An Overview on Vehicle Scheduling Models

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Summary. The vehicle scheduling problem, arising in public transport bus companies, addresses the task of assigning buses to cover a given set of timetabled trips with consideration of practical requirements, such as multiple depots and vehicle types as well as further extensions. An optimal schedule is characterized by minimal fleet size and minimal operational costs. Various publications were released as a result of extensively research in the last decades on this topic. Several model approaches as well as specialized solving strategies were presented for the problem and its extensions. This paper discusses the model approaches for different kinds of vehicle scheduling problems and gives an up-to-date and comprehensive overview on the basis of a general problem definition. Although we concentrate on the presentation of model approaches, also the basic ideas of solution approaches are given.

1 Introduction

The planing process in public transportation consists of different recurrent tasks traditionally maintained subsequently. The process starts on the strategic level with collecting or forecasting data of passenger demand. Based on demand matrices, the infrastructure of the public transportation network should be defined. On this infrastructure planers establish routes and stop points for different lines. In the next planing step particular trips are defined for given lines. For each trip the timetable specifies a departure and an arrival time as well as start and end stations. The further planning process focus on efficient use of resources. It especially assures that vehicles and drivers serve all scheduled trips (see for example [8]). Due to the fact, that optimal usage of resources is a hard problem in combinatorial sense, it has been topic of intense research in the area of Operations Research.

In particular the scheduling of vehicles has become an extensively studied research area in the last 40-50 years. In the following we define the vehicle scheduling problem (VSP) as a task arising in the operational planning process of public transportation.

1.1 Problem definition

Given a set of timetabled trips with fixed travel (departure and arrival) times and start and end locations as well as traveling times between all pairs of end stations, the objective is to find an assignment of trips to vehicles such that

- each trip is covered exactly once,
- each vehicle performs a feasible sequence of trips and
- the overall costs are minimized.

The overall costs can be divided into *fixed costs* of vehicles (like investment and maintenance) and *operational costs* (e.g. fuel and attrition). Operational costs can be interpreted in various ways: driven distance could be considered as well as productive time or waiting time. Since these details are modeled in various ways, this paper only considers the operational costs as a term of "non-fixed" costs. In most practical situations the cost structure reveals the prioritized minimization of the vehicle fixed costs and leaves the operational cost minimization as the secondary objective.

Several extensions for the VSP with different additional requirements were discussed in literature over the last years, like the existence of more than one depot, a heterogeneous fleet with multiple vehicle types, the permission of variable departure times of trips and further restrictions on the routes of the buses.

The VSP is already topic of well-known survey papers. Most of them consider routing and scheduling problems in transportation or decision support systems in general (i.e. [3], [4], [48]). There are also specialized vehicle scheduling surveys which consider some models or solution approaches in detail (i.e. [10], [38]).

Although a lot of surveys on the vehicle scheduling topic exist, there is none which respects the up-to-date research of modeling and solution approaches. Since a lot of new research has been done in the last few years, this paper gives a comprehensive and up-to-date overview on published modeling approaches and the most important related literature. Therefor we propose the following parameter definitions in order to explain the model approaches in the following chapters:

T: set of timetabled trips with $|T| = n$
s_i: start station of trip i
e_i: end station of trip i
d_i: departure time of trip i
d_i: departure time of trip i
t_{uv}: travel time from station u to station v
$i \ \alpha \ j$: compatibility relation - whether trip j can be served after trip i by the same vehicle (whether $a_i + t_{e_i,s_j} \leq d_j$)

1.2 Paper structure

In the first section we have given a general definition of the vehicle scheduling problem and the related data. The formulation will be used in following chapters to show the different network models and Mixed Integer Programming (MIP) formulations and will be extended, if further problem extensions are considered. In section 2 the different model approaches for the vehicle scheduling problem with a single depot are presented. Section 3 describes the problem extension for multiple depots and states the existing model approaches for this case. Since most approaches are based on a Linear Programming (LP) Relaxation, the different quality of LP bounds are compared. Further problem extensions of practical relevance are described in section 4 and existing model approaches are shown.

2 Models for the Single Depot Case

The VSP for a single depot (SD-VSP) is comparatively "easy" to solve in the sense that it could be formulated as a problem for which polynomial time algorithms are known. In this section we present the different approaches to model the single depot case. Since a lot of standard as well as specialized polynomial time algorithms for the solution of the SD-VSP models can be found in literature we only give some hints for solution approaches and do not discuss these in details.

2.1 Minimal Decomposition Model

The first approach for solving the SD-VSP optimally was reported in [44].

The author defines a partial ordered set among the elements of the trip set T. An ordered relationship β is proposed that admits the service of a trip t_2 after trip t_1 if t_2 starts at the end station of t_1 and the departure time t_2 is later or equal to the arrival time of t_1. Since no deadheading is allowed between two end stations, β is a weaker formulation of the compatibility function α presented in 1.1.

The model idea is based on the Dilworth Theorem for partial ordered sets (cmp. [14]) which states that the partial order width is equal to the minimum number of chains needed to cover P. Transferred to the VSP the maximum cardinality set of pairwise incompatible trips is equal to the minimum number of vehicles to cover T.

The resulting model is:

$$\max \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \quad (1)$$

$$s.t. \sum_{j} x_{ij} \leq 1 \qquad \forall \ i = 1, 2, \ldots, n \quad (2)$$

$$\sum_{i} x_{ij} \leq 1 \qquad \forall \ j = 1, 2, \ldots, n \quad (3)$$

$$x_{ij} \geq 0 \quad (4)$$

$$\text{with } c_{ij} = 1 \text{ if } \ i \ \beta \ j \qquad \text{otherwise } c_{ij} = -\infty$$

[44] solved the problem by a reformulation as a network flow problem. A labeling algorithm is used to solve instances of up to 319 trips. In [5] this formulation is used to solve SD-VSP with deadheading (use of relation α instead of β). The problem is solved in a two-phase heuristic approach. The first phase only regards short connections. The second phase solves the whole problem with fixed connections taken from the first phase. It is reported, that instances having up to 650 trips have been solved successfully.

A drawback of the *Minimal Decomposition Model* is that it only solves the minimum fleet size, but operational costs are not respected. Also no upper bound for the fleet size can be set. This is fixed by the assignment model which we will describe in the following section.

2.2 Assignment Model

[40] formulates the SD-VSP as an assignment problem which we visualize in this paper as a complete bipartite graph (cmp. figure 1).

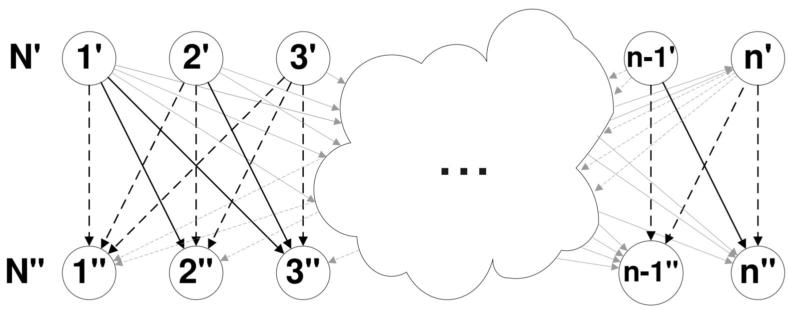

Fig. 1. Assignment Model

Each trip is represented by an arrival node i' and a departure node i''. While in the *Minimal Decomposition Model* only fixed costs are taken into account, each arc a_{ij} has costs equal to the operational costs c_{ij}. In addition to this each arc a_{ij} with $i \ \overline{\alpha} \ j$ gets the fixed costs c_v of a vehicle, because an additional vehicle would be needed to cover both trips.

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{5}$$

$$s.t. \sum_{j} x_{ij} = 1 \qquad \forall\ i \in N' \tag{6}$$

$$\sum_{i} x_{ij} = 1 \qquad \forall\ j \in N'' \tag{7}$$

$$x_{ij} \geq 0 \tag{8}$$

No computational results were published in [40]. The approach was also used later in [27] for dealing with subproblems in the multiple depot case (cmp. section 3).

Although operational costs are considered in this approach, a fixed or maximal number of vehicles could not be modeled.

2.3 Transportation Model

The *Transportation Model* approach was published in [22] and can also be demonstrated by a bipartite graph structure (cmp. figure 2). In contrast to the *Assignment Model*, only arcs a_{ij} with $i\ \alpha\ j$ are inserted into the graph (continuous lines). In addition to this two depot nodes (marked by $n+1$) are connected to each trip by additional depot arcs (dashed lines). Half of the vehicle fixed costs are assigned to each one of them. The depot arcs can be interpreted as empty trips from the depot to a start station (*pull-out*)and back to the depot at the end of a route (*pull-in*).

A transportation problem with demand/supply of one flow unit for each trip node and m units for the vehicle nodes was formulated with m equal to the number of available vehicles. To respect the possibility that not all vehicles are necessary to serve the trips, an arc between the depot nodes has been inserted with zero cost (bold line).

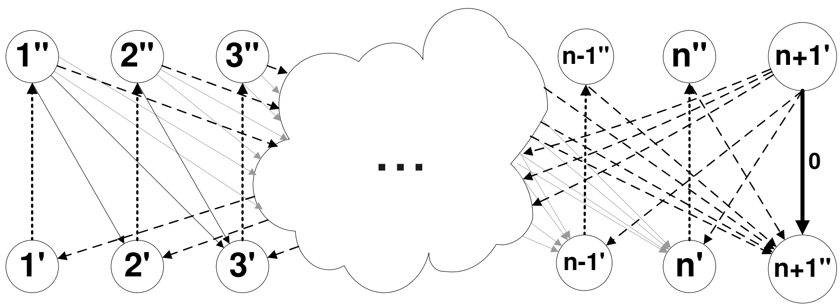

Fig. 2. Transportation Model

For the case of infeasibility the model has been extended by introducing arcs with penalty costs between all trip nodes as shown in figure 2 (dotted lines). These costs represent the penalty of not serving a trip at all. In the case that not enough vehicles are available, a solution is obtained which gives feasible vehicle schedules and a list of unserved trips. This approach was described in [10].

$$\min \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} c_{ij} x_{ij} \tag{9}$$

$$s.t. \sum_{j=1}^{n+1} x_{ij} = 1 \qquad \forall\ i = 1, 2, \ldots, n \tag{10}$$

$$\sum_{i=1}^{n+1} x_{ij} = 1 \qquad \forall\ j = 1, 2, \ldots, n \tag{11}$$

$$\sum_{j=1}^{n+1} x_{n+1,j} = n \tag{12}$$

$$\sum_{i=1}^{n+1} x_{i,n+1} = n \tag{13}$$

$$x_{ij} \geq 0 \tag{14}$$

In literature this *Transportation Model* is often called *Quasi-Assignment Model*, because the supply/demand vectors have only one entry not equal to one. Due to this characteristic specialized assignment algorithms have been adapted to this problem (cmp. [42],[21]). An arc generation approach is presented in [46] where the problem is solved with only short deadhead arcs in the initial master problem and a column generation process is applied to solve the problem to proven optimality.

2.4 Network Flow Model

A network flow approach presented in [4] was motivated by the early works of Dantzig about tanker scheduling (cmp. [12]).

Each trip is represented by two nodes connected via a trip arc. We define the set of trip arcs as AT and the set of all nodes in the network as N. Two additional nodes $n+1'$ and $n+1''$ for the depot are connected by pull-out and pull-in arcs. In this model the depot arcs are provided with operational costs for driving from/to the depot only. The fixed costs are assigned to a single arc leading back from the second depot node to the first one. An example for such kind of graph is shown in figure 3. A feasible flow from nodes $n+1'$ to $n+1''$ represents a feasible route for a vehicle.

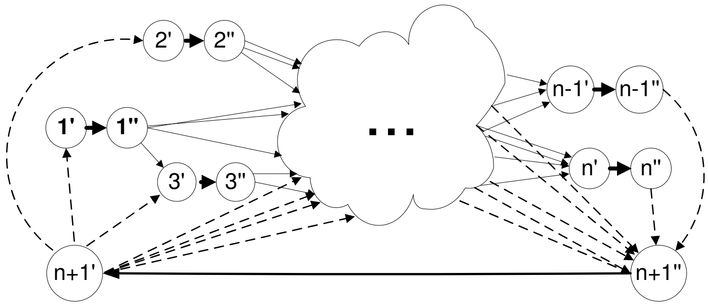

Fig. 3. Network Flow Model

The solution is obtained by solving a minimum cost flow problem considering the transshipment nodes only. Lower and upper bounds on the trip arcs are set to one to assure the serving of all trips and that the upper bound on the arc leading back is equal to the number of vehicles available. Optionally the minimum number of available vehicles can be set as the lower bound on the circulation flow arc.

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{15}$$

$$s.t. \sum_{i:(i,j)\in A} x_{ij} - \sum_{i:(j,i)\in A} x_{ji} = 0 \qquad \forall\ n \in N \tag{16}$$

$$1 \leq x_{ij} \leq 1 \qquad \forall\ (i,j) \in AT \tag{17}$$

$$x_{ij} \geq 0 \text{ and integer} \tag{18}$$

[1] shows a reformulation of a capacitated matching problem into an equivalent variant of this network flow model. The version presented here with a circulation flow arc for vehicle capacities was published in [10].

3 Models for the Multiple Depots Case

In the multiple depot case of the vehicle scheduling problem (MD-VSP) different locations – namely the depots – for starting bus routes are possible. As an additional restriction vehicles have to return to its start depot at the end of their route. This extends the problem to NP-hard complexity which was proven in [1].

We extend our problem formulation from section 1.1 by the definition of the set of depots H with $|H| = h$. Furthermore we define d_j as the number of available vehicles of depot j for each $j = 1, \ldots h$.

In the following we describe different formulations for the main three modeling approaches for MD-VSP given in literature:

1. Single-commodity models
2. Multi-commodity models
3. Set partitioning models

3.1 Single-Commodity Models

In the following formulations the MD-VSP is modeled in a graph with one node per trip and additional nodes for the depots or vehicles (dependent on the formulation). The objective is to find the minimum cost set of elementary circuits such that

1. each node is covered by exactly one circuit,
2. each circuit contains exactly one depot/vehicle node,
3. the number of circuits with a node belonging to depot j never exceeds the depot capacity d_j.

Single-Commodity Model with Subtour Breaking Constraints

[6] proposes a network structure having each available vehicle per depot modeled as one node. For each one of these nodes arcs are inserted to all trip nodes as well as arcs from all trips back to the vehicle node. These arcs are provided with fixed costs for a vehicle plus the operational costs for the empty trip. In order to prevent using unnecessary vehicles, additional arcs with zero costs are inserted for each vehicle node. These arcs point on the same node so that unused vehicles can be indicated. Since the depot nodes and arcs have to be inserted for each possible vehicle, the network has an extremely high amount of elements. Figure 4 shows an example for this graph with two depots.

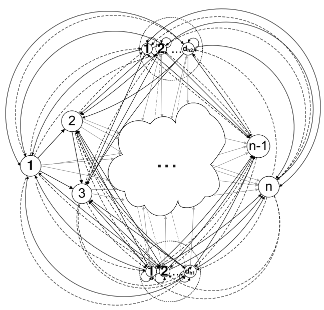

Fig. 4. Single-Commodity Model with Subtour Breaking

The model is formulated as a transportation problem with additional subtour elimination constraints which forbid every elementary path $P \in \Pi$ with more than one depot. The number of such constraints – called *subtour breaking constraints* – is extremely high so that MIP-based solution approaches can not consider these constraints explicitly. We define the node set $V \subset N$ that contains all nodes except the depot nodes and the set A which contains all arcs of the network.

The resulting mathematical model is:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{19}$$

$$s.t. \sum_{j:(i,j)\in A} x_{ij} = 1 \qquad \forall\ i \in V \tag{20}$$

$$\sum_{i:(i,j)\in A} x_{ij} = 1 \qquad \forall\ j \in V \tag{21}$$

$$\sum_{(i,j)\in P} x_{ij} \leq |P| - 1 \qquad \forall\ P \in \Pi \tag{22}$$

$$x_{ij} \in \{0,1\} \qquad \forall\ (i,j) \in A \tag{23}$$

The model was used in [6] to present a branch-and-bound algorithm which was the the first one that could solve the MD-VSP in an exact way. In [18] specialized path elimination cuts are published and used in a branch-and-cut framework. This approach was further extended in [17].

The main drawback of this formulation is the exponential growth of the number of constraints.

Single-Commodity Model with Assignment Variables

Another single-commodity approach was published in [37] where a more comprehensive network structure is used. The vehicle nodes as described above are aggregated and combined to one node per depot. Figure 5 shows an example of the graph.

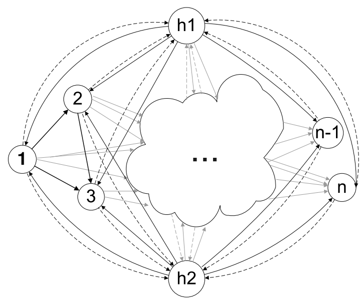

Fig. 5. Single-Commodity Model with Assignment Variables

A second group of variables $y_{i,h}$ is introduced and used to assign a trip i to the depot h. By this the number of constraints as well as the number of variables is reduced in comparison with the formulation in [6].

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{24}$$

$$s.t. \sum_{j:(i,j)\in A} x_{ij} = 1 \qquad \forall\ i \in V \tag{25}$$

$$\sum_{i:(i,j)\in A} x_{ij} = 1 \qquad \forall\ j \in V \tag{26}$$

$$\sum_{j\in N} x_{h,j} \leq d_h \qquad \forall\ h \in H \tag{27}$$

$$x_{h,j} - y_{j,h} \leq 0 \qquad \forall\ h \in H, \forall\ j \in V \tag{28}$$

$$x_{i,h} - y_{i,h} \leq 0 \qquad \forall\ h \in H, \forall\ i \in V \tag{29}$$

$$y_{i,h} + x_{ij} - y_{j,h} \leq 1 \qquad \forall\ h \in H, \forall\ (i,j) \in A \tag{30}$$

$$y_{j,h} + x_{ij} - y_{i,h} \leq 1 \qquad \forall\ h \in H, \forall\ (i,j) \in A \tag{31}$$

$$\sum_{h\in H} y_{i,h} = 1 \qquad \forall\ i \in V \tag{32}$$

$$x_{ij} \in \{0,1\} \qquad \forall\ (i,j) \in A \tag{33}$$

$$y_{i,h} \in \{0,1\} \qquad \forall\ h \in H, \forall\ i \in V \tag{34}$$

3.2 Multi-Commodity Models

In general the multi-commodity formulations are extensions of the network flow approach for the SD-VSP. For each depot an independent network is built. The multi-commodity formulations are then based on the multigraph generated by combining these networks.

We will distinguish two different model approaches. Although the mathematical models are similar, they are based on different underlying graph models.

Connection-Based Networks

In the first model the possible connections between the timetabled trips are modeled by considering all trip compatibilities explicitly. For each possible connection an arc is inserted in the underlying network. The number of connection arcs grows quadratically with the number of trips. We will refer to approaches of this kind as *connection-based approaches.*

An example for a connection-based network is shown in figure 6. The subnetworks are generated in the same way as in the network flow approach

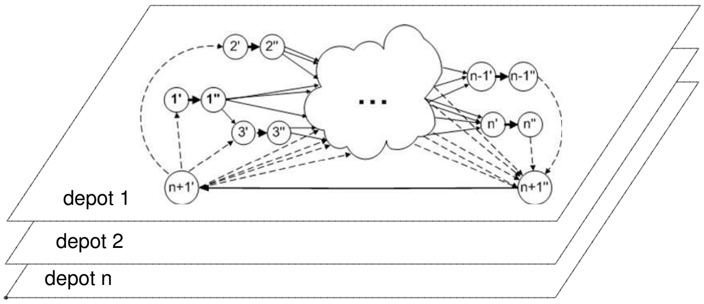

Fig. 6. Connection-based Network

for the single depot case (described in section 2.4). Only the subnetwork of the first depot is visible in this example.

Different mathematical models for this network have similar formulations. Therefore we will show the idea at only one example:

Like in the SD-VSP case the flow conservation constraints (36) are inserted for each node of the multigraph. Since every trip in the model is related to more than one arc, the bounds on the arcs can not just be set to one as in the SD-VSP formulation. Instead new constraints – often called *cover constraints* – have to be added (37). These constraints guarantee the service of all trips by allowing exactly one arc of the trip arcs of a special trip to be chosen in a feasible solution. The depot capacities can be considered by setting the upper bound for the circulation flow arc of a depot h equal to d_h (38).

Let $AT^t \subseteq AT \subset A$ be the set of trip arcs related to trip t. Furthermore we define A^h as the set of circulation flow arcs (in this case only one) of depot h.

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{35}$$

$$s.t. \sum_{i:(i,j)\in A} x_{ij} - \sum_{i:(j,i)\in A} x_{ji} = 0 \qquad \forall\ j \in N \tag{36}$$

$$\sum_{(i,j)\in AT^t} x_{ij} = 1 \qquad \forall\ t \in T \tag{37}$$

$$\sum_{(i,j)\in A^h} x_{ij} \leq d_h \qquad \forall\ h \in H \tag{38}$$

$$x_{ij} \in \{0,1\} \qquad \forall\ (i,j) \in A \setminus \bigcup_{h\in H} A^h \tag{39}$$

$$x_{ij} \geq 0 \text{ and integer} \qquad \forall\ (i,j) \in \bigcup_{h\in H} A^h \tag{40}$$

Several different approaches use this kind of formulation to solve the MD-VSP: in [1] an assignment formulation is extended to the multi-commodity model and heuristically solved by lagrangean relaxation of the cover constraints combined with a repairing procedure. The approach described in [32] uses a bundle of depot nodes per depot to model different daytimes and presents a lagrangean relaxation of depot-related flow conservation constraints in combination with a subgradient algorithm. An exact approach for the multi-commodity model given in [19] use a lagrangean relaxation to run a dual simplex algorithm to obtain a linear programming (LP) solution. The authors of [19] also observe that the potentially fractional solution of the LP is in most cases integer or near-integer for real-life instances. Because of this fact the integer solution was obtained by a standard branch-and-bound algorithm. This near-integer property of the LP was reported also by other authors. The authors of [19] conjecture that the property might arise from the underlying network structure since every detached subnetwork always has an integer solution. An arc generation approach in combination with a branch-and-cut algorithm was proposed in [36]. The decision of which arcs to be added in the master problem was done with a specialized pricing technique called lagrangean pricing. In [26] the amount of arcs needed in the connection-based network was heuristically reduced by defining three daytimes (morning, midday and evening). Each trip is assigned to one time period and the assumption is made that no evening trip will be served directly after a morning trip. This assumption gains a reduction of the model size of up to 40%.

Time-Space Networks

The approach using a multi-commodity formulation with a different underlying network structure was published in [30] and [31]. It avoids the drawback of explicit consideration of all possible connections between compatible trips. The idea is to exploit the transitivity property of partial ordered sets which says that for trips i, j, k the following conclusion applies:

$$(i \ \alpha \ j) \ \wedge \ (j \ \alpha \ k) \ \rightarrow \ i \ \alpha \ k$$

A time-space network (TSN) is constructed in which possible connections between groups of compatible trips are aggregated. Thus the number of compatibility arcs in the network decreases drastically (stable by 97 to 99 %) compared to the connection-based approaches without losing any feasible vehicle schedule. An example for a TSN is shown in figure 7.

The mathematical model is a multi-commodity formulation similar to the one for the connection-based model. Different to the connection-based formulation with mainly binary variables, flow variables in the time-space network based formulation are provided with general integer bounds.

In [31] a standard MIP optimization software is used for solving problems. In [23]) a two-phase heuristic approach is presented which fixes some connections a priori to solve very large-scale instances of MD-VSP.

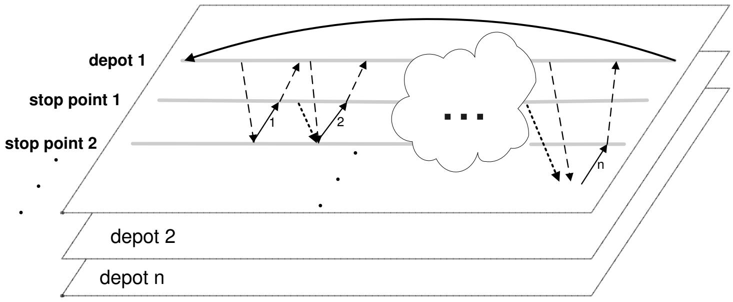

Fig. 7. Time-Space Network

3.3 Set Partitioning Models

The main idea of the set partitioning models is to enumerate all feasible routes for the vehicles and choose a subset of these routes which fulfills all restrictions. This model can be obtained by applying Dantzig-Wolfe decomposition to the multi-commodity model which was shown in [24] so that the same underlying graph structure is used (cmp. figure 6).

The resulting mathematical model is a set partitioning problem (SPP) which has only a few constraints (in fact a constraint for each trip to be covered) but a large number of variables since every feasible path through the network for each depot is a variable in the model. We define Ω as the set of all feasible paths in the multigraph. The constraints guarantee the service of all trips (42) and the adherence to the depot capacities (43):

$$\min \sum_{d \in D} \sum_{p \in \Omega_d} c_p x_p \tag{41}$$

$$s.t. \sum_{d \in D} \sum_{p \in \Omega_d} a_{jp} x_p = 1 \qquad \forall\ j \in T \tag{42}$$

$$\sum_{p \in \Omega_d} x_p \leq k_d \qquad \forall\ d \in D \tag{43}$$

$$x_p \in \{0,1\} \qquad \forall\ p \in \Omega \tag{44}$$

The first SPP approach was published in [43] with a column generation algorithm considering all feasible paths in an implicit way. The column generation process is divided into a master and a subproblem. The master problem solves the partitioning part considering only a subset of all columns. In the subproblem new promising columns are identified by solving a *Shortest Path Problem* on a graph containing dual information from the master problem (cmp. [43] for further details). [2] apply heuristics on the dual problem to eliminate candidate variables by means of a reduced cost criterium. Afterwards the SPP can be solved without column generation. In [24] new valid inequalities for the SPP formulation are introduced and the MD-VSP is solved within a branch-and-cut algorithm. Recently [41] presents a stabilized column generation approach which efficiently handles high degenerate problems.

A further advantage of the SPP formulation – in contrast to the other formulations – is that duty related constraints (like time or fuel restrictions for a bus route) can be easily recognized by excluding the infeasible routes from the route set Ω.

3.4 Comparison of LP Bounds

In this section we present the different qualities of the lower bounds obtained by the different model approaches for the MD-VSP. Since the optimal integer solution is the same, the LP quality is an important factor on a branch and bound algorithm or any other technique to obtain the optimal integer solution.

In [38] it is proven that the lower bound obtained by the LP solution of the single-commodity model with subtour breaking constraints is smaller or equal to the bound of the single-commodity model with assignment variables (cmp. section 3.1). Both single-commodity formulations provide weaker LP bounds than the (connection-based) multi-commodity flow formulation (also proven in [38]). [43] presents the proof that the LP bound of the multi-commodity model and the set partitioning model has the same value.

A summary of the LP bound relationships is given in figure 8 where the approaches are sorted according to their bound quality.

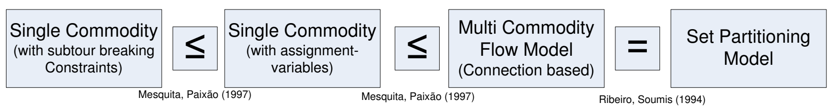

Fig. 8. Quality of LP-Bounds

4 Practical Extensions

This section describes the three extensions that have been reported to be important in practical applications of vehicle scheduling.

4.1 Multiple Vehicle Types and Vehicle Type Groups

For some trips – for example with stations close to hospitals – probably special vehicle types have to be used for service. We will define VT as the set of all given vehicle types. The different types may also have different fixed and/or operational costs or different speeds for deadheading between stations. In addition to this, every vehicle type $j \in VT$ has a limited capacity v_j. This problem is called a vehicle scheduling problem with multiple vehicle types (MVT-VSP). It is (already without multiple depots) NP-hard (cmp. [33]).

A lot of heuristical as well as exact optimization approaches were published. They all consider multiple vehicle types (cmp. [47],[15]). The most common technique to model the extension is to apply the following idea:

As shown in section 3 the multi-commodity and set partitioning models are based on a multigraph with a subnetwork for each depot in the problem. In case of multiple vehicle types the multigraph can be build with a subnetwork for each combination of depot and vehicle type. The costs on the network arcs are build accordingly to the related vehicle type and in the case of different vehicle speeds possibly the number of compatibility arcs in the subnetworks may differ.

This approach was published for the single depot case (SD-MVT-VSP) in [4] and also used in [7].

If the timetabled trips are restricted in that way that they could only be serviced by a subset of all vehicle types, a further extension – called vehicle type groups (VTG) – is inserted. [19] and [35] considers the multiple depot case with vehicle type groups within a connection-based multi-commodity approach and the realization within a time-space network based approach was done in [31].

4.2 Time Windows

A further problem extension in the application of vehicle scheduling is the consideration of variable trip departure and arrival times called time windows.

Typically public transit bus timetables consist of two different types of trips. The first group are regular trips which are timetabled in the line frequency. Additionally there are irregular trips (for example for school buses or additional buses during rush hours). Usually time windows are set especially on irregular trips because a shift will not lead to a disruption of the line frequency. Nevertheless slight changes can also be applied to regular trips.

We will define the time window for a trip i through the earliest and latest possible departure times l_i and u_i for all trips $i \in T$.

The vehicle scheduling problem with time windows (VSP-TW) is an NP-complete problem since already the simplest case with one vehicle and one depot is a traveling salesman problem with time windows (cmp. [45]).

Several solution approaches and practical experiences for dealing with the VSP-TW have been published (see for example [9]). In general two different model approaches are proposed for dealing with time windows for timetabled trips.

Discrete Time Windows

The first approach is to consider only discrete intervals of trip shifting by adding an additional trip arc in the model for each value between l_i and u_i

per trip i. The cover constraints that assure the service of all trips have to be extended by these new variables.

This approach was published in [34] for aircraft fleet routing and later adopted for the connection based multi-commodity flow model (cmp. [16],[11]) and for the time-space network based approach (cmp. [28]).

Continuous Time Windows

An alternative approach for the VSP-TW considers continuous occurrences of time windows. It can be applied to the set partitioning model where all feasible routes are enumerated. In a column generation approach for this model the time windows can be considered within the column generation process by solving a *shortest path problem with time windows* (cmp. [39],[13],[25]).

Both model approaches for time windows cover practical requirements since the timetables are usually also planned in discrete time intervals.

4.3 Route Constraints

A further extension of practical relevance are restrictions that force a special property on the routes of a feasible schedule. We call the resulting problem a *Vehicle Scheduling Problem with Route Constraints (VSP-RC).*

Typical and often discussed occurrences of route constraints are time restrictions on the vehicle routes (cmp. [20]). These are considered for example for fuel restrictions or maintenance intervals. There are three different model approaches for the single depot case (SD-VSP-RC) as well as some approaches for the multiple depot case (MD-VSP-RC) which are compared in the following section.

Route Constraints for SD-VSP

The first approach for the single depot case was given in [4] where the network flow approach as described in section 2.4 is extended. No depot nodes are used, but instead a new set of arcs – called *back arcs* – is used to model the feasible routes. A backing arc between two nodes i and j is inserted for every pair of related trips that could be served by one vehicle within the given time restriction. The mathematical model is extended in a similar way as in the model described in section 3.1: additional constraints guarantee that exactly one back arc is used within each route.

In [20] a publication (in Portuguese) made by I.M. Branco in 1989 is mentioned. The authors explain the approach in the following way: the route time constraints are modeled in an implicit way by building a graph with multiple levels – in particular one level for each timetabled trip. Each level consists of one node for the related trip and a part of the whole compatibility graph containing only the trips that are permitted in a route starting from this related trip. An example for such a graph with four trips is shown in figure 9.

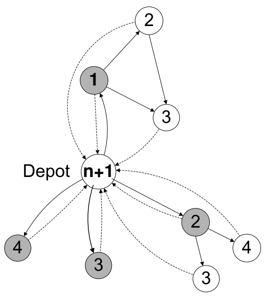

Fig. 9. Multilevel model for the SD-VSP-RC

A third approach is also presented in [20]. Independent network flows for all possible vehicles are modeled and the mathematical model forbids choosing a pair of trips (i, j) violating the time restriction.

Route Constraints for MD-VSP

Considering the case of multiple depot the first publication dealing with route (time) constraints was [39]. It presents a set partitioning formulation for the MD-VSP extended by the new restriction. As mentioned in section 3.3, the master problem of the SPP approach consists of all feasible routes. In the case of the MD-VSP-RC only the subproblem of the column generation process (where new feasible routes are identified) has to be adapted. Instead of solving a standard shortest path problem for identifying a promising route a *Resource Constrained Shortest Path Problem* with time as a resource is solved.

[26] considers time restrictions in a multi-commodity formulation by solving iterations of the MD-VSP and adding violated time constraints in the model. They also proposed two heuristic approaches for dealing with the MD-VSP-RC.

All the discussed approaches have in common that they consider only time as a resource for a route constraint. [29] reports a further route constraint with practical relevance: some bus companies prefer to pose a restriction for bus line changes in vehicle routes. Different flow decomposition techniques for the time-space network model generating cost minimal solutions with a "good" behavior according to the route constraints (line changes) are presented. Furthermore a model extension is presented having additional arcs with negative costs which may favor solutions with less line changes. The approaches are not able to deal with route aspects as hard constraints but instead allow a trade-of between costs and route behavior.

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