An Overview On Intuitionistic Fuzzy Sets

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Abstract: We present a brief overview on Intuitionistic fuzzy sets which cuts across some definitions, operations, algebra, modal operators and normalization on Intuitionistic fuzzy set.

Keywords:algebra, fuzzy sets, intuitionistic fuzzy sets, modal operators, normalization.

Introduction

The theory of fuzzy sets (FS) introduced by [1] has showed meaningful applications in many field of studies. The idea of fuzzy set is welcomed because it handles uncertainty and vagueness which Cantorian set could not address. In fuzzy set theory, the membership of an element to a fuzzy set is a single value between zero and one. However in reality, it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there may be some hesitation degree. Therefore, a generalization of fuzzy sets was introduced by [2, 3] as intuitionistic fuzzy sets (IFS) which incorporated the degree of hesitation called hesitation margin (and is defined as 1 minus the sum of membership and non-membership degrees respectively). We will present a concise overview on IFS viz: some definitions, basic operations, some algebra, modal operators and its normalization.

Brief Introduction of Intuitionistic Fuzzy Sets

Definition 1: Let X be a nonempty set. A fuzzy set A drawn from X is defined as $A = \{\langle x, \mu_A(x)\rangle : x \in X\}$, where $\mu_A(x) : X \longrightarrow [0,1]$ is the membership function of the fuzzy set A.

Definition 2: Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$, where the functions $\mu_A(x), \nu_A(x) : X \longrightarrow [0,1]$ define respectively, the degree of membership and degree of non-membership of the element $x \in X$ to the set$A$, which is a subset of X, and for every element $x \in X$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Furthermore, we have $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ called the intuitionistic fuzzy set index or hesitation margin of $x$in A. $\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0,1]$ i.e.,$\pi_A(x): X \longrightarrow [0,1]$and $0 \leq \pi_A \leq 1$ for every $x \in X$. $\pi_A(x)$expresses the lack of knowledge of whether x belongs to IFS A or not.

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Definition 3: Let $A \in X$ be IFS, then;

1. $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the degree of indetermacy of the element $x \in A$.
2. $\partial_A(x) = \mu_A(x) + \pi_A(x)\mu_A(x)$ is called the degree of favour of $x \in A$.
3. $\eta_A(x) = \nu_A(x) + \pi_A(x)\nu_A(x)$ is called the degree of against of $x \in A$.

For example, let A be an intuitionistic fuzzy set with $\mu_A(x) = 0.5$ and $\nu_A(x) = 0.3$ then, $\pi_A(x) = 0.2$, $\partial_A(x) = 0.6$, and $\eta_A(x) = 0.36$.It can be interpreted as "the degree that the object x belongs to IFS A is 0.5, the degree that the object x does not belong to IFS A is 0.3, the degree of hesitancy or indetermacy of x belonging to IFS A is 0.2, the degree of favour of x belonging to IFS A is 0.6 and the degree of against of x not belonging to IFS Ax is 0.36".

Definition 4 (Similar IFS): Two IFS A and B are said to be similar or cognate if $\exists$ $\mu_A(x) = \mu_B(x)$or $\nu_A(x) = \nu_B(x)$.

Definition 5 (Comparable IFS): Two IFSA and B are said to be equal or comparable if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$.

Definition 6 (Equivalent IFS): Two IFSA and B are said to be equivalent to each other i.e. A is equivalent to B, denoted by $A \sim B$ if $\exists$ functions $f: \mu_A(x) \rightarrow \mu_B(x)$ and $f: \nu_A(x) \rightarrow \nu_B(x)$ which are both injection and surjection (i.e. bijection). Then, the functions define a one-to-one correspondence between A and B.

Definition 7(Inclusive IFS): Let A and B be two IFS, $A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$for $x \in X$. Then A is a subset of B and B is a superset of A.

Definition 8(Proper Subset): A is a proper subset of B i.e. $A \subset B$ if $A \subseteq B$ and $A \neq B$. It means $\mu_A(x) \leq \mu_B(x)$and $\nu_A(x) \geq \nu_B(x)$ but $\mu_A(x) \neq \mu_B(x)$ and $\nu_A(x) \neq \nu_B(x)$for $x \in X$.

Definition 9 (Dominations): An IFS A is dominated by another IFS B (i.e. $A \preccurlyeq B$), if there exist an injection from A to B. A is strictly dominated by B (i.e. $A \prec B$), if (i) $A \preccurlyeq B$ and (ii) A is not equinumerous with B.

Definition 10 (Relations): Let A, B and C be IFSs. Then;

i. $A \preccurlyeq A$ i.e. A is reflexive relation,
ii. $A \preccurlyeq B$ and $B \preccurlyeq A$ i.e. symmetric relation,
iii. $A \preccurlyeq B$ and $B \preccurlyeq C \Rightarrow A \preccurlyeq C$ i.e. transitive relation.

Corollary1: For any IFS A and B, if $A \preccurlyeq B$ and $B \preccurlyeq A \Rightarrow A \sim B$.

Corollary 2: For any IFS A and B, if $A \preccurlyeq A$, $A \preccurlyeq B$ and $B \preccurlyeq A \Rightarrow A$ and B are compatible to each other.

Note:

1. If a relation is reflexive, symmetric and transitive, such a relation is called an *"equivalence relation"*.
2. The proofs of Cor.1 and Cor.2 are obvious.

Basic Operations on Intuitionistic Fuzzy Sets

[Inclusion] $A \subseteq B \leftrightarrow \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)\ \forall x \in X$

[Complement] $A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle : x \in X\}$

[Union]
$A \cup B = \{\langle x, max(\mu_A(x), \mu_B(x)), min(\nu_A(x), \nu_B(x)) \rangle : x \in X\}$

[Intersection]
$A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_B(x)) \rangle : x \in X\}$

[Addition]
$A \oplus B = \{\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \nu_A(x)\nu_B(x) \rangle : x \in X\}$ [Multiplication]
$A \otimes B = \{\langle x, \mu_A(x)\mu_B(x), \nu_A(x) + \nu_B(x) - \nu_A(x)\nu_B(x) \rangle : x \in X\}$

Algebra Laws in Intuitionistic Fuzzy Sets

Let A, B and C be IFS in X, then the following are the algebra:

1. $(A^c)^c = A$ i.e. complementary law.
2. $(i) A \cup A = A$ $(ii) A \cap A = A$ i.e. idempotent law.
3. (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$ i.e. commutative law.
4. $(i)(A \cup B) \cup C = A \cup (B \cup C)$ $(ii)(A \cap B) \cap C = A \cap (B \cap C)$ i.e. associative law.
5. $(i) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $(ii) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ i.e. distributive law. It also holds for right distributive law.
6. (i) $(A \cup B)^c = A^c \cap B^c$ (ii) $(A \cap B)^c = A^c \cup B^c$ i.e. DeMorgan's laws.
7. (i) $A \cap (A \cup B) = A$ (ii) $A \cup (A \cap B) = A$ i.e. absorption laws.
8. $(i) A \oplus B = B \oplus A$ $(ii) A \otimes B = B \otimes A$
9. $(i) A \oplus (B \oplus C) = (A \oplus B) \oplus C$ $(ii) A \otimes (B \otimes C) = (A \otimes B) \otimes C$
10. (i) $(A \oplus B)^c = A^c \otimes B^c$ $(ii)(A \otimes B)^c = A^c \oplus B^c$
11. $(i) A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$ $(ii) A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$
    (iii) $A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C)$ (iv) $A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C)$

Note that (11) also holds for right distributive law and allthese laws can be verified numerically.

Some Modal Operators on Intuitionistic Fuzzy Sets

We define over the set of IFS, two modal operators which transform every IFS into fuzzy set. These operators are similar to the operators *'necessity'* and *'possibility'* defined in some modal logics. This idea is drawn from the modal operators on IFS proposed by [3].

Definition 11(Modal operators): Let X be nonempty. If A is an IFS drawn from X, then;

(i) $\Box A = \{\langle x, \mu_A(x) \rangle : x \in X\} = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X\}$

(ii) $\Diamond A = \{\langle x, 1 - \nu_A(x) \rangle : x \in X\} = \{\langle x, 1 - \nu_A(x), \nu_A(x), \rangle : x \in X\}$.

Theorem 1: Let X be nonempty. For every IFS A in X;

(a) $\Box\Box A = A$
(b) $\Box\Diamond A = \Diamond A$
(c) $\Diamond\Box A = A$
(d) $\Diamond\Diamond A = \Diamond A$

Proof

(a) $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$
$\Box$ $A = \{\langle x, \mu_A(x) \rangle : x \in X\} = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X\} = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$,
$\Box\Box$ $A = \langle x, \mu_A(x) \rangle : x \in X\} = \{\langle x, \mu_A(x), 1 -, \mu_A(x) \rangle : x \in X\} = A$
Note: From Def. 11 and the proof of (a), the proofs of (b) - (d) are straightforward.

Theorem 2: Let X be nonempty. For every two IFS A and B in X;

(a) $(A \cap B) = A \cap B$
(b) $\Diamond(A \cap B) = \Diamond A \cap \Diamond B$
(c) $\Box(A \cup B) = A \cup B$
(d) $\Diamond(A \cup B) = \Diamond A \cup \Diamond B$
(e) $\Box(A \oplus B) = A \oplus \Box B$
(f) $\Box(A \otimes B) = A \otimes \Box B$
(g) $\Diamond(A \otimes B) = \Diamond A \otimes \Diamond B$
(h) $\Diamond(A \oplus B) = \Diamond A \oplus \Diamond B$

Proof

$(a) A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_A(x)) \rangle : x \in X\}$
$\Box(A \cap B) = \{\langle x, min(\mu_A(x), \mu_B(x)) \rangle : x \in X\} = \{\langle x, \mu_A(x) \rangle : x \in X\} \cap \{\langle x, \mu_B(x) \rangle : x \in X\} = \boldsymbol{A} \cap \boldsymbol{B}$
$\Box(A \cap B) = \boldsymbol{A} \cap \boldsymbol{B}$
$(b) A \cap B = \{\langle x, min(\mu_A(x), \mu_B(x)), max(\nu_A(x), \nu_A(x)) \rangle : x \in X\}$
$A \cap B = \{\langle x, min(1 - \nu_A(x), 1 - \nu_B(x)), max(1 - \mu_A(x), 1 - \mu_A(x)) \rangle : x \in X\}$
$\Diamond(A \cap B) = \{\langle x, min(1 - \nu_A(x), 1 - \nu_A(x)) \rangle : x \in X\}$
$\Diamond(A \cap B) = \{\langle x, 1 - \nu_A(x) \rangle : x \in X\} \cap \{\langle x, 1 - \nu_B(x) \rangle : x \in X\} = \Diamond A \cap \Diamond B$
$\Diamond(A \cap B) = \Diamond A \cap \Diamond B$

Note: From the proofs of (a) and (b), Def. 11 and the algebra laws in IFS; the proofs of (c) –(h) are straightforward.

Theorem 3: Let X be a nonempty IFS. Let $A, B \in X$, then;
$(a) A \subseteq \Box B$ if and only if $\Box A \subseteq \Box B$,
$(b) A \subseteq \Diamond B$ if and only if $\Diamond A \subseteq \Diamond B$.

Proof

(a)Given that $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x) \rangle : x \in X\}$
$B = \{\langle x, \mu_B(x) \rangle : x \in X\} = \{\langle x, \mu_B(x), 1 - \mu_B(x) \rangle : x \in X\} = \{\langle x, \mu_B(x), \nu_B(x) \rangle : x \in X\} = B$
i.e. $\Box B = B$ and consequently $\Box A = A$ whenever IFS is transformed to fuzzy set by the modal operators.
$A \subseteq \Box B \Rightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)\, \forall x \in X$. Since $\Box A = A \Rightarrow \Box A \subseteq \Box B$.
Conversely, if $\Box A \subseteq \Box B \Rightarrow A \subseteq \Box B$ for the same reason as in above.
(b) $\Diamond B = \{\langle x, 1 - \nu_B(x) \rangle : x \in X\} = \{\langle x, 1 - \nu_B(x), \nu_B(x) \rangle : x \in X\} = \{\langle x, \mu_B(x), \nu_B(x) \rangle : x \in X\} = B \Rightarrow \Diamond B = B$ and also $\Diamond A = A$. Then if $A \subseteq \Diamond B \Rightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x) \forall x \in X$. Obviously, $\Diamond A \subseteq \Diamond B$ since $\Diamond A = A$.
Conversely, if $\Diamond A \subseteq \Diamond B \Rightarrow A \subseteq \Diamond B$ is true for the same reason as in above.□

Normalization of Intuitionistic Fuzzy Sets

Definition 12: Let X be a nonempty universal set. The normalization of an intuitionistic fuzzy set A denoted by $NORM(\mathrm{A})$ is defined as:
$NORM(\mathrm{A}) = \{\langle x, \mu_{NORM(A)}(x), \nu_{NORM(A)}(x) \rangle : x \in \mathrm{X}\}$, where $\mu_{NORM(A)}(x) = \frac{\mu_A(x)}{\sup(\mu_A(x))}$ and $\nu_{NORM(A)}(x) = \frac{\nu_A(x) - \inf(\nu_A(x))}{1 - \inf(\nu_A(x))}$ for $\mathrm{X} = \{x\}$.
Incorporating $\pi_{NORM(A)}(x)$, $NORM(A) = \{\langle x, \mu_{NORM(A)}(x), \nu_{NORM(A)}(x), \pi_{NORM(A)}(x) \rangle : x \in X\}$ for $\pi_{NORM(A)}(x) = 1 - \mu_{NORM(A)}(x) - \nu_{NORM(A)}(x)$. Supriya *et al.* [4] used $NORM(A)$ as operation of normalization of intuitionistic fuzzy set A and gave the following proposition:

Proposition: For IFS A of the universe X;
(i) if $\pi_A(x) = 0$, then $\pi_{NORM(A)}(x) = 0$,
(ii) $NORM(\mathrm{A}) = (NORM(A))$,
(iii) $NORM(\Diamond \mathrm{A}) = \Diamond(NORM(A))$.

Zeng and Li [5] challenged Def.12 and the Proposition given by [4] to be incorrect, because they do not retain the property of IFS. However, to remedy the problem, we subject the hesitation margin $\pi_A(x)$ to tend to zero i.e. $\pi_A(x) \approx 0$.

Example

Let $X = \{x_1, x_2, x_3\}$ and let IFS A be *s.t.* $A = \{\langle 0.6, 0.4\rangle, \langle 0.8, 0.2\rangle, \langle 0.7, 0.3\rangle\}$.

Then $\sup(\mu_A(x)) = 0.8$ and $\inf(\nu_A(x)) = 0.2$, thus; $\mu_{NORM(A)}(x_1) = 0.75$, $\mu_{NORM(A)}(x_2) = 1.0$, $\mu_{NORM(A)}(x_3) = 0.875$, $\nu_{NORM(A)}(x_1) = 0.25$, $\nu_{NORM(A)}(x_2) = 0.0$, $\nu_{NORM(A)}(x_3) = 0.125$. Then $NORM(A) = \{\langle 0.75, 0.25\rangle, \langle 1.0, 0.0\rangle, \langle 0.875, 0.125\rangle\}$.

Obviously, $\mu_{NORM(A)}(x_1) + \nu_{NORM(A)}(x_1) = 1$, $\mu_{NORM(A)}(x_2) + \nu_{NORM(A)}(x_2) = 1$, $\mu_{NORM(A)}(x_3) + \nu_{NORM(A)}(x_3) = 1$, $\Rightarrow$that $\pi_{NORM(A)}(x) = 0$ since $\pi_{NORM(A)}(x) = 1 - \mu_{NORM(A)}(x) - \nu_{NORM(A)}(x)$. Hence $0 \leq \mu_{NORM(A)}(x) + \nu_{NORM(A)}(x) \leq 1$ is satisfied. Therefore $NORM(A)$ is IFS.

Now, we verify the Proposition as follows; if $A = \{\langle 0.6, 0.4\rangle, \langle 0.8, 0.2\rangle, \langle 0.7, 0.3\rangle\}$,

from Def.11, it is clear that $\Diamond \mathrm{A} = \mathrm{A} = A$ since $\pi_A(x) = 0$ and so $\mu_{NORM(A)}(x) + \nu_{NORM(A)}(x) = 1$. Then it is crystal clear that, if $\pi_A(x) = 0 \Rightarrow \pi_{NORM(A)}(x) = 0$.

Next, we show that $NORM(\mathrm{A}) = (NORM(A))$. Since $\Box \mathrm{A} = \mathrm{A}$, then $NORM(\Box \mathrm{A}) = \{\langle 0.75, 0.25\rangle, \langle 1.0, 0.0\rangle, \langle 0.875, 0.125\rangle\}$. If $NORM(\Box \mathrm{A}) = \{\langle 0.75, 0.25\rangle, \langle 1.0, 0.0\rangle, \langle 0.875, 0.125\rangle\}$, then $\Box(NORM(A)) = \{\langle 0.75, 0.25\rangle, \langle 1.0, 0.0\rangle, \langle 0.875, 0.125\rangle\}$. from Def.11, it is certain that $NORM(\mathrm{A}) = (NORM(A))$.

Then, we show that $NORM(\Diamond \mathrm{A}) = \Diamond(NORM(A))$. Since $\Diamond \mathrm{A} = \mathrm{A}$ as IFS is transformed to fuzzy set by the operators, $NORM(\Diamond \mathrm{A}) = \{\langle 0.75, 0.25\rangle, \langle 1.0, 0.0\rangle, \langle 0.875, 0.125\rangle\}$. Then $(NORM(A)) = \{\langle 0.75, 0.25\rangle, \langle 1.0, 0.0\rangle, \langle 0.875, 0.125\rangle\}$, $NORM(\Diamond \mathrm{A}) = \Diamond(NORM(A))$.

Corollary 3: For an IFS A of the universe X;
(I) $NORM(A) = NORM(\Diamond \mathrm{A})$,
(II) $\Box(NORM(A)) = \Diamond(NORM(A))$.

Proof: Given that $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in X\}$. Since the operators transformed IFS to fuzzy set, it means that $\mu_A(x) = 1 - \nu_A(x)$ and $\nu_A(x) = 1 - \mu_A(x)$ for every $x \in X$. From Def.11, $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : \mathrm{x} \in \mathrm{X}\} = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in \mathrm{X}\} = A$. Also, from Def.11, $\Diamond A = \{\langle x, 1 - \nu_A(x), \nu_A(x) \rangle : \mathrm{x} \in \mathrm{X}\} = \{\langle x, \mu_A(x), \nu_A(x) \rangle : x \in \mathrm{X}\} = A$. Then, we conveniently say that $\Box A = \Diamond A = A$ as $\pi_A(x) = 0$. So $NORM(A) = NORM(\Diamond \mathrm{A})$ *is true.*

Next, we show that $\Box(NORM(A)) = \Diamond(NORM(A))$. From Def.12, $NORM(A) = \{\langle x, \mu_{NORM(A)}(x), \nu_{NORM(A)}(x), \pi_{NORM(A)}(x) \rangle : x \in \mathrm{X}\}$, Since $\pi_A(x) = 0$, then $\pi_{NORM(A)}(x) = 0$, it means that $NORM(A) = \{\langle x, \mu_{NORM(A)}(x), \nu_{NORM(A)}(x) \rangle : x \in \mathrm{X}\}$.
So $\mu_{NORM(A)}(x) + \nu_{NORM(A)}(x) = 1$. From Def.11, $\Box(NORM(A)) = \{\langle x, \mu_{NORM(A)}(x), 1 - \mu_{NORM(A)}(x) \rangle : \mathrm{x} \in \mathrm{X}\} = \{\langle x, \mu_{NORM(A)}(x), \nu_{NORM(A)}(x) \rangle : \mathrm{x} \in \mathrm{X}\} = NORM(\mathrm{A})$. *Also,* $\Diamond(\mathrm{NORM(A)}) = \{\langle x, 1 - \nu_{NORM(A)}(x), \nu_{NORM(A)}(x) \rangle : \mathrm{x} \in \mathrm{X}\} = \{\langle x, \mu_{NORM(A)}(x), \nu_{NORM(A)}(x) \rangle : \mathrm{x} \in \mathrm{X}\} = \mathrm{NORM(A)} \Rightarrow \Box(NORM(A)) = \Diamond(NORM(A))$.□

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