An Unified Numerical Approach for Computing the Outage Probability for Mobile Radio Systems

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***Abstract*— A general finite-range integral for the probability of outage in mobile radio systems is derived. The method handles noninteger Nakagami-fading indexes, unequal Rice factors, unequal shadowing spreads, and unequal transmitted powers as well as all the common fading distributions (Rayleigh, Rice, Nakagami-m, Nakagami-q, lognormal-Rice, Suzuki, and lognormal-Nakagami-m). The integral expression can also be approximated by a Gauss–Chebychev quadrature (GCQ) formula requiring the knowledge of moment generating function at only a small number of points. An estimate of the remainder term is also derived. This numerical technique allows computing the outage with arbitrary precision and it is extremely easy to program.**

***Index Terms*— Cochannel interference, Gauss–Chebysev approximation, mobile radio systems, Nakagami fading, Rician fading, shadowing.**

I. Introduction

In cellular radio systems, the spectrum utilization efficiency may be increased by reducing the cluster size but at the expense of increased cochannel interference (CCI). The probability of outage is a useful statistical measure of performance in the presence of CCI [1]. The outage performance has been studied extensively (see [2]–[7] among many others). The statistical fluctuations of the signal amplitude are often modeled by a Rayleigh, Rician, Nakagami distribution, or compound distributions like lognormal-Nakagami-m. These distributions can model most fading environments.

Consider evaluating the probability of outage (outage) in a mobile fading environment. The instantaneous signal powers are modeled as random variables (RV's) $\mathbf{p_k}$, $k = 0, \cdots, L$, with mean $\overline{p}_k$. Subscript $k = 0$ denotes the desired signal and $k = 1, \cdots, L$ are for interfering signals. The outage is given by

$$P_{\text{out}} = \Pr\{qI > \mathbf{p_0}\} \tag{1}$$

where $I = \mathbf{p_1} + \cdots + \mathbf{p_L}$ and q is the power protection ratio, which is fixed by the type of modulation and transmission technique employed and the quality of service desired. Typically, $9 < q < 20$ dB. For instance, $q = 9.5$ dB for the digital pan-European GSM system using GMSK modulation [1].

Numerous *ad hoc* attempts have been made to obtain closed-form expressions for the outage. To do so often requires making *restrictive* assumptions (e.g., positive integers for the Nakagami fading figure or identical statistical distributions for all interferers) or *approximations* (replacing a Rician RV by a Nakagami RV). If the probability density function (pdf) of I is known, then the outage can readily be obtained. The PDF, $f_I(\xi)$, can be expressed as an L-fold convolutional integral. However, there is no analytical solution to this integral in general. Another approach is to use the moment generating function (MGF). If the RV's are independent, the MGF of I, $\phi_I(s)$ is the product of the individual MGF's. While, in principle, inverting $\phi_I(s)$ gives $f_I(\xi)$, closed-form expressions are difficult or impossible under general conditions. From (1), we have

$$P_{\text{out}} = \int_0^\infty F_0(q\xi) f_I(\xi)\, d\xi \tag{2}$$

where $F_0(\xi)$ is the cumulative density function (cdf) of $\mathbf{p_0}$. If $F_0(\xi)$ consists of terms of the form $\xi^k e^{-\xi}$, then the derivatives of $\phi_I(s)$ yield the outage. By approximating a Rice RV by a Nakagami RV, the above method can also be used for the Rice-faded desired signal.

In this letter we unify the previous results by expressing the outage as a finite-range integral for all the common fading distributions. The MGF's of the desired and interfering signal powers constitute the integrand. Exact analytical evaluation of the integral appears to be impossible, except for some isolated cases (e.g., Rayleigh fading). However, using mathematical packages such as Maple and Matlab, it is extremely simple to evaluate (numerically) the integral with high accuracy. Whereas explicit closed-form solutions tend to require much more programming effort. Our approach here is partly motivated by this consideration. Moreover, we advocate the use of a GCQ formula for the integral. This turns out to be remarkably accurate. For instance, in some cases, the GCQ sum evaluates the outage with an error of less than 10^{-8} percent using only eight samples.

II. MGF's for Common Fading Distributions

Given an RV X, the MGF is given by $\phi(s) = E[e^{-sx}]$. We next identify several common MGF's of the signal power, $\mathbf{p}$.

A. Rician and Rayleigh Fading

The MGF of $\mathbf{p}$ for a noncentralized chi-squared RV (Rice distribution) is given by [8, p. 44], which can be expressed in terms of K, the usual Rician parameter. The MGF for the Rayleigh fading case is obtained by setting $K = 0$.

B. Nakagami-m and Nakagami-q (Hoyt) Fading

The MGF of $\mathbf{p}$ for the Nakagami-m fading channel can be obtained from [2, eq. (44)]. The MGF for Nakagami-q fading

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is [9]

$$\phi_{\mathbf{p}}(s) = \frac{1}{\sqrt{1+2s\overline{p}+s^2(1-b^2)\overline{p}^2}}, \qquad -1 \le b \le 1, \quad (3)$$

where $b = (1-q^2)/(1+q^2)$ and $0 \le q \le \infty$ is the fading parameter. In particular, the Nakagami-q distribution reverts to the Rayleigh fading case when $b = 0$.

C. Lognormal-Rice and Suzuki Fading

Expressing the received fading envelope as the product of independent Rice and lognormal distributions, applying Hermitian integration, we can show

$$\phi(s) = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \frac{w_i}{1+s\mu' e^{\sqrt{2}\sigma x_i}} \exp\left(\frac{-Ks\mu' e^{\sqrt{2}\sigma x_i}}{1+s\mu' e^{\sqrt{2}\sigma x_i}}\right) + R_H \quad (4)$$

where σ is the logarithmic standard deviation of shadowing, μ is the local mean power, and $\mu' = \mu/(1+K)$. The abscissas x_i (ith root of an Nth order Hermite polynomial) and weights w_i are tabulated in [10] for $N \le 20$ and R_H is a remainder term. The MGF for the Suzuki distribution is obtained by setting $K = 0$ in (4).

D. Lognormal-Nakagami-m Fading

Similar to the derivation of (4), the MGF of the received signal power in a Nakagami-m fading channel with lognormal shadowing can be expressed as

$$\phi(s) = \frac{1}{\sqrt{\pi}} \sum_{i=1}^{N} \frac{w_i}{\left(1+s\mu e^{\sqrt{2}\sigma x_i}/m\right)^m} + R_H. \quad (5)$$

III. Outage Computation

A. Exact Formula

Let

$$\gamma = \mathbf{p_0}/q - \left(\sum_{k=1}^{L} \mathbf{p}_k\right) \quad (6)$$

and the MGF of γ is given by

$$\phi_\gamma(s) = \phi_0(s/q) \prod_{k=1}^{L} \phi_k(-s) \quad (7)$$

where $\phi_k(s)$ is the MGF of $\mathbf{p_k}$ and can be any one of the MGF's in Section II. Since the outage probability is $P_{\text{out}} = \Pr(\gamma < 0)$, this can be written as

$$P_{\text{out}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_\gamma(c+j\omega)}{c+j\omega}\, d\omega \quad (8a)$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{\text{Real}[(c-j\omega)\phi_\gamma(c+j\omega)]}{c^2+\omega^2}\, d\omega \quad (8b)$$

where $j^2 = -1$, $0 < c < a_{\min} = \min\{a_i | 1 \le i \le L\}$ with a_i being the ith pole of $\phi_\gamma(s)$ in the left half plane (i.e., $a_i > 0$). Substituting $\omega = c\tan(\theta/2)$ in (8b), we get

$$P_{\text{out}} = \frac{1}{2\pi} \int_{0}^{\pi} \tilde{\phi}(\theta)\, d\theta \quad (9)$$

where $\tilde{\phi}(\theta) = \text{Real}[(1-j\tan(\theta/2))\phi_\gamma(c+jc\tan(\theta/2))]$. This form is both easily evaluated and well suited to numerical integration since it only involves finite integration limits and knowledge of the MGF.

B. Numerical Formula Using a GCQ Rule

Substituting $x = \cos\theta$ and using the GCQ formula [10, p. 889] gives

$$P_{\text{out}} = \frac{1}{2n} \sum_{i=1}^{n} \tilde{\phi}\left[\frac{(2i-1)\pi}{2n}\right] + R_n \quad (10)$$

where the remainder term R_n vanishes rapidly. Although c can be anywhere between zero and $a_{\min}$, the optimal location ensures that $|\phi_\gamma(c+j\omega)|$ decays as rapidly as possible for $|\omega| \to \infty$. This rapid decay occurs if $s = c$ is the saddle point; i.e., at $s = c$, $s^{-1}\phi_\gamma(s)$ achieves its minimum on the real axis. While this optimal c requires a numerical search for the root of $s\phi'_\gamma(s) - \phi(s) = 0$, it sufficient to use $c = a_{\min}/2$. For the MGF's in the previous section, this value can be obtained at once.

The use of the GCQ approach for computing the error rate of coding schemes can be found in [11]. To the best of our knowledge, the GCQ approach has not previously been applied to the outage problem. We also have derived a new, simpler expression for R_n,

$$R_n = \frac{\pi^2}{6n^2}\tilde{\phi}''(\zeta), \qquad \text{for some } 0 < \zeta < \pi. \quad (11)$$

The proof is sketched in the Appendix. Equation (11) only involves a second order derivative of the MGF instead of $2n$th order derivative using the formula in [10, p. 889].

IV. Numerical Results

We now provide a limited set of numerical results. Since the application of (10) for complicated mobile radio scenarios is straightforward, the main aim is to verify its accuracy. $\hat{P}_{\text{out}}$ is the GCQ sum in (10), excluding the remainder term, and the signal-to-interference ratio (SIR) is defined as the ratio between the desired user mean signal power to the sum of all interfering mean signal powers (i.e., $\overline{p}_0/\sum_k \overline{p}_k$). For comparison purposes, the exact outage P_{out} is computed by evaluating (9) using Matlab's *quad8* function with an absolute tolerance of 10^{-15}.

Table I examines the relationship between n and c for a specified tolerance (% error $< \epsilon$). The suboptimal choice of c does not preclude the use of (10) because n does not grow too large as to become unmanageable (even with a relatively large deviation from the optimum value). Interestingly, the rule of thumb eliminates the need for a precise numerical search yet works very well at most instances. Since a percentage accuracy of $\epsilon \approx 10^{-2}$ may be adequate in practice, only six samples are required.

In Table II we investigate the accuracy of the GCQ sum for a Rician-fading channel with two unequal interferers. Our numerical results reveal that the truncated series in this scenario is converging very rapidly. Hence, an extremely high accuracy can be easily attained with only a few number of samples! Also, one may require a slightly larger n to

TABLE I
SENSITIVITY OF THE SELECTION OF PARAMETER c ON $\hat{n}$ (THE NUMBER OF SAMPLES REQUIRED TO ACHIEVE THE DESIRED ACCURACY) FOR $\hat{P}_{\text{out}}$ WITH $L = 2$, $m_0 = 1$, $m_k = [1.3, 2.1]$, $\overline{p}_k = [2.2, 1.6]$, AND $\text{SIR}/q = 15$ dB

c	ϵ	$\hat{P}_{\text{out}}$	$\hat{n}$	P_{out}
$\frac{1}{2}a_{\min}$	$1(-2)$	3.096 086 3(-2)	6	3.096 352 47(-2)
	$1(-3)$	3.096 382 8(-2)	13	
	$1(-5)$	3.096 352 7(-2)	33	
$\frac{1}{4}a_{\min}$	$1(-2)$	3.096 200 2(-2)	15	3.096 352 47(-2)
	$1(-3)$	3.096 378 6(-2)	16	
	$1(-5)$	3.096 352 7(-2)	67	
$\frac{3}{4}a_{\min}$	$1(-2)$	3.096 117 6(-2)	8	3.096 352 47(-2)
	$1(-3)$	3.096 344 8(-2)	10	
	$1(-5)$	3.096 352 7(-2)	12	

TABLE II
COMPARISON BETWEEN (9) AND (10) FOR DIFFERENT SIR/q AND VARIOUS n VALUES. $L = 2$, $K_0 = 2$, $K_k = [1, 1.3]$, $\overline{p}_k = [1.2, 1.7]$, AND $c = (1/2)a_{\min}$

SIR/q	P_{out}	n	$\hat{P}_{\text{out}}$	% error
10 dB	4.756 337 365(-2)	4	4.747 656 219(-2	2(-1)
		5	4.756 119 646(-2)	$5(-3)$
		8	4.756 337 365(-2)	$7(-9)$
15 dB	1.362 473 755(-2)	4	1.359 866 044(-2)	$2(-1)$
		5	1.362 408 269(-2)	$5(-3)$
		8	1.362 473 755(-2)	$7(-9)$
20 dB	4.141 616 632(-3)	4	4.133 519 716(-3)	$2(-1)$
		5	4.141 616 632(-3)	$5(-3)$
		8	4.141 616 632(-3)	$7(-9)$

TABLE III
COMPARISON BETWEEN (9) AND (10) IN A MIXED FADING ENVIRONMENT. $L = 4$, $m_1 = 0.5$, $m_2 = 0.8$, $K_3 = 1$, $K_4 = 1.3$, $\overline{p}_k = [0.6, 1.1, 1.2, 1.7]$, $\text{SIR}/q = 20$ dB, AND $c = (1/2)a_{\min}$

m_0	P_{out}	n	$\hat{P}_{\text{out}}$	% error
1.4	2.157 650 942 95(-3)	5	2.157 111 560 44(-3)	$3(-2)$
		10	2.157 650 680 41(-3)	$1(-5)$
		50	2.157 650 942 82(-3)	$6(-9)$
2.1	1.722 972 597 01(-4)	5		
		10	1.722 972 862 17(-4)	$2(-5)$
		50	1.722 972 597 04(-4)	$1(-9)$
2.8	1.570 986 559 28(-5)	5	1.570 949 124 75(-5)	$2(-3)$
		10	1.570 986 673 81(-5)	$7(-6)$
		50	1.570 986 559 28(-5)	$2(-10)$

achieve the same level of accuracy as the error rates decreases. However, the increase (if required) is minimal.

Finally in Table III, we examine the accuracy of $\hat{P}_{\text{out}}$ in a mixed fading scenario. The received signal amplitude of the desired user and the first two interferers are Nakagami faded, while the signal amplitude of the other two interferers are Rician faded. We observe that the error performance improves as the fading severity parameter of the desired user increases, as anticipated. The GCQ sum is simple yet yields remarkably accurate results even with only ten sampling points of the MGF over a wide range of fading severity parameter.

V. CONCLUSIONS

In the research literature, much effort has been expended to find closed-form expressions for outage in mobile radio systems. This *ad hoc* development has led to various formulas. In contrast, we have developed a unified outage expression (a single finite-range integral) for all common fading distributions. This exact outage expression requires the knowledge of the MGF. We also provide a new closed-form expression (based on the GCQ approximation) that offers a convenient method to evaluate the outage. This numerical method allows for arbitrary fading parameters as well as dissimilar signal strengths, shadowing spreads and so on. It is a powerful tool for outage analysis, not imposing any restrictions while being easy to program.

APPENDIX

Consider the class of integrals $P = \int_0^\pi f(x)\,dx$. We can use a midpoint trapezoidal rule to evaluate P. The basic element of this rule is, as h tends to zero, $\int_a^{a+h} f(x)\,dx = hf(a+h/2) + (h^3/3)f''(a+h/2)$ by expanding $f(x)$ near $x = a+h/2$ using the Taylor series. This rule for an extended interval is

$$\int_0^\pi f(x)\,dx = \frac{\pi}{n}\sum_{k=0}^{n-1} f\left(k\frac{\pi}{n} + \frac{\pi}{2n}\right) + R_n \tag{12}$$

where h is replaced by π/n. If $f''(x)$ is continuous, then

$$R_n = \frac{\pi^3}{3n^2}\,f''(\zeta) \tag{13}$$

for some $a < \zeta < b$. Now if we apply a GCQ formula to P using the substitution $x = \cos^{-1} t$, we will get

$$P = \frac{\pi}{n}\sum_{k=1}^{n} f\left(\frac{(2k-1)\pi}{2n}\right) + R_n \tag{14}$$

where the remainder has the form $f^{2n}(\cos^{-1}\xi)$ [10]. Comparing (12) and (14), we see that the GCQ sum collapses to a simple midpoint trapezoidal sum for P. Thus, either formula for the remainder can be used.

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