

An Unconditionally Stable Staggered Algorithm for Transient Finite Element Analysis of Coupled Thermoelastic Problems

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AN UNCONDITIONALLY STABLE STAGGERED ALGORITHM FOR TRANSIENT FINITE ELEMENT ANALYSIS OF COUPLED THERMOELASTIC PROBLEMS

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Abstract — An unconditionally stable second order accurate implicit-implicit staggered procedure for the finite element solution of fully coupled thermoelasticity transient problems is proposed. The procedure is stabilized with a semi-algebraic augmentation technique. A comparative cost analysis reveals the superiority of the proposed computational strategy to other conventional staggered procedures. Numerical examples of one and two-dimensional thermomechanical coupled problems demonstrate the accuracy of the proposed numerical solution algorithm.

I. INTRODUCTION

Transient response prediction of thermally loaded structures is of considerable importance in many aerospace engineering problems, and it has been the subject of intense research. Finite element formulations of the classical heat conduction problem without mechanical coupling have been presented by Wilson and Nickell [1]. Ritz type methods for the solution of linear dynamic problems in coupled thermoelasticity were given by Nickell and Sackman [2]. Oden [3] has formulated finite element models for the analysis of a class of nonlinear problems in dynamic coupled thermoelasticity, and Oden and Armstrong [4] have developed explicit

quadratic numerical schemes for the integration of nonlinear unpartitioned systems of difference equations arising from the analysis of dynamic coupled thermoviscoelastic problems. Recently, Ting and Chen [5] have introduced a unified numerical approach for the analysis of thermal stress waves. They have derived their algorithm from the concept of heat displacement and a variational formulation in Lagrangian form. They have proposed to integrate the resulting semi-discrete equations with conditionally stable explicit schemes. Liu and Zhang [6] have described an implicit-explicit procedure for the prediction of thermal stress waves in coupled thermoelasticity problems. They have adopted the explicit rational Runge-Kutta method [7, 8] for approximately solving the heat conduction equation and have claimed that their solution procedure is unconditionally stable. However, their computational strategy requires the manipulation of a full matrix. In a sequel note, Liu and Chang [9] have slightly modified the original procedure of Liu and Zhang to involve a banded rather than full matrix, and have numerically verified the unconditional stability on one dimensional problems.

However, several practical issues must be resolved before unconditionally stable explicit rational Runge-Kutta schemes can become suitable for the analysis of real thermomechanical coupled problems. First, when unconditional stability is achieved for explicit time integration algorithms, typically consistency becomes conditional (see for example Hughes and Belytschko [10]). Second, most rational Runge-Kutta algorithms involve some divide operations by the difference between intermediate solution quantities, which can significantly damage accuracy. Finally, these algorithms do not appear to accommodate staggered solution procedures for thermal/structure interaction problems, as they are not implemented in many existing production-level thermal computer programs.

The semi-discrete equations governing soil-pore fluid interaction dynamic problems are similar to those governing thermoelastic coupled transient problems. In this sense, the work of Park [11], and very recently that of Zienkiewicz, Paul and Chan [12] can be extended to the response analysis of thermally loaded structures.

In the present work, we present an unconditionally stable and robust implicit-implicit partitioned procedure for the solution of transient thermoelastic coupled problems. In Section II, we briefly review the basic equations for the linearized coupled thermoelasticity theory. A conventional implicit-implicit staggered solution procedure is summarized in Section III. The thermal coupling term in the structural dynamics equation is treated as an applied force. However, while being very simple to implement, the resulting time integration algorithm suffers from

conditional stability. In Section IV, we introduce an augmented implicit-implicit staggered solution procedure for the partitioned problem. We establish the unconditional stability and second order accuracy of the resulting numerical algorithm in Section V. In Section VI, we discuss the computer implementation aspects of the proposed computational strategy; we conduct a comparative cost analysis which demonstrates the superiority of the proposed solution procedure to other conventional staggered schemes. Finally in Section VII, we apply our partitioned algorithm to the solution of the one-dimensional Second Danilovskaya [13] and two-dimensional Youngdahl-Sternberg [14] problems. For both problems, the results generated by the proposed stabilized procedure are shown to be in excellent agreement with the analytical "exact" solutions.

II. FINITE ELEMENT FORMULATION

Let B denote the body of the structure to be analyzed, and $\partial B = \partial B_u \cup \partial B_s \cup \partial B_\theta \cup \partial B_q$ the surface enclosing it. The basic equations for the linearized isotropic coupled thermoelasticity theory are:

$$\begin{aligned}\rho \ddot{\mathbf{u}} &= \text{div} \boldsymbol{\sigma} + \mathbf{b} \quad \text{in } B \\ c \dot{\theta} &= -\text{div}(-k \nabla \theta) - \alpha(3\lambda + 2\mu)\theta_0 \text{tr}(\dot{\boldsymbol{\epsilon}}) + r \quad \text{in } B \\ \boldsymbol{\sigma} &= 2\mu \boldsymbol{\epsilon} + \lambda(\text{tr} \boldsymbol{\epsilon})\mathbf{I} - \alpha(3\lambda + 2\mu)(\theta - \theta_0)\mathbf{I} \\ \boldsymbol{\epsilon} &= \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)\end{aligned}$$

and (1)

$$\begin{aligned}\mathbf{u} &= \hat{\mathbf{u}} \quad \text{on } \partial B_u \\ \boldsymbol{\sigma} \mathbf{n} &= \hat{\mathbf{s}} \quad \text{on } \partial B_s \\ \theta &= \hat{\theta} \quad \text{on } \partial B_\theta \\ -k \nabla \theta &= \hat{q} \quad \text{on } \partial B_q\end{aligned}$$

where \mathbf{u} , $\boldsymbol{\epsilon}$, $\boldsymbol{\sigma}$, θ , θ_0 , \mathbf{b} , and r are the displacement, strain, stress, temperature, reference temperature chosen such that $(\theta - \theta_0)/\theta_0 \ll 1$, body force, and heat supply fields, respectively, while μ , λ , c , α , ρ , k and \mathbf{n} are the Lamé' moduli, the shear modulus, the specific heat, the coefficient of thermal expansion, the mass per unit volume, the thermal diffusivity, and the normal to the surface at a given

point, respectively. \mathbf{I} is the identity tensor. The dot and T superscripts denote a time derivative and a transpose operation, and tr denotes the trace of a given tensor.

If now we express the dependent variables \mathbf{u} and θ by suitable shape functions as:

$$\mathbf{u} = \mathbf{N}\bar{\mathbf{u}} \quad \text{and} \quad \theta = \hat{\mathbf{N}}\bar{\theta}$$

then a standard Galerkin procedure transforms (1) in the following algebraic coupled system of differential equations:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{D}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \mathbf{C}\theta &= \mathbf{f} \\ \mathbf{Q}\dot{\theta} + \mathbf{H}\theta + \theta_0\mathbf{C}^T\dot{\mathbf{u}} &= \mathbf{r} \end{aligned} \quad (2)$$

where \mathbf{M} , \mathbf{D} and \mathbf{K} are the usual mass, damping and stiffness matrices, \mathbf{f} is the prescribed structural loading vector, and \mathbf{Q} , \mathbf{H} , and \mathbf{r} are respectively the capacity and conductivity matrices and the nodal source vector. If \mathbf{L} denotes the differential operator corresponding to strain, the coupling matrix is expressed as $\mathbf{C} = \int_B (\mathbf{L}\mathbf{N})^T [1, 1, 1, 0, 0, 0] \hat{\mathbf{N}} dB$.

III. CONVENTIONAL IMPLICIT-IMPLICIT PROCEDURE

In many applications, the coupling term $\mathbf{C}^T\dot{\mathbf{u}}$ that appears in the heat equation and which is induced by the effect of the strain rate is negligible. Therefore, one expects the second of equations (2) to remain parabolic and the temperature response to remain close to the uncoupled solution. Consequently, the dependent variable θ is easier to predict than the displacement \mathbf{u} , so that the most natural way of solving (2) would be:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{u}}^{n+1} + \mathbf{D}\dot{\mathbf{u}}^{n+1} + \mathbf{K}\mathbf{u}^{n+1} &= \mathbf{f}^{n+1} + \mathbf{C}\theta^{n+1P} \\ \mathbf{Q}\dot{\theta}^{n+1} + \mathbf{H}\theta^{n+1} &= \mathbf{r}^{n+1} - \theta_0\mathbf{C}^T\dot{\mathbf{u}}^{n+1} \end{aligned} \quad (3)$$

where θ^{n+1P} is the predicted temperature. Unfortunately, the above numerical procedure is only conditionally stable, even when each field is integrated with an unconditionally stable algorithm. Proofs of this result are given by Dubois-Pelerin [15] for various consistent predictors. Next, we introduce an augmentation technique that stabilizes the staggered solution of (2).

IV. AN AUGMENTED IMPLICIT-IMPLICIT PARTITIONED PROCEDURE

Park, Felippa and DeRuntz [16] have introduced a differential augmentation concept that was successfully used in the stabilization of staggered solution procedures for fluid-structure interaction problems. Basically, one of the coupled equations is injected into the other in order to “soften” the system, either by reducing the large eigenvalues of the uncoupled stiff equation, or by introducing some damping into it. Here, we adopt a different strategy. We perform a semi-algebraic augmentation — that is, we augment one of the two coupled equations while integrating both fields.

First, the structural equation is integrated with the trapezoidal rule:

$$\begin{aligned}
 \dot{\mathbf{u}}^{n+1} &= \dot{\mathbf{u}}^n + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^{n+1} + \ddot{\mathbf{u}}^n) \\
 &= \dot{\mathbf{u}}^n + \frac{\Delta t}{2}[\ddot{\mathbf{u}}^n + \mathbf{M}^{-1}(\mathbf{f}^{n+1} - \mathbf{D}\dot{\mathbf{u}}^{n+1} - \mathbf{K}\mathbf{u}^{n+1} + \mathbf{C}\boldsymbol{\theta}^{n+1})] \\
 \mathbf{u}^{n+1} &= \mathbf{u}^n + \frac{\Delta t}{2}(\dot{\mathbf{u}}^{n+1} + \dot{\mathbf{u}}^n) \\
 &= \mathbf{u}^n + \Delta t\dot{\mathbf{u}}^n + \frac{\Delta t^2}{4}[\ddot{\mathbf{u}}^n + \mathbf{M}^{-1}(\mathbf{f}^{n+1} - \mathbf{D}\dot{\mathbf{u}}^{n+1} - \mathbf{K}\mathbf{u}^{n+1} + \mathbf{C}\boldsymbol{\theta}^{n+1})]
 \end{aligned} \tag{4}$$

and the velocity vector is extracted as:

$$(\mathbf{I} + \frac{\Delta t}{2}\mathbf{M}^{-1}\mathbf{D})\dot{\mathbf{u}}^{n+1} = \dot{\mathbf{u}}^n + \frac{\Delta t}{2}[\ddot{\mathbf{u}}^n + \mathbf{M}^{-1}(\mathbf{f}^{n+1} - \mathbf{K}\mathbf{u}^{n+1} + \mathbf{C}\boldsymbol{\theta}^{n+1})] \tag{5}$$

Next, the heat equation is also integrated with the trapezoidal rule:

$$\begin{aligned}
 \boldsymbol{\theta}^{n+1} &= \boldsymbol{\theta}^n + \frac{\Delta t}{2}(\dot{\boldsymbol{\theta}}^{n+1} + \dot{\boldsymbol{\theta}}^n) \\
 &= \boldsymbol{\theta}^n + \frac{\Delta t}{2}[\dot{\boldsymbol{\theta}}^n + \mathbf{Q}^{-1}(\mathbf{r}^{n+1} - \mathbf{H}\boldsymbol{\theta}^{n+1} - \theta_0\mathbf{C}^T\dot{\mathbf{u}}^{n+1})]
 \end{aligned} \tag{6}$$

Finally, the system is augmented by recasting (5) in (6) to obtain:

$$\theta^{n+1} = \theta^n + \frac{\Delta t}{2} \left\{ \dot{\theta}^n + \mathbf{Q}^{-1} [\mathbf{r}^{n+1} - \mathbf{H}\theta^{n+1} - \theta_0 \mathbf{C}^T (\mathbf{I} + \frac{\Delta t}{2} \mathbf{M}^{-1} \mathbf{D})^{-1} (\dot{\mathbf{u}}^n + \frac{\Delta t}{2} (\ddot{\mathbf{u}}^n + \mathbf{M}^{-1} (\mathbf{f}^{n+1} - \mathbf{K}\mathbf{u}^{n+1} + \mathbf{C}\theta^{n+1})))] \right\} \quad (7)$$

Substituting (5) into the second of equations (4) and re-arranging (7) leads to:

$$\begin{aligned} (\mathbf{I} + \frac{\Delta t^2}{4} \mathbf{B}(\Delta t) \mathbf{M}^{-1} \mathbf{K}) \mathbf{u}^{n+1} - \frac{\Delta t^2}{4} \mathbf{B}(\Delta t) \mathbf{M}^{-1} \mathbf{C} \theta^{n+1} &= \mathbf{F}^{n+1} \\ (-\frac{\Delta t^2}{4} \theta_0 \mathbf{A} \mathbf{K}) \mathbf{u}^{n+1} + (\mathbf{I} + \frac{\Delta t}{2} \mathbf{Q}^{-1} \mathbf{H} + \frac{\Delta t^2}{4} \theta_0 \mathbf{A} \mathbf{C}) \theta^{n+1} &= \mathbf{R}^{n+1} \end{aligned} \quad (8)$$

where

$$\begin{aligned} \mathbf{A} &= \mathbf{Q}^{-1} \mathbf{C}^T \mathbf{M}^{-1} \\ \mathbf{B}(\Delta t) &= (\mathbf{I} + \frac{\Delta t}{2} \mathbf{M}^{-1} \mathbf{D})^{-1} \\ \mathbf{F}^{n+1} &= \mathbf{u}^n + \frac{\Delta t}{2} (\mathbf{I} + \mathbf{B}(\Delta t)) \dot{\mathbf{u}}^n + \frac{\Delta t^2}{4} (\mathbf{B}(\Delta t) \ddot{\mathbf{u}}^n + \mathbf{M}^{-1} \mathbf{B}(\Delta t) \mathbf{f}^{n+1}) \\ \mathbf{R}^{n+1} &= \theta^n + \frac{\Delta t}{2} [\dot{\theta}^n + \mathbf{Q}^{-1} (\mathbf{r}^{n+1} - \theta_0 \mathbf{C}^T \mathbf{B}(\Delta t) \dot{\mathbf{u}}^n)] \\ &\quad - \frac{\Delta t^2}{4} [\theta_0 \mathbf{Q}^{-1} \mathbf{C}^T (\mathbf{B}(\Delta t) \ddot{\mathbf{u}}^n + \mathbf{B}(\Delta t) \mathbf{M}^{-1} \mathbf{f}^{n+1})] \end{aligned} \quad (9)$$

Now, a displacement predicted staggered procedure for the solution of (8) is:

1. Predict the displacement field:

$$\mathbf{u}^{n+1P} = \mathbf{u}^n \quad (10)$$

2. Solve for the temperature field:

$$(\mathbf{I} + \frac{\Delta t}{2} \mathbf{Q}^{-1} \mathbf{H} + \frac{\Delta t^2}{4} \theta_0 \mathbf{A} \mathbf{C}) \theta^{n+1} = \mathbf{R}^{n+1} + \frac{\Delta t^2}{4} \theta_0 \mathbf{A} \mathbf{K} \mathbf{u}^{n+1P} \quad (11)$$

3. Correct the displacement field:

$$\left(\mathbf{I} + \frac{\Delta t^2}{4}\mathbf{B}(\Delta t)\mathbf{M}^{-1}\mathbf{K}\right) \mathbf{u}^{n+1} = \mathbf{F}^{n+1} + \frac{\Delta t^2}{4}\mathbf{B}(\Delta t)\mathbf{M}^{-1}\mathbf{C} \boldsymbol{\theta}^{n+1} \quad (12)$$

4. Compute velocity, acceleration and flux fields:

$$\begin{aligned} \dot{\mathbf{u}}^{n+1} &= \mathbf{B}(\Delta t)\left\{\dot{\mathbf{u}}^n + \frac{\Delta t}{2}[\ddot{\mathbf{u}}^n + \mathbf{M}^{-1}(\mathbf{f}^{n+1} - \mathbf{K}\mathbf{u}^{n+1} + \mathbf{C}\boldsymbol{\theta}^{n+1})]\right\} \\ \ddot{\mathbf{u}}^{n+1} &= \mathbf{M}^{-1}(\mathbf{f}^{n+1} + \mathbf{C}\boldsymbol{\theta}^{n+1} - \mathbf{D}\dot{\mathbf{u}}^{n+1} - \mathbf{K}\mathbf{u}^{n+1}) \\ \dot{\boldsymbol{\theta}}^{n+1} &= \mathbf{Q}^{-1}(\mathbf{r}^{n+1} - \theta_0\mathbf{C}^T\dot{\mathbf{u}}^{n+1} - \mathbf{H}\boldsymbol{\theta}^{n+1}) \end{aligned} \quad (13)$$

Remarks:

1. the predictor \mathbf{u}^{n+1^P} is simply the previous step solution. It has been found (see Park [17]) that this is the most stable predictor when used in conjunction with the trapezoidal rule, while still maintaining a second-order accuracy.
2. the injection of (5) into (6) is not arbitrary. It will be shown in Section VI that this is more economical than injecting (6) into (5).
3. equations (13) define the computational path of the staggered procedure.

V. STABILITY AND ACCURACY ANALYSES

In this section, we establish that equations (10)-(13) result in a unconditionally stable second order accurate transient algorithm for the time integration of the coupled system (2). To avoid lengthy expressions, we consider the undamped ($\mathbf{D} = 0$) and unforced ($\mathbf{f} = \mathbf{r} = 0$) case. Note however that even when $\mathbf{D} = 0$, the quantity $\mathbf{C}\boldsymbol{\theta}$ still transmits a rate dependent damping effect to the structural equation.

Stability. The stability of the proposed staggered procedure can be examined by seeking a nontrivial solution in the form:

$$\begin{bmatrix} \mathbf{u}^{n+1} \\ \dot{\mathbf{u}}^{n+1} \\ \ddot{\mathbf{u}}^{n+1} \\ \boldsymbol{\theta}^{n+1} \\ \dot{\boldsymbol{\theta}}^{n+1} \end{bmatrix} = \frac{1+z}{1-z} \begin{bmatrix} \mathbf{u}^n \\ \dot{\mathbf{u}}^n \\ \ddot{\mathbf{u}}^n \\ \boldsymbol{\theta}^n \\ \dot{\boldsymbol{\theta}}^n \end{bmatrix} \quad (14)$$

and determining under what condition the real part of z is positive. Substitution of (10) into (11) and (14) into (11)-(13) yields, after some algebraic manipulations:

$$\begin{bmatrix} z^2\mathbf{I} + \frac{\Delta t^2}{4}\mathbf{M}^{-1}\mathbf{K} & -\frac{\Delta t^2}{4}\mathbf{M}^{-1}\mathbf{C} \\ -(1-z^2)\frac{\Delta t^2}{4}\theta_0\mathbf{A}\mathbf{K} & z^2\mathbf{I} + z\frac{\Delta t}{2}\mathbf{Q}^{-1}\mathbf{H} + \frac{\Delta t^2}{4}\theta_0\mathbf{A}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{u}^n \\ \boldsymbol{\theta}^n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (15)$$

Therefore, the characteristic equation associated with (15) is:

$$|\mathbf{M}z^3 + \mathbf{V}\mathbf{M}\frac{\Delta t}{2}z^2 + (\mathbf{K} + \theta_0\mathbf{C}\mathbf{Q}^{-1}\mathbf{C}^T + \frac{\Delta t^2}{4}\theta_0\mathbf{C}\mathbf{Q}^{-1}\mathbf{C}^T\mathbf{M}^{-1}\mathbf{K})\frac{\Delta t^2}{4}z + \mathbf{V}\mathbf{K}\frac{\Delta t^3}{8}| = 0 \quad (16)$$

where

$$\begin{aligned} \mathbf{V} &= \mathbf{C}\mathbf{U}\mathbf{C}^T \\ \mathbf{U} &= \mathbf{Q}^{-1}\mathbf{H}(\mathbf{C}^T\mathbf{C})^{-1} \end{aligned}$$

and $|\cdot|$ denotes the matrix determinant. If the matrices \mathbf{M} , \mathbf{K} , \mathbf{Q} and \mathbf{H} are positive definite, and the coupling matrix \mathbf{C} has full column rank, then \mathbf{U} , \mathbf{V} and each matrix coefficient of the determinant expression (16) is positive definite. If \mathbf{C} is column rank deficient, \mathbf{U} and \mathbf{V} are positive semi-definite. In any case, all coefficients of the stability polynomial are non-negative. Consequently, the first part of the Routh-Hurwitz criterion [18] for unconditional stability is satisfied. In order to check the second component of this criterion, we consider a 2-d.o.f. model problem for (2). The corresponding scalar form of (16) is:

$$a_3z^3 + a_2z^2 + a_1z + a_0 = 0 \quad (17)$$

where

$$a_3 = 1, \quad a_2 = \frac{\Delta th}{2q}, \quad a_1 = \frac{\Delta t^2}{4}\left[\omega^2 + \frac{\theta_0 c^2}{qm}\left(1 + \frac{\Delta t^2}{4}\omega^2\right)\right], \quad a_0 = \frac{\Delta t^3 h}{8q}\omega^2$$

Since $\Delta t, h, q, \omega^2, \theta_0, c^2$, and $m > 0$, then all the coefficients of the polynomial (17) in z are positive. Moreover, the quantity

$$a_1a_2 - a_0a_3 = \frac{\theta_0 hc^2 \Delta t^3}{8mq^2}\left(1 + \frac{\Delta t^2}{4}\omega^2\right)$$

is also positive, which demonstrates that the staggered solution procedure is unconditionally stable for the 2-d.o.f. model problem.

For the general multi-dimensional case, it turns out that the limiting case $\mathbf{K} = \mathbf{0}$ which states that the structural system will grow quadratically in time, provides a sufficient test. For this case, (16) reduces to:

$$\left| \mathbf{M}z^2 + \frac{\Delta t}{2}\mathbf{V}\mathbf{M}z + \frac{\Delta t^2}{4}\theta_0\mathbf{C}\mathbf{Q}^{-1}\mathbf{C}^T \right| = 0$$

Since \mathbf{M} is positive definite and $\mathbf{V}\mathbf{M}$ and $\mathbf{C}\mathbf{Q}^{-1}\mathbf{C}^T$ are at least positive semi-definite, the procedure is unconditionally stable for the limiting case $\mathbf{K} = \mathbf{0}$, as discussed in Bellman [19]. This argument has been extensively utilized in [12] during the analysis of several partitioned procedures. Therefore, we conclude that the procedure given by (10)-(13) is unconditionally stable.

Remarks:

1. the characteristic equation (16) reveals that the proposed procedure (10)-(13) is algorithmically identical to the one obtained by first differentiating the second of equations (2):

$$\mathbf{Q}\ddot{\theta} + \mathbf{H}\dot{\theta} + \theta_0\mathbf{C}^T\ddot{\mathbf{u}} = \dot{\mathbf{r}}$$

then substituting $\ddot{\mathbf{u}}$ from the first of equations (2) into the above equation:

$$\mathbf{Q}\ddot{\theta} + \mathbf{H}\dot{\theta} + \theta_0\mathbf{C}^T\mathbf{M}^{-1}\mathbf{C}\theta = \dot{\mathbf{r}} - \theta_0\mathbf{C}^T\mathbf{M}^{-1}(\mathbf{f} - \mathbf{K}\mathbf{u})$$

However, differentiating the nodal source vector may be not practical, for example, if \mathbf{r} is a discontinuous function of time. In our present derivation (11)-(13) we avoid this problem.

2. the first-order thermal equation is algorithmically modified to behave as a damped second-order system. It should be emphasized that the described stabilization technique has *not* introduced any artificial damping. The only augmentation that is used is part of the governing equation of motion itself.

Accuracy. After differentiation, the third of equations (13) in the unforced case reads:

$$\ddot{\theta}^{n+1} = -\theta_0 \mathbf{Q}^{-1} \mathbf{C}^T \ddot{\mathbf{u}}^{n+1} - \mathbf{Q}^{-1} \mathbf{H} \dot{\theta}^{n+1} \quad (18)$$

Expanding the various terms in (8) around the time $n\Delta t$ and injecting (13) and (18) when needed leads to:

$$\begin{aligned} \mathbf{M} \ddot{\mathbf{u}}^n + \mathbf{K} \mathbf{u}^n &= \mathbf{C} \theta^n + O(\Delta t^2) \\ \mathbf{Q} \dot{\theta}^n + \mathbf{H} \theta^n &= -\theta_0 \mathbf{C}^T \dot{\mathbf{u}}^n + O(\Delta t^2) \end{aligned} \quad (19)$$

Comparing (2) and (19) demonstrates that the staggered procedure is second order accurate. The same result can be proved for the damped ($\mathbf{D} \neq 0$) and forced ($\mathbf{f} \neq 0$, $\mathbf{r} \neq 0$) case.

VI. COMPUTATIONAL ASPECTS

In the remainder of this paper, we consider the case where the structure is undamped ($\mathbf{D} = 0$) and the mass and capacity matrices are lumped (\mathbf{M}, \mathbf{Q} are diagonal). The unconditionally stable staggered procedure (10)-(13) can be implemented as:

1. Form:

$$\mathbf{R}^{n+1*} = \frac{\Delta t}{2} \mathbf{r}^{n+1} + \mathbf{Q} \left[\theta^n + \frac{\Delta t}{2} (\dot{\theta}^n - \theta_0 \mathbf{C}^T \dot{\mathbf{u}}^n) \right] - \frac{\Delta t^2}{4} \theta_0 \mathbf{C}^T \left[\ddot{\mathbf{u}}^n + \mathbf{M}^{-1} (\mathbf{f}^{n+1} - \mathbf{K} \mathbf{u}^{n+1P}) \right] \quad (20)$$

2. Solve:

$$\left(\mathbf{Q} + \frac{\Delta t}{2} \mathbf{H} + \frac{\Delta t^2}{4} \theta_0 \mathbf{C}^T \mathbf{M}^{-1} \mathbf{C} \right) \theta^{n+1} = \mathbf{R}^{n+1*} \quad (21)$$

3. Form:

$$\mathbf{F}^{n+1*} = \mathbf{M} \left[\mathbf{u}^n + \Delta t (\dot{\mathbf{u}}^n + \frac{\Delta t}{4} \ddot{\mathbf{u}}^n) \right] + \frac{\Delta t^2}{4} (\mathbf{f}^{n+1} + \mathbf{C} \theta^{n+1}) \quad (22)$$

4. Solve:

$$(\mathbf{M} + \frac{\Delta t^2}{4}\mathbf{K})\mathbf{u}^{n+1} = \mathbf{F}^{n+1*} \quad (23)$$

5. Update:

$$\begin{aligned} \ddot{\mathbf{u}}^{n+1} &= \mathbf{M}^{-1}(\mathbf{f}^{n+1} + \mathbf{C}\boldsymbol{\theta}^{n+1} - \mathbf{K}\mathbf{u}^{n+1}) \\ \dot{\mathbf{u}}^{n+1} &= \dot{\mathbf{u}}^n + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^n + \ddot{\mathbf{u}}^{n+1}) \\ \dot{\boldsymbol{\theta}}^{n+1} &= \mathbf{Q}^{-1}(\mathbf{r}^{n+1} - \theta_0\mathbf{C}^T\dot{\mathbf{u}}^{n+1} - \mathbf{H}\boldsymbol{\theta}^{n+1}) \end{aligned} \quad (24)$$

Equations (20) to (24) involve algebraic computations that are common to most implicit algorithms, when applied to the uncoupled problem. Only the quantity $\mathbf{C}^T\mathbf{M}^{-1}\mathbf{C}$ deserves special attention. In particular, it is important to note that:

- $\mathbf{C}^T\mathbf{M}^{-1}\mathbf{C}$ is not a full matrix. It is a symmetric banded operator. Let n_s , n_h , b_s and b_h denote the sizes and the semi-bandwidths of the structural and heat matrices, respectively. Typically, n_s and b_s are two to six times larger than n_h and b_h . The matrix product $\mathbf{C}^T\mathbf{M}^{-1}\mathbf{C}$ is n_h by n_h and has a semi-bandwidth close to $2b_h$. Therefore, equation (21) entails the solution of an n_h by n_h symmetric banded system. On the other hand, if equation (6) had been injected into equation (5) — that is, if the temperature field had been eliminated from the structural equation — the resulting augmentation term would have been $\mathbf{C}\mathbf{Q}^{-1}\mathbf{C}^T$ which is n_s by n_s and has a semi-bandwidth close to $2b_s$. The latter would have entailed the solution of a symmetric system that is several times larger and denser than (21). For a rectilinear mesh composed of two-dimensional truss elements, the patterns of matrices \mathbf{C} , \mathbf{C}^T , $\mathbf{C}^T\mathbf{M}^{-1}\mathbf{C}$ and $\mathbf{C}\mathbf{Q}^{-1}\mathbf{C}^T$ are depicted in figure 1.
- the additional cost incurred by the augmentation term is restricted to the factorization and subsequent solutions of equation (21). The precise value of this additional cost (with respect to the conventional procedure (3)) depends on the cleverness of the implementation.

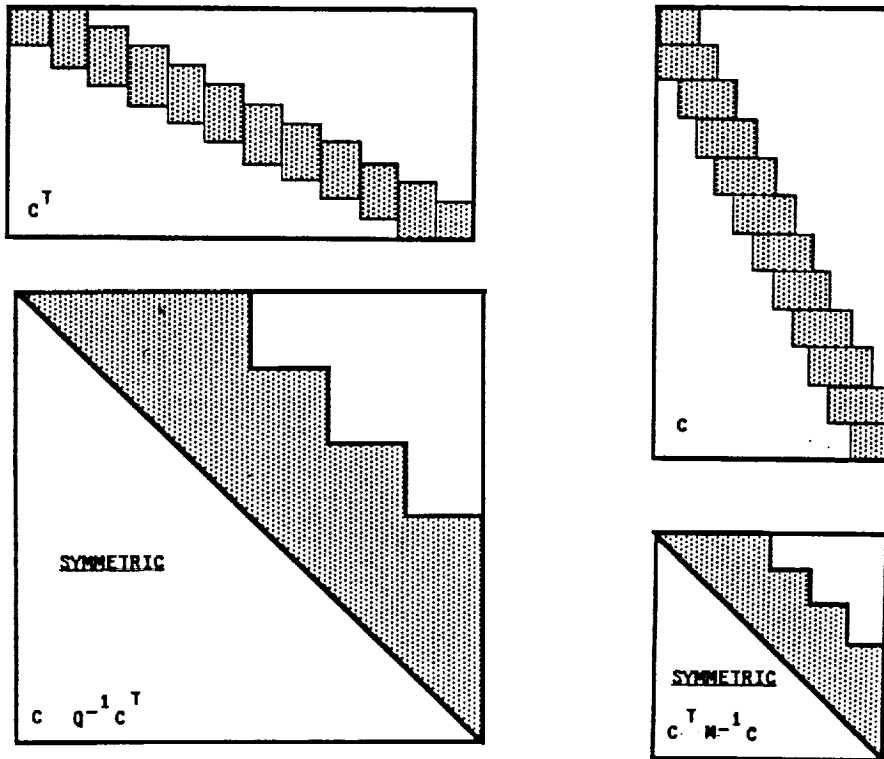


FIG. 1 Patterns of the coupling matrices for a rectilinear mesh with 2D truss elements

In order to illustrate the computational costs of the proposed numerical procedure, we consider the problem of a clamped square plate where the edges are exposed to a sudden heating. The finite element mesh is composed of N by N 4-node regular elements. The stiffness and conductivity matrices \mathbf{K} and \mathbf{Q} are assumed to be stored in banded form so that operation counting is facilitated. In practice, these matrices are compacted in skyline data structures. We denote by d and p , respectively the number of structural degrees of freedom per node ($d \leq 6$) and the number of integration steps.

The assumption of an N by N regular mesh with a number of fixed degrees of freedom at each node is unlikely in practice. However, it is the worst case as far as the computational effort required for the evaluation of the product $\mathbf{C}^T \mathbf{M}^{-1} \mathbf{C}$.

For the above problem, the formation and factorization of equations (21) and (23) require respectively $(2 + d)N^4$ and $d^3 N^4 / 2$ multiplications. The resolution of equations (20)-(24) requires $(7d^2 + 6d + 3)N^3$ multiplications for each time step. Therefore, the total computational effort needed for the transient coupled solution using the proposed stabilized procedure is:

$$E^s \sim \left(\frac{d^3}{2} + d + 2 \right) N^4 + p^s (7d^2 + 6d + 3) N^3 \quad (25)$$

For the same problem, the computational cost associated with a conventional second-order accurate conditionally stable procedure (3) is:

$$E^c \sim \left(\frac{d^3 + 1}{2} \right) N^4 + p^c (7d^2 + 6d + 3) N^3 \quad (26)$$

Clearly, unconditional stability is obtained at the cost of $(d + 3/2)N^4$ additional floating point operations. For linear problems, this computational effort is needed once. In the following, we show that this overhead is compensated by a much larger time step.

The natural frequencies of the clamped square plate are given by:

$$\omega_{mn} = \pi^2 \sqrt{\frac{El^3}{12(1 - \nu^2)\rho}} \left(\frac{m^2 + n^2}{a^2} \right) \quad (27)$$

where E , ν , l , and a are respectively Young modulus, Poisson's coefficient, the plate thickness and its edge size [20]. Therefore, the lowest frequency is:

$$\omega_{min} = \frac{2\pi^2}{a^2} \sqrt{\frac{El^3}{12(1-\nu^2)\rho}} \quad (28)$$

and a good approximation of the highest element frequency is:

$$\omega_{max}^{(e)} = \frac{2\pi^2}{a^2} N^2 \sqrt{\frac{El^3}{12(1-\nu^2)\rho}} \quad (29)$$

An adequate time step for the stabilized procedure is given by $\omega_{min}\Delta t^s = \pi/10$. For the conventional conditionally stable staggered procedure where both \mathbf{u} and θ are integrated with the trapezoidal rule, the stable time step is expressed as a multiple of the time step based on the Courant condition associated with the hyperbolic structural equation. Hence, $\Delta t^c = m \times 2/\omega_{max}^{(e)}$, where $m \geq 1$. Using (28) and (29) we have:

$$\begin{aligned} \Delta t^s &= \frac{a^2}{20\pi} \sqrt{\frac{12(1-\nu^2)\rho}{El^3}} \\ \Delta t^c &= \frac{ma^2}{\pi^2 N^2} \sqrt{\frac{12(1-\nu^2)\rho}{El^3}} \end{aligned} \quad (30)$$

so that

$$\begin{aligned} p^s &= 40 \\ p^c &= \frac{2\pi N^2}{m} \end{aligned} \quad (31)$$

are the number of steps which would cover twice the largest period of the problem. The computational costs for both procedures become:

$$\begin{aligned} E^s &\sim \left(\frac{d^3}{2} + d + 2\right)N^4 \\ E^c &\sim \frac{2\pi}{m}(7d^2 + 6d + 3)N^5 \end{aligned} \quad (32)$$

which demonstrates the superiority of the proposed stabilized staggered procedure for N sufficiently large ($N > m/14$).

VII. NUMERICAL EXAMPLES

First, we consider the Second Danilovskaya problem [13]. An elastic half-space ($x > 0$) with the surface plane $x = 0$ assumed free of tractions for all time is exposed to a sudden high ambient temperature θ_∞ . The continuum is assumed to be mechanically constrained and thermally insulated so that the displacement and temperature fields are given by:

$$\begin{aligned} u_x &= u_x(x, t) \\ u_y &= 0 \\ u_z &= 0 \\ \theta &= \theta(x, t) \end{aligned} \tag{33}$$

The boundary and initial conditions for this problem are:

$$\begin{aligned} \sigma_{xx}(0, t) &= 0 \\ k \frac{\partial \theta}{\partial x}(0, t) &= h(\theta(0, t) - \theta_\infty) \end{aligned}$$

and

$$\tag{34}$$

$$\begin{aligned} u_x(x, 0) &= 0 \\ \dot{u}_x(x, 0) &= 0 \\ \theta(x, 0) &= \theta_0 \end{aligned}$$

where h is the boundary-layer conductance. The following dimensionless variables are introduced:

$$\begin{aligned} \bar{x} &= \frac{ax}{\kappa} \\ \bar{t} &= \frac{a^2 t}{\kappa} \\ \bar{\sigma} &= \frac{\sigma_{xx}}{\beta \theta_0} \\ \bar{\theta} &= \frac{\theta - \theta_0}{\theta_0} \\ \bar{u} &= \frac{a(\lambda + 2\mu)u_x}{\kappa \beta \theta_0} \end{aligned} \tag{35}$$

where

$$\begin{aligned}\kappa &= \frac{k}{\rho c} \\ a^2 &= \frac{\lambda + 2\mu}{\rho} \\ \beta &= \frac{\alpha}{3\lambda + 2\mu}\end{aligned}\tag{36}$$

The thermomechanical coupling parameter is defined by:

$$\delta = \frac{\beta^2 \theta_0}{\rho c (\lambda + 2\mu)} = \frac{\beta^2 \theta_0}{\rho^2 a^2 c}\tag{37}$$

The exact solution for this problem can be obtained using the Laplace transform (see Nickell and Sackman [21]). The finite element solution is carried out using 2-node linear elements. The ratio $\kappa h/ak$ is fixed to 0.5 and the thermomechanical coupling parameter δ is set to 1. We report on the generated results for two time integration steps, $\Delta t^{(1)} = \pi/5\omega_{min}$ and $\Delta t^{(2)} = \Delta t^{(1)}/2 = \pi/10\omega_{min}$. These correspond to sampling the largest period of the mechanical problem into 10 and 20 steps, respectively. Figure 2 depicts the dimensionless temperature $\bar{\theta}$ at $\bar{x} = 1.0$ as a function of the dimensionless time \bar{t} , for $\Delta t = \Delta t^{(1)}$. Figure 3 reports the dimensionless displacement $\bar{u}(\bar{t})$ at $\bar{x} = 1.0$, for $\Delta t = \Delta t^{(2)}$. As expected, the results for $\Delta t = \Delta t^{(2)}$ are more accurate than those for $\Delta t = \Delta t^{(1)}$. However in both cases, the generated solutions are in good agreement with the exact ones.

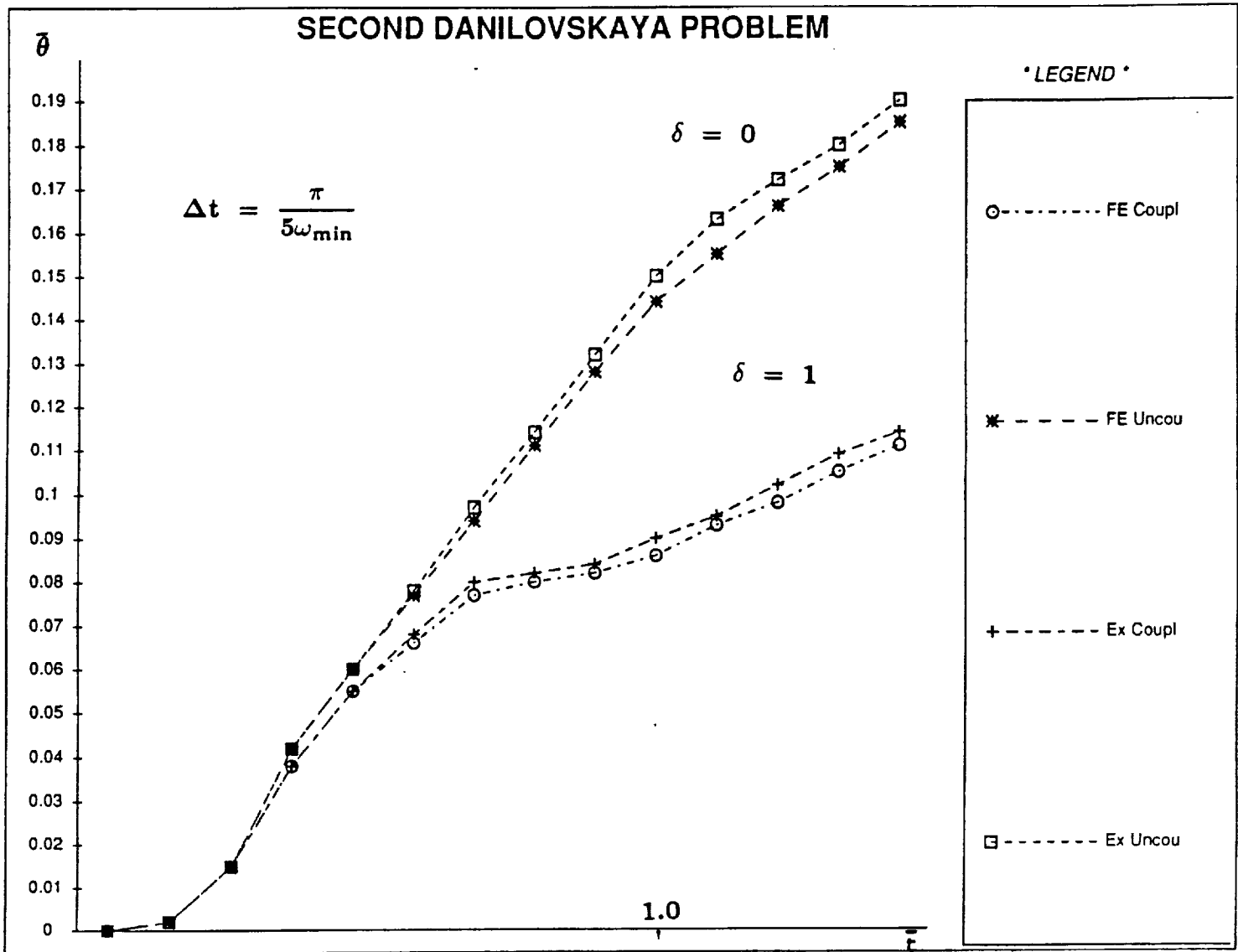


FIG. 2 Dimensionless temperature at $\bar{x} = 1.0$

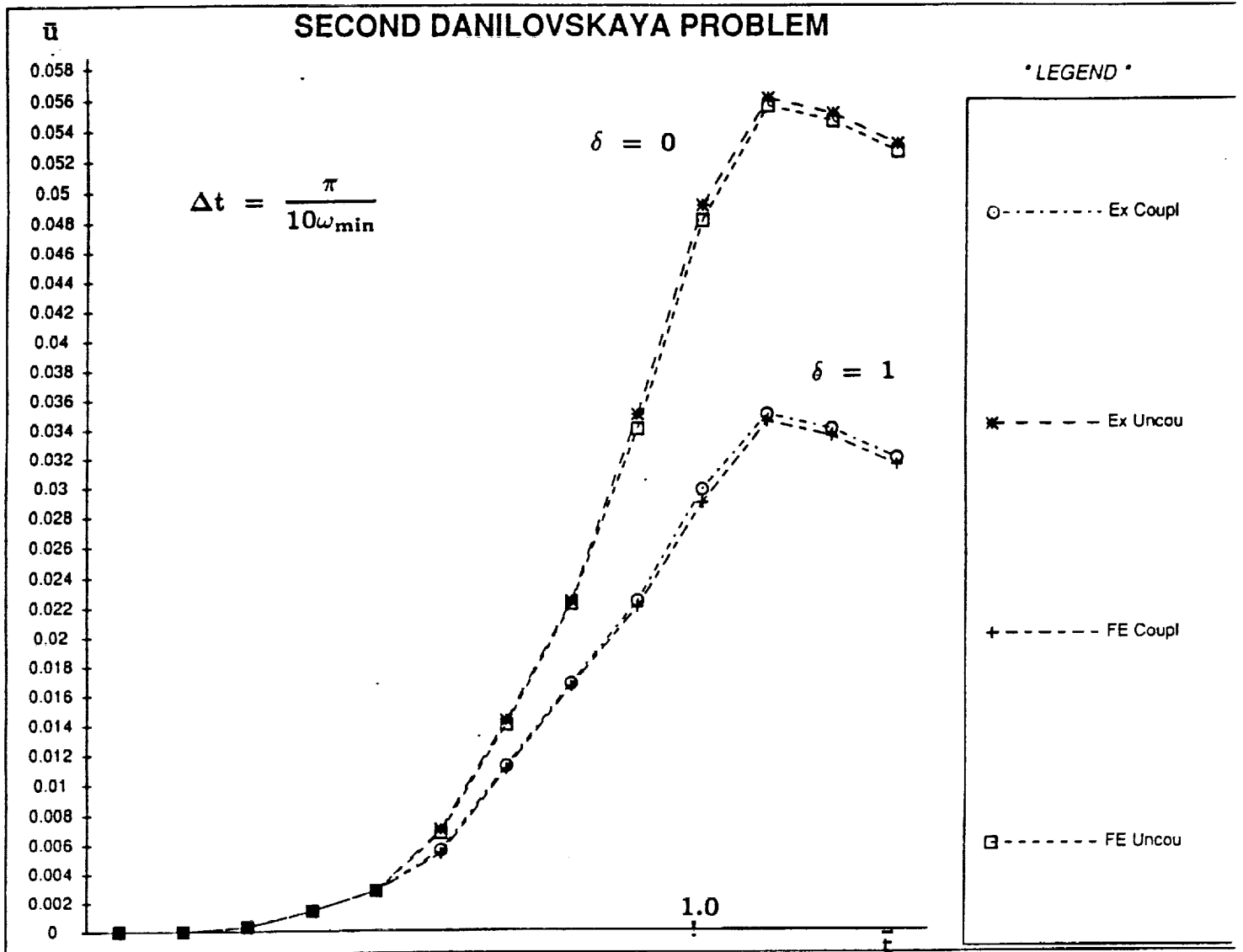


FIG. 3 Dimensionless displacement at $\bar{x} = 1.0$

Next, we consider the case of an infinitely long elastic circular shaft of radius R , where the surface temperature undergoes a sudden uniform change over a finite band of length Z , and is steadily maintained thereafter (fig. 4). Youngdahl and Sternberg have presented in [14] an exact solution for the transient temperature and thermal stresses distributions in the shaft, when thermomechanical coupling is neglected, in the form of definite integrals and infinite series. In cylindrical co-ordinates (r, ϕ, z) , the axisymmetric torsionless displacement and temperature fields are given as:

$$\begin{aligned}
u_r &= u_r(r, z, t) \\
u_\phi &= 0 \\
u_z &= u_z(r, z, t) \\
\theta &= \theta(r, z, t)
\end{aligned} \tag{38}$$

The boundary and initial conditions for this problem are:

$$\begin{aligned}
\sigma_{rr}(R, z, t) &= 0 \\
\sigma_{rz}(R, z, t) &= 0 \\
\sigma_{rr} &\rightarrow 0 \text{ as } |z| \rightarrow \infty \\
\sigma_{\phi\phi} &\rightarrow 0 \text{ as } |z| \rightarrow \infty \\
\sigma_{zz} &\rightarrow 0 \text{ as } |z| \rightarrow \infty \\
\sigma_{rz} &\rightarrow 0 \text{ as } |z| \rightarrow \infty \\
\theta(R, z, t) &= \theta_\infty \quad |z| < \frac{Z}{2} \\
\theta(R, z, t) &= 0 \quad |z| > \frac{Z}{2}
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
u_r(r, z, 0) &= 0 \\
u_z(r, z, 0) &= 0 \\
\dot{u}_r(r, z, 0) &= 0 \\
\dot{u}_z(r, z, 0) &= 0 \\
\theta(r, z, 0) &= 0
\end{aligned}$$

The following new dimensionless variables (there should be no confusion over the

present definition of these variables and their earlier use) are introduced:

$$\begin{aligned}
 \rho &= \frac{r}{R} \\
 \xi &= \frac{2z}{L} \\
 \bar{\theta} &= \frac{\theta}{\theta_{\infty}} \\
 \bar{t} &= \frac{kt}{R^2}
 \end{aligned}
 \tag{40}$$

For all computations, we set $L = 2R$ and $\nu = 1/4$. The finite element solution is carried out using 4-node axisymmetric linear elements, and a time step $\Delta t = \pi/10\omega_{min}$. Figure 5 compares the predicted temperatures at the center of the shaft ($\rho = 0$) with the exact ones for $\delta = 0$, and reports on the effect of thermocoupling ($\delta = 0.5$) on temperature distribution. Clearly, the stabilized procedure provides accurate solutions. The variations of the radial stress at $\xi = 0.1$ for $\delta = 0$ and $\delta = 0.5$ are depicted in figure 6. All numerical results are reported at $\bar{t} = 0.2$. It is interesting to note that when the thermocoupling effect is neglected the temperature field is overestimated, but the radial stress distribution is underestimated.

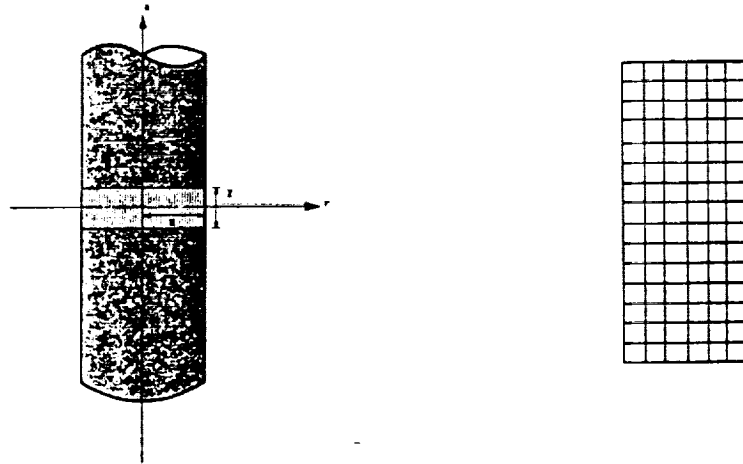


FIG. 4 Problem geometry and finite element discretization

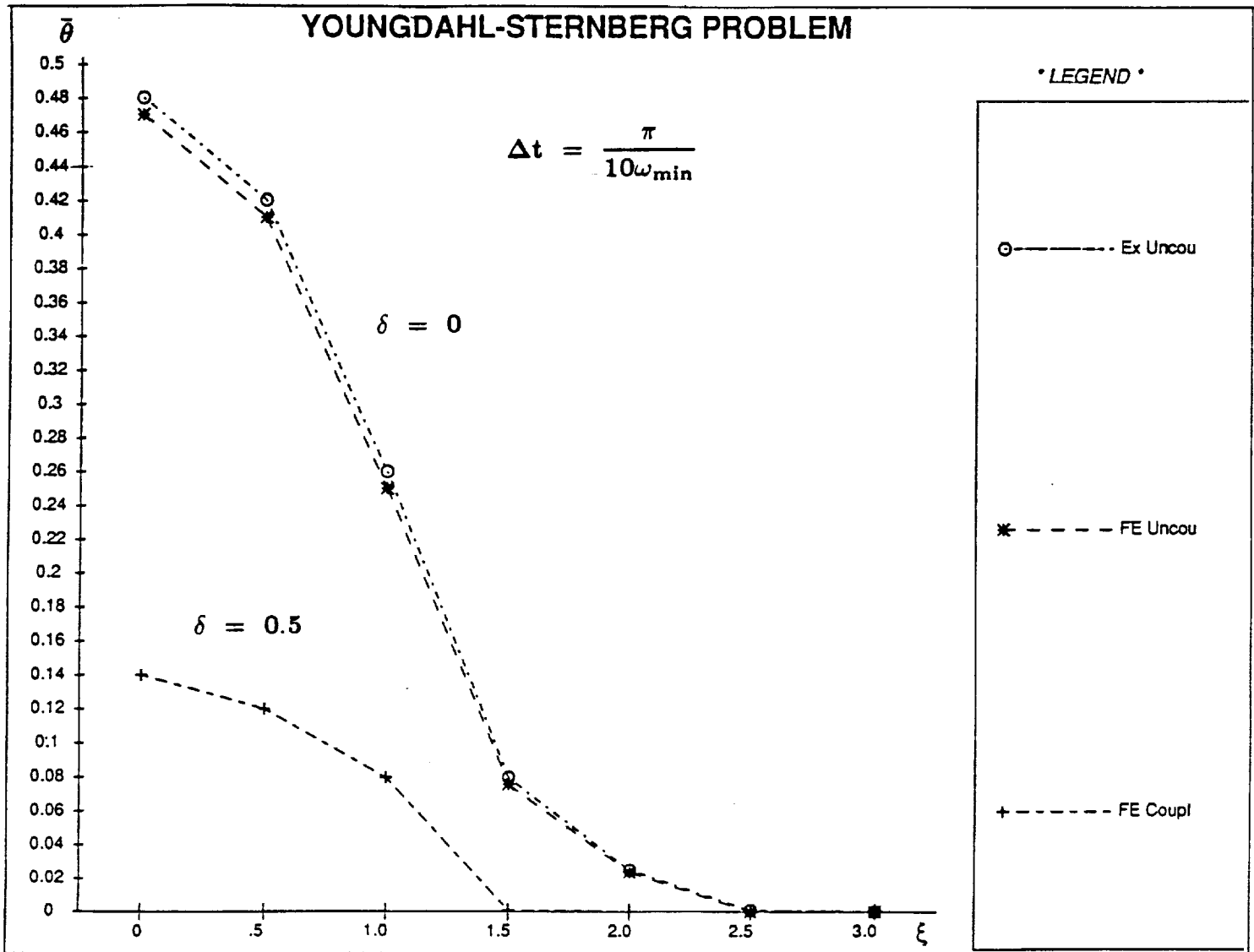


FIG. 5 Dimensionless temperature at $\rho = 0$ for $\bar{t} = 0.2$

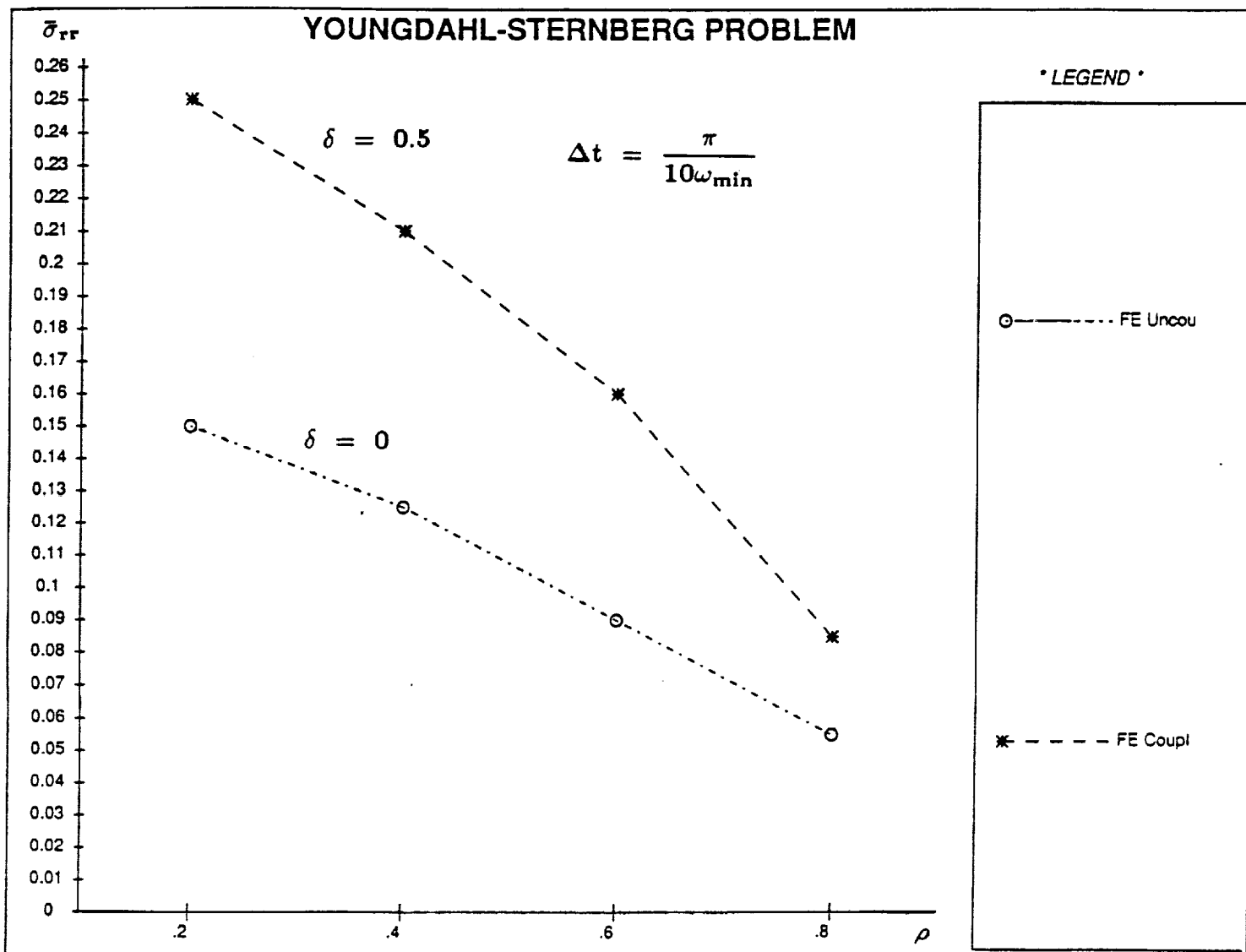


FIG. 6 Dimensionless radial stress at $\xi = 0.1$ for $\bar{t} = 0.2$

VIII. CONCLUSION

An implicit-implicit staggered procedure for the solution of thermoelastic problems is presented. It is stabilized with a cost-effective semi-algebraic augmentation scheme. The resulting transient algorithm is unconditionally stable and second-order accurate.

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