy environment. Zhao and Xu (2014) presented some new synthesized interval-valued intuitionistic fuzzy aggregation operators. An approach to GDM based on interval-valued intuitionistic preference relations by using these proposed operators was developed, and a MCDM method was then established in which the criterion values for alternatives are IVIFSs.

4.2 Entropy and cross-entropy measures

Here we mainly discussed some entropy and cross-entropy measures for decision-making with interval-valued intuitionistic fuzzy environment:

Zhao et al. (2013) extended the VIKOR method based on cross-entropy for MCGDM with interval-valued intuitionistic fuzzy environment. Jin et al. (2014) introduced the interval-valued intuitionistic fuzzy continuous weighted entropy and established an approach to MCGDM. One emergency risk management (ERM) evaluation was provided to illustrate the application of the developed approach. Ye (2011) utilized the IVIFWA operator to aggregate the interval-valued intuitionistic fuzzy information corresponding to each alternative. Then, with the ideal and anti-ideal information measures of cross-entropy of IVIFSs, an objective function was constructed to derive the optimal evaluation for the weight of each alternative and used to an investment company for investing a sum of money in the best option. Wei and Zhang (2015) proposed two entropy measures based on the cosine function for IVIFSs. These entropy measures were applied to assess the experts' weights and to solve the MCGDM problems. Gupta et al. (2015) introduced a cross-entropy for IVIFNs and applied it to deal with the MCGDM problem.

4.3 Supplier selection

Under the interval-valued intuitionistic fuzzy environment, many scholars have studied the supplier selection examples by developing different decision-making methods. For example, Xu and Shen (2014) proposed a new outranking choice method for MCGDM and gave an illustration of the proposed method using a practical supplier selection example. A soft computing model for the MAGDM problems was presented by Yue and Jia (2013), which aggregates all individual decisions on an attribute into an IVIFN and one practical for supplier selection was given. Moreover, Chen et al. (Ciucci 2016) developed a method to tackle the MCGDM problems in the context of IVIFSs. An illustrative supplier selection problem was used to demonstrate how to apply the proposed approach and to observe the computational consequences resulting from various aggregation operators. Xiao and Wei (2008), and Zhang et al. (2013) investigated the MADM problems to deal with the supplier selection in supply chain management and with completely unknown attribute weights in the framework of IVIFS, respectively.

4.4 Some other practical applications

In this subsection, we consider some other practical applications, such as emergency risk management (ERM), software quality evaluation, material selection, investment company selection, and supply chain management, etc. Based on the induced forms, Yang and Yuan (2014) defined the I-IVIFEOWG and I-IVIFEHG operators to deal with the MADM problems under interval-valued intuitionistic fuzzy environments. Based on them, a method was given to deal with software quality evaluation. Additionally, Cai and Han (2014) introduced the I-IVIFEOWA operator and applied it to MADM based on the data mining, one illustrative example about selecting an ERP system was given to verify the developed approach. Considering the Bonferroni mean, Xu and Chen (2011) defined the IVIFBM and the WIVIFBM. Based on which, a procedure was given for MCDM under interval-valued intuitionistic fuzzy environments. The school of management in a Chinese university can select overseas outstanding teachers based on this procedure. Under the situation of Gaussian and power operators, Qi et al. (2013) proposed the Gaussian-GIVIFOWA, Gaussian-GIVIFOWG, P-GIVIFOWA and P-GIVIFOWG operators. They applied these proposed operators to exploitation investment evaluation of tourist spots. Based on the Choquet integral, Xu (2010) proposed the IVIFCA and IVIFCG operators to aggregate interval-valued intuitionistic fuzzy information, and applied them to a practical decision-making problem involving the prioritization of information technology improvement projects. Yu (2014) established a decision-making model for evaluating hydrogen production technologies in China based on the IVIFIOWA and IVIFIOWG operators.

By establishing some Hamacher operations, Li (2014) developed the IVIFHCA operator, and then applied to MADM with interval-valued intuitionistic fuzzy information. A practical example for evaluating the enterprise financial performance was used to illustrate the developed procedures. Similarly, based on the Einstein operations, Chen et al. (2015) developed the dependent interval-valued intuitionistic fuzzy Einstein ordered weighted average (DIVIFEOWA) operator, and then applied them to develop some approaches for MADM with IVIFNs, an illustrative example for evaluating the computer network security was given to verify the developed approach and to demonstrate its practicality and effectiveness. Gou et al. (2015) also defined two exponential operational laws of IVIFNs as well as discussed the interval-valued intuitionistic fuzzy weighted exponential aggregation (IVIFWEA) operator and the dual interval-valued intuitionistic fuzzy weighted exponential aggregation (DIVIFWEA) operator. Meanwhile, they applied these two operators to handle a practical MADM problem involving the choice of the optimal powered roof support for coal extraction with a high recovery rate. Yue (2014) developed an approach for aggregating the attribute values in MAGM with IVIFNs. A practical example was given for assessing the masses' satisfaction with respect to their leading group at Chinese universities. Chen and Li (2013) presented a new method for students' answerscripts evaluation based on IVIFSs, where the fuzzy marks awarded to the answers of students' answerscripts were represented by IVIFSs. They also presented a generalized method for students' answerscripts evaluation, which can overcome the drawbacks of the existing methods for students' answerscripts evaluation. Chen and Chiou (2015) proposed a new MADM method based on IVIFSs, particle swarm optimization (PSO) techniques, and the evidential reasoning methodology. The proposed method uses the evidential reasoning methodology to construct objective functions of the programming models and uses the PSO techniques to get the optimal weights of the attributes and the aggregated IVIFN of each alternative.

5 Further research directions

In spite of the fruitful research results, lots of work on interval-valued intuitionistic fuzzy information needs to be done in the future:

Firstly, some new aggregation operators can be derived by combining fuzzy measure theory, graph theory or dynamic programming theory, and the Hamacher, Einstein and interactive operational laws.

Moreover, some novel methods for the decision-making problems with interval-valued intuitionistic fuzzy information can be established, such as the alternative queuing methods, the interactive decision-making methods, and the methods based on preference relations, etc.

Finally, the applications in other practical fields, such as medical management based on big data, artificial intelligence, data mining, machine learning, etc., are also the interesting and important research subjects in the future. After doing so, we can develop a much more complete knowledge system of interval-valued intuitionistic fuzzy aggregation theory and applications.

6 Conclusions

In this paper, we have mainly provided a survey of the aggregation techniques of interval-valued intuitionistic fuzzy information, and their applications in various fields. Firstly, some useful aggregation operators have been discussed. Then, we have made the classifications for the applications of these aggregation operators and methods based on the different kinds of decision-making (MADM, MCDM and GDM, etc.), entropy measures, supplier selection, and some other practical applications. Finally, we have pointed out some possible directions for future research.

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