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An Upper Bound for Self-Dual Codes

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Gleason has described the general form that the weight distribution of a self-dual code over $GF(2)$ and $GF(3)$ can have. We give an explicit formula for this weight distribution when the minimum distance d between codewords is made as large as possible. It follows that for self-dual codes of length n over $GF(2)$ with all weights divisible by 4, $d \leqslant 4[n/24] + 4$; and for self-dual codes over $GF(3)$, $d \leqslant 3[n/12] + 3$; where the square brackets denote the integer part. These results improve on the Elias bound. A table of this extremal weight distribution is given in the binary case for $n \leqslant 200$ and $n = 256$.

I. PRELIMINARIES

Let $\mathbf{C}$ be a linear code over $GF(q)$ of block length n, containing q^k codewords at a minimum distance of d apart. We call $\mathbf{C}$ an $[n, k, d]$ code. The dual code $\mathbf{C}^{\perp}$ consists of all vectors $\mathbf{x}$ such that

$$\mathbf{x} \cdot \mathbf{y} = \sum_{r=0}^{n-1} x_r y_r = 0$$

for all $\mathbf{y} \in \mathbf{C}$. Then $\mathbf{C}$ is self-dual if $\mathbf{C} = \mathbf{C}^{\perp}$.

The weight $wt(\mathbf{u})$ of a vector $\mathbf{u}$ is the number of its nonzero components. The weight enumerator of a code $\mathbf{C}$ is

$$W(X, Y) = \sum_{\mathbf{u} \in \mathbf{C}} X^{n-wt(\mathbf{u})} Y^{wt(\mathbf{u})}.$$

We consider self-dual codes in 3 cases:

Case 1. Over $GF(2)$ with all weights divisible by 2,

Case 2. Over $GF(2)$ with all weights divisible by 4,

Case 3. Over $GF(3)$ with all weights divisible by 3.

Case 1 includes all binary self-dual codes, since such a code must have all weights divisible by 2. Similarly Case 3 includes all ternary self-dual codes.

II. Gleason's Theorem

Gleason (1971) has shown that the weight enumerator $W(X, Y)$ of a self-dual code of length n is a polynomial in the polynomials f and g where

Case 1. $f = X^2 + Y^2$, $g = X^2Y^2(X^2 - Y^2)^2$, and so n must be even;

Case 2. $f = X^8 + 14X^4Y^4 + Y^8$, $g = X^4Y^4(X^4 - Y^4)^4$, and so n must be divisible by 8;

Case 3. $f = X^4 + 8XY^3$, $g = Y^3(X^3 - Y^3)^3$, and so n must be divisible by 4.

See Berlekamp *et al.* (1972) and MacWilliams, Mallows and Sloane (1972) for alternative proofs, examples, and generalizations of this theorem.

To obtain a unified notation for the 3 cases we replace X by 1 and Y^w by y, and make the following definitions:

Case 1. $w = 2$, $R = 4$, $S = 2$, $\alpha = 1$, $f = 1 + \alpha y$, $g = y(1 - y)^w$;

Case 2. $w = 4$, $R = 3$, $S = 8$, $\alpha = 14$, $f = 1 + \alpha y + y^2$, $g = y(1 - y)^w$;

Case 3. $w = 3$, $R = 3$, $S = 4$, $\alpha = 8$, $f = 1 + \alpha y$, $g = y(1 - y)^w$.

Here R is the ratio of the original degrees of f and g, and n must be a multiple of S.

With the unified notation Gleason's theorem now states that, in all 3 cases, the weight enumerator of a code $\mathbf{C}$ of length $n = Sj$ is given by

$$W(y) = \sum_{k=0}^{m} a_k f^{j-Rk} g^k = \sum_{k=0}^{n/w} A_{wk} y^k, \tag{1}$$

where $m = [j/R] = [n/RS]$, the a_k are integers, and A_i is the number of codewords in $\mathbf{C}$ of weight i.

III. Extremal Weight Enumerators

Let the integers a_k in Eq. (1) be chosen so as to make $A_0 = 1$, $A_1 = A_2 = \cdots = A_r = 0$, where r is as large as possible (regardless of whether or not a code exists with this weight enumerator). The resulting

$W(y)$ is called an extremal weight enumerator. If a code does exist with this weight enumerator, it has the largest possible minimum distance between codewords of any self-dual code in which all weights are divisible by w.

There are m integers $a_1, \ldots, a_m$ to be chosen because a_0 is always 1. The smallest power of y remaining in the extremal weight enumerator is therefore y^{m+1}, unless we are lucky and $A_{w(m+1)}$ is accidentally zero. But Corollary 3 says this never happens. The minimum distance of a self-dual code is therefore at most:

Case 1. $2[n/8] + 2$,

Case 2. $4[n/24] + 4$,

Case 3. $3[n/12] + 3$.

We now study the properties of extremal weight enumerators.

IV. An Explicit Form for the Extremal Weight Enumerator

THEOREM 1. *The extremal weight enumerator is given by*

$$W(y) = \sum_{k=0}^{m} a_k f^{j-Rk} g^k$$

where $a_0 = 1$ *and* a_k, $1 \leqslant k \leqslant m$, *is equal to*

Cases 1 *and* 3:

$$\frac{j}{k} \sum_{r=0}^{k-1} (-\alpha)^{r+1} \binom{j - Rk + r}{r} \binom{(w+1)k - r - 2}{k - r - 1};$$

Case 2:

$$\frac{j}{k} \sum_{r=0}^{k-1} (r+1) \binom{5k - r - 2}{k - r - 1} \sum_{i=0}^{[(r+1)/2]} \frac{(-1)^i (-14)^{r+1-2i} (j - 3k + r - i)!}{(j - 3k)! \, (r + 1 - 2i)! \, i!}.$$

Proof. From Eq. (1) a_k must be chosen so that

$$W(y) = \sum_{k=0}^{m} a_k f^{j-Rk} g^k = 1 + \sum_{k=m+1}^{n/w} A_{wk} y^k, \tag{2}$$

which becomes, upon dividing by f^j,

$$f^{-j} = \sum_{k=0}^{m} a_k \phi^k + O(\phi^{m+1}), \tag{3}$$

where $\phi = \phi(y) = g/f^R$. Using Bürmann's Theorem (Whittaker and Watson (1963), p. 128) we expand f^{-j} in powers of ϕ and obtain

$$\begin{aligned} a_k &= \frac{1}{k!}\left[\frac{d^{k-1}}{dy^{k-1}}\frac{df^{-j}}{dy}\left(\frac{y}{\phi}\right)^k\right]_{y=0} \\ &= -\frac{j}{k!}\left[\frac{d^{k-1}}{dy^{k-1}} f'f^{-(j+1-Rk)}(1-y)^{-wk}\right]_{y=0} \\ &= -\frac{j}{k!}\left[\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{d^r}{dy^r}\{f'f^{-(j+1-Rk)}\}\frac{d^{k-r-1}}{dy^{k-r-1}}(1-y)^{-wk}\right]_{y=0}, \end{aligned}$$

by the Leibniz formula for the derivative of a product (Hardy (1944), p. 229),

$$= \frac{j}{(j-Rk)\,k!}\left[\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{d^{r+1}}{dy^{r+1}}f^{-(j-Rk)}\frac{d^{k-r-1}}{dy^{k-r-1}}(1-y)^{-wk}\right]_{y=0}. \quad (4)$$

The theorem now follows from the formulae

$$\left[\frac{d^r}{dy^r}(1+\alpha y)^{-s}\right]_{y=0} = \frac{(s-1+r)!}{(s-1)!}(-\alpha)^r,$$

$$\left[\frac{d^r}{dy^r}(1+\alpha y+y^2)^{-s}\right]_{y=0} = \sum_{i=0}^{[r/2]}\frac{(-1)^i(-\alpha)^{r-2i}\,r!\,(s-1+r-i)!}{(s-1)!\,(r-2i)!\,i!}.$$

(The second of these is easily obtained from di Bruno's formula for the derivative of a composite function (Riordan, 1958, p. 36)).

V. Number of Codewords of Minimum Weight

THEOREM 2. *The number $A_{w(m+1)}$ of codewords of minimum nonzero weight in the extremal weight enumerator is equal to:*

Case 2.

$$\binom{n}{5}\binom{5m-2}{m-1}\Big/\binom{4m+4}{5}, \qquad \text{if } n = 24m;$$

$$\frac{1}{4}n(n-1)(n-2)(n-4)\frac{(5m)!}{m!\,(4m+4)!}, \qquad \text{if } n = 24m+8;$$

$$\frac{3}{2}n(n-2)\frac{(5m+2)!}{m!\,(4m+4)!}, \qquad \text{if } n = 24m+16;$$

Case 3.

$$2\binom{n}{5}\binom{4m-2}{m-1}\Big/\binom{3m+3}{5}, \qquad \text{if} \quad n = 12m;$$

$$2n(n-1)(n-2)\frac{(4m)!}{m!\,(3m+3)!}, \qquad \text{if} \quad n = 12m+4;$$

$$6n\frac{(4m+2)!}{m!\,(3m+3)!}, \qquad \text{if} \quad n = 12m+8.$$

Remarks. (1) It follows from Theorem 4.2 of Assmus and Mattson (1969) that (a) in Case 2, if n is a multiple of 24, the codewords of any fixed weight form a 5-design; and (b) in case 3, if n is a multiple of 12 and v is in the range $\frac{1}{4}n + 3 \leqslant v \leqslant \frac{1}{2}n + 3$, the nonzero coordinates of the codewords of weight v form a 5-design. We have written $A_{w(m+1)}$ in these cases in terms of binomial coefficients to emphasize this combinatorial interpretation.

(2) The corresponding expressions for Case 1 are omitted, since these weight enumerators usually do not correspond to codes—see the next section.

(3) The proof of the theorem can be used to give an explicit expression for any A_i.

Proof. In Eq. (3) let f^{-j} be expanded further as

$$f^{-j} = \sum_{k=0}^{m} a_k\phi^k + \sum_{k=m+1}^{n/w} b_k\phi^k + O(\phi^{1+n/w}), \tag{5}$$

where b_k is also given by Eq. (4). From Eqs. (2), (5),

$$\sum_{k=m+1}^{n/w} A_{wk}y^k = -f^j \sum_{k=m+1}^{n/w} b_k\phi^k + O(\phi^{1+n/w}),$$

$$= -\sum_{k=m+1}^{n/w} b_k y^k (1-y)^{wk} f^{j-Rk} + O(y^{1+n/w}),$$

and A_{wk} is obtained by expanding the right-hand side in powers of y. In particular $A_{w(m+1)} = -b_{m+1}$, and the theorem follows from Eq. (4).

COROLLARY 3. *The number $A_{w(m+1)}$ of codewords of minimum nonzero weight in the extremal weight enumerator is never zero. Therefore the minimum distance of a self-dual code is at most $w(m+1)$, i.e.,*

Case 1. $d \leqslant 2[n/8] + 2$,

Case 2. $d \leqslant 4[n/24] + 4$,

Case 3. $d \leqslant 3[n/12] + 3$.

VI. Existence of Codes

In this section we consider the question of whether an extremal weight enumerator is in fact the weight enumerator of a code. In Cases 1 and 3 the answer is no if n is large:

THEOREM 4. *In Cases* 1 *and* 3, *for all n sufficiently large, there is no code corresponding to the extremal weight enumerator.*

Proof. Case 1. From Corollary 3 such a code would have $d/n \sim \frac{1}{4}$, violating the Elias bound which is $d < .196n$ at rate $\frac{1}{2}$ for n large [Berlekamp (1968), p. 321].

Case 3. We show that for n large the extremal weight enumerator always contains a negative coefficient, either $A_{3(j+m)}$ (the coefficient of the highest power of y) or $A_{3(j+m-1)}$ (the next-to-highest coefficient).

From Theorem 1, a_k is the coefficient of θ^{k-1} in

$$-(8j/k)(1 + 8\theta)^{-(j-3k+1)}(1 - \theta)^{-3k};$$

i.e.,

$$a_k = -(8j/2\pi ik) \oint (1 + 8z)^{-(j-3k+1)} (1 - z)^{-3k} \, dz/z^k,$$

where the path of integration is a small circle around the origin. The integral around a very large circle is negligible, so

$$\begin{aligned}
a_k &= -\text{ sum of residues at } +1 \text{ \& at } -(1/8) \\
&= \frac{8j}{2\pi ik}\left[\oint \frac{d\omega}{(9 + 8\omega)^{j-3k+1} (-\omega)^{3k} (1 + \omega)^k} \right. \\
&\qquad \left. + \oint \frac{d\omega}{(8\omega)^{j-3k+1} (9/8 - \omega)^{3k} (-1/8 + \omega)^k}\right] \\
&= \frac{-8j}{k}\left[\left(\frac{1}{9}\right)^{j+1-3k} \sum_{s=0}^{3k-1} \left(\frac{8}{9}\right)^s \binom{j - 3k + s}{j - 3k}\binom{4k - 2 - s}{k - 1}\right. \\
&\qquad \left. + (-8)^{k-1} \left(\frac{8}{9}\right)^{3k} \sum_{s=0}^{j-3k} \left(\frac{1}{9}\right)^s \binom{j - 2k - 1 - s}{k - 1}\binom{3k - 1 + s}{3k - 1}\right]
\end{aligned}$$

Let $j - 3k = a$ be fixed and let $k \to \infty$; then

$$a_k \approx -\frac{1}{3^{a-1}}\left\{\frac{1}{\sqrt{6\pi k}}\left(\frac{256}{27}\right)^k - \frac{(4k)^a}{a!}\left(\frac{-4096}{729}\right)^k\right\}.$$

Therefore for $m = [j/3]$ and j large, a_{m-1} and a_m are both negative.

Now from Eq. (1) we have

$$A_{3(j+m)} = (-1)^m \, 8^{j-3m} a_m$$

$$A_{3(j+m-1)} = (-1)^{m-1} \, 8^{j-3m+3} a_{m-1} + (-1)^m \, 8^{j-3m-1} (j - 27m) \, a_m$$

and for j large one of these is always negative.

Corollary 5 (Asymptotic bounds). *For that self-dual code of length n over $GF(2)$ with all weights divisible by 4 which has the largest possible minimum distance d,*

$$H^{-1}\left(\frac{1}{2}\right) \approx 0.1100 < \frac{d}{n} \leqslant \frac{1}{6} + \frac{4}{n},$$

for all n sufficiently large. For that self-dual code of length n over $GF(3)$ which has the largest possible minimum distance d,

$$0.1595 < \frac{d}{n} \leqslant \frac{1}{4}$$

for all n sufficiently large.

Proof. The upper bounds follow from Corollary 3 and Theorem 4, and the lower bounds from MacWilliams, Sloane and Thompson (1972) and Pless and Pierce (1973).

Corollary 5 improves on the Elias bound, which at rate $\frac{1}{2}$ is $d/n \leqslant 0.196$ ($GF(2)$) and 0.281 ($GF(3)$).

VII. Numerical Results

A computer program was written in the rational function manipulating language ALTRAN (Brown (1971), Hall (1970)) to compute the extremal weight enumerator W_e. The results are as follows:

Case 1. For $n = 32, 40, 42, 48, 50, 52$ and $\geqslant 56$, W_e contains a negative coefficient. From the table in Pless (1972a), for $n = 2, 4, 6, 8, 12, 14, 24$ a self-dual code exists with weight enumerator W_e, but for $n = 10, 16, 18, 20$ no such (linear) code exists. However, W_e for $n = 16$ is realized by the Nordstrom–Robinson nonlinear code. In the remaining cases it is not known if a code exists.

Case 2. This is the most important case, since as far as we know at the present time codes may exist corresponding to all of the extremal weight enumerators W_e. These were computed for $n \leqslant 496$, and found to be non-negative: we conjecture that this is always the case.

Codes are known to exist corresponding to W_e for $n = 8, 16, 24$ (the Golay code), 32, 40, 48 (a quadratic residue code [Pless (1963)]), 56, 64, 80, 88, and 104 (a quadratic residue code (Karlin (1969)).

Case 3. The coefficient of the highest power of y is negative for $n = 24i$ $(i \geqslant 3)$, $24i + 4$ $(i \geqslant 7)$,..., and the next-to-highest coefficient is negative for $n = 24i + 12$ $(i \geqslant 11)$,... . The negative coefficient at $n = 72$ was first observed by J. N. Pierce (see Gleason (1971)). The exact value of n beyond which W_e always contains a negative coefficient (in accordance with Theorem 4) is not known; it is greater than 320.

Codes exist corresponding to W_e for $n = 4, 8, 12$ (the Golay code), and 24, 36, 48, 60 (Pless's symmetry codes [Pless (1969), (1970), (1972)]).

VIII. Table of Extremal Weight Enumerators

Because of the importance of case 2, we have included a table of the extremal weight enumerator in this case for $n \leqslant 200$ and $n = 256$. For some values of n (see Section VII) the corresponding codes are known, and it is useful to have the enumerators on record; in the other cases it is hoped that knowledge of the enumerator will assist in deciding the existence of the codes.

Thus the table gives the weight distribution $\{A_i\}$ of the (hypothetical) binary self-dual code of length n, in which all weights are divisible by 4, and having the greatest possible minimum distance. When n is a multiple of 24 these codes correspond to 5-designs (Section V).

For each value of n, the first column of the table gives A_i, the number of codewords of weight i, and the second column gives i. Only the first half of each enumerator is given, since it is symmetrical about $n/2$. The tables were checked by verifying that $\sum A_i = 2^{n/2}$.

TABLE

Extremal Weight Enumerators

n=8

1	0
14	4

n=16

1	0
28	4
198	8

n=24

1	0
759	8
2576	12

n=32

1	0
620	8
13888	12
36518	16

n=40

1	0
285	8
21280	12
239970	16
525504	20

n=48

1	0
17296	12
535095	16
3995376	20
7681680	24

n=56

1	0
8190	12
622314	16
11699688	20
64909845	24
113955380	28

n=64

1	0
2976	12
454956	16
18275616	20
233419584	24
1041971008	28
1706719014	32

n=72

1	0
249849	16
18106704	20
462962955	24
4397342400	28
1•6602715899	32
2•5756721120	36

n=80

1	0
97565	16
12882688	20
590073120	24
1•0588174080	28
7•9707678050	32
26•3303738880	36
39•1106339008	40

n=88

1	0
32164	16
6992832	20
535731625	24
1•6623384448	28
22•5426781470	32
140•5590745152	36
416•3803131796	40
596•8212445440	44

Table continued

TABLE *(continued)*

n=96

1	0
3217056	20
369844880	24
1·8642839520	28
42·2069930215	32
455·2866656416	36
2429·2689565680	40
6572·7011639520	44
9144·7669224080	48

n=104

1	0
1138150	20
206232780	24
1·5909698064	28
56·7725836990	32
991·5185041320	36
8835·5709788905	40
41354·3821457520	44
103637·8989344140	48
140604·4530294756	52

n=112

1	0
355740	20
95307030	24
1·0847290300	28
58·2017237802	32
1562·7131952432	36
21938·0334493320	40
166257·6783018430	44
695846·0336232405	48
1633110·8474136456	52
2168210·1997880004	56

n=120

1	0
39703755	24
6101289120	28
47·5644139425	32
1882·4510698240	36
39745·0513031544	40
453051·2364732800	44
3053159·9026535880	48
11602397·7311397120	52
25725776·6776517715	56
33520028·0030755776	60

n=128

1	0
13228320	24
2940970496	28
32·0411086380	32
1807·2021808640	36
55252·3816524960	40
949111·5264030720	44
9411607·2808107840	48
54982777·3219608576	52
192059473·5166941760	56
405198299·5220321280	60
519357685·1944293670	64

n=136

1	0
3997890	24
1228844320	28
18·2985731775	32
1428·3914414016	36
61287·5802567105	40
1499765·4299809440	44
21536530·7912371890	48
185504911·9250976000	52
974521281·7192721004	56
3160731597·6754469952	60
6382267580·0631219615	64
8062541713·9398579840	68

Table continued

TABLE *(continued)*

n=144	
1	0
4810008528	28
9·0184804281	32
954·2972508784	36
55945·6467836112	40
1895022·5255363376	44
38188857·3363657355	48
468600680·3807297232	52
3564874587·3701148864	56
1·7047372906·6542803616	60
5·1759224213·6399518331	64
10·0538652205·9285093728	68
12·5378917521·2713133280	72

n=152	
1	0
153921850	28
3·9456539335	32
549·9476963240	36
43091·8793394170	40
1971495·6096238900	44
54298748·1413723850	48
922236272·9811216648	52
9845887834·3059002345	56
6·7074030848·6254520870	60
29·4967451865·4707220975	64
84·4602552379·7234712400	68
158·4056485586·6405013660	72
195·2736455236·8598482648	76

n=160	
1	0
45453440	28
1·5387022365	32
278·4234793600	36
28580·9635147520	40
1729496·5003180800	44
63642773·8348698400	48
1460056742·6564289280	52
2·1303554630·4326480640	56
20·0880636349·3271558528	60
123·9735481963·6041047650	64
505·7008276304·1180720000	68
1373·4644538224·9784512000	72
2496·3604326205·1942679040	76
3045·5177418359·7643539648	80

n=168	
1	0
5776211364	32
125·1098739072	36
16606·8570988089	40
1304707·1967014400	44
62904967·6288183920	48
1908712210·2289097472	52
3·7209973263·3702386736	56
47·3929136607·8578079232	60
399·7367376940·1063697390	64
2256·9667675038·3595333248	68
8602·4111073466·0092710580	72
22273·9068076872·9820388352	76
39350·9959008035·4173030112	80
47557·4740865723·2763578880	84

n=176	
1	0
1795555300	32
51·0825469440	36
8566·9933912640	40
860476·9428057600	44
53432271·3203704425	48
2105517330·2285337600	52
5·3778712060·0587763840	56
90·5941959566·2783226880	60
1020·9334862912·4662016350	64
7786·0399257528·6718579200	68
40558·4498446433·6108337600	72
145359·7013087919·5398912512	76
360391·5671092513·1155424340	80
620474·6047546118·0564838400	84
743475·1292567009·7822508800	88

n=184	
1	0
521332812	32
18·9454896384	36
3974·7982400504	40
503015·2585975296	44
39629129·9765668216	48
1995739666·9226585856	52
6·5654101729·6827297297	56
143·6315990341·1816340480	60
2120·7034666204·0308706550	64
21391·1934330300·1678644480	68
148879·9325941144·8475067080	72
720720·5618879136·5661706752	76
2442121·1445081548·7193234248	80
5819733·0447223791·5285331712	84
9786260·3761511921·1340708140	88
11634846·8566948262·3348705280	92

Table continued

TABLE *(continued)*

n=192

1	0
6•9065734464	36
1668•1003659936	40
263818•1865286080	44
26011870•7412159120	48
1650620412•8755716672	52
6•8891956345•8768198624	56
192•3156702196•3529559744	60
3662•9234679278•3194741815	64
47982•3029129154•9388046400	68
437537•3270369432•0252103840	72
2801442•7417808971•5889150656	76
12682897•0918971772•1455882224	80
40824643•7392952797•3794806080	84
93822240•3866579312•9097020640	88
154396045•6403677997•4450436032	92
182248321•4906983687•7698945680	96

n=200

1	0
2•1005534550	36
646•7522952660	40
125297•5498471200	44
15287262•0852751800	48
1206936450•5468120400	52
6•3061514767•0747950200	56
222•1591577969•8502141280	60
5359•9985166299•6527356550	64
89733•1217536072•4436541800	68
1053884•6782935099•5361897825	72
8763102•7466336654•8170765600	76
51978949•1575731101•3178267720	80
221292819•4255035083•6000132400	84
679496375•8320473071•3462120200	88
1510377799•7026804996•1942408800	92
2436591083•1314624778•4654076100	96
2857207329•5182769043•0040227204	100

n = 256

1	0
81•5550677760	44
33706•7577283360	48
9427197•0895660800	52
1798287443•9644012032	56
23•8542954832•3567173120	60
2238•5884204514•7954264620	64
150828•4455480530•3640645120	68
7389744•2146785696•2342366720	72
266206617•0725206080•6263057152	76
7119266411•4446504138•7096346272	80
14•2536162882•6739768348•4469876480	84
215•2117330790•6063595407•4076846080	88
2466•1642294029•7641248565•8537134080	92
21566•9482758782•5232703692•5022134080	96
144620•3933891460•9893983218•9770909696	100
746630•7582377592•9521023097•5874856960	104
2977839•1982437159•5752588037•0387043840	108
9201125•7461996373•8123501375•3162661440	112
22075361•7026193050•4385578928•5509721600	116
41195923•3273193846•8520953898•6394444800	120
59870289•7233756335•4620771908•3818931200	124
67810258•5878568295•7328259340•8656117030	128

Note added in proof. J.-M. Goethals has communicated to us that (in Case 1) an extremal self-dual code exists for $n = 22$ but does not exist for $n = 26$.

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