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An Unreliable Batch Arrival Retrial Queueing System With Bernoulli Vacation Schedule and Linear Repeated Attempts:
Unreliable Retrial System With Bernoulli Schedule

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ABSTRACT

This article deals with the steady-state behavior of an $M^X/G/1$ retrial queue with the Bernoulli vacation schedule and unreliable server, under linear retrial policy. Breakdowns can occur randomly at any instant while the server is providing service to the customers. Further, the concept of Bernoulli admission mechanism is introduced. This model generalizes both the classical $M^X/G/1$ retrial queue with unreliable server as well as the $M^X/G/1$ retrial queue with the Bernoulli vacation model. The authors carry out an extensive analysis of this model. Namely, the embedded Markov chain, the stationary distribution of the number of units in the orbit, and the state of the server are studied. Some important performance measures and reliability indices of this model are obtained. Finally, numerical illustrations are provided and sensitivity analyses on some of the system parameters are conducted.

KEYWORDS

Classical Retrial Policy, Random Breakdown, Reliability Indices, Stationary Distribution

1. INTRODUCTION

Retrial queues (or queues with repeated attempts) are characterized by the feature that a customer that finds, upon arrival, the server busy, is obliged to leave the service area and repeat his demand for service after some time called "retrial time." Between trials, the blocked customer joins a pool of unsatisfied customers called "orbit." Queues in which customers are allowed to conduct retrials have been widely used to model many practical problems in telephone switching systems, telecommunication networks and computers competing to gain service from a central processing unit. Moreover, retrial queues are also used as mathematical models for several computer systems such as packet switching networks, shared bus local area networks operating under the carrier-sense multiple access protocol and collision avoidance star local area networks etc. For a review of the main results and methods, the reader is referred to the survey papers of Yang and Templeton (1987), Falin (1990), Kulkarni

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and Liang (1997) and the book by Falin and Templeton (1997). For more recent references, see the bibliographical overviews in (Artalejo 2010; Artalejo, 1999; Artalejo, 1999). Further, a comprehensive comparison between retrial queues and their standard counterparts with classical waiting lines can be found in Artalejo and Falin (2002).

Many of the queueing systems with repeated attempts operate under the classical retrial policy, where each block of customers generates a stream of repeated attempts independently of the rest of the customers in the orbit, i.e., the intervals between successive repeated attempts are exponentially distributed with rate $n\theta$ (say), when the number of customers in the orbit is n. However, there is a second kind of policy, called constant retrial policy, which arises naturally in problems where the server is required to search for customers (Sengupta 1990) and in communication protocols of type carrier sense multiple access (CSMA). The latter discipline was introduced by Fayolle (1986), who investigated an M/M/1 retrial queue in which the repeat customers form a queue and only the head customers of the orbit queue can request a service after an exponentially distributed retrial time with some parameter γ (say), i.e., the retrial rate is $(1-\delta_{0,n})\gamma$, where $\delta_{i,j}$ denotes the Kronecker's delta, when the number of units in the orbit is n. Farahmand 1990) called this discipline a retrial queue with FCFS orbit retrial policy. Choi et al. (1992) generalized this retrial policy by considering an M/M/1 retrial queue with general retrial times. Artalejo and Gomez-Corral (1997) introduced a more general kind of retrial incorporating both possibilities by assuming that when there are n customers in the system, the time intervals between successive repeated attempts are exponentially distributed random variables with parameter $\theta_n = (1-\delta_{0,n})\gamma$, where θ can be considered as the retrial per customer and γ the rate at which the server seeks service for customers whenever it is idle. Such a type of retrial policy is known as a linear retrial policy. Recently, Choudhury (2008) investigated such a queueing model for two phases of service under Bernoulli vacation schedule.

The classical vacation scheme with Bernoulli service discipline was originated and significantly developed by Keilson and Servi (1986) and co-workers. Kella (1990) suggested a generalized Bernoulli scheme according to which a single server goes on i consecutive vacations with probability p_i if the queue is empty upon his return. At the end of a vacation period, service begins if a customer is present in the queue. Otherwise, the server waits for the first customer to arrive. A wide class of retrial policies for governing the vacation mechanism has also been discussed in the literature. Most of the analyses for retrial queues concerns the exhaustive service schedule (Artalejo, 1997), gated service policy (Langaries, 1999) and recently modified vacation policy (Ke & Chang 2009). A number of papers (Ke & Chang, 2009; Krishnakumar & Arivudainambi, 2002; Krishnakumar et al., 2002; Wenhui, 2005) have recently appeared in the queueing literature in which the concept of Bernoulli vacation schedule has been introduced under the FCFS orbit retrial policy. Such type of queueing models occurs in many real-life situations where the server may be used for other secondary jobs, for instance to serve customers in other systems. Allowing the server to take vacations makes the queueing model more realistic and flexible in studying real-world queueing situations. Applications arise naturally in call centers with multi-task employees, customized manufacturing, telecommunication and computer networks, maintenance activities, production and quality control problems, etc.

The study of queueing models with service interruptions goes back to the 1950s. Among some early papers on service interruptions, we refer the readers to see the papers by Gaver (1962), Avi-ltzhak and Naor (1963), Thirurengadan (1963) and Mitrany and Avi-ltzhak (1968) for some fundamental works. Li et al. (1997), Sengupta (1990), Takin and Sengupta (1998), Tang (1997), among others, have studied some queueing systems with interruptions where, in one of the underlying assumptions, the service channel undergoes repair instantaneously, as soon as it fails. Recently, Lee (2018) considered a model where the breakdowns/repair process is non-stationary in the number of breakdowns/repairs. On the other hand, retrial queues that take into account servers failures and repairs were introduced by Aissani (1988) and Kulkarni and Choi (1990). As related literature, we should mention some papers by Aissani (1994; 1993), Aissani and Artalejo (1998) and Anisimov and Atadzhanov (1994). Wang et al. (2001) studied a repairable M/G/1 retrial queueing model from the viewpoint of reliability for the first time, and both of the queueing indices and reliability characteristics are obtained. Atencia et al. (2008) investigate a similar type of batch arrival retrial model under FCFS orbit retrial policy. Choudhury and Deka (2008) investigate such a repairable M/G/1 retrial queueing model with two phases of service under the classical retrial policy. Although some aspects have been discussed separately on queueing systems with service interruptions, second optional service, repeated attempts, however, no work has been found that combines all these features together for batch arrival queueing systems, even in the most recent studies. Hence to fill up to this gap, in this article an attempt has been made to study an $M^X/G/1$ retrial queue with Bernoulli vacation schedule which is subject to server's breakdown. Further, we introduce the concept of control of the admission policy to the retrial group in the form of Bernoulli admission mechanism.

In the Bernoulli admission mechanism, we assume that each individual blocked customer is admitted to join the retrial group with a probability $\varpi\,(0 \le \varpi \le 1)$, independently of the admission of the rest of the customers arriving in the same batch and/or of the actual size of the retrial group. This type of mechanism for the admission to the retrial group was introduced recently by Artalejo and Atencia (2004) and Choudhury (2007) for continuous time queueing models and Artalejo et al. (2005) for a discrete time queueing model. The consideration of the admission probability ϖ can be viewed as a first step to extend the existing control mechanism for admission of customers in the standard waiting lines to queues with repeated attempts.

The first study of a batch arrival retrial queue was introduced by Falin (1976), who assumed the following operating rule: "If the server is busy at the arrival epoch, then whole batch joins the retrial group, whereas if the server is free, then one of the arriving units starts its service and the rest joins the retrial group". This kind of policy is applicable to the performance evaluation of Local Area Networks operating under transmission protocols like the CSMA/CD (Carrier Sense Multiple Access with Collision Detection); see Choi et al. (1992). In such a context, messages of variable length arrive at the stations and then they are divided into a number of packets in order to be transmitted to the destination station. If the transmission medium (i.e., a bus in the engineering terminology) is idle, then one packet is selected to be transmitted automatically and the rest is stored in a buffer (i.e., the retrial group). On the other hand, if the bus is busy, then the entire packet must be stored in the buffer and the station will retry the transmission later on. A more complete description of this mechanism can be found in Yang and Templeton (Yang & Templeton, 1987). Some recent papers (Aissani, 2000; Artalejo & Atencia, 2004; Artalejo et al., 2005; Choudhury, 2007; Choudhury & Deka, 2013; Ke & Chang, 2009; Kulkarni, 1986) discussed more complicated queueing situations with retrials and batch arrivals. However, our objective in this paper is to extend the analysis of the main $M^X/G/1$ retrial queue under Bernoulli vacation schedule with linear retrial policy and Bernoulli admission mechanism for an unreliable server with a view to unify several classes of related batch arrival queueing systems. To this end, the methodology will be based on a combination of embedded Markov chain and inclusion of supplementary variables techniques.

The rest of the paper is organized as follows. In Section 2, we give a brief description of the mathematical model. Section 3 deals with the derivations of the stability criteria for existence of the stationary regime and studies the embedded Markov chain describing the behavior of the system size distribution at a departure epoch. Section 4 deals with the derivations of the stationary distribution of the state of the server and the number of customers in the orbit. Some important performance measures are derived in Section 5. Numerical illustrations are presented in Section 6.

2. THE MATHEMATICAL MODEL

We consider an $M^X/G/1$ queueing system, where the number of individual primary customers arrives to the system according to a compound Poisson process with arrival rate λ. The size of successive arriving batches is $X_1, X_2, \cdots$ where $X_1, X_2, \cdots$ are iid random variables, distributed with probability mass function (pmf) $a_n = P\{A = n\}, n \geq 1$, probability generating function (PGF) $a(z) = E\left[z^X\right]$ and finite factorial moments $a_{[k]} = E[X(X-1)\cdots(X-k+1)]$. Let $\varpi \in (0,1]$ be the probability of admission for each individual customer and b_n be the probability that a batch of *n* units joins the system. Then, for $n \geq 0$, we have (Artalejo & Atencia, 2004):

$$b_0 = \sum_{k=1}^{\infty} a_k (1-\varpi)^k$$

$$b_n = \sum_{k=n}^{\infty} a_k \binom{k}{n} \varpi^n (1-\varpi)^{k-n}, n \geq 1$$

such that the relationship between the PGFs of the sequences $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 0\}$ is given by:

$$b(z) = \sum_{n=0}^{\infty} z^n b_n = a\left((1-\varpi) + \varpi z\right)$$

In particular, if $\varpi = 1$, (i.e., there is no control of admission to the system), then $a(z) = b(z)$. Further, if we denote by $b_{[k]}$ the *k*th factorial moment of $b(z)$, then we have $b_{[k]} = \varpi^k a_{[k]}$.

The server provides a preliminary service denoted by B to all arriving customers. The service time random variable follows a general law with probability distribution function (df) $B(x)$, Laplace Stieltjes Transform (LST) $\beta^*(s) = E\left[e^{-sB}\right]$ and finite *k*th moment $\beta^{(k)}$. While the server is working with the primary customers, it may breakdown at any time and the service channel will fail for a short interval of time. The service interruptions, i.e., server's life times, are generated by an exogenous Poisson processes with rate α. As soon as a breakdown occurs, the server is sent for repair, during which time it stops providing service to the arriving batch of customers. The customer just being served before server breakdown waits for the server to complete its remaining service. The *repair time* (denoted by G) distribution is assumed to be arbitrarily distributed with df $G(y)$, LST $G^*(s) = E\left[e^{-sG}\right]$ and finite *k*th moment $g^{(k)}$. Immediately after the server is repaired, the server is ready to resume its remaining service to the primary customers and in this case the service times are cumulative, which we may refer to as generalized service times. Now if we define $\left\{B_{c,n}; n \geq 1\right\}$ as a sequence of iid random variables for generalized service time with df $B_c(x)$ and its LST $H^*(s) = E\left[e^{-sB_c}\right]$, then we have:

$$H^*(s) = \sum_{n=0}^{\infty} \int_0^{\infty} e^{-sx} e^{-\alpha_i x} \left[\frac{(\alpha x)^n}{n!}\right] \left[G^*(s)\right]^n dB(x) = \beta^*\left(s + \alpha\left(1 - G^*\left(s\right)\right)\right) \quad (2.1)$$

After each service completion the server takes a Bernoulli vacation, i.e., after each service completion the server may go for a vacation of random length V with probability p or, with probability q, he may serve the next unit, if any, where $p + q = 1$. The vacation time distribution is assumed to be a general law with df $V(y)$, LST $\vartheta^*\left(s\right) = E\left[e^{-sV}\right]$ and finite *k*th moment $V^{(k)}$. This type of model is known as a batch arrival queue with unreliable server and Bernoulli vacation schedule based on a single vacation policy. It should be noted here that a model of similar nature was studied by Li et al. (1997) for the single unit arrival case. Now, for further development for such a type of model, we may further introduce the concept of repeated attempts under a linear retrial policy with Bernoulli admission mechanism, where primary customers finding the server free upon arrival automatically start their service. However, if the primary customer finds the server busy, on vacation or down (attending the repair job), then he joins a group of unsatisfied customers, i.e., orbit, to seek the service again and again, until he finds the server free. The time interval between successive repeated attempts is assumed to be exponentially distributed with rate $\theta_n = n\theta + \gamma\left(1-\delta_{n,0}\right)$, when the number of customers in the retrial group, i.e., orbit size, is $n \geq 0$.

Further, we assume that the input process, the intervals between successive repeated attempts, the server's life time, the server's repair time, the server's vacation time and the service time random variables are mutually independent of each other.

3. EMBEDDED MARKOV CHAIN

Let t_n be the time instant at which the *n*th service completion occurs, i.e., we consider the epoch at which the generalized service time requested by a customer expires, and $N(t)$ be the orbit size at time *t*. Then, the sequence $X_n = N(t_n)$ forms a Markov chain which is an embedded Markov renewal process. The sequence $\left\{X_n; n \geq 0\right\}$ is a homogeneous Markov chain with respect to the following transitions:

$$\left\{X_n \middle| X_{n-1} = j\right\} = \begin{cases} j - 1 + S_n & \text{with probability } \dfrac{\theta_j}{\lambda_0 + \theta_j} \\ j + W_n - 1 + S_n & \text{with probability } \dfrac{\lambda}{\lambda_0 + \theta_j} \end{cases}$$

where S_n is the number of customers that arrive during the *n*th modified service time and if the *n*th customer in service proceeds from a batch arrival then W_n represents the number of customers admitted to join the system. The rate λ_0 is equal to $\lambda\left(1-b_0\right)$.

Now it is not difficult to see that $\left\{X_n; n \geq 0\right\}$ is irreducible and aperodic. To prove its ergodicity, we shall use Foster's criterion, which states that an irreducible and aperiodic Markov chain is ergodic if there exist a non-negative function $f(e), e \geq 0$ and $\varepsilon > 0$ such that:

$$\phi_e = E\left[f\left(X_{n+1}\right) - f\left(X_n\right) \middle| X_n = e\right]$$

is finite for all $e \geq 0$ and $\varphi_e \leq -\varepsilon$ for all $e \geq 0$, except perhaps a finite number. In our case, we take $f(e) = e$ to obtain:

$$\phi_j = \begin{cases} \rho_H & j = 0 \\ \rho_H - \dfrac{\gamma + j\theta}{\lambda_0 + \gamma + j\theta} & j = 1,2 \end{cases}$$

where:

$$\rho_H = \lambda b_{[1]}\beta^{(1)}\left\{1+\alpha g^{(1)}\right\} + p\lambda b_{[1]}v^{(1)}$$

Clearly, if ρ_H satisfies following conditions viz:

1. If $\gamma > 0$ and $\theta = 0$, then:

$$\rho_H < \frac{\lambda_0 + \gamma}{\lambda_0 + \left[\beta^{(1)}\left(1+\alpha g^{(1)}\right) + p\nu^{(1)}\right]^{-1} + \gamma}$$

2. If $\gamma \geq 0$ and $\theta > 0$, then $\rho_H < 1$ and then we have $\lim_{j\to\infty}\varphi_j < 0$. Hence, the embedded Markov chain $\left\{X_n; n \geq 0\right\}$ is ergodic.

The necessary condition follows from Kaplin's condition as noted in Sennott et al. (1983) namely, $\rho_H < \infty$ as $j > 0$ and there exists j_0 such that ρ_H for $j \geq j_0$. It should be pointed out that Kaplin's condition is fulfilled if there is a k such that $p_{ij} = 0, \{j < i-k, i > 0\}$, where $P = (p_{ij}), \{i, j = 0, 1, 2, \cdots\}$ is the transition probability matrix associated with $\{X_n; n \geq 0\}$.

Next, we assume that $\{X_n; n \geq 0\}$ is recurrent-positive to guarantee that the limiting probabilities:

$$\pi_j = \lim_{n\to\infty} P_r\left\{X_n = j\right\}; j \geq 0$$

exist and are positive. The one-step transition probability matrix $P = (p_{ij})$, associated with $\{X_n; n \geq 0\}$ has the elements:

$$p_{i,j} = \begin{cases} \dfrac{\theta_i}{\lambda_0 + \theta_i}\left(qh_0 + pm_0\right) & \text{if} \quad i \geq 1, j = i-1 \\ \dfrac{\lambda}{\lambda_0 + \theta_i}\displaystyle\sum_{n=1}^{j-i+1} b_n\left(qh_{j-i+1-n} + pm_{j-i+1-n}\right) + \dfrac{\theta_i}{\lambda_0 + \theta_i}\left(qh_{j-i+1} + pm_{j-i+1}\right) & \text{if} \quad 0 \leq i \leq j \end{cases}$$

where $m_{ij} = \sum_{i=0}^{j} h_j l_{j-i}$ are the probabilities that several batches totaling j customers arrive during the service period plus vacation period. Here h_j and l_j are defined as follows:

$$h_j = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda_0 x} dB_c\left(x\right) & \text{if} \quad j = 0 \\ \displaystyle\sum_{k=1}^{\infty}\int_0^\infty \frac{b_j^{(k)} e^{-\lambda x}\left(\lambda x\right)^j}{j!} dB_c\left(x\right) & \text{if} \quad j \geq 1 \end{cases}, \quad l_j = \begin{cases} \displaystyle\int_0^\infty e^{-\lambda_0 x} dV\left(x\right) & \text{if} \quad j = 0 \\ \displaystyle\sum_{k=1}^{\infty}\int_0^\infty \frac{b_j^{(k)} e^{-\lambda x}\left(\lambda x\right)^j}{j!} dV\left(x\right) & \text{if} \quad j \geq 1 \end{cases}$$

and $\left\{b_j^{(k)}; j \geq 0\right\}$ is the *n*fold convolution of the sequence $\left\{b_j; j \geq 0\right\}$ with itself. Then, the Kolmogorov equations associated with the Markov chain $\{X_n; n \geq 0\}$ can be written as:

$$\pi_j = \sum_{n=0}^{j} \left(\frac{\lambda \pi_n}{\lambda_0 + \theta_n} \right) \sum_{i=1}^{j-n+1} b_i \left(q h_{j-n+1-i} + p m_{j-n+1-i} \right) + \sum_{i=1}^{j+1} \left(\frac{\theta_i \pi_i}{\lambda_0 + \theta_i} \right) \left(q h_{j-i+1} + p m_{j-i+1} \right); j \geq 0 \tag{3.1}$$

We now introduce the following generating functions:

$$\pi\left(z\right) = \sum_{j=0}^{\infty} z^j \pi_j \text{ and } \phi\left(z\right) = \sum_{j=0}^{\infty} \frac{\pi_j z^j}{\lambda_0 + \theta_j}$$

such that:

$$\pi\left(z\right) = \theta z \phi^{/}\left(z\right) + \left(\lambda_0 + \gamma\right)\phi\left(z\right) - \lambda_0^{-1}\gamma\pi_0 \tag{3.2}$$

Now because of convolution, Equation (3.1) can be transformed with the help of the following generating functions:

$$H\left(z\right) = \sum_{j=0}^{\infty} z^j h_j \text{ , } M\left(z\right) = \sum_{j=0}^{\infty} z^j m_j \text{ and } L\left(z\right) = \sum_{j=0}^{\infty} z^j l_j$$

Note that:

$$H\left(z\right) = \beta^*\left(\lambda\left(1 - b\left(z\right)\right) + \alpha\left(1 - G^*\left(\lambda - \lambda b\left(z\right)\right)\right)\right), L\left(z\right) = \vartheta^*\left(\lambda - \lambda b\left(z\right)\right)$$

and $M\left(z\right) = H\left(z\right)L\left(z\right)$. Then, Equation (3.1) becomes:

$$\pi\left(z\right) = \left[\theta z \phi^{/}\left(z\right) + z^{-1}\left\{\lambda_0 + \gamma - \lambda\left(1 - b\left(z\right)\right)\right\}\phi\left(z\right) - \left(\lambda_0 z\right)^{-1}\gamma\pi_0\right]$$

$$\times\left[q + p\vartheta^*\left(\lambda - \lambda b\left(z\right)\right)\right]\beta^*\left(\lambda - \lambda b\left(z\right) + \alpha\left(1 - G^*\left(\lambda - \lambda b\left(z\right)\right)\right)\right) \tag{3.3}$$

We now combine Equations (3.2) and (3.3) to get:

$$\theta z \phi^{/}(z) + \left[\lambda_0 + \gamma - \frac{\lambda\left(z\right)\left(\left[q + p\vartheta^*\left(\lambda\left(z\right)\right)\right]\beta^*\left(A\left(z\right)\right)\right)}{\left[\left(q + p\vartheta^*\left(\lambda\left(z\right)\right)\right)\beta^*\left(A\left(z\right)\right)\right] - z}\right]\phi\left(z\right) = \frac{\gamma\pi_0}{\lambda_0} \tag{3.4}$$

and:

$$\pi\left(z\right)=\frac{\lambda\left(z\right)\left[q+p\vartheta^{*}\left(\lambda\left(z\right)\right)\right]\beta^{*}\left(A\left(z\right)\right)}{\left[q+p\vartheta^{*}\left(\lambda\left(z\right)\right)\right]\beta^{*}\left(A\left(z\right)\right)-z}\phi\left(z\right) \tag{3.5}$$

where $\lambda(z)=\lambda\left(1-b(z)\right)$ and $A(z)=\lambda(z)+\alpha\left(1-G\left(\lambda(z)\right)\right)$. Since $\pi(1)=1$, Equation (3.5) yields the initial condition:

$$\phi\left(1\right)=\frac{1-\rho_{H}}{\lambda b_{[1]}} \tag{3.6}$$

Now solving the above differential Equation (3.4), we get the following:

Theorem 3.1: Under the stability condition, the PGF $\psi(z)$ is given by:

1. If $\lambda>0$ and $\theta>0$, then:

$$\phi(z)=z^{-\frac{\gamma}{\theta}}\exp\left\{\frac{1}{\theta}\int_{z}^{1}\left(\lambda_{0}-\frac{\lambda\left(u\right)\left\{q+p\vartheta^{*}\left(\lambda\left(u\right)\right)\right\}\beta^{*}\left(A\left(u\right)\right)}{\left[\left\{q+p\vartheta^{*}\left(\lambda\left(u\right)\right)\right\}\beta^{*}\left(A\left(u\right)\right)-u\right]}\right)dy\right\}\times$$

$$\left[\frac{\left(1-\rho_{H}\right)}{\lambda b_{[1]}}-\frac{\gamma\pi_{0}}{\lambda_{0}\theta}\int_{z}^{1}u^{\frac{\gamma}{\theta}-1}\exp\left\{-\frac{1}{\theta}\int_{u}^{1}\frac{1}{y}\left(\lambda_{0}-\frac{\lambda\left(y\right)\left\{q+p\vartheta^{*}\left(\lambda\left(y\right)\right)\right\}\beta^{*}\left(A\left(y\right)\right)}{\left[\left\{q+p\vartheta^{*}\left(\lambda\left(y\right)\right)\right\}\beta^{*}\left(A\right)\left(y\right)-y\right]}\right)dy\right\}du\right] \tag{3.7}$$

where:

$$\pi_{0}=\frac{\left(1-\rho_{H}\right)\lambda_{0}\theta}{\gamma b_{[1]}}\left[\int_{0}^{1}u^{\frac{\gamma}{\theta}-1}\exp\left\{-\frac{1}{\theta}\int_{u}^{1}\frac{1}{y}\left(\lambda_{0}-\frac{\lambda\left(y\right)\left\{q+p\vartheta^{*}\left(\lambda\left(y\right)\right)\right\}\beta^{*}\left(A\left(u\right)\right)}{\left[\left\{q+p\vartheta^{*}\left(\lambda\left(y\right)\right)\right\}\beta^{*}\left(A\left(u\right)\right)-y\right]}\right)dy\right\}du\right]^{-1} \tag{3.8}$$

2. If $\gamma=0$ and $\theta>0$, then:

$$\phi(z)=\frac{\left(1-\rho_{H}\right)}{\lambda b_{[1]}}\exp\left\{\frac{1}{\theta}\int_{z}^{1}\left(\lambda_{0}-\frac{\lambda(u)\left\{q+p\vartheta^{*}\left(\lambda(u)\right)\right\}\beta^{*}\left(A(u)\right)}{\left\{q+p\vartheta^{*}\left(\lambda(u)\right)\right\}\beta^{*}\left(A(u)\right)-u}\right)\frac{du}{u}\right\} \tag{3.9}$$

3. If $\gamma>0$ and $\theta=0$, then:

$$\phi\left(z\right)=\frac{\gamma\pi_{0}}{\lambda_{0}}\left[\gamma+\lambda_{0}-\frac{\lambda\left(z\right)\left\{q+p\vartheta^{*}\left(\lambda\left(z\right)\right)\right\}\beta^{*}\left(A\left(z\right)\right)}{\left[\left\{q+p\vartheta^{*}\left(\lambda\left(z\right)\right)\right\}\beta^{*}\left(A\left(z\right)\right)-z\right]}\right]^{-1} \tag{3.10}$$

where:

$$\pi_0 = \frac{\lambda_0}{\rho_H \gamma}\left[\left(1-\rho_H\right)\left(\lambda_0+\gamma\right)\left\{\beta^{(1)}\left(1+\alpha g^{(1)}\right)+p\nu^{(1)}\right\}-\rho_H\right]$$

Remark 3.1: By putting $z=0$ in (3.4) for $\gamma=0$ and $\theta>0$, since $\pi_0=\lambda\varphi(0)$ then utilizing it in (3.19) for $z=0$, we get:

$$\pi_0 = \frac{\left(1-\rho_H\right)\lambda_0}{\lambda b_{[1]}}\exp\left\{\frac{1}{\theta}\int_0^1\left(\lambda_0-\frac{\lambda(u)\left\{q+p\vartheta^*\left(\lambda(u)\right)\right\}\beta^*\left(A(u)\right)}{\left\{q+p\vartheta^*\left(\lambda(u)\right)\right\}\beta^*\left(A(u)\right)-u}\right)\frac{du}{u}\right\} \tag{3.11}$$

It should be pointed here that the limiting probabilities $\left\{\pi_j; j\geq 0\right\}$ can be computed recursively from Equation (3.1) and the expression (3.8) or (3.10) or (3.11) for π_0, depending upon the nature of retrial policies. The computation of the integrals h_j and m_j are reduced to explicit expressions in the case of many standard service time and vacation time distributions.

Remark 3.2: The result in this section is quite general and covers many practical situations. For example, let us consider the situation when $\alpha=0$, then $\beta^*(A(z))=\beta^*(\lambda-\lambda b(z))$, $\rho_H=\lambda b_{[1]}\left\{\beta^{(1)}+pv^{(1)}\right\}$, and therefore, we have:

$$\pi(z) = \frac{\left(1-\rho_H\right)\left[1-b\left(z\right)\right]\left[q+p\vartheta^*\left(\lambda-\lambda b\left(z\right)\right)\right]\beta^*\left(\lambda-\lambda b\left(z\right)\right)}{b_{[1]}\left[\left\{q+p\vartheta^*\left(\lambda-\lambda b\left(z\right)\right)\right\}\beta^*\left(\lambda-\lambda b\left(z\right)\right)-z\right]}$$

$$\times\exp\left\{\frac{1}{\theta}\int_z^1\left(\lambda_0-\frac{\lambda(u)\left\{q+p\vartheta^*\left(\lambda-\lambda b(u)\right)\right\}\beta^*\left(\lambda-\lambda b(u)\right)}{\left\{q+p\vartheta^*\left(\lambda-\lambda b(u)\right)\right\}\beta^*\left(\lambda-\lambda b(u)\right)-u}\right)\frac{du}{u}\right\}$$

and:

$$\pi_0 = \frac{\left(1-\rho_H\right)}{b_{[1]}}\exp\left\{\frac{1}{\theta}\int_0^1\left(\lambda_0-\frac{\lambda(u)\left\{q+p\vartheta^*\left(\lambda-\lambda B\left(u\right)\right)\right\}\beta^*\left(\lambda-\lambda b\left(u\right)\right)}{\left\{q+p\vartheta^*\left(\lambda-b\left(u\right)\right)\right\}\beta^*\left(\lambda-\lambda b(u)\right)-u}\right)\frac{du}{u}\right\}$$

which is consistent with the result obtained by Choudhury (2007) for $\beta_2^*(\lambda-\lambda b(z))=1$ and the result obtained by Falin and Templeton (1997) for $p=0$ (i.e., there is no Bernoulli vacation schedule) and $\varpi=1$ (i.e., there is no control of admission to join in the retrial group).

4. STATIONARY DISTRIBUTION OF THE NUMBER OF UNITS IN THE ORBIT AND STATE OF THE SERVER

In this section, we first set up the system state equations for its stationary system size distribution by treating the *elapsed service time*, the *elapsed vacation time* and the *elapsed repair time* of the server as supplementary variables. Then we solve the equations and derive the PGFs of the stationary system size distribution. Assume that the system is in steady-state conditions. Let $N(t)$ be the orbit size (i.e., the number of customers in the retrial group) at time *t*, $B^0(t)$ be the elapsed service time of the customer at time *t*. In addition, let $V^0(t)$ and $R^0(t)$ be the *elapsed vacation time* and *elapsed repair time* of the server during which breakdown occurs in the system at time *t*. Further, we introduce the following random variable:

$$Y(t) = \begin{cases} 0, & \text{if the system is idle at time } t, \\ 1, & \text{if the server is busy at time } t, \\ 3, & \text{if the server is on vacation at time } t, \\ 4, & \text{if the system is under repair during service at time } t \end{cases}$$

So that the supplementary variables $B^0(t), V^0(t)$ and $R^0(t)$ are introduced in order to obtain a bivariate Markov process $\{N(t), X(t)\}$, where $X(t) = 0$ if $Y(t) = 0$, $X(t) = B^0(t)$ if $Y(t) = 1$, $X(t) = V^0(t)$ if $Y(t) = 2$, and $X(t) = R^0(t)$ if $Y(t) = 4$. Next, we define the following limiting probabilities for $n \geq 0$:

$$\psi_n = \lim_{t\to\infty} P_r\left\{N(t) = n, X(t) = 0\right\}$$

$$P_n(x)dx = \lim_{t\to\infty} P_r\left\{N(t) = n, X(t) = B^0(t); x < B^0(t) \leq x + dx\right\}; x > 0$$

$$Q_n(y)dy = \lim_{t\to\infty} P_r\left\{N(t) = n, X(t) = V^0(t); y < V^0(t) \leq y + dy\right\}; y > 0$$

$$R_n(x,y)dy = \lim_{t\to\infty} P_r\left\{N(t) = n, X(t) = R^0(t); y < R^0(t) \leq y + dy \middle| B^0(t) = x\right\}; (x,y) > 0$$

Further, it is assumed that $B(0) = 0, B(\infty) = 1, G(0) = 0, G(\infty) = 1$ and that $B(x)$ is continuous at $x = 0$ and $V(y)$ and $G(y)$ are continuous at $y = 0$, respectively, so that:

$$\mu(x)dx = \frac{dB(x)}{1 - B(x)};\ \eta(y)dy = \frac{dV(y)}{1 - V(y)};\ \zeta(y)dy = \frac{dG(y)}{1 - G(y)}$$

are the first order differential (hazard rate) functions of *B*, *V* and *G*, respectively.

First of all, let us investigate the stability condition of our model. Let $\left\{t_n; n \in Z^+\right\}$ be the sequence of epochs of the *n*th total service completion epoch, i.e., epoch at which the service requested by a customer expires. Then, the sequence $N_n = N(t_n^+)$ forms a Markov chain, which is embedded in our queueing system.

Since the arrival process is a Poisson process, it can be shown from Burke's theorem (Cooper, 1981) that the steady-state probabilities of the bivariate Markov process $\left\{N(t), X(t)\right\}$ exist and are positive under the same condition as $\left\{X_n; n \geq 0\right\}$.

4.1. The Steady-State Equations

The Kolmogorov forward equations to govern the system under steady-state conditions (Cox, 1955) can be written as follows:

$$\frac{d}{dx}P_n(x) + \left[\lambda + \alpha + \mu(x)\right]P_n(x) = \lambda\sum_{k=0}^{n} b_k P_{n-k}(x) + \int_0^\infty \zeta(y)R_n(x,y)dy; n \geq 0 \tag{4.1}$$

$$\frac{d}{dy}Q_n(y) + \left[\lambda + \eta(y)\right]Q_n(y) = \lambda\sum_{k=0}^{n} b_k Q_{n-k}(y); n \geq 0 \tag{4.2}$$

$$\frac{d}{dy}R_n(x,y) + \left[\lambda + \zeta(y)\right]R_n(x,y) = \lambda\sum_{k=0}^{n} b_k R_{n-k}(x;y); n \geq 0 \tag{4.3}$$

$$\left(\lambda_0 + \theta_n\right)\psi_n = \int_0^\infty \eta\left(y\right)Q_n(y)dy + q\int_0^\infty \mu(x)P_n(x)dx; n \geq 0 \tag{4.4}$$

These sets of equations are to be solved under the boundary conditions at $x = 0$:

$$P_n(0) = \lambda\sum_{i=1}^{n+1} b_i\psi_{n-i+1} + \theta_{n+1}\psi_{n+1}; n \geq 0 \tag{4.5}$$

at $y = 0$:

$$Q_n\left(0\right) = p\int_0^\infty \mu\left(x\right)P_n\left(x\right)dx; n \geq 0 \tag{4.6}$$

and at $y = 0$ for fixed values of x:

$$R_{i,n}(x,0) = \alpha P_n\left(x\right); x > 0, n \geq 0 \tag{4.7}$$

with the normalizing condition:

$$\sum_{n=0}^{\infty}\left[\psi_n + \int_0^\infty P_n\left(x\right)dx + \int_0^\infty Q_n\left(y\right)dy + \int_0^\infty\int_0^\infty R_n\left(x;y\right)dxdy\right] = 1 \tag{4.8}$$

4.2. The Model Solution

To solve the system of Equations (3.1) - (3.7), let us introduce the following PGFs for $|z|<1$:

$$R(x,y;z)=\sum_{n=0}^{\infty}z^nR_n(x;y),\ R(x,0;z)=\sum_{n=0}^{\infty}z^nR_n(x;0),\ \psi(z)=\sum_{n=0}^{\infty}z^n\psi_n$$

$$Q(y;z)=\sum_{n=0}^{\infty}z^nQ_n(y),\ Q(0;z)=\sum_{n=0}^{\infty}z^nQ_n(0),$$

$$P(x,z)=\sum_{n=0}^{\infty}z^nP_n(x),\ P(0,z)=\sum_{n=0}^{\infty}z^nP_n(0)$$

Let $\lambda(z)=\lambda(1-b(z))$, then proceeding in the usual manner with Equation (4.2) and (4.3), we get a set of differential equations of Lagrangian type whose solutions are given by:

$$Q(y;z)=Q(0;z)[1-V(y)]\exp\{-\lambda(z)y\};y>0 \tag{4.9}$$

$$R(x,y;z)=R(x,0;z)[1-G(y)]\exp\{-\lambda(z)y\};y>0 \tag{4.10}$$

where $R(x,0;z)$ can be obtained from Equations (4.7). Simplification yields:

$$R(x,0;z)=\alpha P(x;z) \tag{4.11}$$

Solving the differential Equations (4.1), we get:

$$P(x;z)=P(0;z)[1-B(x)]\exp\{-A(z)x\};x>0 \tag{4.12}$$

where $A(z)=\lambda(z)+\alpha\left(1-G\left(\lambda(z)\right)\right)$. Now, multiplying both sides of Equations (4.5) and (4.6) by z^n and then taking the summation over all possible values of $n\geq 0$, we get on simplification:

$$P(0,z)=\psi(z)\left(\lambda_0+\gamma-\lambda(z)\right)z^{-1}+\theta\psi'(z)-\frac{\gamma\psi_0}{z} \tag{4.13}$$

$$Q\left(0;z\right)=pP\left(0;z\right)\beta^*\left(A\left(z\right)\right) \tag{4.14}$$

Similarly, from Equation (4.4) after utilizing (4.14), we get:

$$\left(\lambda_0+\gamma\right)\psi\left(z\right)-\gamma\psi_0+\theta z\psi'\left(z\right)=\left[q+p\vartheta^*\left(\lambda\left(z\right)\right)\right]\beta^*\left(A\left(z\right)\right)P\left(0;z\right) \tag{4.15}$$

We now combine (4.13) and (4.15) to get:

$$P(0,z)=\frac{\psi(z)\lambda(z)}{\left[q+p\vartheta^{*}\left(\lambda\left(z\right)\right)\right]\beta^{*}\left(A\left(z\right)\right)-z} \tag{4.16}$$

and:

$$\theta z\psi^{/}(z)+\left[\lambda_{0}+\gamma-\frac{\lambda\left(z\right)\left(\left[q+p\vartheta^{*}\left(\lambda\left(z\right)\right)\right]\beta^{*}\left(A\left(z\right)\right)\right)}{\left[\left(q+p\vartheta^{*}\left(\lambda\left(z\right)\right)\right)\beta^{*}\left(A\left(z\right)\right)\right]-z}\right]\psi\left(z\right)=\gamma\psi_{0} \tag{4.17}$$

which is almost a similar type of expression with expression (3.4) of Section 3. Hence, the solution of Equation (4.17) can also be obtained from Theorem 3.1. To obtain the solution of (4.17), we utilize the classical limiting theorem of Markov regenerative processes established in Çinlar (1975) and we may write:

$$\psi_{j}=\frac{\sum_{n=0}^{\infty}\pi_{n}\tau_{n}\left(j\right)}{\sum_{n=0}^{\infty}\pi_{n}\tau_{n}\left(1\right)};j\geq 0 \tag{4.18}$$

where the expected amount of time $\tau_{n}\left(j\right)$ spent by a test unit in the retrial group is j, when the system is idle during the service cycle, given that at the beginning of the interval the number of units in the retrial group was $'n'$. Similarly, $\tau_{n}\left(1\right)$ is the expected length of the service cycle given that at the beginning of this interval the number of units in the retrial group was $'n'$. Now for our model:

$$\tau_{n}\left(1\right)=\frac{1}{\lambda_{0}+\theta_{n}}+\beta^{(1)}\left(1+\alpha g^{(1)}\right)+p\nu^{(1)};n\geq 0$$

Hence the mean service cycle is given by:

$$\sum_{n=0}^{\infty}\pi_{n}\tau_{n}\left(1\right)=\left[\lambda b_{[1]}\right]^{-1} \tag{4.19}$$

Also:

$$\tau_{n}\left(j\right)=\frac{1}{\lambda_{0}+\theta_{n}}\delta_{j,n} \tag{4.20}$$

where $\delta_{i,j}$ denotes Kronecker's delta. Now utilizing (4.19) and (4.20) in (4.18), we get:

$$\psi_{j}=\frac{\lambda b_{[1]}\pi_{j}}{\lambda_{0}+\theta_{j}};j\geq 0 \tag{4.21}$$

which can also be considered to be a stable recursive scheme to calculate the limiting probabilities $\psi_j\left(j \geq 0\right)$ in terms of $\pi_j\left(j \geq 0\right)$ from expression (4.21). Now multiplying both sides of (4.21) by z^j and then taking the summation over all possible values of $j \geq 0$, we get finally:

$$\psi\left(z\right) = \lambda b_{[1]}\phi\left(z\right) \tag{4.22}$$

which is the relationship between the two PGFs $\psi\left(z\right)$ and $\phi\left(z\right)$. Hence with the help of this relationship, one can easily obtain the solution of (4.17). Further, let $z \to 1$ in (4.16), we obtain by L'Hospital's rule:

$$P(0,1) = \frac{\lambda b_{[1]}\psi(1)}{1-\rho_H} \tag{4.23}$$

where $\rho_H = \rho_1\left\{1 + \alpha g^{(1)}\right\} + p\lambda b_{[1]}\nu^{(1)}$ is the utilization factor of the system and $\rho_1 = \lambda b_{[1]}\beta^{(1)}$. This gives:

$$P(x,1) = \frac{\lambda b_{[1]}\psi(1)\left[1 - B(x)\right]}{1-\rho_H} \tag{4.24}$$

$$Q(y,1) = \frac{p\lambda b_{[1]}\psi\left(1\right)\left[1 - V\left(y\right)\right]}{1-\rho_H} \tag{4.25}$$

and:

$$R(x,y,1) = \frac{\alpha\lambda b_{[1]}\psi(1)\left[1 - B(x)\right]\left[1 - G(y)\right]}{1-\rho_H} \tag{4.26}$$

Hence from the normalizing condition (4.8), we get:

$$\psi(1) = 1 - \rho_H \tag{4.27}$$

The joint distribution of the state of the server and the number in the orbit results is summarized in Theorem 4.1 below.

Theorem 4.1: Under the stability conditions, the joint distribution of the server's state and the number of units in the orbit has the following partial PGFs:

$$P(x;z) = \frac{\lambda(z)\psi(z)[1 - B(x)]\exp\left\{-\left(A(z)\right)x\right\}}{\left\{q + p\vartheta^*\left(\lambda(z)\right)\right\}\beta^*\left(A(z)\right) - z} \tag{4.28}$$

$$Q(y;z) = \frac{p\lambda(z)\psi(z)\beta^*\left(A\left(z\right)\right)[1-V(y)]\exp\left\{-\left(\lambda(z)\right)y\right\}}{\left\{q+p\vartheta^*\left(\lambda(z)\right)\right\}\beta^*\left(A(z)\right)-z} \tag{4.29}$$

$$R(x,y;z) = \frac{\alpha_1\lambda(z)\psi(z)[1-B(x)]\exp\left\{-\left(A(z)\right)x\right\}\times[1-G(y)]\exp\left\{-\left(\lambda(z)\right)y\right\}}{\left\{q+p\vartheta^*\left(\lambda(z)\right)\right\}\beta^*\left(A(z)\right)-z} \tag{4.30}$$

where $A(z) = \lambda(z) + \alpha\left(1 - G^*\left(\lambda(z)\right)\right)$ and $\lambda(z) = \lambda\left(1 - b(z)\right)$.

Next, we are interested in investigating the marginal orbit size distributions due to the state of the server.

Theorem 4.2: Under the stability conditions, the marginal PGFs of the server's state orbit size distributions are given by:

$$P(z) = \frac{\lambda(z)\psi(z)\left[1-\beta^*\left(A(z)\right)\right]}{A(z)\left[\left\{q+p\vartheta^*\left(\lambda(z)\right)\right\}\beta^*\left(A(z)\right)-z\right]} \tag{4.31}$$

$$Q(z) = \frac{p\psi(z)\beta^*\left(A\left(z\right)\right)\left[1-\vartheta^*\left(\lambda(z)\right)\right]}{\left[\left\{q+p\vartheta^*\left(\lambda(z)\right)\right\}\beta^*\left(A(z)\right)-z\right]} \tag{4.32}$$

and:

$$R(z) = \frac{\alpha\left(1-G^*\left(\lambda(z)\right)\right)\psi(z)\left[1-\beta^*\left(A(z)\right)\right]}{A(z)\left[\left\{q+p\vartheta^*\left(\lambda(z)\right)\right\}\beta^*\left(A(z)\right)-z\right]} \tag{4.33}$$

Proof: Integrating (4.28) and (4.29) with respect to x and y, respectively, then using the well-known results of renewal theory:

$$\int_0^\infty e^{-sx}\left(1-B_i(x)\right)dx = \frac{1-\beta^*(s)}{s} \text{ and } \int_0^\infty e^{-sy}\left(1-V\left(y\right)\right)dy = \frac{1-\vartheta^*\left(s\right)}{s}$$

we get formulae (4.31) and (4.32). Similarly, integrating equations (4.30) with respect to y, we get:

$$R(x,z) = \int_0^\infty R(x,y;z)dy = \frac{\alpha\left(1-G^*\left(\lambda\left(z\right)\right)\right)\psi\left(z\right)\left[1-B\left(x\right)\right]\exp\left\{-A\left(z\right)x\right\}}{\left\{q+p\vartheta^*\left(\lambda\left(z\right)\right)\right\}\beta^*\left(A\left(z\right)\right)-z} \tag{4.34}$$

Further integrating expression (4.34) with respect to x, we claimed in formulae (4.33).

Theorem 4.3: See the following:

a. Let Ω_j be the stationary distribution of the number of customers in the orbit, then its corresponding PGF $\Omega(z) = \sum_{j=0}^{\infty} z^j P_j$ is given by:

$$\Omega(z) = \frac{\psi(z)(1-z)}{\left[q + p\vartheta^*\left(\lambda - \lambda b(z)\right)\right]\beta^*\left\{\left(\lambda - \lambda b(z)\right) + \alpha\left(1 - G^*\left(\lambda - \lambda b\left(z\right)\right)\right)\right\} - z} \tag{4.35}$$

b. Let Φ_j be the stationary distribution of the total number of customers in the system at a random epoch i.e., $\Phi_j = \psi_j + \left(1 - \delta_{j,0}\right)\left\{P_{j-1} + Q_{j-1} + R_{j-1}\right\};\ j \geq 0$, then its corresponding PGF is given by:

$$\Phi(z) = \frac{\psi(z)(1-z)\left[q + p\vartheta^*\left(\lambda - \lambda b(z)\right)\right]\beta^*\left\{\lambda - \lambda b(z) + \alpha\left(1 - G^*\left(\lambda - \lambda b\left(z\right)\right)\right)\right\}}{\left[q + p\vartheta^*\left(\lambda - \lambda b(z)\right)\right]\beta^*\left\{\lambda - \lambda b(z) + \alpha\left(1 - G^*\left(\lambda - \lambda b\left(z\right)\right)\right)\right\} - z}$$

$$= \frac{\psi\left(z\right)}{\psi\left(1\right)}\xi\left(z\right) \tag{4.36}$$

where $\xi\left(z\right)$ is the PGF of the number of customers present in the system in an $M^X/G/1$ queue with Bernoulli vacation schedule for an unreliable server under Bernoulli admission mechanism, which is given by:

$$\xi(z) = \frac{\left(1 - \rho_H\right)(1-z)\left[q + p\vartheta^*\left(\lambda - \lambda b(z)\right)\right]\beta^*\left\{\lambda - \lambda b(z) + \alpha\left(1 - G^*\left(\lambda - \lambda b\left(z\right)\right)\right)\right\}}{\left[q + p\vartheta^*\left(\lambda - \lambda b(z)\right)\right]\beta^*\left\{\lambda - \lambda b(z) + \alpha\left(1 - G^*\left(\lambda - \lambda b\left(z\right)\right)\right)\right\} - z} \tag{4.37}$$

Note that for $\alpha = 0$ and $\varpi = 1$, the above formula (4.37) is consistent with expression (3.16) of Madan and Choudhury (2004).

Proof: The result follows with the help of PGFs $\psi(z)$, $P(z)$, $Q(z)$ and $R(z)$, we get the distribution of the PGF of the number of customers in the orbit as:

$$\Omega(z) = \psi(z) + P\left(z\right) + Q\left(z\right) + R\left(z\right)$$

By direct calculation we can obtain (4.35). Similarly, result (4.36) follows by calculating:

$$\Phi(z) = \phi(z) + z\left\{P(z) + Q(z) + R(z)\right\}$$

Remark 4.1: It is important to note here that the stationary distribution of the number of customers present in the system at a random point of an $M^X/G/1$ retrial queue with Bernoulli vacation schedule and linear repeated attempts for unreliable server under Bernoulli admission mechanism given in Equation (4.36) in terms of a generating function decomposes into the distributions of two independent random variables:

a. The system size distribution of an $M^X/G/1$ queue with Bernoulli vacation schedule for unreliable server under Bernoulli admission mechanism [*represented by the second term of equation* (4.36)]; and
b. The conditional distribution of the number of customers in the retrial group given that the system is idle [*represented by the first term of equation* (4.36)].

This confirms the decomposition property of Fuhrmann and Cooper (1985). It should be pointed out that our retrial model can also be viewed as a special type of the non-exhaustive vacation model where the vacations begin at the service completion times. Also, we note that a similar model with two phases of service and Bernoulli admission mechanism was investigated recently by Choudhury and Deka (2013).

Remark 4.2: If we compare expression (4.31) with the expression (3.5), then we have:

$$\pi(z)=\frac{1-b(z)}{b^{/}\left(1\right)(1-z)}\Phi\left(z\right)=B\left(z\right)\Phi\left(z\right) \tag{4.38}$$

as expected, where:

$$B\left(z\right)=\frac{1-b\left(z\right)}{b^{/}\left(1\right)\left(1-z\right)}$$

is the PGF of the number of customers that are before an arbitrary test customer (tagged customer) in an admission batch in which the tagged customer arrived. This number is given as the backward recurrence time in the discrete time renewal process where renewal points are generated by the arrival size random variable. Note that this is consistent with the result of Falin and Templeton (1997) of the main $M^X/G/1$ retrial queue by taking $p=0$, $\alpha=0$ and $\varpi=1$.

5. SOME PERFORMANCE MEASURES

Our next objective is to provide explicit expressions for system size probabilities and performance measures of the system. The results are summarized in the following theorems.

Theorem 5.1: If the system is in steady-state conditions, then:

a. The probability that the server is idle is $P_I=1-\lambda b_{[1]}\beta^{(1)}\left\{1+\alpha g^{(1)}\right\}-p\lambda b_{[1]}\gamma^{(1)}$;
b. The probability that the server is busy is $P_B=\lambda b_{[1]}\beta^{(1)}$;
c. The probability that the server is on vacation is $P_V=p\lambda b_{[1]}\gamma^{(1)}$;
d. The probability that the server is under repair is $P_R=\lambda\alpha b_{[1]}\beta^{(1)}g^{(1)}$.

Proof: Noting that:

$$P_B=\lim_{z\to 1}P(z),\ P_V=\lim_{z\to 1}Q(z)$$

and:

$$P_R = \lim_{z \to 1} R(z)$$

and:

$$P_I = 1 - P_B - P_V - P_R$$

the stated formulae follow by direct calculations.

Next, we are interested in the mean orbit size and the mean system size of this model.

Theorem 5.2: Let $E\left(N_O\right)$, $E\left(N_S\right)$ and $E(N_D)$ be the expected number of units in the retrial groups, system size at random epoch and system size at departure epoch, respectively. Then, under the stability condition, we have:

$$E\left(N_O\right) = L_0 + \frac{M_0}{1 - \rho_H}$$

where:

$$L_0 = \frac{\left[\lambda b_{[1]}\right]^2 \left[\beta^{(2)} \left(1 + \alpha g^{(1)}\right)^2 + \alpha \beta^{(1)} g^{(2)} + p\nu^{(2)} + 2p\nu^{(1)} \beta^{(1)} \left(1 + \alpha g^{(1)}\right)\right]}{2\left(1 - \rho_H\right)} + \frac{\rho_H b_R}{1 - \rho_H} \tag{5.1}$$

and $M_0 = \psi^{/}\left(1\right)$. Now:

1. If $\gamma > 0$ and $\theta > 0$, then:

$$M_0 = \frac{\lambda b_{[1]} \left(1 + \gamma \lambda_0^{-1} \pi_0\right) + \left(\lambda_0 + \gamma\right)\left(\rho_H - 1\right)}{\theta} \tag{5.2}$$

2. If $\gamma > 0$ and $\theta = 0$, then:

$$M_0 = \frac{\left[\lambda b_{[1]}\right]\left(L_0 + \rho_H + b_R\right)}{\lambda_0 + \gamma - \lambda b_{[1]} \left(1 - \rho_H\right)^{-1}} \tag{5.3}$$

$$E\left(N_S\right) = E\left(N_O\right) + \rho_H \tag{5.4}$$

$$E\left(N_D\right) = E\left(N_S\right) + b_R \tag{5.5}$$

where $b_R = \dfrac{b_{[2]}}{2b_{[1]}}$.

Proof: The proof follows by routine differentiation. So, we just indicate a few steps. Furthermore, (4.35), (4.36) and (4.38) are valid for any retrial policy. They are obtained by taking the first derivatives with respect to z and then taking the limit $z \to 1$ by using L'Hospital's rule. With the help of (4.22) and (3.4), we can obtain the following differential equation:

$$\theta z \psi^{/}(z) + \left[\lambda_0 + \gamma - \frac{\lambda\left(1 - b(z)\right)}{\left(1-\rho_H\right)\left(1-z\right)} \varsigma(z) \right] \psi(z) = \frac{\gamma \lambda b_{[1]} \pi_0}{\lambda_0} \tag{5.6}$$

By putting $z = 1$ in (5.6), we get formula (5.2) on simplification. Formula (5.3) follows from (5.6) after putting $\theta = 0$ and suitable differentiation of (4.33). Finally, (5.1), (5.4) and (5.5) follow by routine differentiation in (4.35), (4.36) and (4.38), respectively.

Remark 5.1: The result obtained in this section is consistent with the existing literature. For example, if we take $\gamma = 0$, then for $\theta > 0$, formula (5.4) reduces to:

$$E\left(N_S\right) = \frac{\left[\lambda b_{[1]}\right]^2 \left[\beta^{(2)}\left(1+\alpha g^{(1)}\right)^2 + \alpha\beta^{(1)} g^{(2)} + p\nu^{(2)} + 2p\nu^{(1)}\beta^{(1)}\left(1+\alpha g^{(1)}\right)\right]}{2\left(1-\rho_H\right)}$$
$$+ \frac{\rho_H b_{[2]}}{2\left(1-\rho_H\right) b_{[1]}} + \frac{\lambda_0 b_{[1]} + \lambda_0\left(\rho_H - 1\right)}{\theta\left(1-\rho_H\right)} + \rho_H \tag{5.7}$$

which is the expression for the mean number of units present in the system for the existing model under classical retrial policy. Now suppose that $p = 0$ (i.e., there is no Bernoulli schedule in the system) and $\varpi = 1$ (i.e., there is no control of admission to join the retrial group), then Equation (5.7) yields:

$$E\left(N_S\right) = \rho_H + \frac{\left[\lambda a_{[1]}\right]^2 \left[\beta^{(2)}\left(1+\alpha g^{(1)}\right)^2 + \alpha\beta^{(1)} g^{(2)}\right]}{2\left(1-\rho_H\right)} + \frac{\rho_H a_{[2]}}{2\left(1-\rho_H\right) a_{[1]}} + \frac{\lambda}{\theta}\left(\frac{a_{[1]} + \rho_H - 1}{1-\rho_H}\right)$$

which is the expression for the mean number of units present in the system for an $M^X/G/1$ retrial queue with classical retrial policy and unreliable server. Note that for $\alpha = 0$ the expression is consistent with the results of the $M^X/G/1$ queue under classical retrial policy, e.g. see Falin and Templeton (1997). Similarly, by taking the limit $\theta \to \infty$ in expression (5.7), we get:

$$E\left(N_S\right) = \rho_H + \frac{\left[\lambda b_{[1]}\right]^2 \left[\beta^{(2)}\left(1+\alpha g^{(1)}\right)^2 + \alpha\beta^{(1)} g^{(2)} + p\nu^{(2)} + 2p\nu^{(1)}\beta^{(1)}\left(1+\alpha g^{(1)}\right)\right]}{2\left(1-\rho_H\right)}$$
$$+\frac{\rho_H b_{[2]}}{2\left(1-\rho_H\right) b_{[1]}}$$

Note that for $\alpha = 0$, the above expression is consistent with the result obtained by Madan and Choudhury (2004), without restricted admissibility in their result.

Finally, we consider two reliability indices of the system viz- the system availability and the failure frequency under the steady state conditions. Let $A_V(t)$ be the system availability at time '*t*', that is, the probability that the server is either working for a customer or is in an idle period. The steady-state availability of the server will be $A_V = \lim_{t\to\infty} A_V(t)$.

Theorem 5.3: The availability of the server and failure frequency of the server under the steady-state conditions are respectively given by:

$$A_V = 1 - \alpha\lambda b_{[1]}\beta^{(1)} g^{(1)} - p\lambda b_{[1]}\nu^{(1)} \tag{5.8}$$

and:

$$M_f = \alpha\lambda\beta^{(1)} \tag{5.9}$$

Proof: The result follows directly by considering the following equations:

$$A_V = \sum_{n=0}^{\infty}\psi_n + \int_0^{\infty} P\left(x;1\right)dx \text{ and } M_f = \alpha\int_0^{\infty} P\left(x;1\right)dx$$

Now, since:

$$\int_0^{\infty}\left[1-B(x)\right]dx = \int_0^{\infty} x dB(x) = \beta^{(1)}$$

from Equations (4.24) and (4.27), we get (5.8) and (5.9), respectively.

6. NUMERICAL ILLUSTRATIONS

We present in this section some numerical illustrations to show the effect of using the Bernoulli admission mechanism on the system performance, in particular on the expected number of units in the retrial group $E[N_0]$. For simplicity, assume that the service time, the vacation time, and the repair time follow the exponential distribution with respective means $\beta^{(1)} = 1/\mu$, $v^{(1)} = 1/v$, and $g^{(1)} = 1/r$. Also, assume the size of arriving batches of units follows the displaced geometric distribution with mean $a_{[1]} = 1/(1-\varepsilon)$. Now let us start with the case when the Bernoulli admission mechanism is not implemented in the system. We consider the following base values for the system parameters: $\lambda = 0.1, \mu = 3.5, \varepsilon = 0.5, r = 2\mu, v = 10\mu$. We also take $\alpha = 0.5$ and $p = 0.5$. In this case, we obtain a mean number of units in the retrial group $E[N_0] = 0.3001$. Also, the availability of the server is $A_v = 99.31\%$ and the failure frequency of the server is $M_f = 2.86\%$. The effect of the failure rate α and the Bernoulli vacation schedule p on the expected number of units in the retrial group is shown in Figure 1 below. As can be seen, $E[N_0]$ increase as either α increases or p increases.

Now to observe the effect of the Bernoulli admission mechanism on the expected number of units in the retrial group, we compute $E[N_0]$ for various values of ϖ. The effect of the failure rate and the Bernoulli vacation schedule on employing Bernoulli admission mechanism is investigated by changing the values of α and p. For each combination of the parameters, the value of $E[N_0]$ is computed and compared to the value of $E[N_0]$ when there is no Bernoulli admission mechanism. Tables 1–3 below show the value of the effect:

$$\Delta = | E(N_0)_{\text{with Bernoulli admission mechanism}} - E(N_0)_{\text{without Bernoulli admission mechanism}} |$$

when $\theta > 0$ (we took $\theta = 0.5$). Tables 4 – 6 show the same quantity Δ when $\theta = 0$ (in this case, we took $\gamma = 0.5$). Tables 1 – 6 all show that for a fixed value of ϖ, the effect Δ increases as p increases. This is also shown in Figures 2 and 3 for sample values of ϖ. However, for a fixed value of p, Tables 4 – 6 show that Δ increases when ϖ increases, while Tables 1 – 3 show that Δ first increases and then decreases as ϖ increases, making Δ a concave function of ϖ. This is also shown in Figure 4 for sample values of p.

7. CONCLUSION

We have considered in this paper a complex queueing system that combines many of the well-known queueing theory features: batch arrivals, Bernoulli vacation schedule, potential server breakdowns, retrials under linear retrial policy, and Bernoulli admission mechanism. A thorough steady state analysis yields eventually to various performance measures of the system. We

Figure 1. Effect of α and p for ϖ = 1

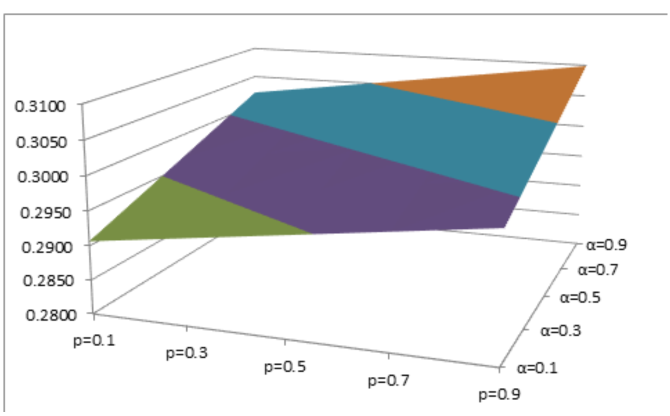

Figure 2. Variations of Δ as a function of p when θ > 0 for ϖ = 0.1, ϖ = 0.5, and ϖ = 0.9

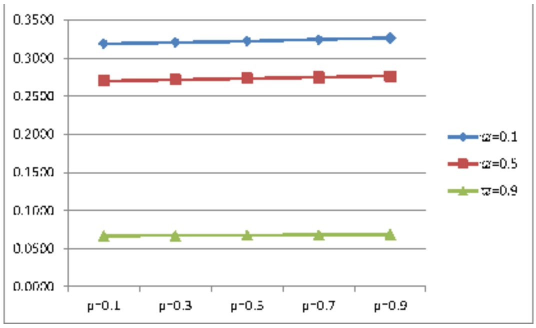

Figure 3. Variations of Δ as a function of p when θ = 0 for ϖ = 0.1, ϖ = 0.5, and ϖ = 0.9

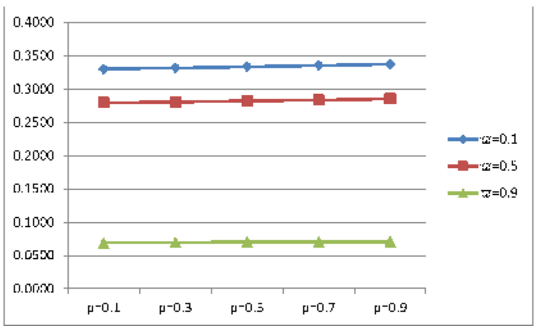

Figure 4. Variations of Δ as a function of ϖ when θ > 0 for p = 0.1 and p = 0.9

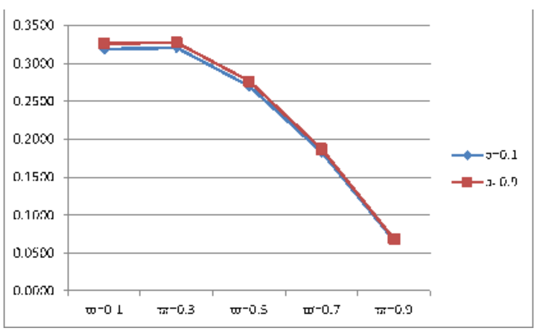

Table 1. Effect of ϖ and p when θ > 0 for α = 0.1

	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
$\varpi = 0.1$	0.3191	0.3210	0.3229	0.3247	0.3266
$\varpi = 0.3$	0.3209	0.3227	0.3244	0.3262	0.3280
$\varpi = 0.5$	0.2707	0.2722	0.2737	0.2752	0.2767
$\varpi = 0.7$	0.1833	0.1843	0.1854	0.1864	0.1875
$\varpi = 0.9$	0.0672	0.0676	0.0680	0.0684	0.0688

Table 2. Effect of ϖ and p when θ > 0 for α = 0.5

	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
$\varpi = 0.1$	0.3248	0.3267	0.3285	0.3305	0.3324
$\varpi = 0.3$	0.3262	0.3280	0.3298	0.3316	0.3333
$\varpi = 0.5$	0.2752	0.2767	0.2782	0.2797	0.2812
$\varpi = 0.7$	0.1864	0.1875	0.1885	0.1896	0.1906
$\varpi = 0.9$	0.0684	0.0688	0.0692	0.0696	0.0700

Table 3. Effect of ϖ and p when θ > 0 for α = 0.9

	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
$\varpi = 0.1$	0.3305	0.3324	0.3343	0.3362	0.3382
$\varpi = 0.3$	0.3316	0.3334	0.3352	0.3370	0.3388
$\varpi = 0.5$	0.2797	0.2812	0.2827	0.2843	0.2858
$\varpi = 0.7$	0.1896	0.1907	0.1917	0.1928	0.1939
$\varpi = 0.9$	0.0696	0.0700	0.0704	0.0708	0.0712

Table 4. Effect of ϖ and p when θ = 0 for α = 0.1

	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
$\varpi = 0.1$	0.6771	0.6810	0.6850	0.6889	0.6928
$\varpi = 0.3$	0.6357	0.6394	0.6432	0.6469	0.6507
$\varpi = 0.5$	0.5450	0.5483	0.5516	0.5549	0.5583
$\varpi = 0.7$	0.3920	0.3944	0.3970	0.3995	0.4020
$\varpi = 0.9$	0.1570	0.1580	0.1591	0.1602	0.1613

Table 5. Effect of ϖ and p when θ = 0 for α = 0.5

	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
$\varpi = 0.1$	0.6888	0.6927	0.6967	0.7007	0.7047
$\varpi = 0.3$	0.6468	0.6506	0.6544	0.6582	0.6620
$\varpi = 0.5$	0.5548	0.5582	0.5616	0.5649	0.5683
$\varpi = 0.7$	0.3994	0.4019	0.4044	0.4070	0.4096
$\varpi = 0.9$	0.1601	0.1612	0.1623	0.1634	0.1645

Table 6. Effect of ϖ and p when θ = 0 for α = 0.9

	$p = 0.1$	$p = 0.3$	$p = 0.5$	$p = 0.7$	$p = 0.9$
$\varpi = 0.1$	0.7006	0.7046	0.7086	0.7126	0.7166
$\varpi = 0.3$	0.6581	0.6619	0.6658	0.6696	0.6735
$\varpi = 0.5$	0.5648	0.5682	0.5716	0.5751	0.5785
$\varpi = 0.7$	0.4069	0.4095	0.4121	0.4147	0.4173
$\varpi = 0.9$	0.1633	0.1644	0.1656	0.1667	0.1678

presented numerical examples illustrating the effect of some system parameters on the expected number of units in the orbit.

This paper can be generalized in various ways. For example, it may be worth investigating how the N policy would improve the optimal management of this system. Another possible extension would be to consider units that require two consecutive phases of service.

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