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AN UNFOLDING OF THE TAKENS-BOGDANOV SINGULARITY

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Abstract. A complete analysis for $|\mu|, |\nu| \ll 1$ of the equation

$$\dot{x} = y, \ \dot{y} = \mu x + \nu y + Mx^2 + \Gamma xy$$

describing a particular unfolding of the Takens-Bogdanov singularity is presented.

1. Introduction. In this paper we provide a complete bifurcation analysis of a nilpotent singularity of the Takens-Bogdanov type with a two-parameter unfolding given by

$$\dot{x} = y + \mathcal{O}(3), \tag{1.1a}$$

$$\dot{y} = \mu x + \nu y + Mx^2 + \Gamma xy + \mathcal{O}(3), \tag{1.1b}$$

subject to the nondegeneracy hypotheses $M \neq 0$, $\Gamma \neq 0$. When $\mu = \nu = 0$ the vector field (1.1) is the normal form for a generic vector field near a singular point with a double zero eigenvalue. In the absence of any constraints, the origin $(x, y) = (0, 0)$ will not remain a singular point under perturbation. The corresponding (universal) unfolding,

$$\dot{x} = y + \mathcal{O}(3), \tag{1.2a}$$

$$\dot{y} = \mu + \nu x + Mx^2 + \Gamma xy + \mathcal{O}(3), \tag{1.2b}$$

has been extensively studied (cf. Takens [9], Bogdanov [1]). The problem with Z_2 symmetry, described by

$$\dot{x} = y + \mathcal{O}(5), \tag{1.3a}$$

$$\dot{y} = \mu x + \nu y + Mx^3 + \Gamma x^2 y + \mathcal{O}(5), \tag{1.3b}$$

is also understood (Carr [2], Knobloch and Proctor [6]). The unfolding (1.1) is the (universal) unfolding for a system in which the origin is constrained to remain a singular point, but no symmetry is present.

In applications the unfoldings are generated by center manifold reduction from a larger, perhaps infinite-dimensional set of ordinary differential equations. It is in this context that Eqs. (1.1) have been obtained and partially analyzed (Merryfield et al.

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[8], Keyfitz [4, 5]). In this paper we complete this analysis with particular emphasis on the global bifurcations that take place.

Without loss of generality we restrict attention to $M > 0$, $\Gamma > 0$, and $\mu > 0$. The remaining possibilities are obtained by straightforward reflections (including time reversal). These are summarized in Table 1.

TABLE 1. Transfomations to obtain (1.1) with $M > 0$, $\Gamma > 0$, $\mu > 0$.

Case	Transformation
$M > 0$, $\Gamma > 0$	–
$M < 0$, $\Gamma > 0$	$x \to -x$, $t \to -t$, $\nu \to -\nu$
$M > 0$, $\Gamma < 0$	$y \to -y$, $t \to -t$, $\nu \to -\nu$
$M < 0$, $\Gamma < 0$	$x \to -x$, $y \to -y$
$\mu < 0$	$x \to x - \frac{\mu}{M}$, $\nu \to \nu + \frac{\Gamma}{M}\mu$

By a simple scaling we can set $M = \Gamma = 1$. In addition, the $\mathcal{O}(3)$ terms may be made arbitraily small relative to the terms retained, and the normal form truncated at second order. Although we do not prove it the phase portraits we obtain are structurally stable (cf. [1]) and the omitted terms introduce no qualitatively new effects.

2. The analysis. The fixed points of the resulting vector field

$$\dot{x} = y, \qquad \dot{y} = \mu x + \nu y + x^2 + xy \tag{2.1}$$

are at $(x, y) = (0, 0)$, $(-\mu, 0)$. The origin undergoes a transcritical bifurcation at $\mu = 0$, and is a saddle for $\mu > 0$. When $\mu > 0$ the nontrivial fixed point $(-\mu, 0)$ is a sink in $\nu < \mu$ and a source in $\nu > \mu$, and undergoes a Hopf bifurcation along the half-line $\mu = \nu$, $\mu > 0$. When $\mu < 0$ the nontrivial fixed point is a saddle while the trivial fixed point $(0, 0)$ is a sink in $\nu < 0$ and a source in $\nu > 0$ and undergoes a Hopf bifurcation along the half-line $\nu = 0$, $\mu < 0$. *Both* Hopf bifurcations are subcritical and so lead to unstable limit cycles.

(a) *Limit cycles.* Kopell and Howard [7] show for a class of two-parameter vector fields in the plane that includes (2.1) that there exists at least one limit cycle for some sufficiently small values of $|\nu|$, $|\mu|$. To verify this result we let $\varepsilon \equiv \sqrt{\mu}$, $0 < \mu \ll 1$, and introduce the scaled quantities

$$\tau = \varepsilon t, \quad x = \varepsilon^2 u, \quad y = \varepsilon^3 v, \quad \nu = \varepsilon^2 \delta. \tag{2.2}$$

In terms of these variables the system (2.1) takes the form of a perturbed Hamiltonian system

$$u' = v, \tag{2.3a}$$

$$v' = u + u^2 + \varepsilon(\delta v + uv), \tag{2.3b}$$

where the prime denotes differentiation with respect to the slow time τ. The unperturbed Hamiltonian system $H_o(u, v) \equiv v^2/2 - u^2/2 - u^3/3$ has a one-parameter

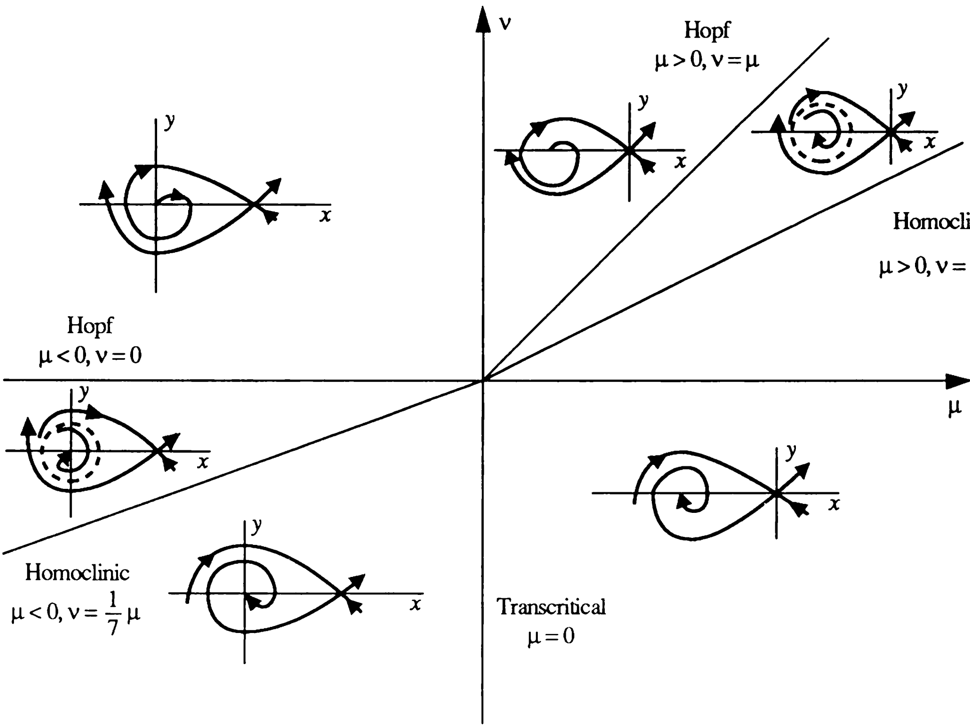

FIG. 1. Phase portraits of the system (2.1) with $M = \Gamma = 1$ in the (ν, μ) plane. Dashed lines indicate the unstable limit cycles.

family of closed orbits, $H_o(u, v) = \alpha$, bounded by the homoclinic orbit

$$u_h(\tau) = -\tfrac{3}{2}\operatorname{sech}^2(\tau/2), \tag{2.4a}$$

$$v_h(\tau) = \tfrac{3}{2}\operatorname{sech}^2(\tau/2)\tanh(\tau/2). \tag{2.4b}$$

When $0 < \varepsilon \ll 1$ this one-parameter family is destroyed, but every orbit that remains closed can be approximated by particular orbits of the unperturbed system for a time $\tau = \mathscr{O}(\varepsilon^{-1})$ (see e.g., Gluckenheimer and Holmes [3]). The corresponding values of α follow from the requirement that for a closed orbit $\Delta H \equiv \oint \dot{H}\, dt = 0$, i.e.,

$$\int_0^{T_\alpha} [\delta v_\alpha^2(\tau) + u_\alpha(\tau) v_\alpha^2(\tau)]\, d\tau = \mathscr{O}(\varepsilon), \tag{2.5}$$

where T_α is the period of the closed orbit of $H_o(u, v) = \alpha$. In terms of the original variables this condition gives

$$\nu = -\mu f(\alpha) + \mathscr{O}(\mu^{3/2}), \tag{2.6}$$

where

$$f(\alpha) \equiv \frac{\int_0^{T_\alpha} u_\alpha v_\alpha^2\, d\tau}{\int_0^{T_\alpha} v_\alpha^2\, d\tau}. \tag{2.7}$$

If the equation $f(\alpha) = -\frac{\nu}{\mu}$ has a unique solution α, then the averaging theorem (see, e.g., [3]) guarantees that the perturbed system (2.3) has a unique periodic orbit for sufficiently small ε.

To show that this is the case we show that $f'(\alpha) > 0$ for all α, $-\frac{1}{6} < \alpha < 0$. This condition implies, in addition, that the closed orbit is unstable. Kopell and Howard [7] demonstrate this result by placing a positive lower bound on $f'(\alpha)$ without actually evaluating the integrals directly. This result may be verified directly by evaluating $f(\alpha)$. The periodic solution of $H_o(u, v) = \alpha$ has the form

$$u_\alpha = a + (b - a)\,\mathrm{sn}^2(\beta\tau, \kappa), \tag{2.8}$$

where a, b, and β are functions of κ, and $\kappa = \kappa(\alpha)$ increases monotonically from 0 to 1 as α goes from $-\frac{1}{6}$ to 0 (see appendix for explicit relations).

The integrals to be calculated are:

$$\int_0^{T_\alpha} v_\alpha^2\, d\tau = 4\beta(b-a)^2 \int_0^{4K(\kappa)} \mathrm{sn}^2(x)\,\mathrm{cn}^2(x)\,\mathrm{dn}^2(x)\, dx \equiv 4\beta(b-a)^2 I(\kappa), \tag{2.9}$$

$$\begin{aligned} &\int_0^{T_\alpha} v_\alpha^2 u_\alpha\, d\tau \\ &\quad = 4\beta(b-a)^2 \int_0^{4K(\kappa)} a\,\mathrm{sn}^2(x)\,\mathrm{cn}^2(x)\,\mathrm{dn}^2(x) + (b-a)\,\mathrm{sn}^4(x)\,\mathrm{cn}^2(x)\,\mathrm{dn}^2(x)\, dx \\ &\quad \equiv 4\beta(b-a)^2\{aI(\kappa) + (b-a)J(\kappa)\}, \end{aligned} \tag{2.10}$$

where $T_\alpha = \frac{4K(\kappa)}{\beta}$. Both $I(\kappa)$ and $J(\kappa)$ can be expressed in terms of the complete elliptic integrals of the first and second kind:

$$K(\kappa) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \kappa^2 \sin^2 x}}, \qquad E(\kappa) = \int_0^{\pi/2} \sqrt{1 - \kappa^2 \sin^2 x}\, dx. \tag{2.11}$$

These expressions for I and J can also be found in the appendix.

Substituting these integrals into (2.7) we have:

$$f(\kappa) = \frac{aI(\kappa) + (b-a)J(\kappa)}{I(\kappa)}. \tag{2.12}$$

Figures 2a and b show plots of $f(\kappa)$ and $f'(\kappa)$. At $\kappa = 0$, $f = -1$ and increases monotonically to $f = -\frac{6}{7}$ at $\kappa = 1$ as expected by Koppel and Howard's results.

From the transformation for $\mu < 0$ in Table 1 there is a $(\mu < 0, \nu)$ region with a similar unstable limit cycle.

(b) *Homoclinic tangencies.* We now show that the limit cycle disappears in a homoclinic bifurcation. Kopell and Howard [7] show for their two-parameter family of vector fields that for sufficiently small $|\mu|$, $|\nu|$ a curve of homoclinic tangencies exists and borders a region of limit cycles. They further show that for (μ, ν) for which (2.1) has neither a closed nor homoclinic orbit the two fixed points are connected by a unique trajectory. The existence of a line of homoclinic orbits can also be demonstrated using Melnikov's integral (see [3]). In addition, this procedure determines the location of this line in the (μ, ν) plane to leading order in ε.

The Melnikov function $M(\delta)$ is the integral of the time rate of change of H_o in the perturbed system (2.3) computed along the homoclinic trajectory (2.4) of the

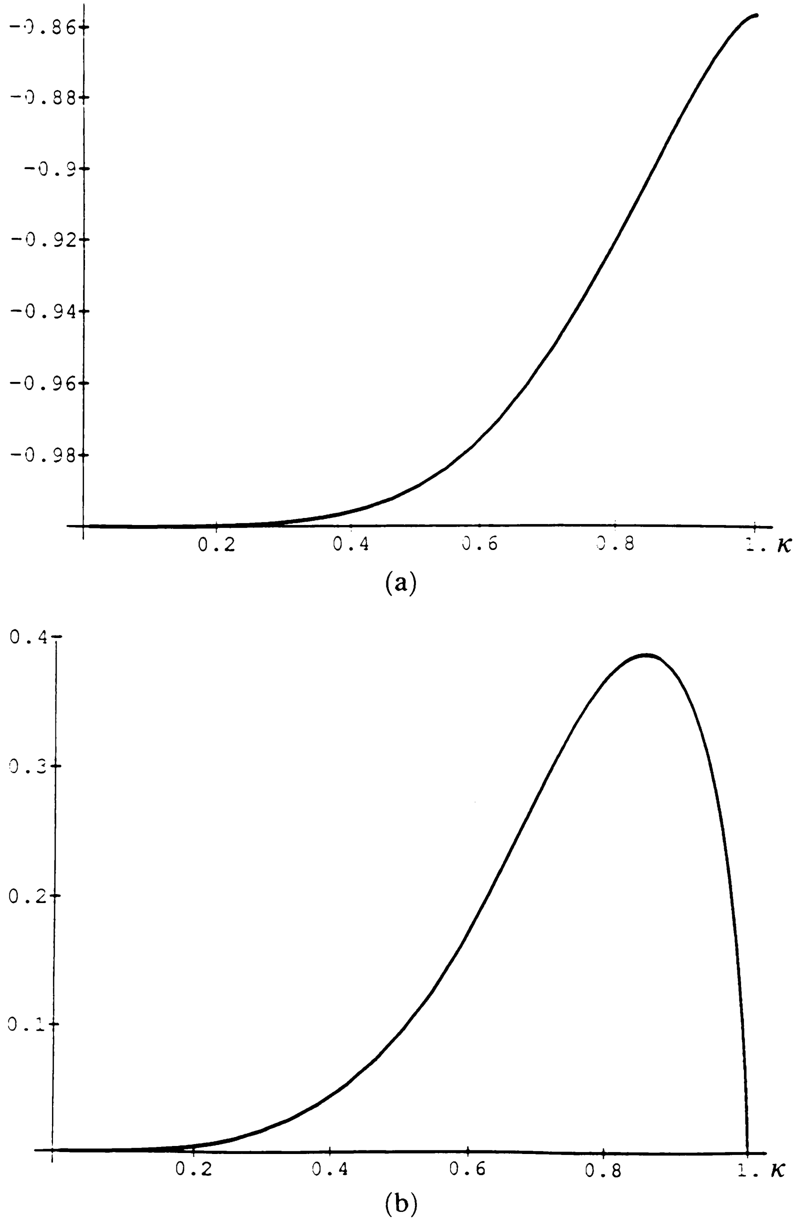

FIG. 2. (a) The function $f(\kappa)$ defined by Eq. (2.12). (b) The derivative of $f(\kappa)$ with respect to κ.

unperturbed Hamiltonian system:

$$M(\delta) = \int_{-\infty}^{\infty} [\delta v_h^2(\tau) + u_h(\tau) v_h^2(\tau)]\, d\tau = \{\tfrac{6}{5}\delta - \tfrac{36}{35}\}. \tag{2.13}$$

The Melnikov integral vanishes for $\delta = \frac{6}{7}$ or in the unscaled variables $\nu = \frac{6}{7}\mu$, $\mu > 0$ in agreement with (2.6) evaluated at $\kappa = 1$. This implies for small nonzero $\varepsilon \equiv \sqrt{\mu}$ that (2.1) has a homoclinic orbit for parameter values $\nu = \frac{6}{7}\mu + \mathcal{O}(\mu^{3/2})$,

$\mu > 0$. Using the $\mu < 0$ transformation in Table 1 we find that another homoclinic bifurcation boundary approaches the $(\mu, \nu) = (0, 0)$ origin along the half-line $\nu = \frac{1}{7}\mu + \mathscr{O}(\mu^{3/2})$, $\mu < 0$.

For $M(\delta) < 0$, $\mu > 0$ we have $\nu < \frac{6}{7}\mu$ and the stable manifold of the origin passes to the left of the unstable one as shown in Fig. 1. For $M(\delta) > 0$, $\mu > 0$ the stable manifold now passes to the right of the unstable one. Using this information and the $\mu < 0$ transformation in Table 1 we complete the bifurcation diagram of Fig. 1.

3. Conclusion. Our results are summarized in the (μ, ν) plane in Fig. 1. Note that for both $(M, \Gamma) > 0$ and $(M, \Gamma) < 0$ the limit cycles are unstable. Of particular interest in applications are the two cases $(M < 0, \Gamma > 0)$ and $(M > 0, \Gamma < 0)$ in which the limit cycles are asymptotically stable and the homoclinic orbits are approached as a limit of stable periodic orbits of larger and larger period. The results for these cases can be deduced using Table 1. The necessity of the global bifurcation can also be deduced on topological grounds (cf. [7]) and consequently the structure of the (μ, ν) plane can be deduced without the detailed calculations presented here (see [4, 5]). The results of this paper show, however, that at each (μ, ν) the limit cycle is unique, and determine its stability and the location of the global bifurcations for $|\mu|, |\nu| \ll 1$.

Note that because of the $\mu < 0$ transformation in Table 1 each flow type on the $\mu > 0$ half-plane appears (with its origin shifted) in the $\mu < 0$ half-plane. Since this transformation does not involve time reversal we find that both sets of limit cycles have identical stability properties for a given M and Γ. In this respect our results differ from those of Merryfield et al. [8].

Appendix. The integrals $I(\kappa)$, $J(\kappa)$ in Eqs. (2.9), (2.10) are:

$$I(\kappa) = \frac{8(1-\kappa^2+\kappa^4)}{15\kappa^4}E(\kappa) - \frac{4(1-\kappa^2)(2-\kappa^2)}{15\kappa^4}K(\kappa), \tag{A.1}$$

$$J(\kappa) = \frac{1}{7\kappa^2}\left[4(1+\kappa^2)I(\kappa) + \frac{4(1-\kappa^2)}{3\kappa^2}K(\kappa) - \frac{4(1+\kappa^2)}{3\kappa^2}E(\kappa)\right]. \tag{A.2}$$

The expressions for a, b, and β in Eq. (2.8) are:

$$a(\kappa) = -\tfrac{1}{2}\left[1 + \frac{(1+\kappa^2)}{\sqrt{1-\kappa^2+\kappa^4}}\right], \tag{A.3}$$

$$b(\kappa) - a(\kappa) = \frac{3\kappa^2}{2\sqrt{1-\kappa^2+\kappa^4}}, \tag{A.4}$$

$$[\beta(\kappa)]^2 = \frac{1}{4\sqrt{1-\kappa^2+\kappa^4}}. \tag{A.5}$$

and α and κ are related by:

$$\alpha(\kappa) = -\tfrac{1}{2}a^2(\kappa) - \tfrac{1}{3}a^3(\kappa). \tag{A.6}$$

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