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An Overview of Rough Set Semantics for Modal and Quantifier Logics*

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In this paper, we would like to present some logics with semantics based on rough set theory and related notions. These logics are mainly divided into two classes. One is the class of modal logics and the other is that of quantifier logics. For the former, the approximation space is based on a set of possible worlds, whereas in the latter, we consider the set of variable assignments as the universe of approximation. In addition to surveying some well-known results about the links between logics and rough set notions, we also develop some new applied logics inspired by rough set theory.

Keywords: Rough set, modal logic, epistemic logic, rough quantifiers.

1. Introduction

The rough set theory is invented by Pawlak[3,4,5] to account for the definability of a concept in terms of some elementary ones in an approximation space. It captures and formalizes the basic phenomenon of information granulation. The finer the granulation is, the more concepts are definable in it. For those concepts not definable in an approximation space, the lower and upper approximations for them can be defined. These approximations construct a representation of the given concept in the approximation space. Pawlak claims that knowledge is deep-seated in the classificatory abilities of human beings and other species, so rough set theory is a framework for discussions about knowledge, in particular when imprecise knowledge is of primary concern([4], p.2). Thus the theory is in particular effective in extracting knowledge from data tables and it has been successfully and widely

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applied to domains such as intelligent data analysis (data mining and knowledge discovery in database), decision making, machine learning, pattern recognition and conflict analysis[6,7,8,9,10]. An important common feature of these applications is the classification of objects and the representation of the classificatory knowledge in rough set notions.

On the other hand, philosophers and logicians have spent centuries of time in developing formal tools for the representation and reasoning of knowledge. The results are various of classical and nonclassical logics [11]. In particular, first order logic[12] and modal logics[13] have been widely used as a formalism for knowledge representation in artificial intelligence and an analysis tool in computer science[14,15].

Since set theory and logic systems are strongly coupled in the development of modern logic, it is expected that the rough set-based knowledge representation framework and the logic-based one have close relationship. This expectation is verified almost from the very beginning stage of the invention of rough set theory. The most well-known relationship between rough set theory and logics is the connection of approximation space with possible world semantics for the modal epistemic logic $S5$. The connection is so obvious that it is difficult to point out who is the first one discovering it. However, undoubtedly, E. Orłowska explores the connection in full details and develop different knowledge representation logics in a sequence of papers [16,17,18,19,20,21,22]. There are of course other works along the same direction, for example, those reported in [23,24,25,26]. The relationship between more general rough set model and modal logics have also been examined in [27,28,29]. As for the first order logic, different logical systems incorporating rough quantifiers and approximation operators have been developed [30,31,32,33,34,35,36,37,38]. The common semantic intuition behind these logics is to view the first order interpretation as an approximation space. These results all show the cross fertilization between rough set theory and logics, so it is worthwhile to investigate the further relationship between different generalizations of rough set models and logic systems. Based on this background, the purpose of this paper is twofold. The first is to survey and present the current results in a uniform way. The second is to fill some missing links between the existing general rough set models and logical formalisms and develop further some applied logics inspired by rough set theory.

In what follows, we will first review the rough set theory and some of its main generalizations. Then the paradigms of possible worlds as approximation space and variable assignments as approximation space are discussed respectively in the following two sections. Finally, some concluding remarks are given.

2. Review of Rough Set Theory

2.1. *Pawlak approximation space and generalizations*

Let U be a set of objects (the universe)[†]and R be an equivalence relation on U,

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[†]In pawlak's original definition, U is assumed to be finite

then for any $X \subseteq U$, we can associate two subsets with X,

$$\underline{R}X = \{x \in U \mid [x]_R \subseteq X\}$$

$$\overline{R}X = \{x \in U \mid [x]_R \cap X \neq \emptyset\},$$

where $[x]_R$ denotes the equivalence class containing x. $\underline{R}X$ and $\overline{R}X$ are called the R-lower and R-upper approximation of X respectively. From a practical viewpoint, R can be considered as an indiscernibility relation, so for a given concept X, we can only know that X contains at least all elements in $\underline{R}X$ and does not contain any element outside $\overline{R}X$. The pair $(\underline{R}X, \overline{R}X)$ is called the rough approximation of X and any such pair is called a rough set. The pair (U, R) defined as above is thus called a Pawlak approximation space(PAS).

For a given approximation space (U, R), the equivalence classes of R and the empty set are called the *elementary sets* , and the union of some elementary sets is called a *composed set* . The composed sets are those definable in the space. Note that both lower and upper approximations of a concept are composed sets, so they can be seen as the definable approximations of the concept.

A direct generalization of the above-mentioned idea is to relax the constraints on R. To allow R to be an arbitrary binary relation, we can get different useful generalizations of the Pawlak rough set model. For example, the case where R is a tolerance (reflexive and symmetry) relation has been considered in [39]. To distinguish the Pawlak approximation space and the generalized one, we will refer the latter as relational approximation space(RAS). When (U, R) is an RAS, the lower and upper approximations of a set X are modified as

$$\underline{R}X = \{x \in U \mid R(x) \subseteq X\}$$

$$\overline{R}X = \{x \in U \mid R(x) \cap X \neq \emptyset\},$$

where $R(x) = \{y \in U \mid (x, y) \in R\}$.

Even further generalization of RAS is possible. The most well-known one is the neighborhood systems proposed by Lin[40]. A neighborhood system(NS) is a pair (U, N), where $N : U \to 2^{2^U}$ satisfies the following constraints:

1. $\emptyset \notin N(x)$ for all $x \in U$,

2. for all $x \in U$ and $X \subseteq Y \subseteq U$, if $X \in N(x)$, then $Y \in N(x)$.

The universe U is open if $U \in N(x)$ for all $x \in U$, or equivalently, $N(x) \neq \emptyset$ for all $x \in U$. The lower and upper approximations of a set X in an NS (U, N) is based on the definition of interior and closure in topology[41].

$$\underline{N}X = \{x \in U \mid \exists Y \in N(x), Y \subseteq X\}$$

$$\overline{N}X = \{x \in U \mid \forall Y \in N(x), Y \cap X \neq \emptyset\}.$$

Given an RAS (U, R), we can define an NS (U, N) by

$$N(x) = \{S \subseteq U \mid R(x) \subseteq S\},$$

so the latter is indeed a generalization of the former.

2.2. *Probabilistic approximation space*

For a PAS or RAS (U, R), and $X \subseteq U$, the accuracy of the approximation of X is defined by[3]

$$\rho(X) = \frac{|\underline{R}X|}{|\overline{R}X|}.$$

Furthermore, the rough membership function associated with X is defined by $\mu_X : U \to [0, 1]$

$$\mu_X(u) = \frac{|X \cap R(u)|}{|R(u)|}.$$

This provides a numeric characterization of rough sets. Based on the definition of rough membership function, a variable precision rough set model is proposed in [42,43]. For $0 \leq \alpha < \beta \leq 1$, the α and β-approximation of X is defined by

$$\underline{R_\alpha} X = \{u \in U \mid \mu_X(u) \geq 1 - \alpha\}$$

$$\overline{R_\beta} X = \{u \in U \mid \mu_X(u) > 1 - \beta\}.$$

Though the rough membership function and the accuracy of approximation are well-defined for finite universe U, it uses the cardinality which may be not finite in the infinite case. To cope with this situation, we extend the RAS to probabilistic approximation space(PRAS). A PRAS is just a triple (U, R, Pr), where (U, R) is still an RAS and Pr is a probability distribution on U. Then we can replace the definition of accuracy and rough membership function by the following equations:

$$\rho(X) = \frac{Pr(\underline{R}X)}{Pr(\overline{R}X)},$$

$$\mu_X(u) = \frac{Pr(X \cap R(u))}{Pr(R(u))}.$$

For convenience, $\mu_X(u) = 1$ if $Pr(R(u)) = 0$. When U is finite and Pr is a uniform distribution, the definitions just reduced to the original ones.

2.3. *Fuzzy approximation space*

In the last subsection, we have considered the combination of probability and rough set theory. It is also possible to combine fuzzy and rough set theory. First, we can define a fuzzy relational approximation space(FRAS) as a pair (U, R), where R is now a fuzzy binary relation on U, i.e., $R : U \times U \to [0, 1]$. Then, there are essentially two approaches to incorporate the notion of fuzzy sets into rough set models[44]. The first one is to consider the lower and upper approximations of a fuzzy concept in an ordinary RAS. The result is called a rough fuzzy set. Let (U, R) be an RAS and F be a fuzzy subset of U, then $\underline{R}F, \overline{R}F : U \to [0,1]$ are defined by

$$\underline{R}F(u) = \inf_{v \in R(u)} F(v),$$

$$\overline{R}F(u) = \sup_{v \in R(u)} F(v).$$

The other approach is to consider the approximations of a crisp or fuzzy concept in an FRAS. The result is called a fuzzy rough set. Let (U, R) be a FRAS and F be a fuzzy subset of U, then $\underline{R}F, \overline{R}F : U \to [0,1]$ are defined by

$$\underline{R}F(u) = \inf_{v \in U} R(u,v) \to_* F(v),$$

$$\overline{R}F(u) = \sup_{v \in U} R(u,v) * F(v),$$

where $* : [0,1] \times [0,1] \to [0,1]$ is a t-norm and $\to_*$: $[0,1] \times [0,1] \to [0,1]$ is the S-implication with respect to $*$ defined by $a \to_* b = 1 - (a * (1-b))$. In particular, when F is a crisp subset of U, the above two equations are reduced to

$$\underline{R}F(u) = \inf_{v \notin F} 1 - R(u,v),$$

$$\overline{R}F(u) = \sup_{v \in F} R(u,v).$$

2.4. *Multiple relations approximation space*

An example of PAS is derivable from data table based knowledge representation systems (KRS). A KRS or data table is a pair $S = (U, A)$, where U is a nonempty, finite set (the universe) and A is a nonempty, finite set of primitive attributes. Every $a \in A$ is a total function $a : U \to V_a$, where V_a denotes possible values of a. An equivalence relation $IND(B)$ is associated with every subset of attributes $B \subseteq A$, and defined by

$$xIND(B)y \Leftrightarrow a(x) = a(y) \forall a \in B.$$

$IND(B)$ is called an indiscernibility relation. We will write $IND(a)$ instead of $IND(\{a\})$ for all $a \in A$. Obviously, $IND(B) = \bigcap_{a \in B} IND(a)$. Since $IND(B)$ is an equivalence relation, we can define $IND(B)$-lower and $IND(B)$-upper approximation of X for any $X \subseteq U$.

The definitions are used in the analysis of dependency between attributes in a data table. Let us say that attribute B_2 depends on B_1, denoted by $B_1 \Rightarrow B_2$, iff $IND(B_1) \subseteq IND(B_2)$, i.e., any two objects in U with same values in their attributes B_1 will have also same ones in B_2. It is easily to show that $B_1 \Rightarrow B_2$ iff $\underline{B_1}X = X$ for all X that is an equivalence class of $IND(B_2)$.

The data table example shows that there may be more than one indiscernibility relations in an approximation space, so motivates the definition of multiple relations approximation space(MRAS). An MRAS is a pair $(U, \{R_i \mid 1 \leq i \leq n\})$, where each R_i is a binary relation. In an MRAS, each primitive relation R_i is a relation expression, and if R and S are relational expressions, so are $R \cap S$ and $R \cup^* S$, where $R \cap S$ is the intersection of R and S and $R \cup^* S$ is the transitive closure of the union of R and S for all relations R and S. The rough approximation in an MRAS is the same as that in PAS or RAS except that it can now be defined with respect to any relational expressions.

2.5. *Multiple domains approximation space*

Rough set theory based on data table has a strong relationship with Dempster-Shafer evidence theory[45], and this has been investigated by different researchers[46,47,48]. Wong, Wang, and Yao have developed the rough set model based on interval structure[49,50]. An interval structure is analogous to a rough set on an evidential structure—the basic construct underlying Dempster-Shafer theory. An evidential structure is a triple (U, V, R), where U and V are two universes and $R \subseteq U \times V$ is a compatibility relation between U and V. It is also assumed that for any $u \in U$, there exists a $v \in V$ with $(u, v) \in R$, and vice versa. A compatibility relation can also be expressed as a multi-valued mapping from U to 2^V such that $R(u) = \{v \in V \mid (u, v) \in R\}$. Then an interval structure is a pair of mapping $\underline{R}, \overline{R} : 2^V \to 2^U$ such that for any $X \subseteq V$,

$$\underline{R}X = \{u \in U \mid R(u) \subseteq X\}$$

$$\overline{R}X = \{u \in U \mid R(u) \cap X \neq \emptyset\}.$$

The interval structure based on the idea of approximating a concept in V by the definable ones in U, so in this way, an evidential structure can be viewed as a two-domain approximation space. In general, we can consider more than two different domains and there may not exist compatibility relation between some domains. The existence of a compatibility relation between two domains means they are communicable, so a connection structure on the set $\{1, 2, \ldots, n\}$ is a binary relation $C \subseteq \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. Now, a multiple domains approximation space(MDAS) based on a connection structure C is a pair $(\{U_i \mid 1 \leq i \leq n\}, \{R_{ij} \mid (i, j) \in C\})$, where U_i's are different universes and R_{ij} is a binary relation from U_i to U_j. For each R_{ij}, we can define $\underline{R_{ij}}, \overline{R_{ij}} : 2^{U_j} \to 2^{U_i}$ as

$$\underline{R_{ij}}X_j = \{u \in U_i \mid R(u) \subseteq X_j\}$$

$$\overline{R_{ij}}X_j = \{u \in U_i \mid R(u) \cap X_j \neq \emptyset\}.$$

The MDAS will be used to define a new applied logic for reasoning about knowledge and communication. Note that when $n = 1$ and C is just the reflexive structure, then an MDAS is reduced to an RAS.

3. Possible Worlds as the Universe

In the last section, we have reviewed the basic rough set theory and some of its generalizations with emphasis on the notion of approximation space. However, except in the data table example to motivate the development of MRAS, we do not specify what the universe is. In this section, we will view the universe as a set of possible worlds and see what the result is. The notion of possible worlds may be traced back to the Leibniz times. It is first used in the semantic account of modern modal logic by Rudolf Carnap (though he use the term state-description instead of possible world), and finally lead to the now widely-accepted notion of possible world models by S.A. Kripke, Jaakko Hintikka and Stig Kanger independently(see [51] for a historical sketch). Essentially, a possible world may be interpreted as a time point, a state of a running computer program, or anything like this in which the truth values of the well-formed formulas(wff) in a logical language can be evaluated.

In this section, we restrict the attention to propositional language, i.e., no quantifiers are involved. In the following subsections, some modal logics will be considered. For each logic, we will first define its syntax, in particular, its wffs, then the possible world semantics of the logic will be given, and finally the link between the semantics and the corresponding rough set model be examined.

3.1. *Normal modal logic systems and RAS*

The alphabet of propositional modal logic(PML) consists of a set of propositional symbols, PV, and the logical symbols $\neg$(negation), $\wedge$(and), $\vee$(or), $\supset$(material implication), $\Box$(necessity modal operator), and $\Diamond$(possibility modal operator).

The set of wffs of PML is the smallest set containing PV and satisfying the following conditions:

- if φ is a wff, then $\neg\varphi, \Box\varphi, \Diamond\varphi$ are wffs,
- if φ and ψ are wffs, then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \supset \psi$ are wffs.

As usual, let $\varphi \equiv \psi$ be an abbreviation for $(\varphi \supset \psi) \wedge (\psi \supset \varphi)$, $\top$ for any tautology $p \vee \neg p$, and $\perp$ for any contradiction $p \wedge \neg p$. A Kripke model for PML is a triple $M = (W, R, V)$, where W is a set of possible worlds, R is a binary relation on W, called an accessibility relation, and $V : W \times PV \to \{0, 1\}$ is a truth assignment evaluating the truth value of each propositional symbol in each world. The function V can be extended to all wffs recursively in the following way:

1. $V(w, \neg\varphi) = 1 - V(w, \varphi)$
2. $V(w, \varphi \wedge \psi) = \min(V(w, \varphi), V(w, \psi))$
3. $V(w, \varphi \vee \psi) = \max(V(w, \varphi), V(w, \psi))$
4. $V(w, \varphi \supset \psi) = \max(1 - V(w, \varphi), V(w, \psi))$
5. $V(w, \Box\varphi) = \inf\{V(u, \varphi) \mid (w, u) \in R\}$

6. $V(w, \Diamond\varphi) = \sup\{V(u, \varphi) \mid (w, u) \in R\}$

For each model M and wff φ, we can denote $\{w \mid V(w, \varphi) = 1\}$ by $|\varphi|_M$. For brevity, we usually drop the subscript M, and call $|\varphi|$ the truth set of φ (in M).

Obviously, (W, R) is an RAS, and for each wff φ, $|\varphi|$ is a subset of W and denote some concept in the RAS, so we can consider the rough approximation of $|\varphi|$ in the RAS. A direct but interesting relationship between PML and rough set theory is then established in the following proposition.

Theorem 1 *1.* $\underline{R}|\varphi| = |\Box\varphi|$

2. $\overline{R}|\varphi| = |\Diamond\varphi|$

A set containing the wffs true in all Kripke models is called a normal modal logic system. The least normal modal logic system is named K in [13]. The above result shows that RAS provides a semantic account to the system K. An axiomatic system for K consists of the following axioms and inference rules:

1. Axioms
   (a) All instances of propositional tautologies
   (b) $\Diamond\varphi \equiv \neg\Box\neg\varphi$
   (c) K: $\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$
2. Inference rules:
   (a) Modus ponens: $\frac{A, A \supset B}{B}$
   (b) N: $\frac{A}{\Box A}$

The axioms (b), (c) and the rule (b) correspond to the following properties of rough set theory,

1. duality: $\overline{R}X = -\underline{R} - X$,
2. $\underline{R}(X \cup Y) \subseteq (\overline{R}X \cup \underline{R}Y)$,
3. $\underline{R}W = W$,

so these three properties completely characterize those of rough approximations in an RAS.

3.2. *Epistemic logic and PAS*

When more constraints are imposed on the relation R of a Kripke model, we get stronger systems than K. In particular, an epistemic model is a Kripke model (W, R, V) such that R is an equivalence relation. According to the semantics, the intuitive reading of the wff "$\Box\varphi$" is "the agent knows φ". The system containing the wffs true in all epistemic models is called $S5$. The system $S5$ has been shown to be useful in the analysis of knowledge in philosophy, economics, artificial intelligence and distributed systems[52,53].

An axiomatic system for $S5$ is that for K with the addition of the following three axioms:

T: $\Box\varphi \supset \varphi$

4: $\Box\varphi \supset \Box\Box\varphi$

5: $\Diamond\Diamond\varphi \supset \Diamond\varphi$

Obviously, if (W, R, V) is an epistemic model, then (W, R) is a PAS, so according to proposition 1, these three axioms correspond to the well-known properties of Pawlak rough set theory.

1. $\underline{R}X \subseteq X \subseteq \overline{R}X$
2. $\underline{R}X = \underline{R}\underline{R}X$
3. $\overline{R}X = \overline{R}\overline{R}X$

3.3. *Local reasoning and NS*

Though $S5$ is an useful model for reasoning about knowledge of an ideal agent, it suffers from the logical omniscience problem in practice. This means that in the systems from K to $S5$, if an agent knows φ, then she also knows all logical consequence of φ, however, people in the real world may be not so powerful in reasoning about her own knowledge, so there is a big difference between an ideal agent and a real one. Some modifications of the system $S5$ have been proposed to address different aspects of the logical omniscience problem[52]. In particular, an important aspect of the logical omniscience problem is that in $S5$, if an agent has inconsistent beliefs, then her beliefs will crash (i.e. she will believe anything), however, in the real situation, an agent may hold inconsistent beliefs but still not believe all things. An approach that takes this aspect into consideration is called local reasoning. The idea of local reasoning is based on viewing an agent as a society of minds, each with its own knowledge, so a local-reasoning model is a triple (W, C, V), where W and V are as above, and $C : W \to 2^{2^W}$ with $C(w)$ nonempty for all $w \in W$. Each element in $C(w)$ represent a frame of the agent's mind in w, so the evaluation of the wffs $\Box\varphi$ and $\Diamond\varphi$ is modified to

$$V(w, \Box\varphi) = \sup_{S \in C(w)} \inf_{u \in S} V(u, \varphi),$$

$$V(w, \Diamond\varphi) = V(w, \neg\Box\neg\varphi).$$

A local-reasoning model is called serial if we require $\emptyset \notin C(w)$ for each $w \in W$. Now, we have the following result.

Theorem 2 *1. If $M = (W, C, V)$ is a serial local-reasoning model, then we can find an NS (W, N) with W open such that $|\Box\varphi| = \underline{N}|\varphi|$ and $|\Diamond\varphi| = \overline{N}|\varphi|$.*

2. *If* (W, N) *is an NS with* W *open and* $V : W \times PV \to \{0,1\}$ *is a truth assignment, then we can find a serial local-reasoning model* $M = (W, C, V)$ *such that* $|\Box\varphi| = \underline{N}|\varphi|$ *and* $|\Diamond\varphi| = \overline{N}|\varphi|$.

An axiomatic system for serial local-reasoning models consists of the propositional tautologies, the modus ponens rule, the duality axiom, the rule N, and the following axiom and inference rule:

1. D: $\neg\Box\bot$,
2. RM: $\frac{\varphi\supset\psi}{\Box\varphi\supset\Box\psi}$,

so the following is the characterization of rough approximations in NS with open universe:

1. duality: $\overline{N}X = -\underline{N} - X$,
2. $\underline{N}\emptyset = \emptyset$,
3. $\underline{N}W = W$,
4. monotonicity: $X \subseteq Y$ implies $\underline{N}X \subseteq \underline{N}Y$.

3.4. *Probabilistic epistemic logic and PRAS*

Though modal logics are widely used in reasoning about knowledge, it is inadequate in the representation of numeric uncertainty. In many application areas, it is usually important to be able to reason about the probability of certain events as well as agents' knowledge. To cope with the problem, different systems of probabilistic logic have been proposed [54,55,56,57,58,59]. Here, we consider a logic for reasoning about epistemic probability, called probabilistic epistemic logic (PEL), and its semantic basis in PRAS.

The syntax of PEL is an extension of PML with a class of probability modal operators $(> r)$ for all $r \in [0,1]$. The formation rules for PEL include, in addition to those for PML, the following

- if φ is a wff, then $(> r)\varphi$ is a wff, for all $r \in [0,1]$.

Furthermore, the following abbreviations are used

1. $(\leq r)\varphi = \neg(> r)\varphi$,
2. $(< r)\varphi = (> 1 - r)\neg\varphi$,
3. $(\geq r)\varphi = \neg(> 1 - r)\neg\varphi$,
4. $(= r)\varphi = (\leq r)\varphi \wedge (\geq r)\varphi$.

The syntax is the same as that of $P_F D$ introduced in [59].

The intended meaning of the wff "$(> r)\varphi$" is "the probability of φ is greater than r", so a model for PEL is a quadruple (W, R, Pr, V), where (W, R, V) is just a Kripke model and Pr is a probability distribution on W. The valuation function V is extended to all wffs as in the PML case except the addition of the following rule:

$$V(w, (> r)\varphi) = 1 \Leftrightarrow \frac{Pr(|\varphi| \cap R(w))}{Pr(R(w))} > r,$$

if $Pr(R(w)) > 0$, otherwise $V(w, (> r)\varphi) = 1$. Though the syntax of PEL is like that of $P_F D$, the semantics is based on the one proposed by R.J. Aumann in the proof of his well-known "impossibility theorem of agreeing to disagree"[60], so a PEL model will be called an Aumann model.

Obviously, the (W, R, Pr) part of an Aumann model is a PRAS, and we have the following result.

Theorem 3 *1.* $|(> r)\varphi| = \overline{R_{1-r}}|\varphi|$

2. $|(\geq r)\varphi| = \underline{R_{1-r}}|\varphi|$

Though a complete axiomatic system has been provided for $P_F D$ in [59], we do not have one for PEL yet. What is lacking is a characteristic axiom to guarantee that the probability values in all worlds have a common prior. That is, for all $w, u \in W$, if $R(w) = R(u)$, then $V(w, (> r)\varphi) = V(u, (> r)\varphi)$ for all $r \in [0, 1]$ and wffs φ. This statement, involved with the inter-world relationship, seems inexpressible in the language of PEL. Though this disadvantage, PEL is a faithful formalization of Aumann's idea, so it can formulate the "impossibility theorem of agreeing to disagree" when the multiagent logic is considered.

3.5. *Many-valued modal logic and FRAS*

Many-valued logics and modal logics represent two main traditions in the reasoning with incomplete information. The former handles the degree of truth, whereas the latter concerns uncertainty, so the combination of these two kinds of logics would provide more powerful tools for management of incomplete information. There have been different attempts in the merging of finitely many-valued and modal logics[61,62,63,64] and development of fuzzy modal logics[65,66,67,68], and recently, uniform methods for making modal logic fuzzy are also proposed[69]. These logics consider the fuzzification of either the valuation function or the accessibility relation in the Kripke model (or both). Here, we consider one similar to that introduced in [68].

The syntax of many-valued modal logic of this type is still the same as PML. However, the model is now a fuzzy Kripke model, defined by (W, R, V), where W is a set of possible worlds, $R : W \times W \to [0, 1]$ is a fuzzy binary relation on W, and $V : W \times PV \to [0, 1]$ is a fuzzy truth valuation of propositional symbols. Then, V can be extended to all wffs in the following way:

1. $V(w, \neg\varphi) = 1 - V(w, \varphi)$

2. $V(w, \varphi \wedge \psi) = V(w, \varphi) * V(w, \psi))$
3. $V(w, \varphi \vee \psi) = V(w, \varphi) \oplus V(w, \psi))$
4. $V(w, \varphi \supset \psi) = V(w, \varphi) \rightarrow_* V(w, \psi)$
5. $V(w, \Box\varphi) = \inf_{u \in W} R(w, u) \rightarrow_* V(u, \varphi)$
6. $V(w, \Diamond\varphi) = \sup_{u \in W} V(u, \varphi) * R(w, u)$

where $*$ is a t-norm, $\oplus$ is the dual co-t-norm, defined by $a \oplus b = 1 - (1-a) * (1-b)$, and $\rightarrow_*$ is the corresponding S-implication. For each fuzzy Kripke model M and wff φ, $|\varphi|_M$ is now a fuzzy subset of W, with the membership function is defined by

$$\mu_{|\varphi|_M}(w) = V(w, \varphi)$$

for all $w \in W$.

Theorem 4 *Let $M = (W, R, V)$ be a fuzzy Kripke model, then*

$$|\Box\varphi|_M = \underline{R}|\varphi|_M,$$

$$|\Diamond\varphi|_M = \overline{R}|\varphi|_M.$$

Thus, $(|\Box\varphi|_M, |\Diamond\varphi|_M)$ is a fuzzy rough set. Moreover, if R is a crisp relation, then $(|\Box\varphi|_M, |\Diamond\varphi|_M)$ is also a rough fuzzy set.

While the proof theory of finitely many-valued modal logics seems more well-studied, the axiomatization of fuzzy modal logics remains largely unexplored. Recently, Hájek[66] provides a complete axiomatization of fuzzy S5 system where the accessibility relation is the universal relation(i.e., $\forall w, u R(w, u) = 1$), and Godo and Rodríguez[65] give a complete axiomatic system for the extension of Hájek's logic with another modality corresponding to a fuzzy similarity relation. The latter is in particular interesting in the proof of completeness since they make use of the translation of modal formulas into first order logic instead of the ordinary canonical model construction method.

3.6. *Graded modal logics and FRAS*

In the last subsection, we consider many-valued modal logic with fuzzy Kripke model. The syntax of that logic is essentially the same as PML, however the truth values are $[0, 1]$ instead of $\{0, 1\}$. An alternative method to incorporate fuzzy reasoning into modal logics is to enhance the syntax of the logical language while remain it two-valued. This results in a kind of polymodal logics, called graded modal logics. Graded modal logics have been shown to be useful in modeling possibilistic or similarity-based reasoning [70,71,72,73,74,75,76,77,78]. Here, we consider the quantitative modal logic(QML) introduced in [70].

The logical symbols of QML consists of those of PML except that $\Box$ and $\Diamond$ are replaced by four classes of quantitative modal operators, $[c], [c]^+, \langle c\rangle, \langle c\rangle^+$ for $c \in [0, 1]$ and the formation rules of wffs for QML include

- if φ is a wff, then $[c]\varphi, [c]^+\varphi, \langle c\rangle\varphi, \langle c\rangle^+\varphi$ are all wffs,

instead of that for $\Box\varphi$ and $\Diamond\varphi$.

As for the semantics, a QML model is a triple (W, R, V), where W and R are as in fuzzy Kripke model, but $V : W \times PV \to \{0, 1\}$ is a two-valued valuation. For the extension of V to all QML wffs, it follows the same rules as in PML for the classical logical symbols, and for the graded modal wffs, it is defined by the following four rules:

5. $V(w, [c]\varphi) = 1$ iff $\inf_{u \notin |\varphi|}(1 - R(w, u)) \geq c$
6. $V(w, [c]^+\varphi) = 1$ iff $\inf_{u \notin |\varphi|}(1 - R(w, u)) > c$
7. $V(w, \langle c\rangle\varphi) = 1$ iff $\sup_{u \in |\varphi|} R(w, u) \geq c$
8. $V(w, \langle c\rangle^+\varphi) = 1$ iff $\sup_{u \in |\varphi|} R(w, u) > c$.

The semantics is based on possibility theory[79]. For example, the intuitive meaning of $[c]\varphi$ is "the necessity measure of φ is at least c". Because we can associate with each world w a possibility distribution π_w such that $\pi_w(u) = R(w, u)$ for all $u \in W$, the term $\inf_{u \notin |\varphi|}(1 - R(w, u))$ is just $N_w(\varphi)$, where N_w is the necessity measure induced from π_w.

Recall that if F is a fuzzy set of W, the α-cut and strict α cut of F are defined by $F_\alpha = \{w \mid F(w) \geq \alpha\}$ and $F_\alpha^+ = \{w \mid F(w) > \alpha\}$ respectively. Then the the relationship between QML and FRAS is as follows.

Theorem 5 *If (W, R, V) is a QML model, then*

$$|[c]\varphi| = (\underline{R}|\varphi|)_c$$

$$|[c]^+\varphi| = (\underline{R}|\varphi|)_c^+$$

$$|\langle c\rangle\varphi| = (\overline{R}|\varphi|)_c$$

$$|\langle c\rangle^+\varphi| = (\overline{R}|\varphi|)_c^+.$$

3.7. *Multi-agent epistemic logics and MRAS*

Though we have shown that RAS is underlying the semantics of PML, it is only a logic for single-agent epistemic reasoning. To model the real environment, a logic for multi-agent epistemic reasoning is usually needed. The application of multi-agent epistemic logics to the analysis of distributed systems has been provided in [52]. The thorough study of the notions of common and distributed knowledge is also carried out there.

The relationship between rough set theory and the semantics of multi-agent epistemic logics has been explored in[23,24]. Interestingly, a logic for data analysis(DAL) proposed by L. Fariñas del Cerro and E. Orłowska[22], when interpreted in terms of epistemic reasoning terminology, has also strong analogy with those developed in [52]. Since DAL is originally proposed according to the rough set semantics, this shows that rough set theory indeed provides a semantic foundation for multi-agent epistemic reasoning.

Syntactically, the alphabet of DAL consists of a set of propositional symbols PV, a finite set of relational symbols $\{r_1, r_2, \ldots, r_n\}$, the classical logical symbols $\neg, \vee, \wedge$ and $\supset$, the relational forming operations $\cup^*$ and $\cap$, and two modal operators $[\cdot]$ and $\langle\cdot\rangle$. The set ER of relational expressions is the smallest set containing $\{r_1, r_2, \ldots, r_n\}$ and satisfying that if $r, s \in ER$, then $s \cup^* s, s \cap s \in ER$. The set of wffs is the smallest set containing PV, and satisfying the following conditions:

- if φ is a wff and $R \in ER$, then $\neg\varphi, [r]\varphi, \langle r\rangle\varphi$ are wffs,
- if φ and ψ are wffs, then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \supset \psi$ are wffs.

Intuitively, each primitive symbol $r_i (1 \leq i \leq n)$ corresponds to an agent, and $[r_i]\varphi$ is interpreted as "agent i knows φ". For any subset $G \subseteq \{1, 2, \ldots, n\}$, $[\cap_{i\in G} r_i]\varphi$ and $[\cup^*_{i\in G} r_i]\varphi$ are interpreted as the distributed and common knowledge of agents in the group G respectively. More specifically, the wffs $[r_i]\varphi, [\cap_{i\in G} r_i]\varphi, [\cup^*_{i\in G} r_i]\varphi$ correspond exactly to $K_i\varphi, D_G\varphi, C_G\varphi$ in the logic K_n of [52]. In this sense, DAL is also more expressive than K_n since in K_n a wff like $[r_1 \cap (r_2 \cup^* r_3)]\varphi$ is not expressible, though there is no essential difficulty to extend the expressive power of K_n to cover such cases.

A DAL model is $(W, \{R_i \mid 1 \leq i \leq n\}, V)$, where W and V are as in Kripke model, and each R_i is a binary relation on W. To extend V to all wffs, we must first decide the denotation of all expressions in ER. Let $m : ER \to [W \times W \to \{0, 1\}]$ is the denotation function assigning to each expression in ER a binary relation on W, defined by

$$m(r_i) = R_i (1 \leq i \leq n),$$

$$m(r \cap s) = m(r) \cap m(s),$$

$$m(r \cup^* s) = m(r) \cup^* m(s).$$

Then V can be extended to all wffs by the classical rules for PML and the following two

5. $V(w, [r]\varphi) = \inf\{V(u, \varphi) \mid (w, u) \in m(r)\}$,
6. $V(w, \langle r\rangle\varphi) = \sup\{V(u, \varphi) \mid (w, u) \in m(r)\}$.

For a DAL model $(W, \{R_i \mid 1 \leq i \leq n\}, V)$, the first two components $(W, \{R_i \mid 1 \leq i \leq n\})$ clearly form an MRAS, and we have
Theorem 6

$$|[r]\varphi| = \underline{m(r)}|\varphi|$$

$$|\langle r\rangle\varphi| = \overline{m(r)}|\varphi|.$$

3.8. *Epistemic communication logic and MDAS*

In the preceding sections, we survey existing modal logics and their links with different approximation spaces. In this section, we would like to develop new applied logic with semantics based on MDAS. The application domain is in the multi-agent environment. Assume there is a finite set of agents, each with her own knowledge base and some agents can communicate with each other. For simplicity, we assume that the communication structure of the agents are fixed in advance, i.e., the communication channels are static during the time periods we are concerning. Moreover, we consider a multilingual environment in which each agent may have different vocabulary of themselves.

To model the reasoning about knowledge and communication under such environment, we first assume a set of agents $A = \{1, 2, \ldots, n\}$ and a communication structure $CS \subseteq A \times A$ is a reflexive binary relation on A. Now, the alphabet of epistemic communication logic (ECL) consists of n sets of propositional symbols $PV_i (1 \leq i \leq n)$, the classical logical connectives, and the modal operators $\Box_{ij}$ and $\Diamond_{ij}$ for $(i, j) \in CS$. The set of wffs of ECL is $\mathcal{L}_{CS} = \cup_{1 \leq i \leq n} L_i$, where each L_i is the smallest set containing PV_i, closed under classical connectives, and satisfying that

- if $\varphi \in L_j$ and $(i, j) \in CS$, then $\Box_{ij}\varphi, \Diamond_{ij}\varphi \in L_i$.

The wffs $\Box_{ii}\varphi$ and $\Diamond_{ii}\varphi$ are abbreviated as $\Box_i\varphi$ and $\Diamond_i\varphi$ respectively.

Intuitively, $(i, j) \in CS$ means that there is some communication channel from j to i and $\Box_{ij}\varphi$ denotes that the message φ (encoded in j's language) is communicated to i. In particular, $\Box_{ii}\varphi = \Box_i\varphi$ means that i knows φ by herself. Thus, we can represent both the a prior knowledge and acquired information of an agent in the same time.

Formally, an ECL model is defined as a triple

$$(\{W_i \mid 1 \leq i \leq n\}, \{R_{ij} \mid (i, j) \in CS\}, \{V_i \mid 1 \leq i \leq n\})$$

, where

- W_i's are pairwisely disjoint sets of possible worlds,
- $R_{ij} \subseteq W_i \times W_j$ is a binary relation between W_i and W_j, for $(i, j) \in CS$, and
- $V_i : W_i \times PV_i \to \{0, 1\}$ is a truth assignment of PV_i in W_i, for $1 \leq i \leq n$.

Each V_i can be extended to a mapping from $W_i \times L_i$ to $\{0, 1\}$ in the following way:

5. $V(w_i, \Box_{ij}\varphi_j) = \inf\{V(w_j, \varphi_j) \mid (w_i, w_j) \in R_{ij}\}$,
6. $V(w_i, \Diamond_{ij}\varphi_j) = \sup\{V(w_j, \varphi_j) \mid (w_i, w_j) \in R_{ij}\}$,

in addition to the rules 1. - 4. for classical connectives in PML.

Then, it can be seen that the set $|\varphi| = \{w_i \in W_i \mid V(w_i, \varphi) = 1\}$ is a subset of W_i if $\varphi \in L_i$ for all $1 \leq i \leq n$, and since $(\{W_i \mid 1 \leq i \leq n\}, \{R_{ij} \mid (i, j) \in CS\})$ is also an MDAS, we have

Theorem 7

$$|\Box_{ij}\varphi| = \underline{R_{ij}}|\varphi|,$$

$$|\Diamond_{ij}\varphi| = \overline{R_{ij}}|\varphi|.$$

Of course, the logic is only a preliminary proposal, so it still has some problems. First, it also suffers from the logical omniscience problem, thus if the information φ is communicated to agent i by j, so are all logical consequence of φ in j's language. Second, the current logic allows agents to use different languages, so the interaction between a prior knowledge and acquired information is limited. For example, from $\Box_i\varphi$ and $\Box_{ij}(\varphi \supset \psi)$, we can not infer $\Box_i\psi$ even $\psi \in L_i$, though the inference seems intuitively reasonable. Further development of the logic and its improvement will be reported in a forthcoming paper.

Finally, though to our best knowledge, the logic ECL is an innovation, there have been also some attempts in developing the logic of communication [80,81,82,83].

4. Variable Assignments as the Universe

The analogy between quantifiers and modal operators has been noticed by Montague in 1960[84], and recently, the relationship is further investigated in detail[85]. This suggests that the links between modal logics and rough set theory can also be carried through the quantification logics. In this section, we will examine the results of applying rough set notions to quantifiers.

4.1. *First-order logic from the perspective of PAS*

We start with the examination of first-order logic(FOL) from the perspective of rough set theory. The notations used for FOL here are mostly drawn from [86].

The alphabet of a first-order language consists of

- logical symbols: $\neg, \wedge, \vee, \supset, \forall, \exists$,
- individual variables: a countably infinite set $VAR = \{x_0, x_1, x_2, \ldots\}$,
- function symbols: a countable set $FS = \{f_0, f_1, f_2, \ldots\}$, where each f_i has an arity, denoted by $\tau(f_i)$, if $\tau(f_i) = 0$, then f_i is also called a constant symbol
- predicate symbols: a countable set $PS = \{P_0, P_1, P_2, \ldots\}$, where each P_i has an arity, denoted by $\tau(P_i)$, if $\tau(P_i) = 0$, then P_i is also called a propositional symbol

The set of terms is the smallest set containing VAR and satisfying that if $f \in FS$ with $\tau(f) = n$ and $t_1, \ldots, t_n$ are terms, then $ft_1 \ldots t_n$ is a term. If P is a predicate symbol with $\tau(P) = n$ and $t_1, \ldots, t_n$ are terms, then $Pt_1 \ldots t_n$ is an atomic formula. The set of wffs of FOL is the smallest set containing all atomic formulas and satisfying the following conditions:

- if φ is a wff and $x \in VAR$, then $\neg\varphi, \forall x\varphi, \exists x\varphi$ are wffs,
- if φ and ψ are wffs, then $\varphi \wedge \psi, \varphi \vee \psi, \varphi \supset \psi$ are wffs.

In a wff of the form $\forall x\varphi$ or $\exists x\varphi$, the variable x is said to be bound by the quantifier $\forall$ (or $\exists$), so it is called a bound variable (in the wff). A variable not bound in a wff is called a free variable in it. A wff not containing free variables is also called a sentence.

The semantics of FOL is based on first-order structure. An FOL structure is a pair (D, I), where D is a nonempty set, called the domain of the structure, and I is an interpretation function which assigns functions and relations on D to symbols in FS and PS as follows:

1. for each $f \in FS$ with $\tau(f) = n$, $I(f) : D^n \to D$ is an n-ary function, in particular, if $n = 0$, then $I(f)$ is just an element of D,
2. for each $P \in FS$ with $\tau(P) = n$, $I(P) : D^n \to \{0, 1\}$ is an n-ary relation, in particular, if $n = 0$, then $I(P) = 0$ or 1.

Given an FOL language and structure, a *variable assignment* is any function $v : VAR \to D$ from the set of variables to the domain of the structure. The set of all such functions is denoted by $U = [VAR \to D]$. Now, for each variable $x \in VAR$, we can define an equivalence relation $=_x$ on U by

$$u =_x v \Leftrightarrow (\forall y \neq x.u(y) = v(y)),$$

for all $u, v \in U$. Thus $(U, =_x)$ is a PAS for any $x \in VAR$. The relation $=_x$ has been used in the investigation of the relationship between quantifiers and modal operators in [85].

The meaning of a wff is then a subset of U, that is, the set of variable assignments satisfying the wff. To define the meaning function, we first extend a variable assignment to a term assignment inductively. Let $v \in U$, then for any term t, if $t = x \in VAR$, then $v(t) = v(x)$, if $t = ft_1 \dots t_n$, then $v(t) = I(f)(v(t_1), \dots, v(t_n))$. Now, for a given structure $M = (D, I)$, we can define the denotation of a wff φ, $|\varphi|_M \subseteq U$ inductively as follows (we drop the subscript M in the definition):

1. $|Pt_1 \dots t_n| = \{v \in U \mid I(P)(v(t_1), \dots, v(t_n)) = 1\}$
2. $|\neg\varphi| = U - |\varphi|$
3. $|\varphi \wedge \psi| = |\varphi| \cap |\psi|$
4. $|\varphi \vee \psi| = |\varphi| \cup |\psi|$
5. $|\varphi \supset \psi| = (U - |\varphi|) \cup |\psi|$
6. $|\forall x\varphi| = \underline{=_x}|\varphi|$
7. $|\exists x\varphi| = \overline{=_x}|\varphi|$

Thus, the meaning of the wff $\forall x\varphi$ (resp. $\exists x\varphi$) is the lower (resp. upper) approximation of that of φ in the PAS $(U, =_x)$. A wff φ is said to be true in a structure M if $|\varphi|_M = U$, and it is valid if it is true in all structures.

4.2. *Rough quantifiers and MRAS*

We have seen that the quantified wffs can be given a rough set interpretation based on the equivalence relation $=_x$ defined in the space of all variable assignments. Let us now examine the relation $=_x$ more closely. If $\sim = D \times D$, then $\sim$ is obviously an equivalence relation on the domain D. The definition of $=_x$ is in fact equivalent to the following

$$u =_x v \Leftrightarrow u(x) \sim v(x) \wedge (\forall y \neq x. u(y) = v(y)),$$

so $=_x$ is only defined with respect to a special binary relation on D. A generalization of this is to consider any binary relation on D. Let R be a binary relation on D and $x \in VAR$, then we can define a binary relation R_x on $U = [VAR \to D]$ by

$$uR_xv \Leftrightarrow (u(x), v(x)) \in R \wedge (\forall y \neq x. u(y) = v(y)).$$

The idea has been exploited (at least implicitly) in [35,31,32,33,34,30] for the development of rough quantifiers. Here, we present a logic of rough quantifiers more general than that in [35] based on this idea.

The alphabet for the logic of rough quantifiers (LRQ) is that of FOL without the quantifier symbols $\forall$ and $\exists$, but with

- a finite set of primitive quantifiers $Q_0 = \{q_1, \ldots, q_n, u, e\}$, where u and e denote the universal and null quantifiers respectively, and
- two binary operations $\cap$ and $\cup^*$ for forming quantifier expressions.

The definition of terms remains unchanged and the set of quantifier expressions Q is the smallest set containing Q_0 and satisfying that if $q, r \in Q$, then $q \cap r, q \cup^* r \in Q$. The set of wffs of LRQ is the smallest set containing all atomic formulas and satisfying the following condition:

- if φ is a wff, $q \in Q$, and $x \in VAR$, then $\underline{q}x\varphi, \overline{q}x\varphi$ are wffs,

and the formation rules for propositional connectives.

For the semantics of LRQ, an LRQ structure is a triple (D, I, E), where (D, I) is an FOL structure and E assigns binary relations on D to quantifier symbols in Q_0 such that $E(u) = D \times D$ and $E(e) = \{(a, a) \mid a \in D\}$. The assignment E is extended to all quantifier expressions in Q by $E(q \cap r) = E(q) \cap E(r)$ and $E(q \cup^* r) = E(q) \cup^* E(r)$. Then the denotation of the quantified wffs are defined by

$$|\underline{q}x\varphi| = \underline{E(q)_x}|\varphi|,$$
$$|\overline{q}x\varphi| = \overline{E(q)_x}|\varphi|.$$

According to the semantics, the wffs $\underline{u}x\varphi$ and $\overline{u}x\varphi$ correspond exactly to $\forall x\varphi$ and $\exists x\varphi$ in FOL respectively.

The LRQ proposed here is obviously more expressive than that in [35] since we allow more than one rough quantifiers in our language. However, there is another subtle difference between these two logics. This lies on the distinction of free and bound variables. In FOL, the value of a bound variable does not influence the denotation of the wff. In other words, if v is in $|\forall x\varphi|$, then so are any variable assignments different from v only in the value of x. In particular, if φ is a sentence in FOL, then $|\varphi|$ is either the universe U or the empty set. On the other hand, if φ contains some free variables, then $|\varphi|$ may be a proper subset of U. However, our rough quantifiers totally eliminate the difference between free and bound variables. The variable occurring in the wff $\underline{q}x\varphi$ may be free in the sense of FOL. Thus, for example, while $\underline{u}x(\underline{u}x\varphi) \equiv \underline{u}x\varphi$ is valid, it is not so for general quantifier expressions. In fact, the wffs $\underline{e}x\varphi$ and $\overline{e}x\varphi$ are equivalent to φ, so x is a free variable in them in the sense of FOL.

On the other hand, the rough quantifiers $\tilde{\forall}$ and $\tilde{\exists}$ introduced in [35] can be defined in our LRQ as follows:

$$\tilde{\forall} x\varphi = \underline{u}x\overline{q_1}x\varphi,$$

$$\tilde{\exists} x\varphi = \overline{u}x\underline{q_1}x\varphi,$$

for some fixed $q_1 \in Q_0$. So the variable x in $\tilde{\forall}x\varphi$ and $\tilde{\exists}x\varphi$ is indeed bound in the sense of FOL.

Whether the peculiarity of LRQ quantifiers is an advantage remains to be seen, however it indeed provides much flexibility in the representation of imprecise knowledge. In fact, LRQ also encompass another kind of approximation logics presented in [34,37,38,87,36]. For example, in the rough logic of [34], there are wffs of the form $L\varphi$ as well as those of FOL. This kind of wff can be represented in LRQ as $\underline{q}x_1 \ldots \underline{q}x_n\varphi$ for some fixed quantifier expression q if $\{x_1, \ldots, x_n\}$ is the set of free variables occurring in φ.

The transition of the semantics from FOL to LRQ have an analogy on modal logic historically, that is, the development of Carnap's semantics to Kripke's one. In the Carnap's semantics, a wff of the form $\Box\varphi$ or $\Diamond\varphi$ is true in all possible worlds of a model if it is true in one. However, the accessibility relation provided in the Kripke model eliminates this property. The property then holds only when the accessibility relation is a universal one.

4.3. *Generalized quantifiers and NS*

The study of generalized quantifiers has become an important field of logic and linguistics in the last decade. Many fruitful theoretical results and applications to the analysis of natural language have been produced[88]. It is impossible even to give a brief survey of the field here. However, we will consider a logic with generalized quantifiers proposed in [89] and discuss its relationship with NS.

The language of the generalized quantifier logic (GQL) include two generalized quantifiers ∇ and $\clubsuit$ as well as the alphabet of FOL. In addition, the following formation rule is added to those of FOL

- if φ is a wff, then $\nabla x\varphi$ and $\clubsuit x\varphi$ are wffs,too.

The intuitive meaning of $\nabla x\varphi$ is "most x satisfy φ" and $\clubsuit x\varphi$ means that "for at least a few x, φ holds".

A GQL structure is then a triple (D, I, N), where (D, I) is an FOL structure, and (D, N) is an NS satisfying the following conditions:

1. Openess: D is open,
2. Uniformity: N is uniform in the sense that $N(a) = N(b)$ for all $a, b \in D$, so N can be identified with a subset of 2^D, and we will view N in such way from now on,
3. Coherence: if $A, B \in N$, then $A \cap B \neq \emptyset$.

Given a GQL structure (D, I, N) and a variable $x \in VAR$, we can induce an NS (U, N_x), where $U = [VAR \to D]$ and $N_x : U \to 2^{2^U}$ is defined by

$$N_x(v) = \{S \subseteq U \mid (\{u(x) \mid u \in S\} \in N) \wedge (\forall u \in S u =_x v)\}$$

for all $v \in U$. Then the following two rules are added to the denotation function for FOL

8. $|\nabla x\varphi| = \underline{N_x}|\varphi|$,
9. $|\clubsuit x\varphi| = \overline{N_x}|\varphi|$.

This shows that NS can be used to define some useful generalized quantifiers.

4.4. *Probability quantifiers and PRAS*

In [58], two kinds of probability on first-order logics are distinguished. One puts a probability on possible worlds and results in the probabilistic modalities considered in section . The other approach puts a probability on the domain and results in probability quantifiers. In this section, we will consider the relationship between this kind of probabilistic logic and PRAS.

The alphabet of probability quantifier logic (PQL) consists of a class of probability quantifiers $(> r)$ for all $r \in [0, 1]$ as well as that of FOL. For simplicity, we assume that the set of variables VAR is finite. Let VAR^+ denote the set of finite nonempty sequences of variables. The wffs of PQL will include $(> r)\mathbf{x}\varphi$ if φ is a wff and $\mathbf{x} \in VAR^+$.

A PQL structure is a triple (D, I, Pr), where (D, I) is just an FOL structure, and Pr is a probability distribution on D. The probability measure Pr can be extended to a product measure Pr^n on the domain D^n for any $n > 1$ by measure theoretic techniques[58]. For any sequence of variables $\mathbf{x}$, we can define the equivalence relation $=_{\mathbf{x}}$ on $U = [VAR \rightarrow D]$ by

$$u =_{\mathbf{x}} v \Leftrightarrow \forall y \notin \mathbf{x} u(y) = v(y).$$

If the cardinality of VAR is n, then $(U, =_{\mathbf{x}}, Pr^n)$ forms a PRAS for any $\mathbf{x} \in VAR^+$, and the denotation of the wff $(> r)\mathbf{x}\varphi$ is defined by

$$|(> r)\mathbf{x}\varphi| = \overline{(=_{\mathbf{x}})_{1-r}}|\varphi|.$$

We must remarks that PQL is in fact only a fragment of the logic developed in [58] since we do not allow arithmetic expressions formed by probability terms. However, it is still sufficiently expressive to represent statistical knowledge such as "The probability that a randomly chosen bird flies is greater than 0.9". For example, $(> 0.5)xyLike(x, y)$ means that if we pick a pair randomly from the domain, the probability that x likes y is greater than 0.5. Moreover, we consider only the probability quantifiers based on equivalence relations of the form $=_{\mathbf{x}}$. However, it is not difficult to combine the idea of rough quantifiers discussed above, and have a logic of rough probability quantifiers.

5. Concluding Remarks and Future Works

Though all logics discussed in this paper are by themselves a topic of research, we only touch upon the basic aspects of them and concentrate on their relationship with rough set theory. Therefore, there remain many research problems deserving further study. Among them, we list some of the most important ones:

1. A development of complete axiomatization of the PEL discussed in Sec. 3.4.
2. The articulation and further development of the logic for reasoning about knowledge and communication based on MDAS.
3. The study of the logical properties of the rough quantifiers introduced in Sec. 4.2.

Moreover, this paper only surveys the logical aspects of rough set theory that can be fitted into the paradigms of possible worlds or variable assignments as the universe. There are also other related topics to be considered in the future extension, such as the study of consequence relations based on information systems[90,91].

Briefly speaking, in this paper, we survey and establish the correspondence between various approximation spaces and the semantic domains of modal and quantifier logics. On the other hand, in [92,93], the rough set theory is treated uniformly in the relation algebra framework. For an approximation space (U, R) and a subset of U, X, the lower or upper approximation maps X into a subset of U. However, if we identify X with its cylindrical extension $X^c =_{def} X \times U$, then X can be also seen as a binary relation on U. Then, we have $(\overline{R}X)^c = R \circ X^c$, where $\circ$ is the composition operation between two relations. This facilitates the transformation of an approximation space into a relation algebra. By combining his results and ours, the semantics of many logics explored here may have an algebraic interpretation. This seems unsurprising since algebraic semantics ever played an important role before Kripke semantics for modal logic becomes so prevailing. However, to what extent the results can give rise to the algebraic semantics of the logics discussed in this paper need still further investigation.

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