AN OVERVIEW OF MORPHOLOGICAL FILTERING

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Abstract

This paper consists in a tutorial overview of morphological filtering, a theory introduced in 1988 in the context of mathematical morphology. Its first section is devoted to the presentation of the lattice framework. The emphasis is put on the lattices of numerical functions in digital and continuous spaces. The basic filters, namely the openings and the closings, are then described and their various versions are listed. In the third section, morphological filters are defined as increasing idempotent operators, and their laws of composition are proved. The last sections are concerned with two special classes of filters and their derivations: first, the alternating sequential filters allow one to bring into play families of operators depending on a positive scale parameter. Finally, the center and the toggle mappings modify the function under study by comparing it, at each point, with a few reference transforms.

1 Mathematical morphology for complete lattices

1.1 Introduction

We define a morphological filter as an operator ψ, acting on a complete lattice $\mathcal{T}$ [4], which:

(i) preserves the ordering $\leq$ of $\mathcal{T}$, i.e.,

$$X \leq Y \Longrightarrow \psi(X) \leq \psi(Y), \qquad X, Y \in \mathcal{T}$$

(ii) is idempotent, i.e.,

$$\psi(\psi(X)) = \psi(X), \qquad X \in \mathcal{T}.$$

The first condition, which is called *growth* or *increasingness,* just means that, since we deal with lattices, we decide to focus on the transformations ψ which preserve one of the basic lattice features (just as, in vector spaces, one pays attention to the operators which commute with addition and scalar product, e.g., convolutions). The second condition responds to the fact that an increasing operation is non reversible and looses information. Idempotence stops such a simplifying action at its first stage.

The above axiomatics, due to Serra in 1982 [24], did not arise at once, *ex nihilo.* It results from a long series of interactions between theory and practice. In addition, it has opened the way to new types of filters that could probably never have been discovered through experimentation. The shematic diagram shown in Fig. 1 gives an overview of the connections between ideas and of their chronology.

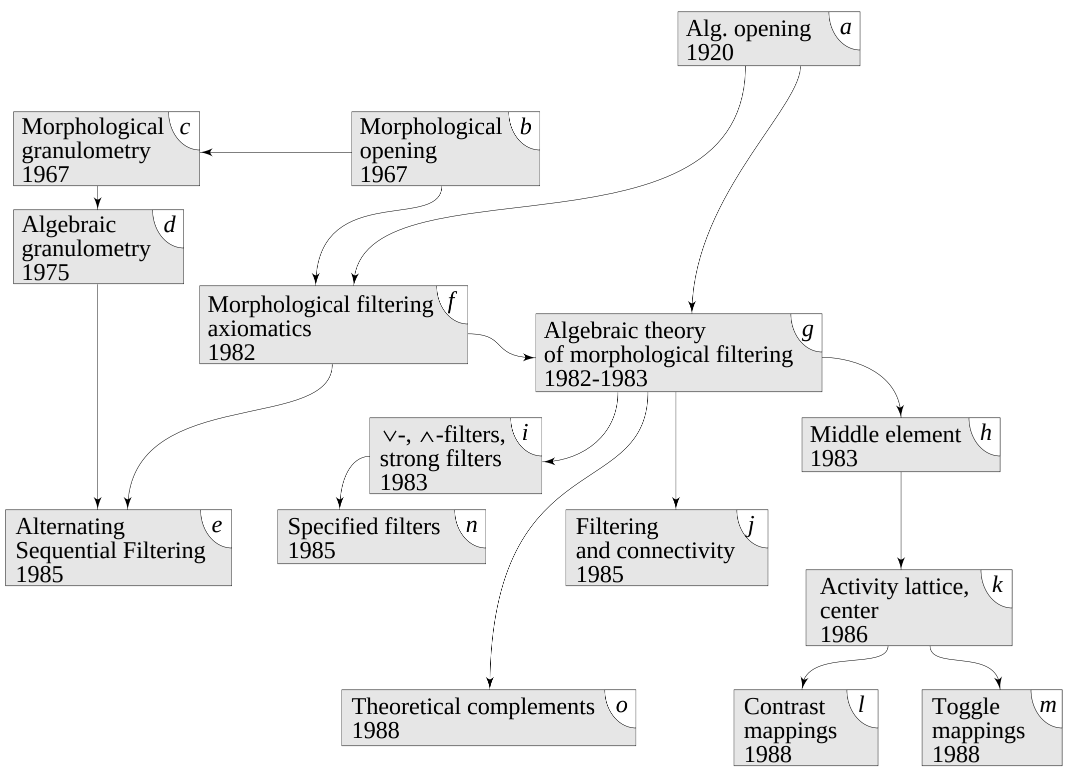


Figure 1: Milestones for the morphological filtering theory : Moore : a ($\approx$ 1920) ; Matheron : b, c [14] ; d [15] ; g, h, i, j [25] ; Serra : e, f, j, k [25] ; l [17] ; m [27] ; Sternberg : e [30] ; Meyer : l [17] ; Ronse and Heijmans : o [21] ; specified filters : rank opening (Ronse [20]), segment based filters (Meyer and Serra [17]) ; orientation dependent filtering (Kurdy and Jeulin [9]) ; filters for graphs (Vincent [31]) ; multispectral filtering (Vitria [35]) ; etc...

During the late seventies, practitioners generalized in two ways the concept of a morphological opening (or closing), initially designed for sets. Firstly, they applied it to functions (Meyer, Sternberg) and to planar graphs (Lantuéjoul). Secondly, they started composing closings with openings. The first extension led to base the theory on complete lattices, whereas the second one is at the origin of the morphological filtering axiomatics, proposed in 1982. The same year and one year later, Matheron established a series of major theoretical results : lattice of the filters, $\vee$- and $\wedge$-filters, strong filters, middle element, the four envelopes..., and extended these results to increasing (but not necessarily idempotent) operators. The part of Matheron's theory which is presented in this overview corresponds to § 3.

Matheron's concept of middle element suggested to Serra in 1986 the ideas of morphologiocal center and activity lattice [25, chapter 8]. A further step led the same author to leave increasingness and to introduce the notion of toggle operations [27]. This approach allows in particular to associate *optimization* criteria to morphological transformations. It is presented below in § 5.

Several other pieces of theory exist, that we shall not develop here. We may quote, among others, the relationships between filtering and connectivity preservation (Matheron and Serra, [25, chapter 7]), an instructive approach to multispectral filtering due to Vitria [35] and the properties of sequential monotone convergence established by Heijmans and Serra [7]. The pedagogical purpose of this document imposes to keep down such derivations and to prune the text of many technicalities. For the same reason, a special effort has been put on figures and on practical comments.

1.2 Algebraic framework of the complete lattices

Inclusion is a set oriented notion. The scenes under study may be modeled by sets, but also by grey-tone functions, by multi-spectral functions, by graphs, each of them acting either on the Euclidean space $\mathbb{R}^n$ or on digital ones, like $\mathbb{Z}^n$. All these situations share a common denominator formed by the two ideas which define the notion of a **complete lattice** $\mathcal{T}$ [4], namely:

1. there exists a partial ordering $\geq$ over $\mathcal{T}$,
2. for any (finite or infinite) family (A_i) in $\mathcal{T}$, there exists:
   - a smallest majorant $\vee A_i$ called the "sup" (for *supremum*),
   - a largest minorant $\wedge A_i$ called the "inf" (for *infimum*).

In particular, $\mathcal{T}$ posesses a greatest element, E, and a smallest one, $\emptyset$. In a lattice, any logical consequence of a choice of ordering remains true when we commute the symbols $\vee$ and $\wedge$, and $\leq$ and $\geq$. This is called the principle of **duality with respect to the order.**

Here is now a review of a few basic lattices:

1.2.1 Boolean lattices

Start from an arbitrary set E. Obviously, the set $\mathcal{P}(E)$ of the subsets of E, which is ordered for the inclusion relationship, is a complete lattice for the operations $\cup$ (union) and $\cap$ (intersection). Moreover, with each $X \in \mathcal{P}(E)$, there exists a unique $X^C \in \mathcal{P}(E)$, called the *complement* of X, such that:

$$X \cap X^C = \emptyset \qquad \text{and} \qquad X \cup X^C = E. \tag{1}$$

Finally, $\mathcal{P}(E)$ also satisfies the important property of **general distributivity** under which, for all $Y \in \mathcal{P}(E)$ and any family (X_i) of elements of $\mathcal{P}(E)$, we have:

$$(\bigcup X_i) \cap Y = \bigcup (X_i \cap Y), \tag{2}$$

$$(\bigcap X_i) \cup Y = \bigcap (X_i \cup Y). \tag{3}$$

1.2.2 Topological lattices

When E is a topological space, its open sets generate a complete lattice for the inclusion, where the sup coincides with the union and where $\inf(X_i)$ is the interior of $\bigcap X_i$. This lattice is not complemented. It satisfies the general distributivity of the type (2), but finite distributivity only of the type (3). Indeed, in the general case of an infinite family (X_i), we have

$$
\begin{aligned}
(\bigcup X_i) \cap Y &= \bigcup (X_i \cap Y), && (4)\\
\text{but only} \quad \overbrace{(\bigcap X_i) \cup Y}^{\circ} &= \overbrace{\bigcap (X_i \cup Y)}^{\circ}. && (5)
\end{aligned}
$$

Similar structures are derived for the *closed* sets and the *compact* sets.

1.2.3 The convex lattice

The class of the convex sets of the Euclidean space $\mathbb{R}^n$ generates a complete lattice where the inf coincides with intersection and where the sup is the *convex hull.*

1.2.4 The partition lattice

In the set of the partitions of an arbitrary set E, we can introduce the following ordering: a partition A is smaller than a partition B when each class of A is included in a class of B. This leads to a lattice which is complete, but neither complemented nor distributive.

1.2.5 Function lattices

Let E be an arbitrary space. The class $\mathcal{F}$ of the "extended" real valued functions $f : E \longrightarrow \overline{\mathbb{R}}$ is obviously ordered by the relation: $f \leq g$ if for each $x \in E$, $f(x) \leq g(x)$ and constitutes a complete lattice. The sup and the inf are given by the relationships:

$$
\begin{array}{rclcrcll}
f & = & \vee f_i & \Longleftrightarrow & f(x) & = & \sup f_i(x), & \forall x \in E, \\
f' & = & \wedge f_i & \Longleftrightarrow & f'(x) & = & \inf f_i(x), & \forall x \in E.
\end{array} \qquad (6)
$$

The lattice is completely distributive but not complemented. Rel. (6) implies that $f(x)$ may equal $+\infty$. However, if we want to restrict ourselves to bounded functions, it suffices to remark that the previous lattice is isomorphic (by anamorphosis) to

- either the class $\mathcal{F}'$ of the non negative functions $f : E \longrightarrow [0, +\infty]$,
- or the class $\mathcal{F}''$ of the functions $f : E \longrightarrow [0, 1]$.

Functions and umbrae: Is it possible to identify the function lattice $\mathcal{F}$ with the **set** class of the associated subgraphs, or umbrae? Remember that with every function $f : E \longrightarrow \overline{\mathbb{R}}$ (and more generally with every set in $E \times \overline{\mathbb{R}}$, see Fig. 2), we can associate the two sets $U^+(f)$ and $U^-(f)$ of $E \times \overline{\mathbb{R}}$ defined by the relations:

$$
\begin{aligned}
U^+(f) &= \{(x, z) \in E \times \overline{\mathbb{R}}, f(x) \leq z\}, && (7)\\
U^-(f) &= \{(x, z) \in E \times \overline{\mathbb{R}}, f(x) < z\}. && (8)
\end{aligned}
$$

Clearly, every umbra comprised between $U^+(f)$ and $U^-(f)$ generates the same function f. Moreover, the associated ordering relations are equivalent, since

$$
f \leq g \quad \Longleftrightarrow \quad U^+(f) \subseteq U^+(g) \quad \Longleftrightarrow \quad U^-(f) \subseteq U^-(g).
$$

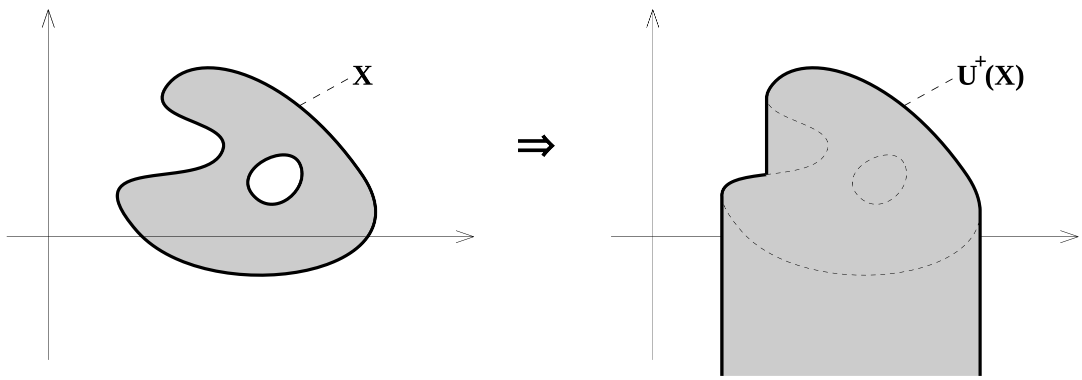


Figure 2: Umbra $U^+(X)$ of a set $X \subset \mathbb{R} \times \overline{\mathbb{R}}$.

But this does not mean that umbrae and functions are interchangeable. For example, consider the *threshold mapping* defined as follows:

$$[\psi(f)](x) = \begin{cases} f(x) & \text{when } f(x) \geq 1, \\ -\infty & \text{otherwise.} \end{cases} \tag{9}$$

This operation is shown on Fig. 3.

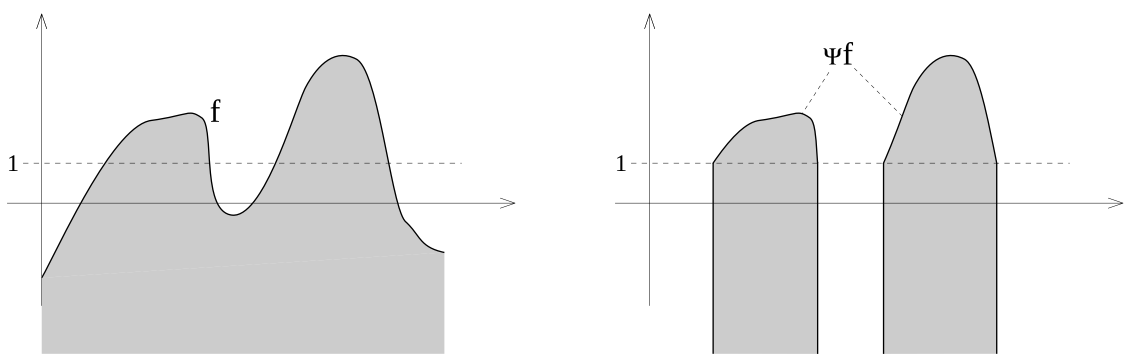


Figure 3: The threshold mapping ψ.

In set terms, the transformation ψ consists in intersecting the umbra $U^+(f)$ by the closed half space

$$E_1 = \{(x, z), x \in E, z \geq 1\},$$

and in taking the upper umbra of the result:

$$U^+(\psi(f)) = U^+[E_1 \cap U^+(f)] \cup E_{-\infty}. \tag{10}$$

If functions and upper umbrae are equivalent, then the two algorithms (9) and (10) must yield the same result. Let's apply both of them to the sup of the following family (see Fig. 4):

$$\begin{cases} f_i(x) &= 1 - 1/i & \text{when } |x| \leq 1, \\ f_i(x) &= -\infty & \text{otherwise.} \end{cases}$$

If the sup f of this family is understood in the sense of the function lattice, it is equal to:

$$\begin{cases} f(x) &= 1 & \text{when } |x| \leq 1, \\ f(x) &= -\infty & \text{otherwise,} \end{cases}$$

and according to the rel. (9), $\psi f = f$. But if the sup is understood in the umbrae lattice, we get the following set :

$$\bigcup_i U^+(f_i),$$

the umbra of the function $f = \sup_i f_i$ in this lattice. Then, from rel. (10), we derive $U^+(\psi(f)) = E_{-\infty}$, i.e. $\forall x \in E, \psi f(x) = -\infty$. In other words, in the Euclidean case, the function lattice and the set oriented lattice of umbrae are not equivalent at all. Nevertheless, in the discrete case of functions $f : \mathbb{Z}^n \longrightarrow \mathbb{Z}$, the two approaches coincide and one can transpose the way of reasoning from sets to functions.

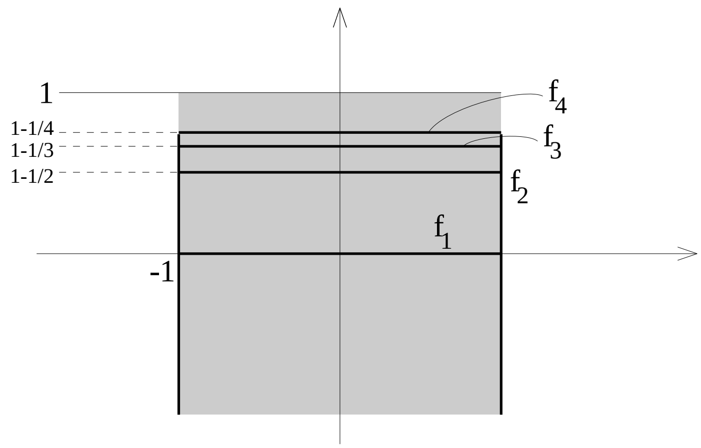


Figure 4: The family of functions (f_i).

Functions and stacks of sets Instead of associating with function $f : E \longrightarrow \overline{\mathbb{R}}$ its set oriented umbra, one can alternatively consider the stack of its horizontal sections $X_z(f)$:

$$X_z(f) = \{x, f(x) \geq z\}.$$

As threshold level z increases, $X_z(f)$ decreases *continuously,* i.e.,

$$X_z(f) = \bigcap_{z'<z} X_{z'}(f) \quad \text{or} \quad X_z(f) = \downarrow X_{z'}(f).$$

Conversely, it is easy to see that every continuously decreasing family $\{X_z\}_{z\in\overline{\mathbb{R}}}$ of sets generates a unique function f, by the algorithm:

$$f(x) = \sup\{z, x \in X_z\}.$$

Here, the monotonic continuity $\downarrow$ is essential (see a counter example in [24, page 427]). It is because of it that we have to find classes of transformations which preserve the $\downarrow$-continuity for some families of stacks of sets. These transformations will be the upper semi-continuous mappings, and these families, the upper semi-continuous functions.

1.2.6 Lattices of semi-continuous functions

A function f is *upper semi-continuous* (u.s.c.) when its umbra $U^+(f)$ is a closed set in $E \times \overline{\mathbb{R}}$. It is *lower semi-continuous* (l.s.c.) when the umbra $U^-(f)$ is an open set in $E \times \overline{\mathbb{R}}$. Moreover, every continuously decreasing stack of closed sets generates an upper semi-continuous function, and conversely. The use of semi-continuity becomes **strictly** necessary as soon as extrema are involved, at least in continuous cases. For example, could we extract the maxima of the following function in $\mathbb{R}$ (see Fig. 5):

$$f(x) = \begin{cases} 1 - x^2 & \text{when } 0 < |x| < 1, \\ 0 & \text{when } |x| \geq 1 \text{ or } x = 0 \end{cases} \qquad ?$$

Actually, the maximum of such a function, although it is bounded, **does not exist.** Conversely, as soon as we refer to the "maximum" of a function over a continuous space, we implicitely introduce the requirement that it is u.s.c. (or l.s.c. when looking for minima). Therefore, in § 3.6, we assume that the functions under study are lower semi continuous.

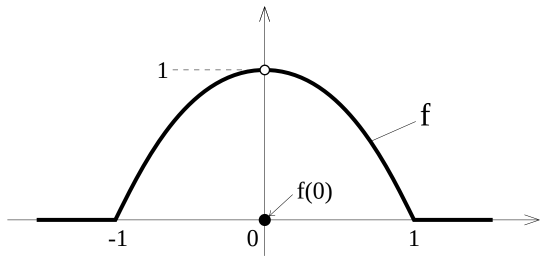


Figure 5: A function without a maximum (it is not u.s.c.).

In the lattice $\mathcal{F}_u$ of the u.s.c. functions, we have:

$$\begin{aligned}\inf_i f_i &= \{f \in \mathcal{F}_u, U^+(f) = \bigcap_i U^+(f_i)\},\\ \sup_i f_i &= \{f \in \mathcal{F}_u, U^+(f) = \overline{\bigcup_i U^+(f_i)}\},\end{aligned}$$

a notation which shows that the lattice $\mathcal{F}_u$ and that of the closed upper umbrae are *isomorphic* (in this case, the identification between sets and functions works.). The use of upper semi-continuous functions is required not only when maxima are studied, but also when one wants to use upper semi-continuous transformations, i.e., when one wants to make sure that every $\downarrow$-continuous stack of sets is transformed into a $\downarrow$-continuous stack of sets. In particular, this property is strictly needed in § 1.5.4 to generate planar increasing mappings.

1.3 Erosion, dilation

In the same way that linear image processing puts the emphasis on the transformations that commute with addition, morphology naturally stresses the transformations that commute with the sup or, by duality, with the inf. This results in the following definition:

Definition 1.1 *Let $\mathcal{T}$ be a complete lattice. The mappings from $\mathcal{T}$ into itself which commute with the* sup *(resp. the* inf*) are called dilations δ (resp. erosions ε):*

$$\delta(\vee X_i) = \vee\delta(X_i), \qquad \varepsilon(\wedge X_i) = \wedge\varepsilon(X_i), \qquad X_i \in \mathcal{T}, \tag{11}$$

with in particular $\delta(\emptyset) = \emptyset$ and $\delta(E) = E$.

The following theorem (see [25, page 24]) characterizes these operations:

Theorem 1.2 *Let $\mathcal{T}$ be a complete lattice. The classes of the dilations and of the erosions on $\mathcal{T}$ are two complete isomorphic lattices, which correspond to one another through the duality relation:*

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y), \qquad X, Y \in \mathcal{T}, \tag{12}$$

To each dilation δ corresponds a unique erosion ε:

$$\varepsilon(X) = \vee\{B \in \mathcal{T}, \delta(B) \leq X\} \tag{13}$$

and to each erosion ε corresponds a unique dilation δ:

$$\delta(X) = \wedge\{B \in \mathcal{T}, \varepsilon(B) \geq X\}. \tag{14}$$

Not only dilations and erosions are themselves increasing mappings, they generate two comprehensive classes of increasing mappings. Indeed, we have the following theorem [25, page 20]:

Theorem 1.3 *Any mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$ such that $\psi(E) = E$ is increasing if and only if it can be written as*

$$\psi = \vee\{\varepsilon_B, B \in \mathcal{T}\}, \tag{15}$$

with the erosions ε_B given by

$$\varepsilon_B(X) = \begin{cases} \psi(B) & \text{if } X \geq B, \\ \emptyset & \text{otherwise.} \end{cases}$$

(dual result for the dilation.)

1.4 Increasing mappings on boolean lattices

In this section and in the following, we would like to compare the two lattices which model the binary and the grey-tone images.

The first one is the boolean lattice $\mathcal{P}(E)$, where E is $\mathbb{R}^n$ or $\mathbb{Z}^n$ for example. We can look at mappings from $\mathcal{P}(E)$ into itself as extensions of mappings from E into $\mathcal{P}(E)$. In the following, lower-case letters such as x, y, a, b denote elements of E, or points, and capital letters denote elements of $\mathcal{P}(E)$. A point $x \in E$, when considered as an element of $\mathcal{P}(E)$, is written as $\{x\}$. The letter δ denotes the mapping $E \longrightarrow \mathcal{P}(E)$, which generates a dilation, as well as the dilation from $\mathcal{P}(E)$ into itself. We define a **structural mapping**[1] on $\mathcal{P}(E)$ as any mapping $\delta : E \longrightarrow \mathcal{P}(E)$. Then, we have [25, page 41]:

Theorem 1.4 *Let E be an arbitrary set. The datum of any mapping $\delta : E \longrightarrow \mathcal{P}(E)$ is equivalent to that of a dilation from $\mathcal{P}(E)$ into itself, again symbolized by δ, and defined by the relation*

$$\delta(X) = \bigcup_{x \in X} \delta(x), \qquad X \in \mathcal{P}(E). \tag{16}$$

Conversely, any dilation of $\mathcal{P}(E)$ into itself determines a unique structural mapping $\delta : E \longrightarrow \mathcal{P}(E)$.

1.4.1 The three dualities

In any boolean algebra $\mathcal{P}(E)$, the duality w.r. to the complementation associates with each mapping ψ the operation $\psi^* = \Theta\psi\Theta$, where Θ designates the complement operator, as expressed by

$$\forall X \in \mathcal{P}(E), \qquad \psi^*(X) = [\psi(X^C)]^C.$$

In the case of the dilation δ, we find

$$\delta^*(X) = [\bigcup_{x \in X^C} \delta(x)]^C = \bigcap_{x \in X^C} [\delta(x)]^C. \tag{17}$$

δ^*, which obviously commutes with the inf, is an erosion. $\delta^*(X)$ consists of the points that are not *descendant* from any point in the complement of X (that are not included in any $\delta(x)$ when $x \in X^C$), i.e:

1. those whose ancestors are all included in X,
2. those that do not have ancestors (a fixed part S, which remains the same for any set X).

[1] *Structural mappings* are often referred to in literature as *structuring functions ;* further to a reviewer's remark, this terminology was abandoned here since it induces a confusion with *grey-level structuring element.* The term "structural mapping" was suggested by P. Maragos.

We form another duality notion by operating on the structural mapping with the **transposition** $\delta \longmapsto \check{\delta}$, i.e:

$$\check{\delta}(x) = \{y \in E, x \in \delta(y)\}.$$

The transpose $\check{\delta}(x)$ of $\delta(x)$ is made of the set of points from which x descends, hence $\check{\check{\delta}} = \delta$. The structural mapping $\check{\delta}$ generates the dilation $\check{\delta}$:

$$\check{\delta}(X) = \{y \in E, \delta(y) \cap X \neq \emptyset\}. \tag{18}$$

From the two relations (17) and (18), we derive the links between these two dualities, and the basic one, namely $\delta \leftrightarrow \varepsilon$ (see rel. (12)):

$$\varepsilon = (\check{\delta})^* \qquad \check{\varepsilon} = \delta^* \qquad \varepsilon^* = \check{\delta}. \tag{19}$$

1.4.2 Translation invariance

We now assume that E is equipped with a translation (e.g. $E = \mathbb{Z}^n$ or $E = \mathbb{R}^n$), and that the dilation δ is translation invariant, i.e. is a t-dilation. In other words, the structural mapping $\delta(h)$ at point h is deduced from that of the origin o (denoted $\delta(o) = B$) by translation: $\delta(h) = B_h = \{B + h, b \in B\}$. The set B is called structuring element. We see from relation (16) that

$$\begin{aligned}
\delta(X) &= X \oplus B = B \oplus X \\
&= \bigcup_{x \in X} B_x \\
&= \{b + x, x \in X, b \in B\} \\
&= \bigcup_{b \in B} X_b.
\end{aligned} \tag{20}$$

The t-dilation δ is classically known as **Minkowski addition** between sets X and B. By duality under complementation, it gives

$$\delta^*(X) = \check{\varepsilon}(X) = \bigcap_{b \in B} X_b = X \ominus B \tag{21}$$

and by lattice duality:

$$\varepsilon(X) = \bigcap_{b \in \check{B}} X_b = X \ominus \check{B}, \qquad \text{with } \check{B} = \{-b, b \in B\}. \tag{22}$$

Both operations are Minkowski subtractions of X by B and $\check{B}$ respectively. According to a classical result due to G. Matheron, any increasing t-mapping is a union of t-erosions, and also an intersection of t-dilations [15, page 221]. More precisely,

Theorem 1.5 *Let ψ be a translation invariant increasing mapping. Then, for any $X \in \mathcal{P}(E)$,*

$$\psi(X) = \bigcup\{X \ominus \check{B}, B \in \mathcal{P}(E), o \in \psi(B)\} = \bigcap\{X \oplus \check{B}, B \in \mathcal{P}(E), o \in \psi^*(B)\}.$$

1.4.3 Examples

We will now present some examples of dilations and erosions in the lattice of sets $\mathcal{P}(\mathbb{R}^2)$. Fig. 6 shows the dilation of a set X by a bipoint B, i.e. the set $X \oplus \check{B}$. Similarly, Fig. 7 illustrates the effect of an erosion of X by a segment S. On Fig. 8, the same set is dilated and eroded by a disc D (*Euclidean* dilation and erosion). The dilation by a disc is then compared, on Fig. 9, to that by an hexagon H of similar size. One can remark that many parts on the boundary of $X \oplus \check{H}$ are parallel to the vertices of H.

Lastly, Fig. 10 illustrates the algorithm which is used for performing a *geodesic* dilation of a set Y inside a set X (see § 2.4.1).

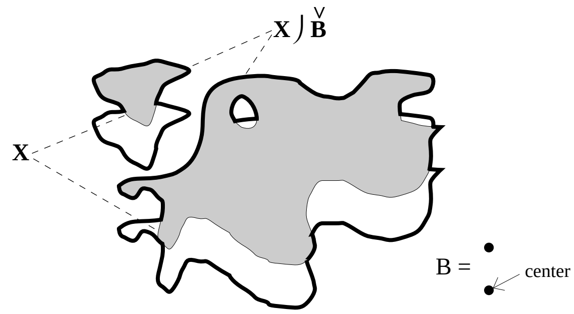


Figure 6: Dilation of a set X by a bipoint $\check{B}$.

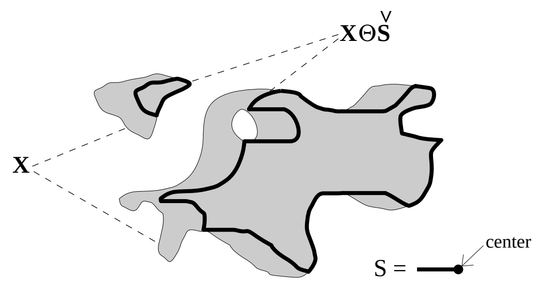


Figure 7: Erosion of a set X by a segment $\check{S}$.

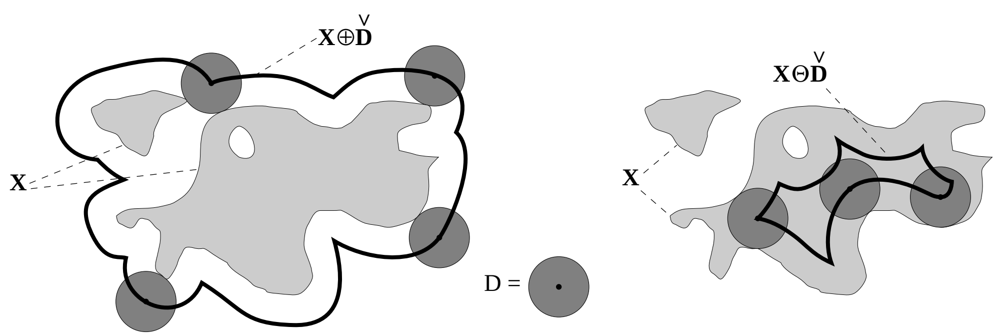


Figure 8: Dilation and erosion of X by a disc $D = \check{D}$.

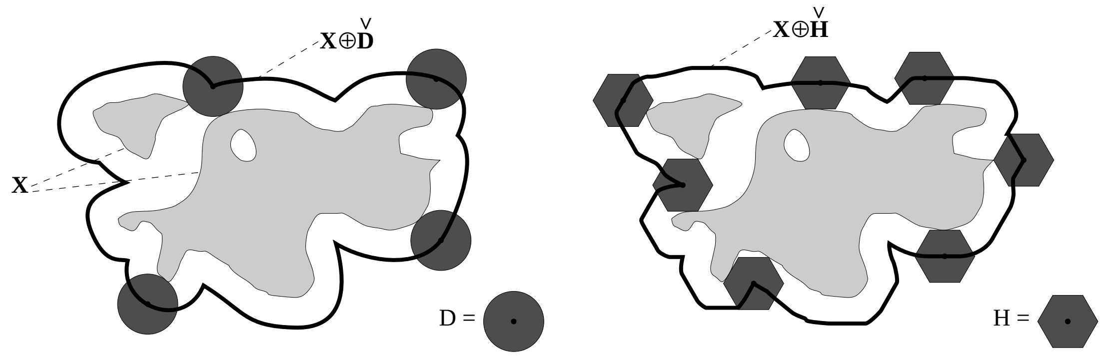


Figure 9: Comparison between the dilations of X by a disc and by an hexagon. Note that these structuring elements are symmetrical: $D = \check{D}$ and $H = \check{H}$.

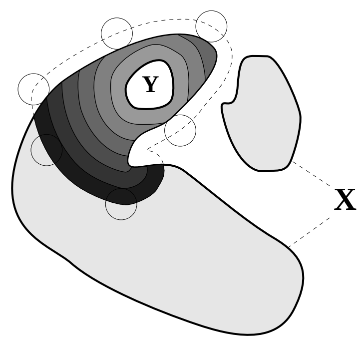


Figure 10: Successive geodesic dilations of set Y inside set X.

1.5 Increasing mappings on function lattices

The lattice $\mathcal{F}(E, \overline{\mathbb{R}})$ of the functions $f : E \longrightarrow \overline{\mathbb{R}}$ shares several properties with the previous one, but it differs from $\mathcal{P}(E)$ by two major aspects:

1. it is not complemented,
2. when additions or subtractions are involved, they may lead to indetermination, of the type $+\infty - \infty$, since the range of variation is $\overline{\mathbb{R}}$.

We will now study $\mathcal{F}(E, \overline{\mathbb{R}})$ by following the same plan as for $\mathcal{P}(E)$.

1.5.1 Generation of dilations from structural mapping

Call *impulse* $u_{h,z}$ a function whose value is z at point $h \in E$, and $-\infty$ elsewhere [6]:

$$\forall x \in E, \quad u_{h,z}(x) = \begin{cases} z & \text{when } x = h, \\ -\infty & \text{otherwise.} \end{cases}$$

The class $\mathcal{I}(E)$ of the impulses is equivalent to that of the points $(h, z) \in E \times \overline{\mathbb{R}}$. Clearly, any function $f \in \mathcal{F}(E, \overline{\mathbb{R}})$ is the sup of upper bounded impulses smaller than itself (just as a set is the union of the points it contains):

$$f = \sup\{u_{h,z}, h \in E, z < f(h)\}.$$

Introduce now a structural mapping on $\mathcal{F}(E, \overline{\mathbb{R}})$ as any upper bounded mapping $\delta : \mathcal{I}(E) \longrightarrow \mathcal{F}(E, \overline{\mathbb{R}})$. We then have [25, page 185]:

Theorem 1.6 *any structural mapping is equivalent to a dilation from $\mathcal{F}(E, \overline{\mathbb{R}})$ into itself, defined by the relation*

$$\delta(f) = \sup\{\delta(u_{h,z}), h \in E, z < f(h)\}. \tag{23}$$

Conversely, any dilation $\delta : \mathcal{F}(E, \overline{\mathbb{R}}) \longrightarrow \mathcal{F}(E, \overline{\mathbb{R}})$ induces a **unique** *structural mapping obtained by restricting δ to $\mathcal{I}(E)$.*

1.5.2 Dualities

The transposition duality extends immediately to functions, by replacing points by impulses. The duality with respect to the complementation is replaced by all those given by the relation

$$\psi^*(f) = m - \psi(m - f), \tag{24}$$

as m spans the class of the real numbers. In practice, ψ often commutes with vertical shifts, i.e. $\psi(f + m) = \psi(f) + m$. Then, all the relations (24) are equal to $\psi^*(f) = -\psi(-f)$, and the three expressions (19) extend to functions.

1.5.3 Translation invariances

We can consider either a translation operation t'_h, by vector $h \in E$, or a translation operation $t_{h,z}$ by a vector $(h,z) \in E \times \overline{\mathbb{R}}$. The two corresponding formulas are:

$$\begin{aligned} (t_{h,z} f)(x) &= f(x-h)+z, \\ (t'_h f)(x) &= f(x-h). \end{aligned}$$

We shall focus on the t-invariant mappings, which are the most useful in practice. Saying that dilation δ is invariant with respect to translations is equivalent to saying that the *structural mapping* δ is the same everywhere, i.e. if $g = \delta u_{0,0}$ is the transform of the origin-impulse $u_{0,0}$, then $\forall x \in E, \delta u_{h,z}(x) = g(x-h)+z$. Then, the expression (23) of the dilation δ takes the following simpler form:

$$(\delta f)(x) = \sup\{g(x-h)+z, z < f(h), h \in E\}, \qquad f \in \mathcal{F}(E, \overline{\mathbb{R}}).$$

Note that the operand $g(x-h)+z$ cannot take the undetermined form $+\infty - \infty$ since, for all h, x and z, each of the two numbers $g(x-h)$ and z is $< +\infty$. Hence, we have finally

$$(\delta f)(x) = \sup\{g(x-h)+f(h), h \in E\}, \qquad f \in \mathcal{F}(E, \overline{\mathbb{R}}). \tag{25}$$

The two dual erosions ε and $\check{\varepsilon}$ of δ are given by the following formulae:

$$\begin{aligned} (\varepsilon f)(x) &= \inf\{g(x+h) - f(h), h \in E\}, \qquad (26) \\ (\check{\varepsilon} f)(x) &= \inf\{g(x-h) - f(h), h \in E\}. \end{aligned}$$

Similarly to theorem 1.5, any increasing mapping $\psi : E \longrightarrow \overline{\mathbb{R}}$ which is t-invariant may be decomposed into a sup of erosions as well as into an inf of dilations (same proof as for theorem 1.5).

1.5.4 Planar increasing mappings

Generally speaking, the representation of a function $f : E \longrightarrow \overline{\mathbb{R}}$ by the stack of its sections allows one to generate a function transformation on f from every set transformation $\psi : E \longrightarrow E$. It suffices to put

$$X_z(\psi(f)) = \bigcap_{z' < z} \psi[X_{z'}(f)].$$

Then, one speaks of *planar transformation,* or again of *stack transformation.* Obviously, the family $X_z(\psi(f))$ is continuously decreasing with z. Hence, it generates a function. This technique has been used for example in [24, page 451] to derive the thickening for functions from that for sets.

However, it is for increasing mappings that the planar increasing transformations have been the more extensively studied. A basic reference is here Serra [24, pages 426–434]. One can refer as well to [25, chapter 9] or to Maragos and Schafer [13]. See also Sternberg [30], Meyer [16], Rosenfeld [22] and Yli-Harja, Astola and Neuvo [37]. Here, in the continuous case, we have to assume that ψ is *upper semi-continuous,* which implies that it preserves the $\downarrow$-continuity, i.e.,

$$\psi[X_z(f)] = \bigcap_{z' < z} \psi[X_{z'}(f)].$$

Then, the definition of the planar mapping ψ leads to:

$$X_z[\psi(f)] = \psi[X_z(f)]. \tag{27}$$

This relation is illustrated by Fig. 11.

In other words, the planarity of the mapping ψ allows us to process f threshold by threshold. A series of results derive directly from the key relation (27), namely:

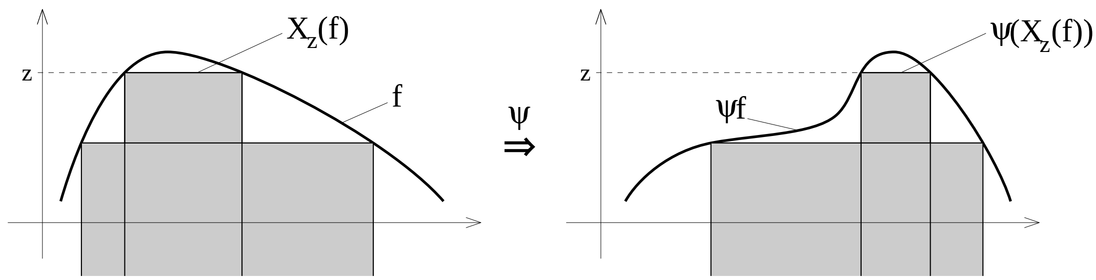


Figure 11: An example of a planar increasing mapping.

Theorem 1.7 *Let ψ be an upper semi-continuous planar increasing mapping from $\mathcal{F}(E, \overline{\mathbb{R}})$ into itself. Then:*

1. *the class of step functions with k levels (k arbitrary) is closed under ψ,*
2. *ψ commutes with anamorphosis.*

An anamorphosis is a strictly increasing and continuous point mapping $s : \overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$. For example, if $s = \exp$, then for any planar increasing mapping ψ, we have:

$$\psi \exp(f) = \exp \psi(f), \qquad f \in \mathcal{F}(E, \overline{\mathbb{R}}),$$

i.e. "vertical" and "horizontal" dimensions are treated **independently.**

A basic example of a planar increasing mapping consists in the dilation of an upper semi-continuous function f by a function b which is equal to 0 on its compact support B and to $-\infty$ elsewhere. In this case, relation (25) becomes:

$$(\delta f)(x) = \sup\{f(x-h), h \in B\} = (f \oplus B)(x). \tag{28}$$

By duality w.r. to lattice, relation (26) yields:

$$(\varepsilon f)(x) = \inf\{f(x+h), h \in B\} = (f \ominus \check{B})(x). \tag{29}$$

These operations are referred to as planar dilations and erosions, or as dilations and erosions with respect to a *planar structuring element* B. An example of such transformations is shown in Fig. 12.

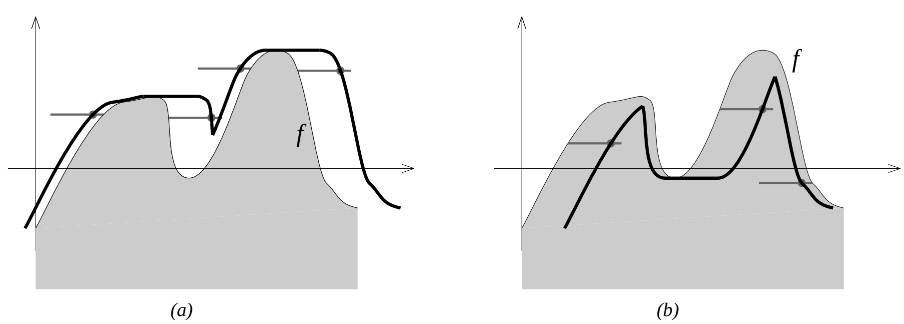


Figure 12: Dilation (a) and erosion (b) of a function f by a planar structuring element B.

Comments

1. In the example of rel. (28), all the horizontal cross-sections are processed by the same operator, namely the dilation by B. Such a vertical invariance is not compulsory, and a stack increasing operator may

well depend on the level z. It just has to fulfill the condition that the ψ_z's associated with each level z decrease continuously, i.e., that for every closed set X, we have:

$$\psi_z(X) = \bigcap_{z'<z} \psi_{z'}(X).$$

As an example, start from a *threshold mapping* ψ_{z_0} similar to that defined in rel. (9) and illustrated in Fig. 3:

$$[\psi_{z_0}(f)](x) = \begin{cases} f(x) & \text{when } f(x) \geq z_0, \\ -\infty & \text{otherwise.} \end{cases}$$

Consider now two circular openings (see § 2.1) γ_1 and γ_2, with $\gamma_2 \geq \gamma_1$ (γ_2 is *less severe* than γ_1). Apply γ_1 to the initial function f and γ_2 to the "threshold version" $\psi_{z_0}(f)$, and take their sup:

$$\gamma(f) = \gamma_1(f) \vee \gamma_2(\psi_{z_0}(f)).$$

The operator γ, which is still an opening, turns out to be a sort of *Dolby* filter for images: it removes the small details when they are dark enough, but preserves them otherwise.

2. Among all the image transformations presented here, the flat mappings are the only ones to preserve the *physical dimensionality* of the imags under study. Physically speaking, the "vertical" axis is rarely homogeneous with the "horizontal" ones: the latter represent the space $[L^2]$ or $[L^3]$ whereas the former is more like the physical intensity $[I]$ of light, electricity, strength, etc... When we dilate such a function by a cuboctahedron, for example, we loose the physical meaning of the axis, and the z-level of the transform no longer represents an intensity $[I]$. On the contrary, the flat mappings preserve this physical meaning.

   Note that the converse of this last proposition is false: a notion such as the maximum involves a pile of successive sections, since a connected component C_t in the cross-section X_t is a maximum iff for all $t' > t$, $X_{t'} \cap C_t = \emptyset$.

1.6 Digital implementations

In this section, we concentrate upon implementations of t-dilations (and t-erosions), which are the basic stones for building up more sophisticated algorithms.

When the dilation is planar, it is produced for functions in the same way as for sets. One has merely to replace union by sup and intersection by inf (e.g. refer to relation (28)). When the dilation is not planar, one can scan the successive levels of the structuring function (i.e., the grey-level structuring element), or use Steiner decomposition. In both cases, we shall use the following notation:

$$[f \oplus (\ 1 \quad \mathbf{0}\ )](x) = \sup\{f(x+1)+1, f(x)\}.$$

The number associated with each point denotes the altitude of the corresponding structuring function (here a function whose support is reduced to an horizontal doublet). When needed, a bold character is used for indicating the location of the origin. The elementary "spherical"—and centered—structuring functions are:

- the cube: 9 pixels, on two successive levels
- the octahedron: 5 pixels, on two successive levels
- the rhombododecahedron: 9 pixels, on two successive levels
- the cuboctahedron: 9 pixels, on two successive levels.

They are represented on Fig. 13.

The elementary *rhombododecahedron* R can be represented (as in Fig. 14) by taking the spacing of the horizontal square grid to be $\sqrt{2}$.

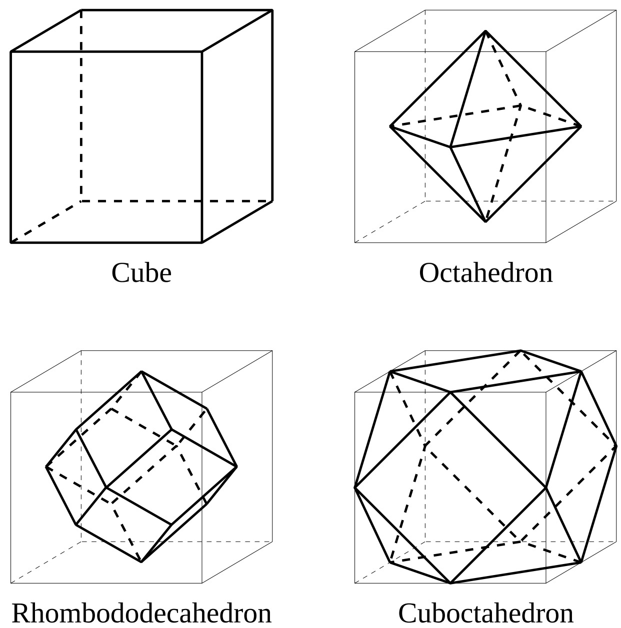


Figure 13: Basic spherical shapes in $\mathbb{Z}^3$. The plane $z = 0$ corresponds to the median horizontal section of the cube. The structuring functions derive from these sets by taking their umbrae.

$$\begin{array}{ccccc} . & . & . & . & . \\ . & 0 & 1 & 0 & . \\ . & 1 & 2 & 1 & . \\ . & 0 & 1 & 0 & . \\ . & . & . & . & . \end{array} \quad = \quad (\,\mathbf{1} \quad 0\,) \oplus (\,0 \quad \mathbf{1}\,) \oplus \begin{pmatrix} \mathbf{-1} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \mathbf{-1} \end{pmatrix}$$

Figure 14: The elementary rhombododecahedron R (i.e. Steiner rhomb of $\mathbb{Z}^3$) and its decomposition in four dilations by segments. The complex shape of the polyhedron has been decomposed into four simple structuring functions, whose implementation is very simple and extremely efficient.

The Steiner rhomb kR of size k is obtained by taking k dilations of R:

$$kR = \underbrace{R \oplus R \oplus \ldots \oplus R}_{k \text{ times}} = R^{\oplus k}.$$

As k increases, the difference between the Steiner rhomb and the ball becomes more apparent, but it is a simple matter to combine R with other Steiner polyhedra, such as the cuboctahedron, or simply with another Steiner rhomb, R^*, that is constructed at 45° to the first one (exactly as we construct octagons in $\mathbb{Z}^2$). This possibility is illustrated in Fig. 15.

$$\underbrace{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{array}}_{R} \oplus \underbrace{\begin{array}{ccccc} & & 0 & & \\ & 1 & . & 1 & \\ 0 & . & 2 & . & 0 \\ & 1 & . & 1 & \\ & & 0 & & \end{array}}_{R^*} = \underbrace{\begin{array}{ccccccc} & & 0 & 1 & 0 & & \\ & 1 & 2 & 2 & 2 & 1 & \\ 0 & 2 & 3 & 3 & 3 & 2 & 0 \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 0 & 2 & 3 & 3 & 3 & 2 & 0 \\ & 1 & 2 & 2 & 2 & 1 & \\ & & 0 & 1 & 0 & & \end{array}}_{R \oplus R^*}$$

Figure 15: Dilation of the Steiner rhomb R by R^* (R^* is obtained in the same way as R, by four dilations by doublets).

The elementary *cuboctahedron* C does not lead to a sequence of segments. It has the decomposition shown in Fig. 16 (with the horizontal spacing being again equal to $\sqrt{2}$).

$$\underbrace{\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}}_{C} = \underbrace{\begin{array}{ccc} 0 & . & 0 \\ . & . & . \\ 0 & . & 0 \end{array}}_{C_0} \cup \underbrace{\begin{array}{ccc} . & 1 & . \\ 1 & 1 & 1 \\ . & 1 & . \end{array}}_{C_1}$$

Figure 16: The elementary cuboctahedron C and its decomposition into two successive horizontal planes.

To dilate a function f by C, it suffices to perform

$$f_1 = f \oplus \begin{pmatrix} 0 & . & 0 \\ . & . & . \\ 0 & . & 0 \end{pmatrix}, \qquad f_2 = f \oplus \begin{pmatrix} . & 0 & . \\ 0 & 0 & 0 \\ . & 0 & . \end{pmatrix}$$

and then to compute the sup between f_1 and $f_2 + 1$:

$$f \oplus C = \sup\{f_1, f_2 + 1\}.$$

2 Openings and closings

This section is a go-between from dilations to morphological filtering. Here, the two basic references are [25, chapters 7 & 17] and [15, chapters 1 & 5]. We shall see how, by looking for an *inverse* to the dilation—i.e. for an impossibility—we find a new operation, the *morphological closing,* whose three basic properties are extremely useful. We shall then try and keep these properties as axioms for the general concept of an (algebraic) closing. The notion of an *opening* is introduced by duality. It satisfies two of the three basic properties of the closing, that will become the two axioms of the morphological filtering in the next section.

2.1 Morphological opening and closing

Generally, in a complete lattice $\mathcal{T}$, the dilation $X \longrightarrow \delta(X)$ and the erosion $X \longrightarrow \varepsilon(X)$ do not admit inverses and there is no way for determining **one** element X from the images $\delta(X)$ or $\varepsilon(X)$. However, starting from a dilation and then performing the dual erosion (or the contrary), we always have either an upper, or a lower bound according to the situation at hand.

Indeed, if we take $\delta(X)$ for the set Y in relation (12), the left inclusion is satisfied, so $X \leq \varepsilon\delta(X)$, and by duality:

$$\delta \circ \varepsilon(X) \leq X \leq \varepsilon \circ \delta(X),$$

or in terms of operators:

$$\delta\varepsilon \leq I \leq \varepsilon\delta.$$

We say that $\varepsilon\delta$ is **extensive** (larger than the identity mapping) and that $\delta\varepsilon$ is **anti-extensive**. Both operations are also **increasing** as the product of increasing mappings. Now, $\varepsilon\delta \geq I$ implies, by growth, that $\delta\varepsilon\delta\varepsilon \geq \delta\varepsilon$, whereas $\delta\varepsilon \leq I$ implies the inverse inequality. Hence $\delta\varepsilon = \delta\varepsilon\delta\varepsilon$, i.e. is **idempotent** (as well as $\varepsilon\delta$, by duality). The three properties of $\varepsilon\delta$ characterize what is called a *closing,* in algebra, and those of $\delta\varepsilon$ an *opening.* We shall call these two operators *morphological* to indicate that they are generated from a dilation and its dual erosion, and we denote:

$$\gamma_m = \delta\varepsilon \qquad\qquad \varphi_m = \varepsilon\delta \tag{30}$$

Fig. 17 shows an example of a morphological opening and of a morphological closing of a set X in the plane. In this 2-D case, a morphological opening may remove three types of features: capes, isthma and islands. By duality, a morphological closing may fill gulfs, channels and lakes.

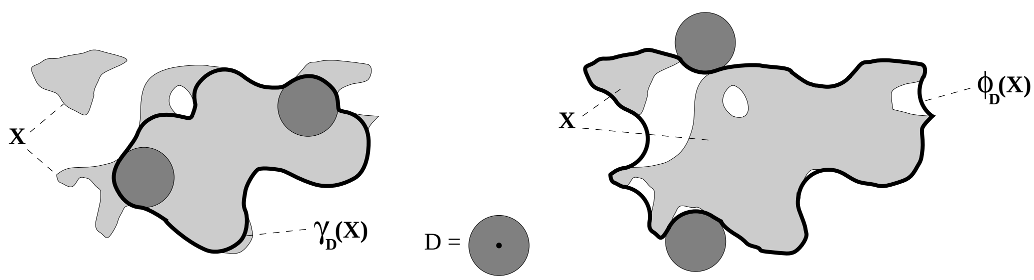


Figure 17: Examples of a morphological opening and of a morphological closing of a set X by a disc D.

Let $Z = \delta(X)$ be the dilation image of an arbitrary element $X \in \mathcal{T}$. We have:

$$\gamma_m(Z) = \delta\varepsilon\delta(X) \quad \begin{array}{ll} \geq \delta(X) & \text{by extensivity of } \varepsilon\delta \\ \leq \delta(X) & \text{by anti-extensivity of } \delta\varepsilon \end{array}$$

Hence $\gamma_m(Z) = Z$, i.e. Z belongs to the class $\mathcal{B}$ of the **invariant elements** of $\mathcal{T}$ under γ_m. Conversely if $Z \in \mathcal{B}$, then $Z = \delta(\varepsilon(Z))$ i.e. is the dilation of an element of $\mathcal{T}$. To summarize, we have the following theorem:

Theorem 2.1 *Given a dilation δ on lattice $\mathcal{T}$ and its dual erosion ε, the composition products $\gamma_m = \delta\varepsilon$ and $\varphi_m = \varepsilon\delta$ are respectively an opening and a closing on $\mathcal{T}$, called morphological. The invariance domain of the former is the image of $\mathcal{T}$ under δ and that of the latter forms the image of $\mathcal{T}$ under ε.*

Corollary 2.2 *Given $X \in \mathcal{T}$, $\gamma_m(X)$ is the smallest inverse image of X under ε, and $\varphi_m(X)$ is the largest one under δ.*

This corollary is illustrated by Fig. 18.

proof: Suppose that $Y \in \mathcal{T}$ is such that $\varepsilon(Y) = \varepsilon(X)$. Then, a fortiori, $Z = \varepsilon(X) \leq \varepsilon(Y)$ and thus, applying rel. (12), $\delta(Z) \leq Y$, or:

$$\gamma_m(X) \leq Y.$$

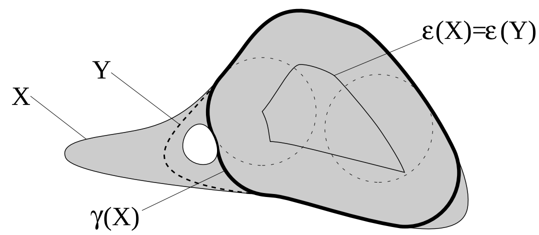


Figure 18: $\gamma_m(X)$ is the smallest element $Y \subseteq \mathcal{T}$ such that $\varepsilon(Y) = \varepsilon(X)$.

By duality, we have also

$$\forall Y \in \mathcal{T}, \delta(Y) = \delta(X) \Longrightarrow Y \leq \psi_m(X),$$

which completes the proof. □

Corollary 2.3 *If $\mathcal{B}$ and $\mathcal{B}'$ stand for the invariance domains of γ_m and φ_m respectively, then*

$$\gamma_m(X) = \vee\{B, B \in \mathcal{B}, B \leq X\} \tag{31}$$

$$\varphi_m X) = \wedge\{B, B \in \mathcal{B}', B \geq X\} \tag{32}$$

proof: From relation (13), we have

$$\begin{aligned} \gamma_m(X) = \delta(\varepsilon(X)) &= \delta(\vee\{B \in \mathcal{T}, \delta(B) \leq X\}) \\ &= \vee\{\delta(B), B \in \mathcal{T}, \delta(B) \leq X\}, \end{aligned}$$

but according to the theorem, $\mathcal{B} = \{\delta(B), B \in \mathcal{T}\}$. Hence, we get relation (31). As concerns relation (32), it has a dual proof. □

Example:

We have seen in § 1.5.4 that **planar** increasing mappings preserve vertical walls. Fig. 19 typically illustrates this point by showing the morphological opening of a 1-D function by an horizontal segment. Unlike this kind of opening, **circular** openings (i.e. openings with discs) do not preserve the vertical parts of the 1-D functions on which they act. In this case, Fig. 19 clearly indicates changes of slope. The same remarks apply in the 2-D case and the experimenter must choose between one approach or the other acording to his purpose. It should be noticed that “planar” structuring elements are most of the time preferred, since the computation of the corresponding openings and closings can be done more efficiently than with 3-D structuring elements.

2.2 Algebraic openings and closings

The important corollary 2.3 directly associates an opening γ_m with its invariant elements, without referring to the intermediary erosion and dilation. Should it be also true for any algebraic opening γ, i.e. for any operation on $\mathcal{T}$ which is increasing, anti-extensive and idempotent? Let $\mathcal{B}$ be the invariance domain of such a γ, and B be an invariant element, $B \leq X$. Then (by growth) $B = \gamma(B) \leq \gamma(X)$, hence $\gamma(X) \geq \vee\{B, B \in \mathcal{B}, B \leq X\}$. But $\gamma(X) \in \mathcal{B}$ (by idempotence) and $\gamma(X) \leq X$ (by anti-extensivity), therefore $\gamma(X)$ is one of the B of the right member. Thus, relation (31) is valid for any opening.

Conversely, start from an arbitrary part $\mathcal{B}_0$ of lattice $\mathcal{T}$ and let $\mathcal{B}$ be the class closed under union generated by $\mathcal{B}_0$. The operation defined by

$$\gamma(X) = \vee\{B, B \in \mathcal{B}_0, B \leq X\} \tag{33}$$

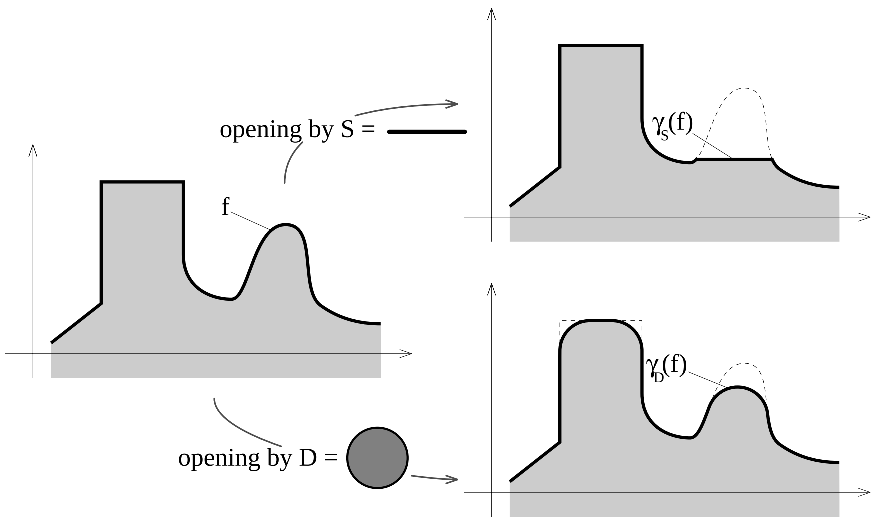


Figure 19: Comparison between the openings of a 1-D function by an horizontal segment S and by a disc D.

is increasing and anti-extensive. Moreover, $\gamma(X) = X$ iff $X \in \mathcal{B}$. The product $\gamma \circ \gamma$ is smaller than γ (growth and anti-extensivity), but also:

$$\begin{aligned} \gamma\gamma(X) &\geq \vee\{\gamma(B), B \in \mathcal{B}_0, B \leq X\} \\ &= \vee\{B, B \in \mathcal{B}_0, B \leq X\} \\ &= \gamma(X). \end{aligned}$$

We may therefore state the following:

Theorem 2.4 *An operation γ (resp. φ) on $\mathcal{T}$ is an opening (resp. a closing) if and only if there exists a class $\mathcal{B} \subseteq \mathcal{T}$, closed under union (resp. intersection) such that*

$$\begin{aligned} \gamma(X) &= \vee\{B, B \in \mathcal{B}, B \leq X\} \\ \varphi(X) &= \wedge\{B, B \in \mathcal{B}, B \geq X\}. \end{aligned}$$

$\mathcal{B}$ is the invariance domain *of γ (resp. φ).*

In other words, we can approach openings and closings either directly or via their invariance domains. Now, what about the *composition,* the *sup* or the *inf* of openings. Are they still operations of the same type? As far as sups are concerned, the anwer is yes. Indeed:

Theorem 2.5 *The sup of a family (γ_i) of openings is again an opening, whose domain of invariance is the class closed under union generated by the union of the $\mathcal{B}_i$ (invariance domains of the γ_i's).*

proof: Clearly, $\vee\gamma_i$ is increasing and anti-extensive. Furthermore, for all i, we have $\gamma_i \circ (\vee\gamma_i) \geq \gamma_i$. Therefore, $(\vee\gamma_i) \circ (\vee\gamma_i) \geq (\vee\gamma_i)$, and also the inverse inclusion, since $(\vee\gamma_i) \leq I$. This gives us the idempotence. The domain of invariance is determined as was done before. □

Unfortunately, the class of the openings is neither closed under $\wedge$, nor under composition. Consider for instance in $\mathbb{Z}$ the following set:

$$X = \texttt{..1111..111111..1111...}$$

and the two structuring elements

$$A = \texttt{.1.....1.} \quad \text{and} \quad B = \texttt{.11111.}$$

Denoting γ_A and γ_B, the associated morphological openings, we have:

$$\gamma_A(X) = X \quad \text{and} \quad \gamma_B(X) = \ \texttt{.111111.}$$

and

$$\gamma_B \circ \gamma_A(X) = \gamma_B(X) \neq \gamma_A \circ \gamma_B(X) = \emptyset.$$

Hence:

$$(\gamma_B \gamma_A)(\gamma_B \gamma_A) \neq (\gamma_B \gamma_A),$$

and

$$(\gamma_B \wedge \gamma_A)(\gamma_B \wedge \gamma_A)(X) = \emptyset \neq (\gamma_B \wedge \gamma_A)(X) = \gamma_B(X).$$

□

Let us quote a last result which clarifies the links between morphological and algebraic openings:

Theorem 2.6 *A mapping $\gamma : \mathcal{T} \longrightarrow \mathcal{T}$ is an opening if and only if it is the sup of a family (γ_i) of morphological openings. Moreover, if a translation is defined over $\mathcal{T}$, γ is translation invariant if and only if the γ_i are translation invariant (dual statement for the closings).*

proof: Easy, refer to [15, page 190], [24, page 161], [25, page 22]. □

2.3 (Non exhaustive) catalog of openings and closings

Although theorem 2.6 is heuristically deep, we may have difficulties in applying it directly, as the number of terms γ_i necessary for generating a given γ becomes prohibitive. Actually, there are four starting points for creating openings, namely:

- *the morphological openings,*
- *the trivial openings,*
- *the connected openings,*
- *the envelope openings.*

...plus any derivation obtained by cross-union of these various types. The first mode has already been developed. We will now present the other three.

2.3.1 Trivial openings

A criterion T is said to be *increasing* when, for all $X \in \mathcal{T}$:

$$\begin{cases} X \text{ satisfies } T \text{ and } Y \geq X & \Longrightarrow \quad Y \text{ satisfies } T, \\ X \text{ does not satisfy } T \text{ and } Y \leq X & \Longrightarrow \quad Y \text{ does not satisfy } T. \end{cases}$$

For example, in $\mathbb{R}^n$, for X to hit a given set A_0, as well as to have a Lebesgue measure larger than a given value λ_0 are both increasing criteria.

Proposition 2.7 *Given an increasing criterion T over lattice $\mathcal{T}$, the operation*

$$\gamma_1(X) = \begin{cases} X & \textit{when } X \textit{ satisfies } T, \\ \emptyset & \textit{otherwise.} \end{cases}$$

(with $\gamma_1(\emptyset) = \emptyset$) is an opening called the trivial opening associated with criterion T.

2.3.2 Connected opening γ_x

We consider a boolean lattice $\mathcal{P}(E)$ and an arbitrary point $x \in E$. A part $\mathcal{C}$ of $\mathcal{P}(E)$ is called a **connected class** on $\mathcal{P}(E)$ when it satisfies:

(i) $\emptyset \in \mathcal{C}$ *and* $\forall x \in E$, $\{x\} \in \mathcal{C}$,

(ii) For every family (C_i) *in* $\mathcal{C}$, $\cap C_i \neq \emptyset \implies \cup C_i \in \mathcal{C}$.

One proves then [25, page 52] that the datum of a connected class $\mathcal{C}$ on $\mathcal{P}(E)$ is equivalent to the family of openings γ_x such that:

(iii) $\forall x \in E$, $\gamma_x(\{x\}) = \{x\}$,

(iv) $\forall A \subseteq E$, $x, y \in E$, $\gamma_x(A)$ *and* $\gamma_y(A)$ *are equal or disjoint, i.e.*

$$\gamma_x(A) \cap \gamma_y(A) \neq \emptyset \implies \gamma_x(A) = \gamma_y(A),$$

(v) $\forall A \subseteq E, \forall x \in E, \quad x \notin A \Longrightarrow \gamma_x(A) = \emptyset$.

At first sight, this theorem just indicates that the operation shown in Fig. 20 is an opening called *the connected component of A that contains point x*, which is somewhat obvious. But it also tells for instance that the operator:

$$\nu_x(A) = \begin{cases} (\gamma_x \circ \delta(A)) \cap A & \text{when } x \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

(where δ is a dilation by a disc) is again a connected opening associated with the connectivity shown in Fig. 21, which is less obvious [25, page 55].

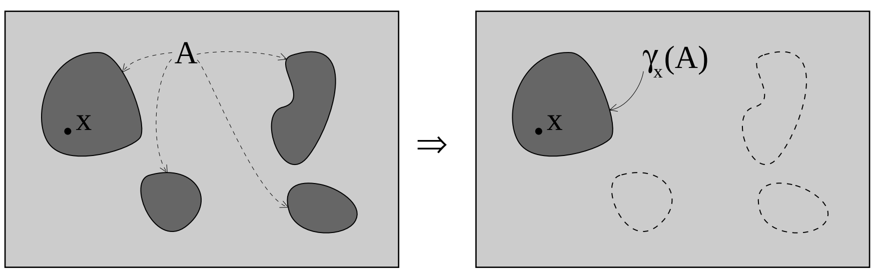


Figure 20: The opening called *the connected component of A that contains point x*.

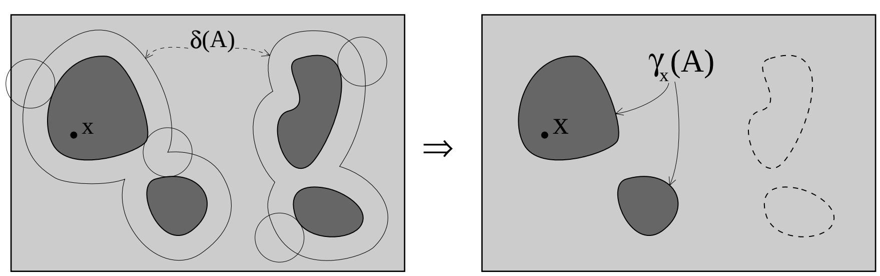


Figure 21: A less obvious connectivity notion, associated with the opening ν_x.

2.3.3 Envelope openings

Consider a **finite** lattice $\mathcal{T}$ and an increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$. Then, for any $X \in \mathcal{T}$, the sequence $[X \cap \psi(X)]^n$ decreases with n, and finally stops for a certain n_0, since $\mathcal{T}$ is finite. The operator

$$\check{\psi} = (I \wedge \psi)^{n_0} \tag{34}$$

is therefore an opening. Moreover, if h is an opening smaller than ψ, then $h \leq I \wedge \psi$. Hence $h = h^n \leq (I \wedge \psi)^n$ for every n and thus $h \leq \check{\psi}$. In other words:

Theorem 2.8 *Let $\mathcal{T}$ be a finite lattice. Then, for every increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$, there exists an upper envelope $\check{\psi}$ of the openings which minorate ψ. It is itself an opening and is given by the relation*

$$\check{\psi} = (I \wedge \psi)^{n_0} \qquad \textit{for a finite } n_0.$$

(dual result with $\hat{\psi} = (I \vee \psi)^{n_0}$.)

Some interesting remarks and counter-examples concerning the case of *non finite* lattices can be found in [5].

Note: *(i)* The iterations may well stop at the first step. In § 3.5.3, the example of the rank-operators illustrates this point.

(ii) Under conditions which are always fulfilled in practice, theorem 2.8 extends to the lattice of the functions $f : \mathbb{Z}^n \longrightarrow \mathbb{Z}$ [26].

2.4 Cross-unions of basic openings

2.4.1 Reconstruction algorithms (combination of types 2 and 3)

A series of algorithms are based on the same approach and resort to the so-called *geodesic methods* [10]. For example, for 2-D binary images: *keep the connected components of X whose circular opening of size k is not empty and filter out the others.* The algorithm is extended to numerical functions via their horizontal sections, but as we will see, it can be directly implemented in terms of grey-scale operators. In this section, we only consider the discrete cases of the lattice $\mathcal{P}(\mathbb{Z}^2)$ and of the lattice $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z}^+)$ (functions from $\mathbb{Z}^2$ into $\mathbb{Z}^+$). In practice, they correspond respectively to binary and grey-scale morphology.

For all $k \in \mathbb{Z}^+/\{0\}$, denote ε_k and δ_k the isotropic erosion and dilation of size k (square, hexagonal, octogonal...) in $\mathcal{P}(\mathbb{Z}^2)$, as well as in $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z}^+)$. We first focuse on the set case: let X be a finite set of $\mathcal{P}(\mathbb{Z}^2)$. We call geodesic dilation of size one in X the mapping δ_X defined on $\mathcal{P}(X)$ as follows:

$$\delta_X \left(\begin{array}{lcl} \mathcal{P}(X) & \longrightarrow & \mathcal{P}(X) \\ Y & \longmapsto & \delta_1(Y) \cap X. \end{array} \right) \tag{35}$$

The geodesic dilation of size $n > 0$ inside X, denoted $\delta_X^{(n)}$, is then simply defined by iterating δ_X n times:

$$\delta_X^{(n)} = \underbrace{\delta_X \circ \delta_X \circ \ldots \circ \delta_X}_{n \text{ times}}. \tag{36}$$

Note that this operation has already been illustrated by Fig. 10. Now, for a given set $Y \subseteq X$, $\delta_X^{(1)}(Y)$, $\delta_X^{(2)}(Y)$,..., $\delta_X^{(k)}(Y)$... constitutes an increasing sequence of sets, which tends toward an idempotent limit. This limit, denoted $\rho_X(Y)$ is called the *reconstruction* of set X from the "marker" Y:

$$\rho_X(Y) = \lim_{k \to +\infty} \delta_X^{(k)}(Y). \tag{37}$$

In practice, this transformation allows us to extract—or to "reconstruct"—those connected components of set X which are said to be "marked" by Y, i.e. whose intersection with Y is not empty.

Let us now take the erosion of size k of X—i.e. $\varepsilon_k(X)$—as marker set. The reconstruction operation allows us to extract the connected components of X that do not completely disappear after an erosion (or after an opening) of size k, i.e. that can hold the isotropic element of size k. Denote μ_k the transformation thus brought to the fore:

$$\forall X \subset \mathbb{Z}^2, \qquad \mu_k(X) = \rho_X(\varepsilon_k(X)) = \rho_X(\gamma_k(X)), \tag{38}$$

where γ_k stands for the opening with respect to the isotropic element of size k. It is very easy to show that μ_k is an increasing, anti-extensive and idempotent operation. It is therefore an algebraic opening, often referred to as *opening by reconstruction.* Contrary to the morphological opening γ_k, μ_k allows us to preserve the shape of the components that are not filtered out. It is illustrated by Fig. 22. Remark that the same technique still yields an opening when replacing ε_k by any increasing mapping in (38).

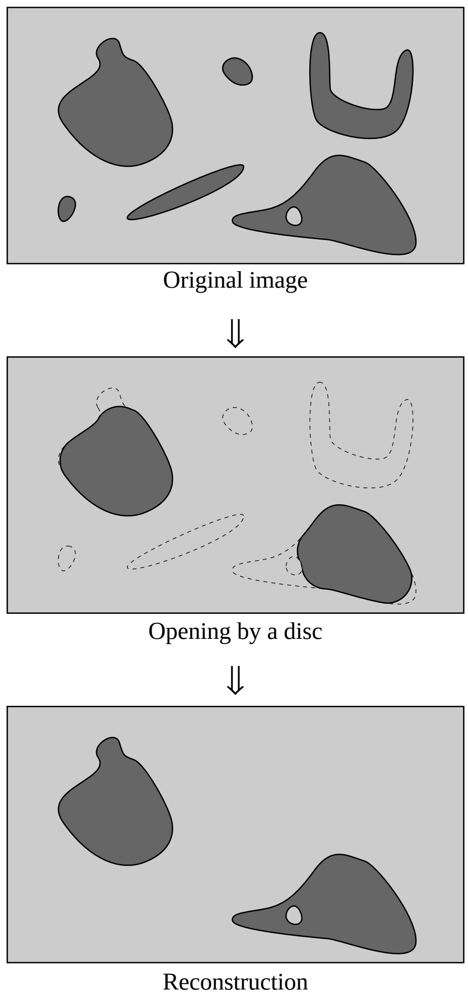


Figure 22: Algebraic opening which is defined as a morphological opening by a disc followed by a reconstruction

As already mentioned, these notions extend directly to the grey-scale case, i.e. to lattice $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z}^+)$. Here, ε_k, δ_k, γ_k and ϕ_k denote respectively the erosion, dilation, opening and closing with respect to the *flat* isotropic element of size k. Given two functions f and g such that $g \leq f$, the geodesic dilation of size 1 of g with respect to f is denoted $\delta_f(g)$ and given by

$$\delta_f(g) = \delta_1(g) \wedge f. \tag{39}$$

Similarly, $\delta_f^{(n)}(g)$ denotes the geodesic dilation of size n of g with respect to f:

$$\delta_f^{(n)}(g) = (\underbrace{\delta_f \circ \delta_f \circ \ldots \circ \delta_f}_{n \text{ times}})(g) \tag{40}$$

whereas the reconstruction operation ρ_f is given by the following equation:

$$\rho_f(g) = \lim_{k \to +\infty} \delta_f^{(k)}(g). \tag{41}$$

Similarly as in the binary case, this grey-scale reconstruction preserves the peaks or crest-lines of f that are initially marked by g. It is illustrated in Fig. 23. Moreover, the grey-scale opening by reconstruction μ_k is defined exactly as for sets and allows one to preserve the peaks, domes or crest-lines of f that do not totally disappear after an erosion of size k. This kind of openings is extremely useful in practice, since it enables very clean and efficient filtering of grey-scale images. Some "structures" are filtered out whereas the remaining ones are left unchanged.

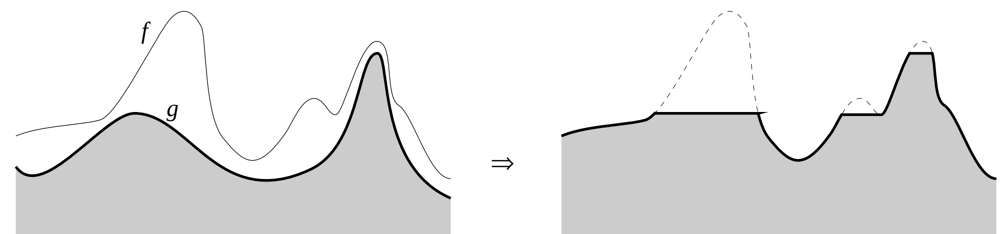


Figure 23: Grey-scale reconstruction of function f from the "marker-function" g: the right peak of f is marked by g and thus preserved in the reconstruction process.

By duality, one can also define grey-scale geodesic erosions, as well as the reconstruction ρ_f^* and the closing by reconstruction μ_k^*. Suppose that $g \geq f$. The geodesic erosion of size 1 of g with respect to f is denoted $\varepsilon_f(g)$ and given by $\varepsilon_f(g) = \varepsilon_1(g) \vee f$, whereas that of a given size n, denoted $\varepsilon_f^{(n)}(g)$ has the following definition:

$$\varepsilon_f^{(n)}(g) = (\underbrace{\varepsilon_f \circ \varepsilon_f \circ \ldots \circ \varepsilon_f}_{n \text{ times}})(g). \tag{42}$$

Now, the "dual" reconstruction of f from the marker-function g is defined by

$$\rho_f^*(g) = \lim_{k \to +\infty} \varepsilon_f^{(k)}(g). \tag{43}$$

This operation preserves the basins and valleys of f that are initially marked by g. It is at the basis of the definition of the *closing by reconstruction* μ_k^* of a function f:

$$\mu_k^*(f) = \rho_f^*(\phi_k(f)). \tag{44}$$

Moreover, as we shall see in § 3.6, the dual reconstruction ρ_f^* enables a very efficient filtering of the minima of a grey-scale image. This turns out to be extremely interesting for the segmentation of decimal images by means of *watershed* techniques [3, 1].

2.4.2 Annular opening (unions of types 1)

Consider the pair of points $B = \{o, b\}$, made of the origin o and a point b in direction α in $\mathbb{R}^2$ or in $\mathbb{Z}^2$. Clearly, the morphological opening γ_b with respect to B is equivalent to

$$\gamma_b = I \wedge \delta_{B'}$$

where $\delta_{B'}$ is the t-dilation by the bi-point $B' = \{-b; +b\}$. Now, make vary b in a certain domain D which does not contain the origin (e.g. three consecutive vertices of an hexagon centered on o, half a circle, ...) and take the sup γ:

$$\gamma = \vee\{\gamma_b, b \in D\} = I \wedge \{\vee \delta_{B'}, b \in D\}$$

i.e. since the dilation commutes with $\vee$:

$$\gamma = I \wedge \delta_{D \cup \check{D}} \tag{45}$$

where $\delta_{D\cup\check{D}}$ is the dilation by $D \cup \check{D} = \bigcup_{b \in D}\{-b; +b\}$. The effect of this *annular opening* γ is shown on Fig. 24. γ eliminates the components of a given set X as a function of their environment more than of their size or shape. On the example presented here, $D \cup \check{D}$ is taken to be a circle and γ eliminates the central particle without touching the others.

To illustrate the specific action of γ, we can compare it with the morphological opening γ' by a disc and with the union γ'' of the morphological openings by segments in various directions (see Fig. 24).

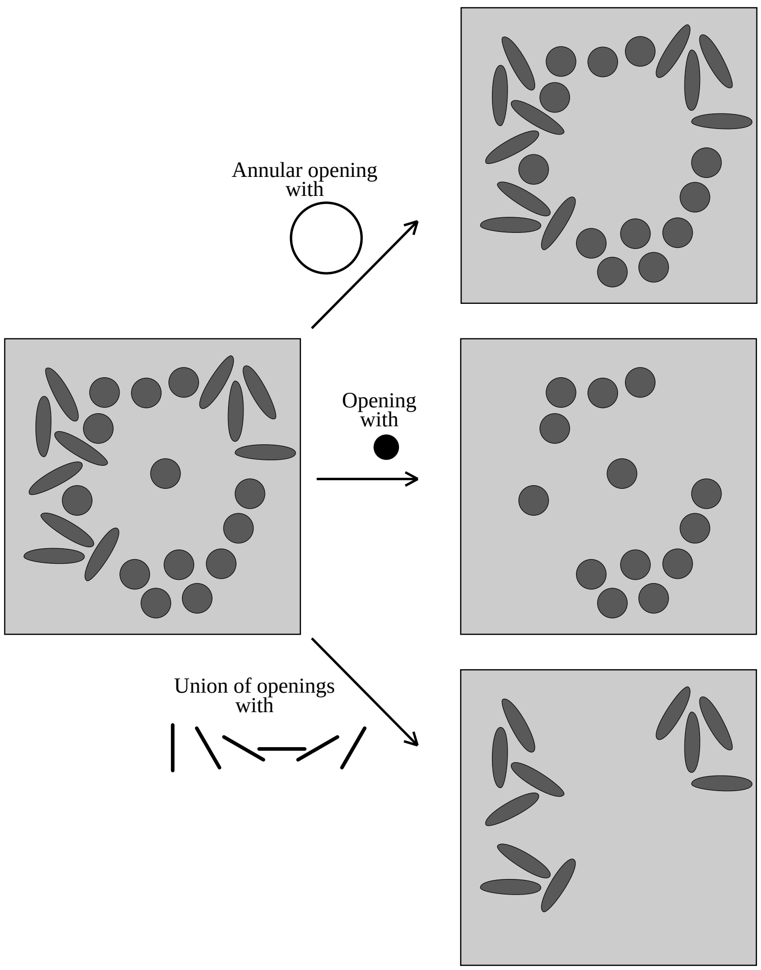


Figure 24: Annular opening γ versus a classical opening by a disc and a union of openings by segments.

3 Morphological filters

3.1 The lattice of the increasing mappings

This section constitutes an overview of the theory of morphological filtering, due to G. Matheron [25, chapter 6]. The lattice examples introduced in § 1 concerned the scenes under study. We will now consider classes of *operations* working on these objects. Let ψ be such an operator, i.e. be a mapping from a complete lattice $\mathcal{T}$ into itself. We assume that ψ is increasing, i.e. that it preserves the ordering relation of $\mathcal{T}$:

$$\forall A, A' \in \mathcal{T}, \qquad A \geq A' \Longrightarrow \psi(A) \geq \psi(A'). \tag{46}$$

The set $\mathcal{T}'$ of the increasing mappings on the complete lattice $\mathcal{T}$ satisfies the following properties:

1. $\mathcal{T}'$ is a semi-group for the composition product $\circ$, with a unit element, namely the identity mapping I $(\forall A \in \mathcal{T}, I(A) = A)$.
2. $\mathcal{T}'$ is a complete lattice for the ordering relation:

   $$f \geq g \iff \forall A \in \mathcal{T}, f(A) \geq g(A),$$

   since the following identities

   $$(\vee_{\mathcal{T}'} f_i)(A) = \vee_{\mathcal{T}}(f_i(A)) \qquad \text{and} \qquad (\wedge_{\mathcal{T}'} f_i)(A) = \wedge_{\mathcal{T}}(f_i(A))$$

   generate a supremum and an infimum in the set $\mathcal{T}'$.

The two basic structures of the semi-group and of the lattice interact with each other, and we have, for all f, g, h and (f_i) in $\mathcal{T}'$:

$$(\vee f_i) \circ g = \vee(f_i \circ g) \qquad ; \qquad g \circ (\vee f_i) \geq \vee(g \circ f_i) \tag{47}$$

$$(\wedge f_i) \circ g = \wedge(f_i \circ g) \qquad ; \qquad g \circ (\wedge f_i) \leq \wedge(g \circ f_i) \tag{48}$$

and

$$f \geq g \Longrightarrow \left\{ \begin{array}{lcl} f \circ h & \geq & g \circ h \\ h \circ f & \geq & h \circ g \end{array} \right.$$

In the following, the two classes of the *overfilters* (i.e. the mappings $f \in \mathcal{T}'$ such that $f \circ f \geq f$) and of the *underfilters* play a major role. Indeed:

Theorem 3.1 *the class of the underfilters (resp. overfilters) is closed under $\wedge$ (resp. $\vee$) and under self-composition.*

For example, let $(f_j)_{j \in J}$ be a family of underfilters. From (46) and (48), we get:

$$(\wedge_{j \in J} f_j) \circ (\wedge_{j \in J} f_j) = \wedge_{i \in J}(f_i \circ \wedge_{j \in J} f_j) \leq \wedge_{i \in J}(f_i \circ f_i) \leq \wedge_{i \in J} f_i,$$

so that $\wedge_{j \in J} f_j$ is an underfilter. Moreover, given an underfilter f, $ff \leq f$ implies, by growth, that $ff \circ ff \leq ff$, so that the self-composition ff is an underfilter.

3.2 Morphological filters: definition

Following G. Matheron and J. Serra [25, chapters 5–6], we define the notion of a **morphological filter** as follows:

Definition 3.2 *The elements of $\mathcal{T}'$ which are both underfilters and overfilters are called (morphological)* **filters.**

Note that in literature, the term "filter" may also be associated with growth only [12, 13], and can even be a synonymous with mapping [37]. Here however, the morphological filters are the transformations acting on the scenes under study (i.e. the lattice $\mathcal{T}$) which are **increasing** and **idempotent.** We shall denote by $\mathcal{V}$ the class of the filters, with $\mathcal{V} \subseteq \mathcal{T}'$. Remark that the class $\mathcal{V}$ is not closed either under $\vee$, or $\wedge$, or under composition (a counter-example, based on openings, has been brought to the fore in § 2.2).

This apparent drawback suggests us to investigate more accurately the possible connections of class $\mathcal{V}$ with the composition product and with extrema. Can we find, for example, pairs (f, g) of filters such that $f \circ g$, $g \circ f$, $f \circ g \circ f$, etc... are surely filters (composition problem)? Can we keep the usual ordering relation in $\mathcal{V}$ and equip $\mathcal{V}$ with **new** sup and inf, such that it turns out to become a complete lattice (extrema problem)? These two sorts of questions constitute the subject of sections 3.3 and 3.4.

3.3 Composition of morphological filters

With any increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$, associate:

1. the **image domain** $\psi(\mathcal{T})$, i.e. the set of the transforms by ψ:

$$\psi(\mathcal{T}) = \{\psi(A), A \in \mathcal{T}\},$$

2. the **invariance domain** $\mathcal{B}_\psi$, i.e. the class of those $B \in \mathcal{T}$ which are left unchanged under ψ:

$$\mathcal{B}_\psi = \{B \in \mathcal{T}, \psi(B) = B\}.$$

When ψ is a filter, $\mathcal{B}_\psi$ is often called the *root* of ψ in literature.

We always have $\mathcal{B}_\psi \subseteq \psi(\mathcal{T})$, an inclusion which becomes an equality

$$\mathcal{B}_\psi = \psi(\mathcal{T})$$

if and only if ψ is **idempotent.** This preliminary remark leads to the following two criteria:

Criterion 3.3 *For any mappings f, g from $\mathcal{T}$ into itself,*

$$fg = g \iff g(\mathcal{T}) \subseteq \mathcal{B}_f.$$

In particular, when g is idempotent:

$$fg = g \iff \mathcal{B}_g \subseteq \mathcal{B}_f. \tag{49}$$

Criterion 3.4 *Two mappings f and g from $\mathcal{T}$ into itself are idempotent and admit the same invariance domain $\mathcal{B}_f = \mathcal{B}_g$ if and only if:*

$$fg = g \qquad \text{and} \qquad gf = f. \tag{50}$$

Proofs: criterion 3.3 is obvious. Now, if rel. (50) is satisfied, then $ff = f \circ gf = gf = f$, i.e. f, and similarly g, are idempotent. Hence, from (49), $\mathcal{B}_f \subseteq \mathcal{B}_g$ and $\mathcal{B}_g \subseteq \mathcal{B}_f$. Conversely, when f and g, idempotent, have the same invariance domain, rel. (50) is nothing but (3.4). □

In these two criteria, the ordering $\leq$ does not intervene. From now on, we shall only consider the **increasing mappings** ψ, i.e. the mappings $\psi \in \mathcal{T}'$. For any filter ψ, the class of the filters ψ' that have the same invariance domain $\mathcal{B}_\psi$ as ψ is denoted $\mathcal{I}d(\psi)$. The following theorem is the key result concerning the composition of filters:

Theorem 3.5 (structural theorem) *Let f and g be two filters on $\mathcal{T}$ such that $f \geq g$. Then:*

(i) $f \geq fgf \geq gf \vee fg \geq gf \wedge fg \geq gfg \geq g$,

(ii) gf, fg, fgf *and* gfg *are filters, and* $fgf \in \mathcal{I}d(fg)$, $gfg \in \mathcal{I}d(gf)$,

(iii) fgf *is the smallest filter greater than* $gf \vee fg$ *and* gfg *is the greatest filter smaller than* $gf \wedge fg$*,*

(iv) the following equivalences hold:

$$\begin{array}{llll} \mathcal{B}_{fg} = \mathcal{B}_{gf} & \iff \mathcal{B}_{fg} = \mathcal{B}_f \cap \mathcal{B}_g & \iff & \mathcal{B}_{gf} = \mathcal{B}_f \cap \mathcal{B}_g \\ & \iff fgf = gf & \iff & gfg = fg \\ & \iff gf \geq fg. & & \end{array}$$

proof: The inequalities *(i)* are obvious.

From the relationships

$$fg = fffg \geq fgfg \geq fggg = fg,$$

we conclude that fg is a filter. By the dual inequalities, gf is also a filter. Now, we have

$$\begin{aligned} fgf \circ fg &= fg(ff)g = fgfg = fg, \\ fg \circ fgf &= fgfg \circ f = (fg \circ fg)f = fgf, \end{aligned}$$

and thus, $fgf \in \mathcal{I}d(fg)$, by criterion 3.4. In the same way, we find that $fgf \in \mathcal{I}d(gf)$, so that *(ii)* is proved.

Now, fgf is a filter (by *(ii))* and $fgf \geq gf \vee fg$ (by *(i))*. Let ψ be a filter such that $\psi \geq fg$ and $\psi \geq gf$. It follows that $\psi = \psi\psi \geq fggf = fgf$. Thus, fgf is the smallest filtering upper bound of fg and gf. Hence *(iii)* is proved.

By criterion 3.4, we have $\mathcal{B}_{fg} = \mathcal{B}_{gf}$ if and only if

$$fg \circ gf = fgf = gf \quad \text{and} \quad gf \circ fg = gfg = fg.$$

These relations actually imply one another. For instance, $fgf = gf$ implies $fgf \circ g = gfgf$, i.e. $fg = gfg$. By *(iii),* these relations are equivalent to $gf \geq fg$.

The inclusions

$$\mathcal{B}_f \cap \mathcal{B}_g \subseteq \mathcal{B}_{fg} \subseteq \mathcal{B}_f$$

always hold, so that $\mathcal{B}_{fg} = \mathcal{B}_f \cap \mathcal{B}_g$ if and only if $\mathcal{B}_{fg} \subseteq \mathcal{B}_g$, i.e, by criterion 3.3, if and only if $gfg = fg$. This completes the proof. □

Examples:

1. Start from an arbitrary opening γ and an arbitrary closing ϕ. Since

$$\gamma \leq I \leq \phi,$$

by theorem 3.5, $\gamma\phi$, $\phi\gamma$, $\gamma\phi\gamma$ and $\phi\gamma\phi$ are filters. The composition products of ϕ by γ, then by ϕ, etc... generates the oscillating sequence

$$\phi \longrightarrow \gamma\phi \longrightarrow \phi\gamma\phi \longrightarrow \gamma\phi \longrightarrow \ldots$$

Remark that when $\gamma\phi \geq \phi\gamma$ (which is generally not the case), we have $\gamma\phi = \phi\gamma\phi$ and the oscillations are stopped after the first step.

2. There is a more particular example, which illustrates point *(iv)* of the theorem. In $\mathcal{P}(\mathbb{R}^n)$ (or $\mathcal{P}(\mathbb{Z}^n)$, or $\mathcal{F}(\mathbb{R}^n, \mathbb{R})$, or $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$), consider the morphological opening γ_l by a segment of length l in the horizontal direction. For a given X, $\gamma_l(X)$ is made of horizontal segments of length $\geq l$. Moreover, closing this set by the dual closing ϕ_l, i.e. determining $\phi_l\gamma_l(X)$ may only suppress intervals between two such segments, hence increase the length of the horizontal intercepts; Therefore, $\gamma_l\phi_l\gamma_l(X) = \phi_l\gamma_l(X)$, and by the theorem, $\phi_l\gamma_l \leq \gamma_l\phi_l$.

3.4 The lattice of the filters

We now go back to the structure of the set $\mathcal{V}$ of the filters acting on lattice $\mathcal{T}$. Clearly, if (ψ_i) is a family of elements of $\mathcal{V}$, then $\vee\psi_i$ is an overfilter,

$$(\vee_i\psi_i)(\vee_i\psi_i) = \vee_i(\psi_i(\vee_j\psi_j)) \geq \vee_i(\psi_i \circ \psi_i) = \vee_i\psi_i,$$

and similarly, $\wedge\psi_i$ is an underfilter. Since $\vee\psi_i$ is an overfilter, the class $\mathcal{C}$ closed under $\vee$ and self-composition generated by $\vee\psi_i$ only comprises overfilters (theorem 3.1). It admits a largest element f, for $\mathcal{T}'$ is a complete lattice. But $f \circ f$ also belongs to $\mathcal{C}$ (closure under self-composition), hence $f \circ f \leq f$, i.e. f is a **filter.**

Consider now a filter ψ' larger than $\vee\psi_i$. The class $\mathcal{C}'$ of the overfilters smaller than ψ' is, in turn, closed under $\vee$ and self-composition, and contains the previous class $\mathcal{C}$. Hence, $f \leq \psi'$, i.e. f turns out to be the smallest filter which majorates the ψ_i's. By duality, we have a similar result for the inf, and we may state:

Theorem 3.6 *The set $\mathcal{V}$ of the filters on $\mathcal{T}$ is a complete lattice. For any family (ψ_i) of filters on $\mathcal{T}$, the smallest filter greater than $\vee\psi_i$ is the largest element f of the class closed under $\vee$ and self-composition generated by (ψ_i). Dual result for the largest filter smaller than $\wedge\psi_i$.*

In particular, when $\mathcal{T}$ is finite, we always have, for a large enough n:

$$f = (\vee\psi_i)^n, \tag{51}$$

a result which provides the algorithm for computing f in practice. Refer to [5] for some counter-examples in the case where $\mathcal{T}$ is not a finite lattice.

Examples:

1. Lattice of the openings: take the class $\mathcal{V}' \subseteq \mathcal{V}$ of the openings on $\mathcal{T}$. We have seen that for every family (γ_i) in $\mathcal{V}'$, $\vee\gamma_i$ is still an opening (theorem 2.5). Moreover, from theorem 3.6, there exists a largest filter g which is smaller than all the γ_i's. g being obviously anti-extensive, $\mathcal{V}'$ is a complete lattice.

2. Start from an arbitrary increasing mapping ψ. Then, the extensive mapping $I \vee \psi$ is an overfilter and the proof of the theorem shows that there exists a smaller filter $\hat{\psi}$—hence a smaller **closing**—that majorates ψ. In the finite case, we find again theorem 2.8.

3.5 $\vee$- and $\wedge$-filters, $\gamma\phi$ and $\phi\gamma$, strong filters

3.5.1 Introduction

We say that a mapping $f : \mathcal{T} \longrightarrow \mathcal{T}$ is a $\vee$-mapping when

$$f = f \circ (I \vee f) \tag{52}$$

and a $\wedge$-mapping when

$$f = f \circ (I \wedge f). \tag{53}$$

Basically, this property is something new and independent from the two axioms which build the definition of the morphological filters. If now f is increasing and satisfies rel. (52), we shall call it a $\vee$-underfilter. Indeed, any $\vee$-underfilter is an underfilter and similarly, any $\wedge$-overfilter is an overfilter. If f is, for instance, a $\vee$-underfilter, then we have:

$$f = f \circ (I \vee f) \geq f \vee ff \geq f.$$

Thus, $f = f \vee ff$ is an underfilter.

A filter which satisfies rel. (52) (resp rel. (53)) will be called a $\vee$-filter (resp. a $\wedge$-filter). When it satisfies both rel. (52) and (53) it will be said to be a **strong filter.** The geometrical interpretation of $\vee$-filtering and of $\wedge$-filtering are very easy. Indeed, ψ is a $\vee$-filter if and only if, for any $A \in \mathcal{T}$, every B between A and $A \vee \psi(A)$ has the same transform as A itself (see Fig. 25), i.e.

$$\psi \ \vee -\text{filter:} \qquad \forall A \in \mathcal{T}, (A \leq B \leq A \vee \psi(A) \Longrightarrow \psi(B) = \psi(A)).$$

Similarly, we have

$$\psi \ \wedge -\text{filter:} \qquad \forall A \in \mathcal{T}, (A \wedge \psi(A) \leq B \leq A \Longrightarrow \psi(B) = \psi(A)).$$

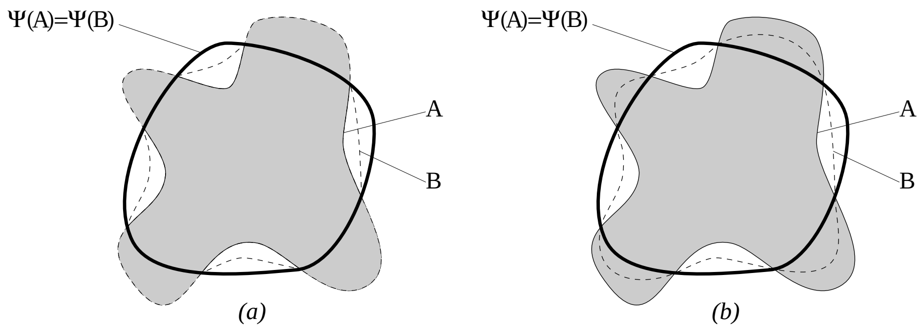


Figure 25: An example of a $\vee$-filter (a) and of a strong filter (b).

The following result corresponds to the theorem 3.1 of the general case—and is proved in the same way:

Theorem 3.7 *the class of the $\vee$-underfilters (resp. $\wedge$-overfilters) is closed under $\wedge$ (resp. under $\vee$) and self-composition.*

(Note the chiasma; it is the sup and not the inf of $\wedge$-overfilters which is still an $\wedge$-overfilter.) This theorem suggests us to approach first the properties of self-composition, and then that of sup and inf, just as we did in the general case.

3.5.2 Composition of $\vee$- and $\wedge$-filters

Theorem 3.8 *Let f and g be two filters on $\mathcal{T}$ and $f \geq g$. Then*

(i) if f is a $\vee$-filter, gf and fgf are $\vee$-filters,

(ii) if f is a $\wedge$-filter, fg and gfg are $\wedge$-filters,

(iii) if gf is a $\wedge$-filter, fgf is a $\wedge$-filter,

(iv) if fg is a $\vee$-filter, gfg is a $\vee$-filter.

proof: Easy [25, page 119]. □

Example: If γ is an *opening* and ϕ a *closing,* we have $\gamma \leq I \leq \phi$, and then

- γ and ϕ are strong filters,
- $\gamma\phi$ and $\phi\gamma\phi$ are $\vee$-filters,
- $\phi\gamma$ and $\gamma\phi\gamma$ are $\wedge$-filters.

Moreover, if $\gamma\phi$ is a $\wedge$-filter, and thus a strong filter, $\phi\gamma\phi$ is a strong filter. In the same way, if $\phi\gamma$ is a strong filter, then $\gamma\phi\gamma$ is a strong filter.

3.5.3 The two envelopes $\psi\hat{\psi}$ and $\psi\check{\psi}$

In § 3.4, we have associated with each $\psi \in \mathcal{T}'$ the largest opening $\check{\psi}$ that minorates ψ and the smallest closing $\hat{\psi}$ which majorates ψ. These two primitives play a central role in the $\vee$- and $\wedge$-characterizations, as is shown by the following theorem:

Theorem 3.9 *An increasing mapping $\psi : \mathcal{T} \longrightarrow \mathcal{T}$ is a $\vee$-underfilter (resp. a $\wedge$-overfilter) if and only if $\psi = \psi\hat{\psi}$ (resp. $\psi = \psi\check{\psi}$).*

<u>proof</u>: If $\psi = \psi(I \vee \psi)$, then

$$(I \vee \psi)(I \vee \psi) = I \vee \psi \vee \psi(I \vee \psi) = I \vee \psi \vee \psi = I \vee \psi.$$

The mapping $I \vee \psi$, which is idempotent and which majorates I, is nothing but $\hat{\psi}$, and $\psi(I \vee \psi) = \psi\hat{\psi}$.

Conversely, start from

$$\hat{\psi} \leq \hat{\psi}(I \vee \psi) \leq \hat{\psi}(I \vee \hat{\psi}) = \hat{\psi},$$

which implies $\hat{\psi} = \hat{\psi}(I \vee \psi)$. Now, if $\psi = \psi\hat{\psi}$, then
$\psi = \psi\hat{\psi} = \psi\hat{\psi}(I \vee \psi) = \psi(I \vee \psi)$. □

Corollary 3.10 *If $\psi \in \mathcal{T}'$ is an overfilter (resp. an underfilter), then $\psi\hat{\psi}$ is the smallest $\vee$-filter which majorates ψ (resp. the largest $\wedge$-filter which minorates ψ).*

<u>proof</u>: If ψ is an overfilter, then

$$\psi\hat{\psi} \leq \psi\psi\hat{\psi} \leq \psi\hat{\psi}\psi\hat{\psi} \leq \psi\hat{\psi}\hat{\psi}\hat{\psi} = \psi\hat{\psi}$$

and $\psi\hat{\psi}$ is a filter. It is also a $\vee$-filter, since

$$\psi\hat{\psi} \leq \psi\hat{\psi}(I \vee \psi\hat{\psi}) \leq \psi\hat{\psi}(I \vee \hat{\psi}\hat{\psi}) = \psi\hat{\psi},$$

which implies $\psi\hat{\psi}(I \vee \psi\hat{\psi}) = \psi\hat{\psi}$. Finally, if ψ' is a $\vee$-filter, with $\psi' \geq \psi$, then $\psi' = \psi'\hat{\psi}' \geq \psi$ implies $\hat{\psi}' \geq \hat{\psi}$, hence $\psi' = \psi'\hat{\psi}' \geq \psi\hat{\psi}$. □

Corollary 3.11 *When $\mathcal{T}$ is a finite lattice, then*

$$\textit{for } n \textit{ large enough,} \qquad \psi\hat{\psi} = \psi(I \vee \psi)^n. \tag{54}$$

Indeed, corollary 3.10 associates two envelopes with each filter ψ, so that ψ is finally surrounded by four extremum filters as follows:

$$\check{\psi} \leq \psi\check{\psi} \leq \psi \leq \psi\hat{\psi} \leq \hat{\psi}. \tag{55}$$

We have seen that the product $\gamma\phi$ of any closing followed by any opening was a $\vee$-filter. We will prove now that the converse is true, so that the $\vee$-property **characterizes** the class of the filters of the type $\gamma\phi$.

Theorem 3.12 *A mapping $\psi \in \mathcal{T}'$ is a $\vee$-filter (resp. a $\wedge$-filter) if and only if there exist an opening γ and a closing ϕ such that $\psi = \gamma\phi$ (resp. $\psi = \phi\gamma$).*

<u>proof</u>: Assume that ψ is a $\vee$-filter and consider its invariance domain $\mathcal{B}$. Denote by $\underset{\sim}{\mathcal{B}}$ the class closed for the sup which is generated by $\mathcal{B}$, and by $\underset{\sim}{I}$ the associated opening. Clearly, we have $\psi \geq \underset{\sim}{I}$. Moreover, according to criterion 3.4, $\mathcal{B} \subseteq \underset{\sim}{\mathcal{B}}$ implies $\psi = \underset{\sim}{I}\,\psi$ and (theorem 3.9) $\psi = \underset{\sim}{I}\,\psi\hat{\psi}$. Thus, we may write:

$$\psi = \underset{\sim}{I}\,\psi = \underset{\sim}{I}\,\psi\hat{\psi} \begin{cases} \geq \underset{\sim}{I}\,\hat{\psi} & \text{for} \quad \psi \geq \underset{\sim}{I}, \\ \leq \underset{\sim}{I}\,\hat{\psi} & \text{for} \quad \psi \leq \hat{\psi}. \end{cases}$$

Hence, $\psi = \underset{\sim}{I}\,\hat{\psi}$, i.e. the composition product of a closing by an opening. □

<u>Remark</u>: the above decomposition is not unique. We also have, for a $\vee$-filter ψ, $\psi = \check{\psi}\hat{\psi}$.

Example: the rank operators We will illustrate the important theorem 3.9 by considering some properties of the rank operators [19, 8, 20]. Let f be a function from $\mathbb{Z}^n$ into $\mathbb{Z}$ and let $B \subset \mathbb{Z}^n$ be a finite and symmetrical set of p points implanted at the origin. B is the (moving) window in which the ranking operation will be implemented. The transform R_k of rank k of f at point $x \in \mathbb{Z}^n$ is obtained by ordering the family $(f(y))_{y \in B_x}$ with decreasing values (for example), and by replacing $f(x)$ by the k-th value of the sequence that is thus constructed ($1 \leq k \leq p$). For $k = p$ and $k = 1$, this leads to Minkowski addition and subtraction. When $k = \frac{1}{2}(p+1)$ and p is an odd number, the resulting operation is sometimes called *median filtering,* [37] in literature.

The rank operator R_k of rank k is increasing, since it can be decomposed into the sup of the t-erosions ε_i by all the $B_i \subseteq B$ which possess k points:

$$R_k(f) = \vee\{\varepsilon_i(f), B_i \subseteq B, \mathrm{Card}(B_i) = k\}. \tag{56}$$

Now, $\psi_i = \delta_B \varepsilon_i$ is a $\wedge$-overfilter. Indeed:

$$\psi_i \gamma_i = \delta_B \varepsilon_i \delta_i \varepsilon_i \begin{cases} \geq \delta_B \varepsilon_i = \psi_i & \text{for} \quad \varepsilon_i \delta_i \geq I, \\ \leq \delta_B \varepsilon_i = \psi_i & \text{for} \quad \delta_i \varepsilon_i \leq I, \end{cases}$$

and $\psi_i \gamma_i = \psi_i$. On the other hand, $\gamma_i \leq \psi_i$ implies $\gamma_i \leq I \wedge \psi_i \leq I$, hence $\gamma_i = \gamma_i(I \wedge \psi_i)$ and finally, $\psi_i = \psi_i(I \wedge \psi_i)$. Rel. (56) shows that $\delta_B R_k = \vee \psi_i$ is still a $\wedge$-overfilter (theorem 3.7), and by application of theorem 3.9, $I \wedge \delta_B R_k$ is an opening (called Ronse opening). In particular, for $k = 1$, one finds the morphological opening by B, and for $k = p$, the identity I. The other openings increase with k.

As another application of the results presented in this section, one can refer to an interesting study of F. Meyer [18], where the author directly transcribes into practice and into algorithms the above theorems. Some of the filters which are thus brought to the fore stem directly from classical median filters [37]. Besides, in two papers by P. Maragos [12, 13] which are among the most interesting ones in the recent literature on morphological filters, one can find a thorough study on the relations between morphological filters and non-morphological ones, namely median filters, rank filters and stack filters...

3.5.4 The lattice of the strong filters

Starting from an $\wedge$-overfilter f', the first corollary of theorem 3.9 ensures that $f = f'\hat{f}'$ is an $\vee$-filter. It would be excellent if it could also keep the $\wedge$-overfiltering property of its primitive f'. In this case, we would have found the key for producing **strong** filters. The answer will actually be positive in the case of modular lattices $\mathcal{T}$, i.e. such that

$$\forall A, B, C \in \mathcal{T}, \qquad B \geq A \implies (A \vee C) \wedge B = A \vee (B \wedge C)$$

(Except the partition lattice, all the lattices used as models in morphology are modular.). Then, we have the following lemma:

Lemma 3.13 *When $\mathcal{T}$ is modular, then*

1. *$f \circ (I \wedge f)$ is a $\vee$-filter for any $\vee$-filter f,*
2. *$g \circ (I \vee g)$ is a $\wedge$-filter for any $\wedge$-filter g.*

proof: easy. Refer to [25, page 124]. □

Theorem 3.14 *When the lattice $\mathcal{T}$ is modular, if f' is a $\wedge$-overfilter (resp. an $\wedge$-underfilter), then $f = f'\hat{f}'$ (resp. $g = g'\check{g}'$) is a strong filter.*

proof: Let f' be a $\wedge$-overfilter and $f = f'\hat{f}'$. We have

$$f' = f'(I \wedge f') \leq f(I \wedge f) \leq f.$$

Now, from the lemma, $f(I \wedge f)$ is a $\vee$-filter. Since it also majorates f', it is larger than the smallest $\vee$-filter which majorates f', namely f. Hence, $f = f(I \wedge f)$ is strong. □

Corollary 3.15 *When $\mathcal{T}$ is modular, the class of the strong filters on $\mathcal{T}$ is a complete lattice based on the usual ordering. The supremum of a family (ψ_i) of strong filters is $f'\hat{f}'$, with $f' = \vee\psi_i$, and the infimum is given by $g'\breve{g}'$, with $g' = \wedge\psi_i$.*

Examples:

1. Theorem 3.14 opens the way for the construction of as many strong filters as we wish, by iterations. It suffices to start from an arbitrary opening γ and an arbitrary closing ϕ: when lattice $\mathcal{T}$ is finite, there exist two integers n and p such that both mappings

   $$\psi = \phi\gamma(I \vee \phi\gamma)^n \qquad \text{and} \qquad \psi' = \gamma\phi(I \wedge \gamma\phi)^n$$

   are strong filters.

2. In some cases, it is not necessary to perform iterations. It is the case, for instance, of the morphological opening γ_l and the closing ϕ_l with respect to a segment l (see § 3.3). The points which change from 0 to 1 in $I \vee \phi_l\gamma_l$ are uniquely those which were initially modified from 1 to 0 by γ_l, and preserved as 0's by ϕ_l (See the example of Fig. 26.). When γ_l acts on $I \vee \phi_l\gamma_l$, we then recover $\phi_l\gamma_l$. Hence $\phi_l\gamma_l$ is strong. However, it is not self-dual, since by theorem 3.5 *(iv)*, we have $\gamma_l\phi_l \geq \phi_l\gamma_l$.

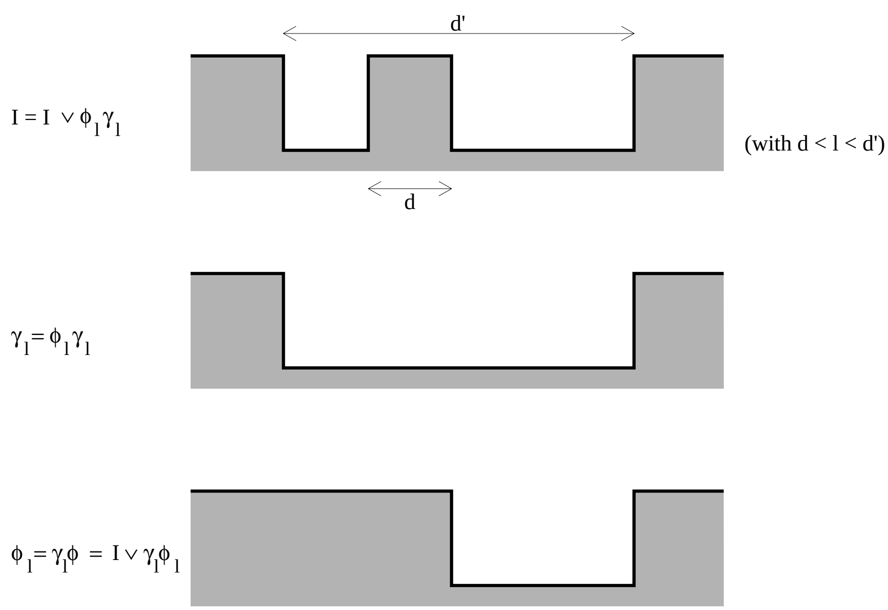


Figure 26: A case where it is not necessary to perform iterations: the opening γ_l and the closing ϕ_l with respect to a segment l.

As a conclusion, we can say that in the framework of morphological filtering, the $\vee$- and $\wedge$-properties are weaker substitutes for extensivity and anti-extensivity (closings and openings are strong filters, but the converse is false). Such a weaker version allows us to combine both properties in filters that are not trivial, whereas the only strong filter to be extensive and anti-extensive at the same time is the identity mapping I. We will see in § 4 how this advantage can be used to produce self-dual filters.

3.6 Filtering as a tool for segmentation

In the present section, we deal with discrete grey-scale images, i.e. we address the lattice $\mathcal{F}(\mathbb{Z}^2, \mathbb{Z}^+)$. One of the main problems with which image analysts are concerned is that of the *segmentation* of these grey-scale images. By segmenting an image, we mean dividing it into regions, one of which standing for the *background* whereas each of the remaining ones represents one of the objects to be extracted. The boundaries of these regions must be as close as possible to the "true" contours of these objects. Therefore, the segmentation task we deal with is nothing but a contour detection problem.

The major difficulty is to define the contours of the image I under study at best. The objects to extract are generally light on a dark background, or are dark on a light background. Therefore, one can imagine defining their contours as the regions of I where the grey values are varying very fast, i.e. as *crest-lines of the gradient* of this image. Depending on the problem, i.e. on the contours that have to be detected, many different gradients may be used. In the field of mathematical morphology, the most commonly used one is often referred to as Beucher's gradient [24]: denote δ and ε respectively the dilation and the erosion with respect to the flat elementary isotropic element (square, hexagon,...). The gradient of I is given by

$$grad(I) = \delta(I) - \varepsilon(I), \tag{57}$$

where $-$ refers to the algebraic difference of two functions. In some particular cases, one can also use directional gradients, disymmetryc ones like $\delta(I) - I$ or $I - \varepsilon(I)$, etc... and various combinations of these different gradients.

Now, to extract the crest-lines of a grey-scale image (namely a gradient image), one of the most appropriate tools is the *watershed transformation* [2]. Let us give here briefly an intuitive introduction to watersheds (for more in-depth presentation, one can refer to [3], [1] or [34, 33]). Consider an arbitrary grey-scale image J, and regard it as a relief or a topographic surface (the grey level of a pixel stands for the elevation at this point). A water drop falling at a given pixel p of J flows down along the relief, following a certain descending path called the *downstream* of p, until it finally reaches a *minimum* M of J, i.e. a connected plateau of pixels at a given level h such that every surrounding pixel has an altitude strictly higher than h. The pixel p is said to belong to the *catchment basin* $C(M)$ associated with minimum M:

Definition 3.16 *The catchment basin associated with a minimum M of a grey-scale image J is the locus of the pixels p such that a drop falling on p eventually reaches M.*

The crest-lines which separate different catchment basins build the *watersheds*—or *dividing lines* for some authors—of image J. These notions are illustrated by Fig. 27.

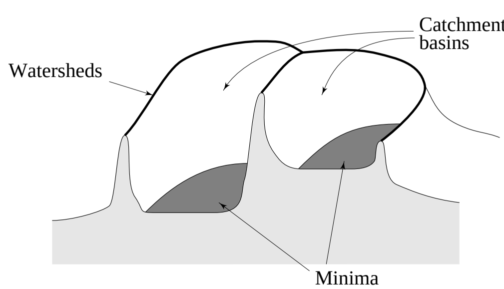


Figure 27: Minima, catchment basins and watersheds.

In fact, the brutal computation of the watersheds of the gradient image $grad(I)$ does not provide a good segmentation of I. Indeed, whatever gradient is used, the simple computation of its watersheds most of the time results in an *over-segmentation:* the correct contours are lost in a mass of irrelevant ones, i.e., I is divided into too many regions. This is true even if one takes the precaution of using some simple filters on the original image I or on its gradient before computing the watershed transformation. This over-segmentation is simply due to the fact that the gradient image exhibits too many minima. To get rid of this problem, one may try to remove the irrelevant contour arcs of the watershed image, but this is usually a difficult task. For some very complex segmentation problems, it may be interesting to work directly on the tessellation of I provided by the above catchment basins: one can consider their associated adjacency graph and resort to region-growing techniques, which can themselves be efficiently approached via watersheds on graphs [31, 32, 33]. However, in most practical cases, there is a much better solution which consists in modifying the gradient function before computing its watersheds. The idea is to make use of *grad*(I) for the construction of a function $\theta(I)$ whose watersheds provide the desired segmentation.

To achieve this goal, a very powerful method was introduced by F. Meyer, which is detailed in [3]. Rather than filtering out the irrelevant minima of the gradient blindly, this technique assumes that markers of the objects of I are available. By marker, we mean a connected component of pixels which is *included* in the object which is marked. The development of a robust automatic marking procedure may well be an extremely complex task, and has to be adapted to each case. Usually, some external knowledge on the collection of images under study has to be used. From now on, we assume that a set of markers is available. We also suppose that a marker of the background has been brought to the fore.

Let S be the set of pixels of I belonging to one of the above markers. Starting from *grad*(I), the construction of $\theta(I)$ is done in two steps:

1. Impose as minima the previously extracted markers.
2. Suppress the unwanted minima.

In step 1, we simply construct the function f defined by:

$$\forall p, \qquad f(p) = \begin{cases} c & \text{when } p \in S, \\ \mathrm{grad}(I)(p) & \text{otherwise,} \end{cases}$$

with c being an arbitrary constant, strictly minorating grad(I).

In the step 2, we have to filter out the unwanted minima of f, without forgetting to fill up their associated watersheds! To do so, we first construct a function, say g, as follows:

$$\forall p, \qquad g(p) = \begin{cases} c & \text{when } p \in S, \\ A & \text{otherwise,} \end{cases}$$

with A being an arbitrary constant majorating grad(I). Then, using the dual grey-scale reconstruction operation ρ_f^* presented in § 2.4.1, we build $\theta(I)$ as follows:

$$\theta(I) = \rho_f^*(g).$$

These two operations are illustrated in Fig. 28.

One can see that the resulting function $\theta(I)$ is such that its dividing lines correspond exactly to the desired contours. Indeed, the highest crest-lines of the gradient separating the markers have been preserved. According to the set of markers available and to the gradient which is being used, we therefore have extracted the best possible contours! Note that, given a set of markers, the transformation $(\mathrm{grad}(I) \longmapsto \theta(I))$ is nothing but *a strong filter:* it is indeed the composition product of a closing by an opening, and can as well be defined as the composition of an opening by a closing... We have thus illustrated how one can develop customized filters adapted to a specific problem—namely the filtering of unwanted minima. The present method turns out to be extremely powerful in a number of complex segmentation cases, since the only problem (which can be itself very complicated!) comes down to detecting the markers of the objects to extract.

4 Granulometries, Alternating Sequential Filters

4.1 Size distributions

Size distributions (also called *granulometries)* deal with families of openings or closings that are parametrized by a positive number (the size) [14]. More precisely, we have the following:

Definition 4.1 *A family (γ_λ) of mappings from $\mathcal{T}$ into $\mathcal{T}$, depending on a positive parameter λ is a size distribution when*

(i) $\forall \lambda > 0, \quad \gamma_\lambda$ is an opening,

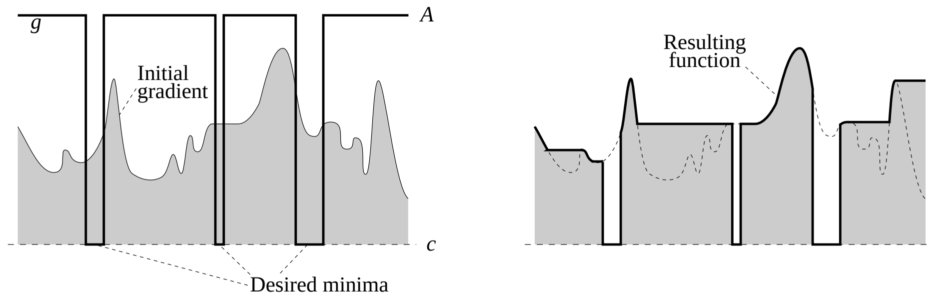


Figure 28: Construction of the function $\theta(I)$ whose watersheds correspond to the desired contours.

and when one of the three equivalent conditions (ii) to (iv) is fulfilled:

(ii) $\lambda, \mu > 0 \implies \gamma_\lambda \gamma_\mu = \gamma_\mu \gamma_\lambda = \gamma_{\sup(\lambda,\mu)}$

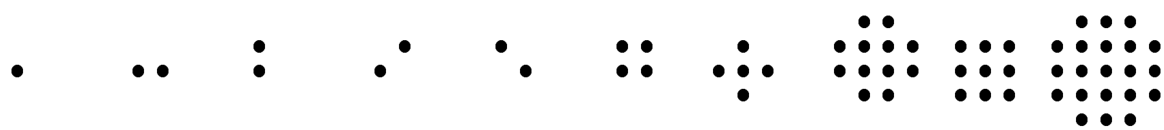
(iii) $\lambda \geq \mu > 0 \implies \gamma_\lambda \leq \gamma_\mu$

(iv) $\lambda \geq \mu > 0 \implies B_\lambda \subseteq B_\mu$.

Similarly, we introduce also *anti-size distributions* as the families of closings (φ_λ), whose dual openings build size distributions.

These conditions are called Matheron's axioms for sized distributions [15, page 192]. It is easy to verify that these conditions are satisfied by every process that common sense would designate as a size distribution.

For example, consider the following structuring elements:



e_0 e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8 e_9

If we denote γ_i the morphological opening with respect to stucturing element e_i, then among the various sequences

γ_0	γ_0	γ_0	γ_0	γ_0	γ_0
$\gamma_1 \vee \gamma_2$	$\gamma_1 \vee \gamma_2 \vee \gamma_3 \vee \gamma_4$	$\gamma_1 \vee \gamma_2$	$\gamma_3 \vee \gamma_4$	$\gamma_1 \vee \gamma_2$	γ_5
γ_5	γ_6	γ_5	γ_5	γ_5	γ_7
γ_7	γ_7	γ_7	γ_8	γ_6	γ_8
γ_9	γ_9	γ_9	γ_9	γ_8	γ_9

the four former ones lead to size distributions, but not the last two ones.

Note: Remark that the convexity of the structuring elements is not a necessity. For instance, the sequence

e_1 e_2 e_3 etc...

induces a size distribution. However, in the Euclidean space $\mathbb{R}^n$, a family $(B_\lambda)_{\lambda \geq 0}$ of structuring elements generates a size distribution $(\gamma_\lambda)_{\lambda \geq 0}$ which is compatible with the magnification, i.e.

$$\forall \lambda \geq 0, \forall X \subset \mathbb{R}^n, \qquad \gamma_\lambda(X) = \lambda \gamma_1(X/\lambda), \tag{58}$$

if and only of the B_λ's are the homotethics of a compact **convex** set B. The signification of rel. (58) is clear: it just means that γ_λ acts on λX just as γ_1 does on X. Such a property, which is always satisfied for convolution products, may not exist for morphological filters. However, in the two important cases of the size distributions and of the alternating sequential filters (See § 4.2), we easily obtain it.

4.2 Alternating Sequential Filters

This section is devoted to the presentation of a particular class of filters, namely the *alternating sequential filters (ASF)*. It provides a simplified version of the results proved in [25, chapter 10]. The theory developed in [25] requires *monotonous sequential continuity* for covering both continuous and discrete cases. For the sake of simplicity, we restrict ourselves here to the latter case (i.e. $\mathcal{P}(\mathbb{Z}^n)$, $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$ or planar graphs), which allows us to avoid monotonous topology. Furthermore, so as not to bore the reader, most of the proofs will be omitted. They are indeed all very similar to one another and the missing ones can be found in [28, chapter 5].

4.2.1 Introduction

The appearance of some alternating sequential filters (of N_i type, see below) in the world of mathematical morphology is due to an experimental work of Sternberg [30] (the corresponding theoretical developments are due to Serra [25, chapter 10]). His study consisted in taking a polyhedric form (namely a cuboctahedron, see § 1.6), altering it by the addition of a largely varying white noise, and then trying to clean the resulting image X (see [25, page 204]). To attain this goal, X was first filtered by a small closing ϕ_1, followed by a small opening γ_1, then by a slightly larger closing ϕ_2 followed by a slightly larger opening γ_2, etc... The final operator produced by this succession of openings and closings was

$$M = (\gamma_i \phi_i) \circ \ldots \circ (\gamma_2 \phi_2) \circ (\gamma_1 \phi_1).$$

The family (ϕ_i) that was used in this example consisted in morphological closings by homothetic structuring elements, whereas (γ_i) was the dual family of openings. After this experiment, a certain number of questions arose: is the operator M a filter? To what extend does it depend on the totality of the sequence of parameters 1, 2,..., i? Is it essential to use a size distribution (γ_i) and its dual (ϕ_i)? Is the product of these operators an operator of the same type?...

In this section, a more formal approach of the above operators is proposed. The ASF's of type M are defined, as well as other classes of ASF's, namely that of type N, R and S. A number of properties of these operators, dealing with composition and order relation, are then given, and the symmetrical sequential filters are also presented. The end of the section is concerned with more concrete problems, such as computation time and practical use of these filters.

4.2.2 Definitions

In the following, $\mathcal{T}$ denotes a discrete lattice such as $\mathcal{P}(\mathbb{Z}^n)$ or $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$. We also consider two families $(\gamma_i)_{i \geq 1}$ and $(\phi_i)_{i \geq 1}$—indexed on $\mathbb{Z}^+/\{0\}$—of operators on $\mathcal{T}$, which are a size distribution and an anti-size distribution respectively, i.e:

$$\forall i \in \mathbb{Z}, i \geq 1, \qquad \gamma_i \text{ is an opening and } \phi_i \text{ is a closing.} \tag{59}$$

$$\forall i, j \in \mathbb{Z}, 1 \leq i \leq j, \qquad \gamma_j \leq \gamma_i \leq I \leq \phi_i \leq \phi_j. \tag{60}$$

Moreover, we have seen in definition 4.1 that the inequalities (60) are equivalent to the following property:

$$\forall i, j \in \mathbb{Z}, \qquad \gamma_i \gamma_j = \gamma_j \gamma_i = \gamma_{\max(i,j)} \text{ and } \phi_i \phi_j = \phi_j \phi_i = \phi_{\max(i,j)} \tag{61}$$

Remark that these two families are chosen **independently** from one another (although they are often taken as dual of each other in practice).

Now, we know (refer to the structural theorem 3.5) that there are four different ways for composing two filters f and g such that $f \leq g$ in order to get new filters. Therefore, we define for all $i \in \mathbb{Z}, i \geq 1$, the following operators

$$\begin{array}{rclcrcl} m_i & = & \gamma_i\phi_i & \qquad & r_i & = & \phi_i\gamma_i\phi_i \\ n_i & = & \phi_i\gamma_i & & s_i & = & \gamma_i\phi_i\gamma_i. \end{array} \tag{62}$$

which are filters (see theorem 3.5). From now on, we give the proofs only for the case of the m_i, the other cases being left to the reader.

Proposition 4.2 *For $i, j \in \mathbb{Z}$ such that $1 \leq i \leq j$, we have:*

$$m_j m_i \leq m_j \leq m_i m_j \qquad \left\{ \begin{array}{c} r_j r_i \\ r_i r_j \end{array} \right\} \leq r_j \tag{63}$$

$$n_i n_j \leq n_j \leq n_j n_i \qquad \left\{ \begin{array}{c} s_j s_i \\ s_i s_j \end{array} \right\} \geq s_j. \tag{64}$$

proof: from $\gamma_i \leq I$, we obtain $\gamma_i\phi_i \leq \phi_i \leq \phi_j$. Hence, $\phi_j\gamma_i\phi_i \leq \phi_j$ and finally, $m_j m_i = \gamma_j\phi_j\gamma_i\phi_i \leq \gamma_j\phi_j = m_j$, which is the first inequality. Similarly, $I \leq \phi_i$ implies $\gamma_i \leq \gamma_i\phi_i$. Thus $\gamma_j = \gamma_i\gamma_j \leq \gamma_i\phi_i\gamma_j$, and $m_j = \gamma_j\phi_j \leq \gamma_i\phi_i\gamma_j\phi_j = m_i m_j$. This completes the proof. □

Proposition 4.3 *Let $(i_k)_{1 \leq k \leq p}$ be p numbers such that*

$$\forall k, i_k \in \mathbb{Z}, 1 \leq i_k \leq i_1 = i_p.$$

then:

$$\left\{ \begin{array}{lclcl} m_{i_p} m_{i_{p-1}} \ldots m_{i_2} m_{i_1} & = & m_{i_p} & = & m_{i_1} \\ n_{i_p} n_{i_{p-1}} \ldots n_{i_2} n_{i_1} & = & n_{i_p} & = & n_{i_1} \end{array} \right.$$

proof: Easy by using rel. (63). Refer to [28, page 47]. □

We can now give the definition of the alternating sequential filters and prove that they are effectively filters:

Definition 4.4 *For all $i \in \mathbb{Z}, i \geq 1$, the following operators:*

$$\boxed{\begin{array}{rclcrcl} M_i & = & m_i m_{i-1} \ldots m_2 m_1 & \qquad & R_i & = & r_i r_{i-1} \ldots r_2 r_1 \\ N_i & = & n_i n_{i-1} \ldots n_2 n_1 & & S_i & = & s_i s_{i-1} \ldots s_2 s_1 \end{array}}$$

are called alternating sequential filters of order i.

Proposition 4.5 *$\forall i \in \mathbb{Z}, i \geq 1$, the operators M_i, N_i, R_i and S_i are filters.*

proof: These operators are increasing as compositions of increasing mappings. Moreover, we have:

$$\begin{array}{rcll} M_i M_i & = & (m_i m_{i-1} \ldots m_2 m_1 m_i)(m_{i-1} \ldots m_2 m_1) & \\ & = & m_i(m_{i-1} \ldots m_2 m_1) & \qquad \text{(by prop. 4.3)} \\ & = & M_i. & \end{array}$$

The idempotence of M_i is thus proved. One can also easily show that N_i, R_i and S_i are filters. □

4.2.3 Properties

We have already said that there is no need for duality between the γ_i's and the ψ_i's. Anyway, an ASF *cannot be self dual,* since $N_i \neq M_i$. We present below some properties of the ASF's with respect to the composition product and the order relationship between operators : the "absorption laws". We then deal with new filters derived from the previous ones and called *symmetrical alternating filters.*

Proposition 4.6 (Absorption laws) *For $i, j \in \mathbb{Z}$ such that $1 \leq i \leq j$, we have the following relations:*

$$M_j M_i = M_j \leq M_i M_j \qquad R_i R_j \leq R_j = R_j R_i, \tag{65}$$

$$N_i N_j \leq N_j = N_j N_i \qquad S_j S_i = S_j \leq S_i S_j. \tag{66}$$

proof: similar to that of proposition 4.5. □

Definition 4.7 *With for all $i \in \mathbb{Z}, i \geq 1$:*

$$\begin{array}{rclrcl} M_i^t & = & m_1 m_2 \ldots m_{i-1} m_i & R_i^t & = & r_1 r_2 \ldots r_{i-1} r_i \\ N_i^t & = & n_1 n_2 \ldots n_{i-1} n_i & S_i^t & = & s_1 s_2 \ldots s_{i-1} s_i \end{array} \tag{67}$$

the following operators:

$$\begin{array}{rclrcl} \tilde{M}_i & = & M_i^t M_i & \tilde{R}_i & = & R_i^t R_i \\ \tilde{N}_i & = & N_i^t N_i & \tilde{S}_i & = & S_i^t S_i \end{array} \tag{68}$$

are called alternating symmetrical filters of order i.

One can easily prove that these operators are indeed filters. They also satisfy absorption laws and other properties, which are listed and proved in [28]. In addition, we have the following theorem :

Theorem 4.8 *The set $\{\tilde{M}_i, i \in \mathbb{Z}, i \geq 1\}$ (resp. $\{\tilde{N}_i, i \in \mathbb{Z}, i \geq 1\}$, $\{\tilde{R}_i, i \in \mathbb{Z}, i \geq 1\}$, $\{\tilde{S}_i, i \in \mathbb{Z}, i \geq 1\}$) constitutes a commutative semi-group of filters satisfying the following internal law of composition:*

$$\forall i \geq 1, j \geq 1, \qquad \tilde{M}_i \tilde{M}_j = \tilde{M}_{\sup(i,j)}.$$

Note that, apart from the above ones, the only known semi-groups of this type are the granulometries.

Within each of the four types of operations that we have described in this section (types M, N, R and S), one can find interesting equalities and inequalities. The most important ones are detailed in [28], together with the corresponding proofs. These formulas lead to a better understanding of the ASF's and tell us how they should be composed to get new filters. Moreover, they help simplifying the expression of a given filter until the most "compact" expression is reached. This allows to optimize the computation times.

Compatibility under magnification When the two primitive families (γ_i) and (ϕ_i) are compatible under magnification, i.e.

$$\forall X, \forall k > 0, \quad \left\{ \begin{array}{rcl} \phi_k(kX) & = & k\phi_1(X), \\ \gamma_k(kX) & = & k\gamma_1(X), \end{array} \right. \tag{69}$$

this property is transmitted to the corresponding ASF's. The physical interpretation of this is clear: it means that the ASF of order k works on the k times magnified image exactly as does the ASF of order 1 on the initial image. In fact, it is probably this property which makes the ASF's of such a common use.

4.2.4 Applications, computation time

In practice, the ASF's are among the most efficient morphological filters and can be finely adjusted to each case. Indeed, one can operate on:

- the families (γ_i) and (ϕ_i) (most of the time, this comes down to choosing families of structuring elements),

- the type of filter (alternating sequential filter, alternating symmetrical filter),
- the "size" of the filter.

The only problem with these filters is that of the computation time. Indeed, although the formulas for computing them can be "compacted", a certain number of elementary operations has yet to be performed in each case. As an example, suppose that the computation times of ϕ_i and γ_i are equal to $i \times \Delta t$, for a fixed image size. Then, the computation time of M_i, N_i, etc... is proportional to i^2, as shown by table 1, where we suppose that the most efficient fomulas are used.

Filter	Computation time	Computation time for $i = 5$
M_i, N_i	$i(i+1)\Delta t$	$30 \times \Delta t$
R_i, S_i	$i(i+2)\Delta t$	$35 \times \Delta t$
$\tilde{M}_i, \tilde{N}_i$	$2i(i+1)\Delta t$	$60 \times \Delta t$
$\tilde{R}_i, \tilde{S}_i$	$(2i^2 + 2i + 1)\Delta t$	$65 \times \Delta t$

Table 1: Computation time of some sequential filters, provided that the time required for computing γ_i or ϕ_i equals $i \times \Delta t$ (for a fixed image size).

5 Activity lattice, toggle mappings

5.1 Toggling, optimization and self duality

In § 1, we have seen that, given a set E, the two sets $\mathcal{P}(E) = \{X, X \subseteq E\}$ and $\mathcal{F}(E, \overline{\mathrm{IR}}) = \{f, f : E \longrightarrow \overline{\mathrm{IR}}\}$ are complete and distributive lattices. The first one is even complemented. Later, in § 3, we have associated with any complete lattice $\mathcal{T}$ the set $\mathcal{T}'$ of the increasing mappings applying $\mathcal{T}$ on itself, and shown that $\mathcal{T}'$ is in turn a complete lattice. When starting from $\mathcal{P}(E)$ or $\mathcal{F}(E, \overline{\mathrm{IR}})$, we shall denote this second lattice $\mathcal{P}'$ or $\mathcal{F}'$ respectively. More generally, the set of all the (increasing *or not*) mappings $\psi : \mathcal{T} \longrightarrow \mathcal{T}$ is still a complete lattice $\mathcal{T}''$ (same approach as for $\mathcal{T}'$ in § 3), denoted $\mathcal{P}''$ and $\mathcal{F}''$ in the two cases of sets and of functions.

The idea developed in the present section consists in associating with a function f (or with a set X):

1. *a* series *of possible transforms* $\psi_i f$*,*
2. *a* toggling criterion, *i.e. a decision rule which determines at each point* $x \in E$ *the "best" value among the "candidates"* $(\psi_i f)(x)$*.*

Such an approach allows us to introduce optimality conditions. The "best" choice may be understood in the sense of noise reduction, of contrast enhancement, or of both. It may also be understood, for multispectral images, in the sense of edge matching between channels, etc... The toggling approach corresponds to the various minimization techniques involved in linear processings (e.g. minus squares). But there is no quadratic form here. The candidate mappings ψ_i play the role of the Lagrange coefficients. Moreover, for minimizing the criterion, the inf is used rather than derivatives.

Toggle mappings are not necessarily self dual for the complement (set case) or for the negation (function case). However, thay can *always* admit a self dual version. It is obtained by taking the self dual family (ψ_i, ψ_i^*) generated by any family (ψ_i) of primitives, and by symmetrizing the toggling criteria with respect to duality. We shall give an explicit example of such a procedure in § 5.3.1, where *morphological centers* are concerned. In the following, we take it for granted that all the algorithms presented in this section possess the property of having self dual versions, and we shall not repeat this point after each result.

5.2 Activity lattice and semi-lattice

5.2.1 Boolean case

Let $\mathcal{P}(E)$ be a boolean lattice, and $\mathcal{P}'$, $\mathcal{P}''$ be the two associated operator lattices. Equip $\mathcal{P}''$ with the following relationships:

$$\forall \psi_1, \psi_2 \in \mathcal{P}'', \quad \psi_1 \preceq \psi_2 \Longleftrightarrow \begin{cases} I \cap \psi_1 & \supseteq & I \cap \psi_2, \\ I \cup \psi_1 & \subseteq & I \cup \psi_2. \end{cases} \tag{70}$$

Note that the second inequality may be rewritten

$$\Theta \cap \phi_1 \subseteq \Theta \cap \phi_2,$$

with Θ standing for the complement operator, i.e. $\Theta : X \longmapsto X^C$.)

Obviously, relation $\preceq$ is an ordering relationship. Moreover, it makes $\mathcal{P}''$ a complete lattice. Indeed, any minorant m of a family (ψ_i) in $\mathcal{P}''$ satisfies the following system:

$$\begin{cases} I \cap m & \supseteq & I \cap \zeta, \\ I \cup m & \subseteq & I \cup \eta, \end{cases} \tag{71}$$

with

$$\zeta = \bigcup_i \psi_i \qquad \text{and} \qquad \eta = \bigcap_i \psi_i. \tag{72}$$

Therefore, if there exists a mapping β verifying

$$\begin{cases} I \cap \beta & = & I \cap \zeta \\ I \cup \beta & = & I \cup \eta, \end{cases} \tag{73}$$

then β is larger, for order relation $\preceq$, than any minorant m. Note that we can modify the second equality of (73) by using again the complement operator Θ:

$$I \cup \beta = I \cup \eta \iff \Theta \cap \beta = \Theta \cap \eta.$$

Now, system (73) admits one and one only solution, since every element $\alpha \in \mathcal{P}''$ may be decomposed into:

$$\alpha = (I \cup \Theta) \cap \alpha = (I \cap \alpha) \cup (\Theta \cap \alpha). \tag{74}$$

By applying this remarkable identity to β, we find:

$$\begin{aligned} \beta &= (I \cap \zeta) \cup (\Theta \cap \eta) \\ &= [(I \cap \zeta) \cup \eta] \cap [(I \cap \zeta) \cup \Theta] \\ \Longrightarrow \dot{\wedge} \Psi_i = \beta &= (I \cup \eta) \cap \zeta = (I \cap \zeta) \cup \eta, \end{aligned} \tag{75}$$

a quantity which satisfies system (73) (Note that in the present section, $\dot{\wedge}$ denotes the inf associated with the order relation $\preceq$ of lattices $\mathcal{P}''$ or $\mathcal{F}''$. It is called the *activity* inf. In the following, $\dot{\vee}$ will also denote the *activity* sup.). Similarly, if there exists a *smaller majorant* δ to the family (ψ_i), it must satisfy the system:

$$\begin{cases} I \cap \delta & = & I \cap \eta, \\ I \cup \delta & = & I \cup \zeta \Longleftrightarrow \Theta \cap \delta = \Theta \cap \zeta. \end{cases} \tag{76}$$

By using again identity (74), we find this time:

$$\dot{\vee} \psi_i = \delta = (\Theta \cup \eta) \cap \zeta = (\Theta \cap \zeta) \cup \eta. \tag{77}$$

Therefore, we have a lattice, which turns out to be complete and distributive (easy proof). We go from the inf $\dot{\wedge}$ to the sup $\dot{\vee}$ by replacing the identity I by the complementation Θ. As a matter of fact, the two operators I and Θ are just the null element and the universal element of the lattice generated by order $\preceq$.

Applied to set $A \subseteq E$, the first one does not modify any point of A or of A^C, whereas the second one modifies all the points of A and of A^C.

Given two mappings ψ_1 and ψ_2 in $\mathcal{P}''$, we shall say that ψ_2 is more **active** than ψ_1 when $\psi_2 \succeq \psi_1$. By acting on a set $A \subseteq E$, ψ_2 removes more points from A than ψ_1 does, and also ψ_2 adds more points to A than ψ_1 does (which is nothing but the geometrical meaning of system (70)). This gives the following:

Theorem 5.1 *Let E be an arbitrary set, and let $\mathcal{P}''$ be the set of the mappings from $\mathcal{P}(E)$ into itself. The two inclusions*

$$\forall \psi_1, \psi_2 \in \mathcal{P}'', \quad \begin{cases} I \cap \psi_1 & \supseteq & I \cap \psi_2, \\ I \cup \psi_1 & \subseteq & I \cup \psi_2, \end{cases}$$

generate an ordering relation $\psi_1 \preceq \psi_2$ on $\mathcal{P}''$, to which is associated a complete and distributive lattice. It is called the activity lattice $\mathcal{A}$. *The null element is the identity mapping I and the universal element is the complementation mapping Θ. If $\eta = \cap \psi_i$ and $\zeta = \cup \psi_i$ denote respectively the intersection and the union of a family (ψ_i) of elements of $\mathcal{P}''$, then the* inf β *and the* sup δ *of family (ψ_i) with respect to the activity are called* **center** *and* **anti-center** *of the ψ_i's and are given by the expressions:*

$$\begin{array}{lllllll} \dot{\wedge}\psi_i & = & \beta & = & (I \cup \eta) \cap \zeta & = & (I \cap \zeta) \cup \eta, \\ \dot{\vee}\psi_i & = & \delta & = & (\Theta \cup \eta) \cap \zeta & = & (\Theta \cap \zeta) \cup \eta. \end{array}$$

5.2.2 Function case

The activity ordering (70) applies obviously to the lattice $\mathcal{F}''$ of the mappings acting on $\mathcal{F}(E, \overline{\mathbb{R}})$, by changing the symbols $\subseteq$, $\supseteq$, $\cup$ and $\cap$ into $\leq$, $\geq$, $\vee$ and $\wedge$ respectively. The existence and the unicity of the activity inf β derive from the fact that $\mathcal{F}''$ is completely distributive [25, page 164]. Thus, $\mathcal{F}''$ turns out to be an inf semi-lattice for the activity order, but *no more.* As a set, the activity sup of subgraphs of functions exists, but it is not itself a subgraph (see Fig. 29).

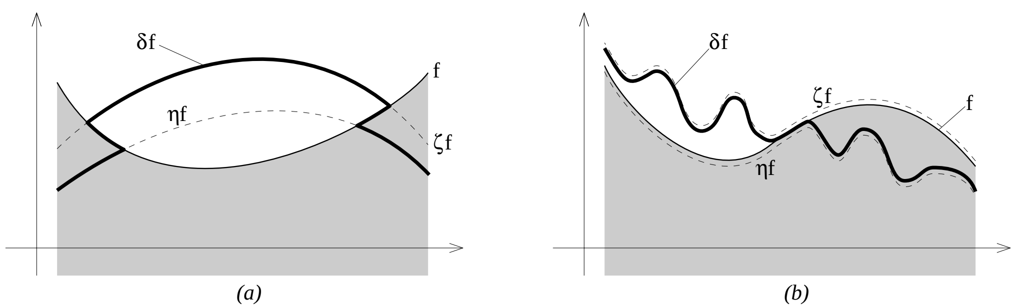


Figure 29: (a) The anti-center $\dot{\vee}$ of ζ and η may no longer be a function. (b) The conditions of theorem 5.2 are satisfied, so that, in this case, δf is a function.

The question which arises then is to know which conditions two mappings η and ζ in $\mathcal{F}''$, $\eta \leq \zeta$, must satisfy to define one and only one δ such that

$$\begin{cases} I \wedge \delta & = & I \wedge \eta, \\ I \vee \delta & = & I \vee \zeta. \end{cases} \tag{78}$$

(i.e. the equivalent of rel. (76)). For the sake of clarity, we shall introduce the following simplification in the notation:

> *when a condition applies to all the elements f of $\mathcal{F}(E, \overline{\mathbb{R}})$, function f is removed from the notation (For example, if at a given point $x \in E$, ηf must always be larger than or equal to f, we shall write $\eta_x \geq I_x$ instead of $\forall f \in \mathcal{F}(E, \overline{\mathbb{R}}), (\eta f)(x) \geq f(x)$.).*

Using this notation, we may state the following:

Theorem 5.2 *Let $\mathcal{F}(E,\overline{\mathrm{IR}})$ be the class of the functions $f : E \longrightarrow \overline{\mathrm{IR}}$, and $\mathcal{F}''$ be the family of the mappings from $\mathcal{F}(E,\overline{\mathrm{IR}})$ into itself. The elements of $\mathcal{F}''$ constitute an inf semi-lattice for the activity ordering $\preceq$. Moreover, the sup $\delta = \zeta\dot{\vee}\eta$ of an ordered pair (ζ,η) of $\mathcal{F}''$ exists if and only if:*

$$\{x \in E, \eta_x < I_x\} \cap \{x \in E, \zeta_x > I_x\} = \emptyset. \tag{79}$$

The activity sup δ is then given by

$$\forall x \in E, \quad \delta_x = \begin{cases} \eta_x & \textit{when} \quad \eta_x < I_x, \\ \zeta_x & \textit{when} \quad \zeta_x > I_x, \\ I_x & \textit{when} \quad \eta_x = \zeta_x = I_x. \end{cases} \tag{80}$$

proof: easy, refer to [17]. □

In particular, we have:

Corollary 5.3 *Let τ and θ be respectively an extensive and an anti-extensive mapping from $\mathcal{F}(E,\overline{\mathrm{IR}})$ into itself. Their activity sup $\delta = \tau\dot{\vee}\theta$ maps also $\mathcal{F}(E,\overline{\mathrm{IR}})$ into itself if and only if, for every $f \in \mathcal{F}(E,\overline{\mathrm{IR}})$, the two domains*

$$X(f) = \{x \in E, (\theta f)(x) \neq f(x)\} \quad \textit{and} \quad Y(f) = \{x \in E, (\tau f)(x) \neq f(x)\}$$

are disjoined.

Finally, we draw two lessons from this section:

1. Any toggle mapping on $\mathcal{F}(E,\overline{\mathrm{IR}})$ using the activity ordering will also apply on $\mathcal{P}(E)$, since the former case is more restrictive. Therefore, in the following, we shall concentrate exclusively on $\mathcal{F}(E,\overline{\mathrm{IR}})$, $\mathcal{F}'$ and $\mathcal{F}''$.

2. The dissymmetry between the activity inf and sup suggests us to treat as a priority the simpler extremum, namely the inf, also called the center.

5.3 Morphological center

5.3.1 Geometrical interpretation

The center β of a family (ψ_i) in $\mathcal{F}''$ is itself a mapping on $\mathcal{F}(E,\overline{\mathrm{IR}})$, i.e. an element of $\mathcal{F}''$, whose definition algorithm derives from relation (75):

$$\dot{\wedge}\psi_i = \beta = (I \vee \eta) \wedge \zeta = (I \wedge \zeta) \vee \eta, \tag{81}$$

with $\eta = \vee\psi_i$ and $\zeta = \wedge\psi_i$. Clearly, algorithm (81) is equivalent to the following system:

$$\begin{cases} I \vee \zeta & \leq & \beta & \leq & \zeta, \\ \zeta & \leq & \beta & \leq & I \vee \eta. \end{cases} \tag{82}$$

When applied, for example, to images $\psi_i(f)$ of a real-valued function f on E, this system states that if at point $x \in E$, all the $(\psi_i f)(x)$ are above $f(x)$, then we take the lowest value. If, on the other hand, they are all below $f(x)$, then we take the highest value. In all other cases, we leave $f(x)$ as it is (see Fig. 30). In particular, when $i = 2$, i.e. when ψ_1 and ψ_2 are the only primitives we have, then the morphological center is nothing but the median of I, ψ_1 and ψ_2. This bridges gap between the approach developed by Yli-Harja, Astola and Neuvo in [37] and the present one.

We are able to produce as many self dual centers as we may desire. It is sufficient to start with a self dual family (ψ_i). In particular, to each element $\psi \in \mathcal{F}''$, there corresponds a self-dual element $\beta(\psi) \in \mathcal{F}'$ that is the inf $\psi\dot{\wedge}\psi^*$, where $\psi^*(f) = -\psi(-f)$. In fact,

$$\beta^* = [I \vee (\psi \vee \psi^*)]^* \wedge (\psi \wedge \psi^*)^* = [I \vee (\psi \wedge \psi^*)] \wedge (\psi \vee \psi^*) = \beta. \tag{83}$$

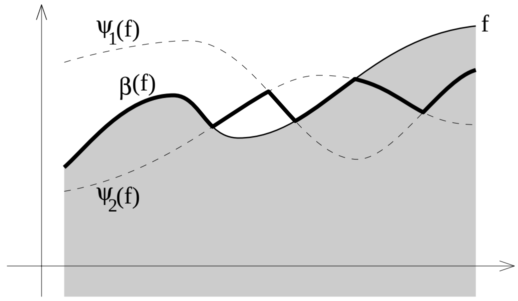


Figure 30: Morphological center β between two increasing mappings ψ_1 and ψ_2, when applied to the real-valued function f. We see the two properties that are common to all "central value mappings" (average, median, etc...) of an image: they wander between $\psi_1(f)$ and $\psi_2(f)$ and are closer to the initial function f.

We now carry on with increasing centers. Indeed, when the ψ_i's are increasing, we see from (81) that they transmit to β their growth property (as well as their possible planarity). Then, the center mapping has the advantage of not damaging high frequencies—as do convolutions—, that of commuting with anamorphosis, etc. Now, the rank operators, and in particular the *median filtering* (see § 3.5.3), are also increasing and planar. However, they may oscillate under iteration (see Fig. 31) and even become periodic for some starting functions [25, page 160].

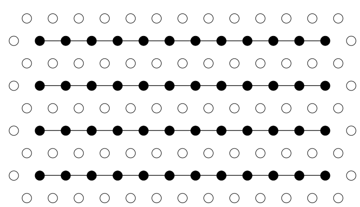


Figure 31: Oscillations of a median filter. The neighborhood B_p of pixel p consists in the seven points of the elementary hexagon $H(p)$ centered at point p. The transformation replaces $f(p)$ by the median of the histogram of values of $H(p)$. When acting on parallel lines, it oscillates...

We shall try now to avoid this drawback by building up increasing centers which do not oscillate under iteration and tend toward morphological filters.

5.3.2 Iterations of increasing centers

A center β, of primitives (ψ_i), does not oscillate if and only if, for any function $f \in \mathcal{F}(E, \overline{\mathbb{R}})$ and for all points $x \in E$, we have

$$\begin{array}{rl} \text{either} & f(x) \leq (\beta f)(x) \leq (\beta \circ \beta f)(x) \leq (\beta^3 f)(x) \leq \ldots \\ \text{or} & f(x) \geq (\beta f)(x) \geq (\beta \circ \beta f)(x) \geq (\beta^3 f)(x) \leq \ldots \end{array}$$

In terms of operators, this point monotonicity means exactly that the activity of the successive powers β^n of β increases with n, i.e. $\forall n \geq 0, \beta^{n+1} \succeq \beta^n$. We now present two classes of β, where such an activity growth is satisfied. In practice, these two classes cover allmost all situations (for a general theory, see [25, page 166]).

5.3.2.1 ζ $\wedge$-overfilters, η $\vee$-underfilters We always have

$$\beta \circ \beta = (\beta \wedge \zeta\beta) \vee \eta\beta = (I \wedge \zeta \wedge \zeta\beta) \vee (\eta \wedge \zeta\beta) \vee \eta\beta. \tag{84}$$

Moreover, when ζ is an $\wedge$-overfilter and η is a $\vee$-underfilter, the inequalities (82) imply that

$$\eta\beta \leq \eta = \eta(I \vee \eta) \leq \zeta = \zeta(I \wedge \zeta) \leq \zeta\beta,$$

and according to rel. (84), $\beta\beta = \beta$. The center β is thus a **morphological filter.** Such a case occurs, among others, when the ψ_i's are strong filters. In such a case indeed, ζ is the sup of $\wedge$-filters and η the inf of $\vee$-filters. Moreover, when the ψ_i's are strong, β admits the following double decomposition:

$$\beta = (I \vee \eta)(I \wedge \zeta) = (I \wedge \zeta)(I \vee \eta) = \hat{\eta}\check{\zeta} = \check{\zeta}\hat{\eta}, \tag{85}$$

i.e., by theorem 3.12, it is itself a strong filter, called the *middle element* between η and ζ.

Examples:

- We have seen in § 3.5.4 that the composition product $\phi_{l,\alpha}\gamma_{l,\alpha}$ of the morphological opening $\gamma_{l,\alpha}$ and closing $\phi_{l,\alpha}$ with respect to a segment of length l and direction α is a strong filter (in $\mathbb{R}^n$ or in $\mathbb{Z}^n$). Put $\psi_\alpha = \phi_{l,\alpha}\gamma_{l,\alpha}$. Then, in $\mathbb{R}^2$ for instance, the family $(\psi_\alpha)_{\alpha\in[0,\pi]}$ admits a strong, isotropic and self-dual filter β as center. Note that, by starting from $\gamma_{l,\alpha}\phi_{l,\alpha}$, we obtain *another* center β', strong itself, isotropic and self-dual, but such that $\beta' \geq \beta$.
- We have also seen in § 3.5.3 that the median transformation m in a given neighborhood B was associated with the two envelopes
$$\delta_B m \quad \text{and} \quad \varepsilon_B m,$$
where δ_B and ε_B stand for the Minkowski addition and subtraction respectively. Now, since $\delta_B m$ is a $\wedge$-overfilter and $\varepsilon_B m$ is a $\vee$-underfilter such that $\delta_B m \geq \varepsilon_B m$, then the center
$$\beta = (I \wedge \delta_B m) \vee \varepsilon_B m$$
is idempotent. Therefore, it is a morphological filter.

5.3.2.2 ζ $\vee$-underfilters, η $\wedge$-overfilters Under these new assumptions, we have $\zeta \leq \zeta(I \vee \eta) \leq \zeta(I \vee \zeta) = \zeta$, and

$$\beta = (I \vee \eta) \wedge \zeta = (I \vee \eta) \wedge \zeta(I \vee \eta) = (I \wedge \zeta)(I \vee \eta),$$

as well as, by duality $\beta = (I \vee \eta)(I \wedge \zeta)$. This gives us

$$\beta^n = (I \wedge \zeta)^n (I \vee \eta)^n = (I \vee \eta)^n (I \wedge \zeta)^n.$$

We know, from theorem 2.8, that when the lattice $\mathcal{F}(E, \overline{\mathbb{R}})$ is finite, then for a certain $n_0 < +\infty$, $(I \vee \zeta)^{n_0} = \check{\zeta}$ and $(I \vee \eta)^{n_0} = \hat{\eta}$. Hence,

$$\beta^{n_0} = \check{\zeta}\hat{\eta} = \hat{\eta}\check{\zeta}.$$

The n_0-th iteration of β yields the middle element between ζ and η, in the sense of rel. (85). It is still a strong filter, since it is written as a *commutative* product of an opening and a closing (theorem 3.12). Moreover, we have for all n:

$$I \wedge \beta^n = (I \wedge \zeta)^n \quad \text{and} \quad I \vee \beta^n = (I \vee \eta)^n,$$

so that the successive powers of β are more and more active. When applied to a function f, the sucessive transforms $\beta^n f$ coincide with $(f \wedge \zeta f)^n$ in the zones where $(\beta f)^n$ decreases, and with $(f \vee \eta f)^n$ in the zones where $(\beta f)^n$ increases.

Example: It suffices to start from an arbitrary opening γ and an arbitrary closing ϕ, and to take $\zeta = \phi\gamma\phi$ and $\eta = \gamma\phi\gamma$. For small structuring elements (size one or two), the range of n_0 goes from two to five. For larger ones (e.g. size five), this range goes up to ten.

5.4 Toggle mappings

A toggle mapping ω is defined, on the one hand, by a family (ϕ_i) of reference mappings called *primitives,* and, on the other hand, by a *decision rule* which makes, at each point x, ω_x equal to one of the primitives $\psi_{i,x}$. The first example of a toggle mapping which comes in mind is that of the thresholding operation. The primitives are the white and the black, and the decision rule involves, at point x the value $f(x)$ and that of a constant, namely the threshold level. Note that thresholding is an idempotent operator. In this simple example, the two primitives are independent from the function f under study. But it is not always the case, and we have just seen toggle mappings such as the center, where the primitives are themselves transformations acting on the initial image. This leads us to the following formal definition:

Definition 5.4 *Let $\mathcal{F}(E,\overline{\mathbb{R}})$ be the class of the functions $f : E \longrightarrow \overline{\mathbb{R}}$, and $\mathcal{F}''$ be that of the mappings from $\mathcal{F}(E,\overline{\mathbb{R}})$ into itself. Given a family (ψ_i) of elements of $\mathcal{F}''$, one calls toggle mapping of primitives (ψ_i) any mapping ω of $\mathcal{F}''$ such that:*

(i) at each point x, ω_x equals one of the $\psi_{i,x}$ or I_x,

(ii) the criterion which affects one of the ψ_i's, say ψ_{i_0} to ω at a given point x depends only on the various primitives ψ_i, on the numerical value I_x and on possible constants,

(iii) if at point x, at least one of the ψ_i's, say ψ_{i_0}, coincides with the identity mapping I, then:

$$\omega_x = I_x = \psi_{i_0,x}. \tag{86}$$

Toggle mappings generate jumps, and the first way for keeping down this effect is to look for idempotent toggles. The following theorem, which involves the class $\mathcal{C}$ of the continuous functions $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$, provides a comprehensive class of such operators [27]:

Theorem 5.5 *Let, in $\mathbb{R}^n$, $\omega : \mathcal{C} \longrightarrow \mathcal{C}$ be a toggle mapping which admits as primitives the family $(\psi_j)_{j\in J}$ of strong filters, and let ρ and σ be the two mappings defined by:*

$$\begin{aligned} \rho_x &= \vee\{\psi_{j,x}, j \in J, \psi_{j,x} < I_x\}, \\ \sigma_x &= \wedge\{\psi_{j,x}, j \in J, \psi_{j,x} > I_x\}. \end{aligned}$$

If, for all points x, $\omega_x \in \{\rho_x, I_x, \sigma_x\}$ and if the family $(\psi_j)_{j\in J}$ is ordered for the order $\geq$, then the toggle mapping ω is idempotent.

Comments

- When dealing with the functions $f : \mathbb{Z}^n \longrightarrow \overline{\mathbb{Z}}$, the distinction between $\mathcal{C}$ and $\mathcal{F}(\mathbb{Z}^n, \mathbb{Z})$ is cumbersome.
- The middle element of equation (85) turns out to be a particular toggle, admitting the strong filters ψ_i, plus the identity, as primitives, and the activity inf as a criterion.
- Associate with ω the two following mappings:

$$\theta_x = \begin{cases} \rho_x & \text{when } \omega_x = \rho_x, \\ I_x & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \tau_x = \begin{cases} \sigma_x & \text{when } \omega_x = \sigma_x, \\ I_x & \text{otherwise.} \end{cases}$$

  Then, the conditions of corollary 5.3 are fulfilled, and the toggle ω is nothing but the activity sup $\tau\dot{\vee}\theta$ (whereas the ψ_i themselves do not have any activity sup).

5.4.1 First example: contrast mapping

Take for primitives an opening γ and a closing ϕ, and the criterion according to which, when $\gamma_x \neq I_x$ and $I_x \neq \phi_x$, the value at point x must move down (to γ_x) or up (to ϕ_x). This generates a *two states contrast* κ which is idempotent (theorem 5.5). In this simple case, the idempotence may be proved directly, since

$$\kappa \geq \gamma\kappa \geq \gamma\gamma = \gamma.$$

Hence, $\gamma_x = \kappa_x \Longrightarrow (\gamma\kappa)_x = \kappa_x$. Now, rel. (86), when applied to κ itself, implies:

$$(\gamma\kappa)_x = \kappa_x \quad \Longrightarrow \quad (\gamma\kappa)_x = (\kappa\kappa)_x,$$

i.e. by combining these two implications as well as their dual versions,

$$(\kappa_x = \gamma_x \ \text{ or } \ \kappa_x = \phi_x) \quad \Longrightarrow \quad (\kappa)_x = (\kappa\kappa)_x,$$

so that κ is idempotent. Fig. 32 presents an example of such a contrast algorithm (for the sake of slarity, the closing and the opening have been replaced in this example by a dilation and an erosion of f).

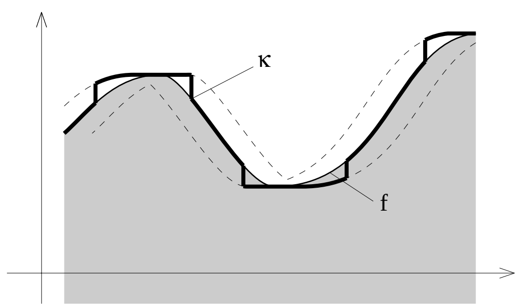


Figure 32: Example of contrast mapping.

5.4.2 Second example: combined toggle

In this second example, we combine the antinomic properties of *noise reduction* of the centers and of *contrast enhancement.* Let ζ and η be two strong filters, and let (γ, ϕ) be a pair made of an opening and a closing such that

$$\gamma \leq \eta \leq \zeta \leq \phi.$$

Fig. 33 illustrates the arrangement of the four transforms γf, ηf, ζf and ϕf around the initial function f. According to theorem 5.5, any toggle mapping ω is allowed to jump up (or down) at each point x, from $f(x)$ to the nearest transform above (or below) f. However, this flexibility is amended by one constraint: if at point x, f equals one of the four transforms under study, one cannot modify it.

As an efficient definition rule for ω, we may proceed as follows:

1. When $(\eta f)(x) \leq f(x) \leq (\zeta f)(x)$, apply a contrast κ, of primitives η and ζ.
2. When $(\gamma f)(x) \leq f(x) < (\eta f)(x)$, go down to $(\gamma f)(x)$, except if $(\zeta f - \eta f)(x)$ is smaller than a fixed value d. In the latter case, go up to $(\eta f)(x)$.
3. When $(\zeta f)(x) < f(x) \leq (\phi f)(x)$, apply a rule similar to rule 2, possibly with a scalar d' different from d.

The reason for this rule is the following. Usually, one takes $\zeta = \phi'\gamma'\phi'$ and $\eta = \gamma'\phi'\gamma'$, with an opening $\gamma' \geq \gamma$ and a closing $\phi' \leq \phi$. One can easily ascertain that ζ and η are close to one another in the narrow dark (resp. clear) features when both of them are also above (resp. below) the initial function. In the technique adopted here, such features are considered as non significant, and are reduced, if their sizes and their depths (their heights) are small enough with respect to the large opening γ and closing ϕ.

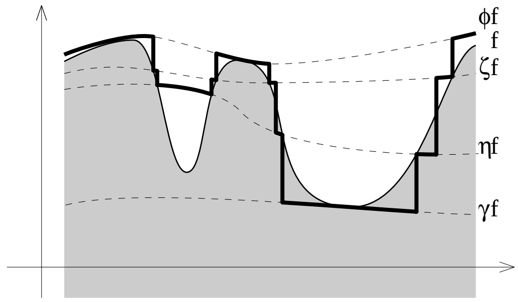


Figure 33: Function f, four primitives ϕf, ζf, ηf and γf. In superimposition, the toggle transform ωf given by the rules 1 to 3 detailed above.

5.5 Weakening of idempotence: the amplifier toggle mappings

We have wished the idempotence assumption because it ensures that the toggling process is well controlled. If not, we would run the two risks of generating, by iterations, oscillations and/or parasitic halos (especially around the peaks and slope changes, see [17]). However, some behaviours under iteration, such as the fixed zones growth, overcome the parasitic effects. The activity of a mapping ψ is said to exhibit a *fixed zone growth* when:

$$\psi \circ \psi \succeq \psi \qquad \text{and} \qquad \psi_x = I_x \Longrightarrow (\psi \circ \psi)_x = I_x. \tag{87}$$

In other words, the successive iterations ψ_x, ψ_x^2,…,ψ_x^n may only strictly increase (or decrease) at point x, or stop. In the latter case, the stop becomes permanent through further iterations. Here is now an example of such a behaviour, which also shows a toggle mapping based on *one* primitive, and called *amplifier* (see Fig. 34).

Starting from a strong filter ψ, define the toggle mapping ω' by

$$\forall x \in E, \quad \omega'_x = \begin{cases} \lambda'(I-\psi)_x + I_x & \text{when } I_x > \psi_x \qquad (\Longrightarrow \omega'_x > I_x > \psi_x) \\ I_x & \text{when } I_x \leq \psi_x \qquad (\Longrightarrow \omega'_x = I_x \leq \psi_x) \end{cases}$$

and similarly, the toggle mapping ω'' by

$$\forall x \in E, \quad \omega''_x = \begin{cases} \lambda''(I-\psi)_x + I_x & \text{when } I_x < \psi_x, \\ I_x & \text{when } I_x \geq \psi_x. \end{cases}$$

Consider the composition product $\omega = \omega''\omega'$. When $\omega'_x < (\psi\omega')_x = \psi_x$, ω_x is equal to:

$$(\omega''\omega')_x = \lambda''(\omega' - \psi\omega')_x + \omega'_x = \lambda''(\omega' - \psi)_x + \omega'_x.$$

But $\omega'_x < \psi_x$ implies that $\omega'_x = I_x$, hence

$$(\omega''\omega')_x = \lambda''(I-\psi)_x + I_x = \omega''_x.$$

When $\omega'_x \geq (\psi\omega')_x$, then ω'' is the identity operator and

$$\omega_x = (\omega''\omega')_x = \omega'_x.$$

Finally, we obtain $\omega''\omega' = \omega''\dot{\vee}\omega'$, and by duality, we can state:

$$\omega = \omega''\omega' = \omega'\omega'' = \omega''\dot{\vee}\omega'. \tag{88}$$

Under iteration, the toggle ω exhibits a fixed zone growth behaviour, since

$$\omega\omega = \omega''\omega'\omega'\omega'' = \omega''[\lambda''(\lambda''+2)]\omega'[\lambda'(\lambda'+2)].$$

ω^2 is of the same type as ω, with changes in the intensity factors λ' and λ'' only. Therefore, the $\omega(\lambda', \lambda'')$'s satisfy a semi-group relationship when ψ is fixed, namely:

$$\omega(\lambda', \lambda'')\omega(\mu', \mu'') = \omega(\lambda' + \mu' + \lambda'\mu', \lambda'' + \mu'' + \lambda''\mu'').$$

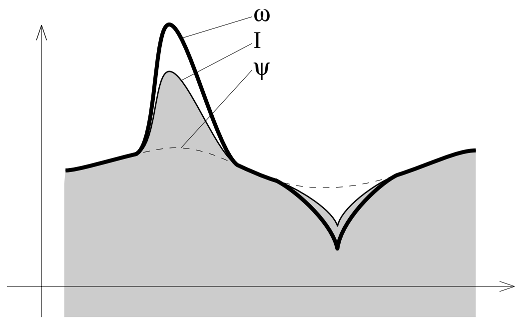


Figure 34: Example of an amplifier $\omega = \omega''\omega'$ based on a strong filter $\psi = \gamma\phi(I \wedge \gamma\phi)^n$ (with γ, ϕ morphological opening and closing).

6 Conclusion

The theory of morphological filtering is recent and first appeared totally formalized in 1988 in [24], where it occupies half of the book. As a result, the practitionners of image analysis are still surprised by it. We have written the present paper to overcome this effect and also to bring to the fore a comprehensive series of practical implementations, from "Dolby" openings to middle elements and toggle contrasts.

Which domains of morphological filtering still lie follow? In which directions may the method evolve? One of them is certainly that of geodesic filters: up to now, the emphasis was put on the translation invariant operators. A suitable extension should consist in the geodesic approach. Moreover, the functions from E (i.e., $\mathbb{R}^d$ or $\mathbb{Z}^d$) onto a vectorial space instead of $\overline{\mathbb{R}}$ or $\mathbb{Z}$ should open the way to color and vector fields filterings. Last, but surely not least, it should be fruitful to question oneself about the use of morphological filters for interpolating. Typically, linear filtering has progressed in both directions of image cleaning and predictor. Will we witness a similar evolution with morphological filters?

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