AN UNSOLVED PROBLEM ON THE POWERS OF $\frac{3}{2}$ *

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Let α be an arbitrary positive number. For every integer $n \geqq 0$ we can write

$$\alpha(\tfrac{3}{2})^n = g_n + r_n,$$

where

$$g_n = [\alpha(\tfrac{3}{2})^n]$$

is the largest integer not greater than $\alpha(\frac{3}{2})^n$, i.e. the integral part of $\alpha(\frac{3}{2})^n$, and r_n is its fractional part and so satisfies the inequality

$$0 \leqq r_n < 1.$$

We say that α is a *Z-number* if

(1) $$0 \leqq r_n < \tfrac{1}{2} \qquad \text{for all suffixes } n \geqq 0.$$

Several years ago, a Japanese colleague proposed to me the problem whether such Z-numbers do in fact exist. I have not succeeded in solving this problem, but shall give here a number of incomplete results. In particular, it will be proved that *the set of all Z-numbers is at most countable.*

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Assume that α is a Z-number. Evidently

$$g_{n+1} + r_{n+1} = \tfrac{3}{2}(g_n + r_n).$$

Here g_n and g_{n+1} are integers, while r_n and r_{n+1} lie in the interval

$$J = [0, \tfrac{1}{2}).$$

Hence one of the following two cases must hold.

(A) g_n is an even number, hence $\frac{3}{2}g_n$ is an integer. Since

$$0 \leqq \tfrac{3}{2}r_n < \tfrac{3}{4},$$

necessarily

$$g_{n+1} = \tfrac{3}{2}g_n \text{ and } r_{n+1} = \tfrac{3}{2}r_n.$$

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(B) g_n is an odd number and so both numbers $\frac{3}{2}g_n \mp \frac{1}{2}$ are integers. Since $\frac{3}{2}r_n + \frac{1}{2}$ cannot lie in J, we now must have

$$g_{n+1} = \tfrac{3}{2}g_n + \tfrac{1}{2} \text{ and } r_{n+1} = \tfrac{3}{2}r_n - \tfrac{1}{2}.$$

Put

$$\varepsilon_n = \begin{cases} 0 & \text{if } g_n \text{ is even,} \\ 1 & \text{if } g_n \text{ is odd.} \end{cases}$$

The two cases (A) and (B) can then be combined in the one formula

(2) $$g_{n+1} = \tfrac{3}{2}g_n + \tfrac{1}{2}\varepsilon_n, \quad r_{n+1} = \tfrac{3}{2}r_n - \tfrac{1}{2}\varepsilon_n.$$

We also see that the case (A) can hold only if

$$0 \leqq r_n < \tfrac{1}{3}$$

and case (B) if

$$\tfrac{1}{3} \leqq r_n < \tfrac{1}{2}.$$

Hence ε_n may also be defined by

$$\varepsilon_n = \begin{cases} 0 & \text{if } 0 \leqq r_n < \frac{1}{3}, \\ 1 & \text{if } \frac{1}{3} \leqq r_n < \frac{1}{2}. \end{cases}$$

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From (2),

$$g_0 = -\tfrac{1}{3}\varepsilon_0 + \tfrac{2}{3}g_1, \quad g_1 = -\tfrac{1}{3}\varepsilon_1 + \tfrac{2}{3}g_2, \cdots, \quad g_{n-1} = -\tfrac{1}{3}\varepsilon_{n-1} + \tfrac{2}{3}g_n.$$

Since

$$g_0 + r_0 = (\tfrac{2}{3})^n (g_n + r_n),$$

it follows from these equations that

(3) $$g_0 = -\tfrac{1}{3}\{\varepsilon_0 + \tfrac{2}{3}\varepsilon_1 + (\tfrac{2}{3})^2\varepsilon_2 + \cdots + (\tfrac{2}{3})^{n-1}\varepsilon_{n-1}\} + (\tfrac{2}{3})^n g_n$$

and similarly also

(4) $$r_0 = +\tfrac{1}{3}\{\varepsilon_0 + \tfrac{2}{3}\varepsilon_1 + (\tfrac{2}{3})^2\varepsilon_2 + \cdots + (\tfrac{2}{3})^{n-1}\varepsilon_{n-1}\} + (\tfrac{2}{3})^n r_n.$$

These equations can be generalised. For this purpose put

$$\alpha_0 = \alpha \text{ and } \alpha_m = (\tfrac{3}{2})^m \alpha.$$

Then

$$(\tfrac{3}{2})^n (g_m + r_m) = (\tfrac{3}{2})^n \alpha_m = (\tfrac{3}{2})^{m+n}\alpha = g_{m+n} + r_{m+n},$$

and it follows in analogy to (3) and (4) that for all suffixes m and n,

(5) $$g_m = -\tfrac{1}{3}\{\varepsilon_m + \tfrac{2}{3}\varepsilon_{m+1} + (\tfrac{2}{3})^2\varepsilon_{m+2} + \cdots + (\tfrac{2}{3})^{n-1}\varepsilon_{m+n-1}\} + (\tfrac{2}{3})^n g_{m+n}$$

and

(6) $$r_m = +\tfrac{1}{3}\{\varepsilon_m + \tfrac{2}{3}\varepsilon_{m+1} + (\tfrac{2}{3})^2\varepsilon_{m+2} + \cdots + (\tfrac{2}{3})^{n-1}\varepsilon_{m+n-1}\} + (\tfrac{2}{3})^n r_{m+n}.$$

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The formula (6) for r_m immediately implies a convergent series for this number. For all r_{m+n} lie in the interval J, while the factor $(\frac{2}{3})^n$ tends to zero as n tends to infinity. It follows therefore that for all suffixes $m \geqq 0$,

$$(7) \qquad 3r_m = \varepsilon_m + \tfrac{2}{3}\varepsilon_{m+1} + (\tfrac{2}{3})^2\varepsilon_{m+2} + \cdots$$

and in particular,

$$(8) \qquad 3r_0 = \varepsilon_0 + \tfrac{2}{3}\varepsilon_1 + (\tfrac{2}{3})^2\varepsilon_2 + \cdots.$$

Here the convergence is in the sense of ordinary real analysis.

Consider next the formula (5) for g_m. The last term $(\frac{2}{3})^n g_{m+n}$ of this formula is a rational number the numerator of which is divisible by at least the n-th power of 2. In the so-called 2-adic analysis in the rational number field one considers numbers as small if they are divisible by a high power of 2 in the numerator, and as large if such a power of 2 occurs in the denominator. In this 2-adic sense the sequence of numbers $(\frac{2}{3})^n g_{m+n}$ tends to zero as n tends to infinity. We may therefore write

$$(9) \qquad -3g_m = \varepsilon_m + \tfrac{2}{3}\varepsilon_{m+1} + (\tfrac{2}{3})^2\varepsilon_{m+2} + \cdots \quad \text{in the 2-adic sense,}$$

and in particular,

$$(10) \qquad -3g_0 = \varepsilon_0 + \tfrac{2}{3}\varepsilon_1 + (\tfrac{2}{3})^2\varepsilon_2 + \cdots \quad \text{in the 2-adic sense.}$$

It is rather interesting that the same series converges in two different senses and to two different limits.

From this we can already deduce the fact *the set of all Z-numbers is at most countable*. For if the integer $g_0 \geqq 0$ is given, then, by § 1, the corresponding sequence of integers $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ is determined uniquely, and so, by (8), also the fractional part r_0. We may express this result as follows.

(11) *For any given non-negative integer g_0 there exists at most one Z-number in the interval $[g_0, g_0+1)$, and this Z-number lies in fact in the first half $[g_0, g_0+\frac{1}{2})$ of this interval.*

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Much more can be said about the possible Z-numbers and their integral parts g_0.

All the fractional parts r_m, where $r = 0, 1, 2, \cdots$, lie by construction in the interval $J = [0, \frac{1}{2})$. This means by (7) that for every suffix m the inequality

$$(12) \qquad \varepsilon_m + \tfrac{2}{3}\varepsilon_{m+1} + (\tfrac{2}{3})^2\varepsilon_{m+2} + \cdots < \tfrac{3}{2}$$

is satisfied. In this set of inequalities each of the numbers $\varepsilon_m, \varepsilon_{m+1}, \varepsilon_{m+2}, \cdots$ can assume only either of the two values 0 or 1.

It is then, firstly, immediately clear that *for no m simultaneously*

$$\varepsilon_m = \varepsilon_{m+1} = 1.$$

For this would imply that

$$\varepsilon_m + \tfrac{2}{3}\varepsilon_{m+1} + (\tfrac{2}{3})^2 \varepsilon_{m+2} + \cdots \geqq \tfrac{5}{3} > \tfrac{3}{2},$$

contrary to (12). Therefore

(13) *if* $m < n$ *and* $\varepsilon_m = \varepsilon_n = 1$, *then* $n \geqq m+2$.

From the inequalities (12) one can deduce restrictions on those suffixes m for which simultaneously $\varepsilon_m = \varepsilon_{m+2} = 1$, $\varepsilon_{m+1} = 0$. We omit this discussion because no use will be made of the results so obtained.

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Denote from now on by

$$M = \{m_1, m_2, m_3, \cdots\}, \quad \text{where} \quad 0 \leqq m_1 < m_2 < m_3 < \cdots,$$

the set of all suffixes m for which $\varepsilon_m = 1$. Thus

$$\varepsilon_m = \begin{cases} 1 & \text{if } m \in M, \\ 0 & \text{if } m \notin M. \end{cases}$$

In other words, g_m is even or odd according as to whether m is, or is not, an element of M.

Further put

$$G_k = g_{m_k} \qquad (k = 1, 2, 3, \cdots),$$

so that all the G_k are odd.

On applying the equation (5) with

$$m = m_k \quad \text{and} \quad m+n = m_{k+1},$$

thus with

$$\varepsilon_m = 1, \quad \varepsilon_{m+1} = \varepsilon_{m+2} = \cdots = \varepsilon_{m+n-1} = 0,$$

it follows that

$$G_k = -\tfrac{1}{3} + (\tfrac{2}{3})^{m_{k+1}-m_k} G_{k+1},$$

hence that

$$G_{k+1} = (\tfrac{3}{2})^{m_{k+1}-m_k-1} \frac{3G_k+1}{2}. \tag{14}$$

This formula leads to the following algorithm connected with our problem.

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We shall use the notation

$$2^a || H$$

to denote that H is divisible by 2^a, but not by 2^{a+1}.

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Put

$$(15) \qquad a_k = m_{k+1} - m_k - 1, \quad H_k = \frac{3G_k + 1}{2}.$$

Then, by (14), the following properties hold.

For every $k \geqq 1$,

(16) G_k is odd; H_k is even; $a_k \geqq 1$; $2^{a_k} || H_k$; and $G_{k+1} = (\frac{3}{2})^{a_k} H_k$ is odd.

Thus, starting with any odd integer G_1, these formulae allow to determine successively the integers

$$H_1, a_1; \ G_2, H_2, a_2: G_3, H_3, a_3; \cdots.$$

If G_1 was the integral part of a Z-number, then this algorithm can be continued indefinitely. It thus provides a necessary (but not a sufficient) condition for G_1 to be the integral part of a Z-number.

By way of example, if we start with $G_1 = 13$, we obtain the following sequence of integers.

$G_1 = 13$	$H_1 = 20$	$a_1 = 2$
$G_2 = 45$	$H_2 = 68$	$a_2 = 2$
$G_3 = 153$	$H_3 = 230$	$a_3 = 1$
$G_4 = 345$	$H_4 = 518$	$a_4 = 1$
$G_5 = 777$	$H_5 = 1166$	$a_5 = 1$
$G_6 = 1749$	$H_6 = 2624$	$a_6 = 6$
$G_7 = 29889$	$H_7 = 44834$	$a_7 = 1$
$G_8 = 67251$	$H_8 = 100877.$	

Since H_8 is odd, the algorithm breaks off, and there is no Z-number between 13 and 14.

In spite of much computer work, no integer G_1 is known for which the algorithm does not break off. It is thus highly problematical whether there do in fact exist Z-numbers.

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If the existence of Z-numbers is assumed, further properties of such numbers can be obtained.

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Let us deal with the possible frequency of Z-numbers! We have already seen that there can be at most one Z-number in each interval between consecutive integers g and $g+1$ where $g \geqq 0$. Thus, for $x > 0$, there are not more than $x+1$ Z-numbers between 0 and x. This estimate can now be replaced by a stronger one.

Let us first consider Z-numbers with *odd* integral parts, say with the integral part G_1. Put

$$b_k = a_k+1 \quad \text{and} \quad c_k = a_k-1 \qquad (k = 1, 2, 3, \cdots),$$

so that by (16),

$$b_k \geqq 2 \text{ and } c_k \geqq 0 \text{ for all } k.$$

By (15) and (16),

$$G_k = -\tfrac{1}{3}+(\tfrac{2}{3})^{b_k}G_{k+1}.$$

On applying this equation repeatedly, we find that

$$(17) \quad G_1 = -\tfrac{1}{3}\{1+(\tfrac{2}{3})^{b_1}+(\tfrac{2}{3})^{b_1+b_2}+\cdots+(\tfrac{2}{3})^{b_1+b_2+\cdots+b_n}\}+(\tfrac{2}{3})^{b_1+b_2+\cdots+b_{n+1}}G_{n+2}.$$

Here

$$B_n = -\tfrac{1}{3}\{1+(\tfrac{2}{3})^{b_1}+(\tfrac{2}{3})^{b_1+b_2}+\cdots+(\tfrac{2}{3})^{b_1+b_2+\cdots+b_n}\}$$

is a rational number with an odd numerator and with a denominator which is a power of 3.

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Let now t be an arbitrarily large positive integer. For the given Z-number there exists just one suffix n such that

$$(18) \qquad b_1+b_2+\cdots+b_n \leqq t < b_1+b_2+\cdots+b_{n+1}.$$

There further is a unique integer s_n satisfying

$$1 \leqq s_n \leqq 2^t-1$$

such that

$$B_n \equiv s_n \pmod{2^t},$$

i.e. that the numerator of B_n-s_n is divisible by 2^t. It is then clear from (17) that also

$$(19) \qquad G_1 \equiv s_n \pmod{2^t}.$$

The rational number B_n, and so also the integer s_n, depend only on t and on the ordered set of integers $b_1, b_2, \cdots, b_n$. Denote by $T(t)$ the number of ordered sets of integers $n, b_1, b_2, \cdots, b_n$ which satisfy the left-hand inequality (18). This number $T(t)$ is then also the number of all residue classes $s_n \pmod{2^t}$ in which there can lie *odd* integral parts G_1 of Z-numbers.

One can easily obtain an upper bound for $T(t)$. The left-hand inequality (18) is equivalent to the inequality

$$c_1+c_2+\cdots+c_n \leqq t-2n;$$

hence $T(t)$ may also be defined as the number of ordered solutions n, $c_1, c_2, \cdots, c_n$ of this inequality where now $c_1, c_2, \cdots, c_n$ may run independently over all non-negative integers. For each separate value of n, this inequality has

$$\binom{[t-2n]+n}{n} = \binom{t-n}{n}$$

solutions, and hence, summing over n,

$$T(t) = \binom{t-1}{1} + \binom{t-2}{2} + \binom{t-3}{3} + \cdots$$

where all terms after the $[\frac{t}{2}]$-th vanish.

This formula may be written as

$$T(t)+1 = \sum_{n=0}^{t} \binom{t-n}{n} = \sum_{n=0}^{t} \binom{n}{t-n}.$$

By the binomial theorem, it implies that $T(t)+1$ is the coefficient of z^t in the power series in powers of z for

$$\sum_{n=0}^{t} \{z(1+z)\}^n = \frac{1-\{z(1+z)\}^{t+1}}{1-z(1+z)},$$

and hence $T(t)+1$ is also the coefficient of z^t in the power series for

$$f(z) = \frac{1}{1-z-z^2}.$$

Put

$$A = \frac{1+\sqrt{5}}{2}, \quad B = \frac{1-\sqrt{5}}{2}, \quad \text{so that } A+B = 1, \quad AB = -1, \quad A-B = \sqrt{5}.$$

Then

$$1-z-z^2 = (1-Az)(1-Bz) \text{ and } f(z) = \frac{1}{\sqrt{5}}\left(\frac{A}{1-Az} - \frac{B}{1-Bz}\right).$$

On developing here $f(z)$ into a series in powers of z, it follows at once that

$$T(t) = \frac{1}{\sqrt{5}}\{A^{t+1}-B^{t+1}\}-1. \tag{20}$$

Actually, $T(t)+1$ is the $(t+1)$-st term of the well known Fibonacci sequence.

Since trivially B^{t+1} has the limit 0 as t tends to infinity, and since further $A < \sqrt{5}$, it also follows from (20) that, for sufficiently large t,

(21) $$T(t) \leqq \left(\frac{1+\sqrt{5}}{2}\right)^t.$$

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By the definition of $T(t)$, there are $T(t)$ distinct residue classes (mod 2^t) in which the integral part G_1 of a Z-number can lie when it is odd.

Consider next a Z-number $\alpha = g_0+r_0$ with even integral part g_0, say

$$2^m || g_0.$$

Then

$$\alpha,\ \tfrac{3}{2}\alpha,\ (\tfrac{3}{2})^2\alpha,\ \cdots,\ (\tfrac{3}{2})^m\alpha$$

likewise are Z-numbers, and they have the integral parts

$$g_0,\ \tfrac{3}{2}g_0,\ (\tfrac{3}{2})^2 g_0,\ \cdots,\ (\tfrac{3}{2})^m g_0,$$

respectively. Here $(\frac{3}{2})^m g_0$, $= G_1$ say, is an odd integer divisible by 3^m, and

$$g_0 = (\tfrac{2}{3})^m G_1,\quad \tfrac{3}{2}g_0 = (\tfrac{2}{3})^{m-1}G_1,\ \cdots,\quad (\tfrac{3}{2})^m g_0 = G_1.$$

These $m+1$ products lie in the residue classes

(22) $$(\tfrac{2}{3})^\mu G_1 \pmod{2^t},$$

respectively, where μ runs over the successive values $\mu = m,\ m-1,\ m-2, \cdots, 1, 0$. If $\mu \geqq t$, then $(\frac{2}{3})^\mu G_1$ lies in the residue class $\equiv 0 \pmod{2^t}$.

Thus to every *odd* residue class G_1 (mod 2^t) containing the integral part of a Z-number there correspond at most t *even* residue classes (22) in which there are likewise integral parts of Z-numbers.

(23) *This implies that there cannot be more than*

$$(t+1)T(t)$$

odd or even residue classes (*mod* 2^t) *containing the integral part of a* Z-*number.*

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Trivially,

$$\frac{1+\sqrt{5}}{2} < 2^{0.7}.$$

Thus, as soon as t is sufficiently large, it follows from (21) that there exist at most

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$$2^{0.7 \cdot t-1}$$

odd or even residue classes (mod 2^t) in which there is the integral part of at least one Z-number.

Denote now by x a sufficiently large positive integer, and choose the integer t such that

$$2^t \leqq x-1 < 2^{t+1}.$$

Then every residue class (mod 2^t) contains at most two integers $\leqq x-1$. Hence there can be at most *two* Z-numbers not greater than x the integral parts of which lie in this residue class. By (23), the number of residue classes which need be considered is only

$$2^{0.7 \cdot t-1} < \tfrac{1}{2}x^{0.7}.$$

We obtain therefore the following result.

(24) *For sufficiently large x there are at most*

$$x^{0.7}$$

Z-numbers satisfying

$$0 \leqq \alpha \leqq x.$$

This paper dealt with the numbers α for which the fractional parts r_n defined in § 1 satisfied the inequalities

$$0 \leqq r_n < \tfrac{1}{2} \qquad (n = 0, 1, 2, \cdots).$$

It is possible to establish a similar theory if all the r_n are assumed to lie in some other subinterval $[c, c+\frac{1}{2})$ of $[0, 1)$. It would be very interesting if a similar theory could be established for subintervals of smaller length, or perhaps even of arbitrarily small length.

Naturally, one can consider analogous problems for the products

$$\alpha \left(\frac{p}{q}\right)^n \qquad (n = 0, 1, 2, \cdots)$$

where α is again a positive number, and p and q are integers satisfying

$$p > q \geqq 2, \qquad (p, q) = 1.$$

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