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An Upper Bound for the Minimum Diameter of Integral Point Sets

Heiko Harborth, Arnfried Kemnitz, and Meinhard Möller

Diskrete Mathematik, Technische Universität Braunschweig,
W-3300 Braunschweig, Federal Republic of Germany

Abstract. For all $n > d$ there exist n points in the Euclidean space E^d such that not all points are in a hyperplane and all mutual distances are integral. It is proved that the minimum diameter of such integral point sets has an upper bound of $2^{c \log n \log \log n}$.

1. Introduction

Sets of n points in the Euclidean space E^d, not all in a hyperplane, do exist for all $n > d$ such that all mutual distances are integral. Proofs are published in [10] and [12] for plane concyclic sets, and in [3] and [4] for sets in higher dimensions, in addition in [4] no three points are allowed to be collinear.

First upper bounds for the minimum diameter $k(d, n)$ of such integral sets of n points in E^d are given in [3] where the diameter means the largest distance of two points of the set. These bounds are of order 2^n for any fixed dimension.

In this note we construct concyclic integral point sets which improve the upper bound of $k(2, n)$ to $2^{c \log n \log \log n}$, that is $n^{c \log \log n}$. Then these constructions are used to find corresponding results for $d \geq 3$. Some exact values are presented for small numbers of points.

2. Construction of Plane Integral Point Sets

We obtain a concyclic integral point set by appropriate rotations of an equilateral triangle around its center.

Theorem 1. *On a circle of diameter* $\frac{2}{3}R\sqrt{3}$ *there exist* $n = 3\tau(R) = 3\prod_{j=1}^{m}(v_j + 1)$ *points with pairwise integral distances if* $R = \prod_{j=1}^{m} p_j^{v_j}$ *for any* m *distinct prime numbers* $p_j \equiv 1 \pmod 3$.

Proof. We consider the Euclidean quadratic number field $K(\rho)$ with $\rho = (-1 + \sqrt{-3})/2$. In $K(\rho)$ the rational primes $p \equiv 1 \pmod 3$ have a unique prime factorization $p = \omega\bar{\omega}$ where $\omega = a + b\rho$ and $\bar{\omega} = a + b(-1 - \sqrt{-3})/2 = a - b - b\rho$ are conjugate primes of $K(\rho)$ with rational integers a and b. For any element $\alpha = a + b\rho$ of $K(\rho)$ its norm is $\alpha\bar{\alpha} = |\alpha|^2 = a^2 - ab + b^2$ (see Chapter 12 of [6], or [7]).

For each of the $\tau(R)$ divisors of R,

$$\prod_{j=1}^{m} p_j^{u_j} = \prod_{j=1}^{m} \omega_j^{u_j}\bar{\omega}_j^{u_j}, \qquad 0 \le u_j \le v_j,$$

and with

$$\eta_{3h} = \prod_{j=1}^{m} \omega_j^{v_j + u_j}\bar{\omega}_j^{v_j - u_j}, \qquad \eta_{3h-1} = \rho\eta_{3h},$$
$$\eta_{3h-2} = \rho^2\eta_{3h}, \qquad 1 \le h \le \tau(R),$$

we consider the vertices ξ_s, $1 \le s \le 3\tau(R)$, of $\tau(R)$ equilateral triangles with their centers in the origin and with circumradius $\frac{1}{3}R\sqrt{3}$, where

$$\xi_{3h-k} = \frac{\sqrt{3}}{3R}\eta_{3h-k}^2, \qquad k = 0, 1, 2, \quad 1 \le h \le \tau(R).$$

All vertices ξ_s are pairwise distinct since the prime factorization in $K(\rho)$ is unique.

For

$$\eta_s = \rho^{k_s}\eta_{3h} = x_s + y_s\rho, \qquad 0 \le k_s \le 2, \quad 1 \le s \le 3\tau(R),$$

and with rational integers x_s, y_s we conclude

$$|\eta_s|^2 = \eta_s\bar{\eta}_s = x_s^2 - x_s y_s + y_s^2 = \prod_{j=1}^{m} \omega_j^{2v_j}\bar{\omega}_j^{2v_j}$$
$$= \prod_{j=1}^{m} p_j^{2v_j} = R^2.$$

This implies that all points ξ_s are concyclic with radius $(R/3)\sqrt{3}$.

We use

$$\eta_s^2 = R^2 - \tfrac{3}{2}y_s^2 + i\sqrt{3}(x_s y_s - \tfrac{1}{2}y_s^2)$$

with $i = \sqrt{-1}$ to determine the distance of any two points ξ_s and ξ_t, $1 \le s$, $t \le 3\tau(R)$. We obtain

$$\begin{aligned} R^2|\xi_t - \xi_s|^2 &= \tfrac{1}{3}|\eta_t^2 - \eta_s^2|^2 \\ &= \tfrac{1}{3}|\tfrac{3}{2}(y_s^2 - y_t^2) + i\sqrt{3}(\tfrac{1}{2}(y_s^2 - y_t^2) + x_t y_t - x_s y_s)|^2 \\ &= (y_s^2 - y_t^2)^2 + (y_s^2 - y_t^2)(x_t y_t - x_s y_s) + (x_t y_t + x_s y_s)^2 \\ &= (y_s^2 - y_t^2)(R^2 - x_s^2 - (R^2 - x_t^2)) + (x_t y_t - x_s y_s)^2 \\ &= (x_t y_s - x_s y_t)^2 \end{aligned}$$

which implies

$$|\xi_t - \xi_s| = \frac{1}{R}|x_t y_s - x_s y_t|.$$

Since

$$\begin{aligned}\eta_s\bar{\eta}_t &= (x_s + y_s\rho)(x_t + y_t\bar{\rho})\\ &= x_s x_t - x_s y_t + y_s y_t + \rho(x_t y_s - x_s y_t),\end{aligned}$$

all distances $|\xi_t - \xi_s|$ are integers if $\eta_s\bar{\eta}_t$ and thus $x_t y_s - x_s y_t$ is divisible by R. We obtain

$$\begin{aligned}\eta_s\bar{\eta}_t &= \rho^{k_s}\bar{\rho}^{k_t}\prod_{j=1}^{m}\omega_j^{v_j+u_j}\bar{\omega}_j^{v_j-u_j}\prod_{j=1}^{m}\bar{\omega}_j^{v_j+w_j}\omega_j^{v_j-w_j}\\ &= \rho^{k_s-k_t}\prod_{j=1}^{m}\omega_j^{2v_j+u_j-w_j}\bar{\omega}_j^{2v_j-u_j+w_j}\\ &= \rho^{k_s-k_t}\prod_{j=1}^{m}p_j^{v_j}p_j^{v_j-(u_j+w_j-2q_j)}\omega_j^{2(u_j-q_j)}\bar{\omega}_j^{2(w_j-q_j)}\\ &= R(A + B\rho),\end{aligned}$$

where $q_j = \min(u_j, w_j)$, and A and B are rational integers since the exponents of the last product are nonnegative. Therefore R divides $\eta_s\bar{\eta}_t$, and the proof of Theorem 1 is complete. □

3. An Upper Bound for $k(2, n)$

We use Theorem 1 to get the following upper bound for the minimum diameter $k(2, n)$.

Theorem 2.

$$k(2, n) < 2^{c\log n\log\log n}, \qquad c = \textit{constant}.$$

Proof. We choose the first m prime numbers $p_j \equiv 1 \pmod 3$ and $v_j = 1$ for $1 \le j \le m$ to get, from Theorem 1,

$$k(2, n) < \tfrac{2}{3}\sqrt{3}\prod_{j=1}^{m}p_j \quad \text{and} \quad n = 3\cdot 2^m.$$

We use the well-known results from number theory (see, for example, Chapters 1 and 22 of [6]) that

$$\prod_{j=1}^{m}p_j < \prod_{j=1}^{c_1 m}q_j, \qquad q_j = j\text{th prime number}, \quad q_j < c_2 j\log j,$$

and

$$\prod_{q \le x} q < c_3^x, \qquad q \text{ is a prime number},$$

with constants c_1, c_2, c_3. We get

$$k(2, n) < \tfrac{2}{3}\sqrt{3} \prod_{q \le c_1 c_2 m \log(c_1 m)} q < 2c_3^{c_1 c_2 m \log(c_1 m)}.$$

Together with $m < c_4 \log n$ and constants c, c_4, c_5 it follows that

$$k(2, n) < c_3^{c_5 \log n \log \log n} = 2^{c \log n \log \log n}. \qquad \square$$

4. Higher Dimensions

The construction used for Theorem 1 also implies an upper bound $k(d, n)$ in general.

Theorem 3.

$$k(d, n) < 2^{c \log n \log \log n}, \qquad c = \textit{constant}.$$

Proof. We choose, corresponding to Theorem 1, $n - d + 2$ points with pairwise integral distances on a circle of radius $R\sqrt{3}$. The remaining $d - 2$ points are chosen as follows: We consider a regular $(d - 2)$-simplex of side length R in that subspace orthogonal to the plane of the circle so that one of its vertices is the center point of the circle. Then all other $d - 2$ vertices of this $(d - 2)$-simplex have distance $2R$ to all points of the circle. The diameter of these n points is less than $2R\sqrt{3}$, and Theorem 2 completes the proof. □

For $d \ge 3$ we know other constructions of integral point sets with smaller diameters, for example, if several two-dimensional circular point sets of Theorem 1 are combined, however, no construction is known to us which implies a smaller order of magnitude.

5. Exact Values for Small Point Sets in the Plane

No reasonable lower bounds are known in general. For the plane the following exact values of $k(2, n)$ are determined with the aid of a computer:

$$k(2, n) = 1, 4, 7, 8, 17, 21, 29 \qquad \text{for} \quad n = 3, 4, \ldots, 9;$$

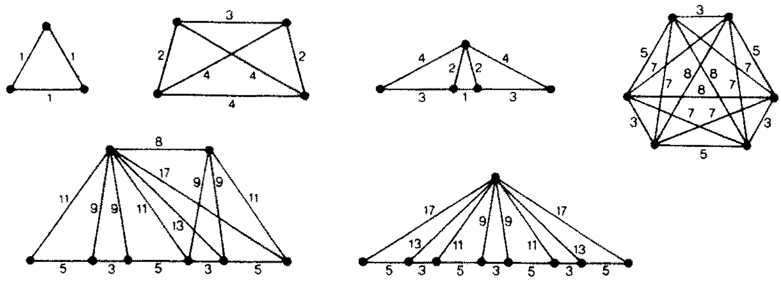

Fig. 1

see Fig. 1 for examples, where the example for $n = 7$ follows from that for $n = 8$ by deletion of one of the two points at distance 21.

The construction in the proof of Theorem 1 always gives concyclic integral point sets. In general, we have determined by computer, for $n \leq 9$, the smallest integral point sets without any three being collinear; see Fig. 2 for those which are not in Fig. 1 (see also [11]). An example for $n = 8$ follows from that for $n = 9$ by deletion of one point.

For $n > 9$ upper bounds follow from constructions by Theorem 1. Some upper bounds, for example for $k(2, 12)$ and $k(2, 24)$, are already given in [1].

If no three points are collinear and, moreover, no four concyclic, then the question for integral point sets is a so-called Erdös problem (see also Section D20 of [2]). For $n \geq 7$ even the existence of such special integral point sets is unknown. The smallest diameters for $n = 4, 5$, and 6 are 8, 73, and 174 (see [3], [4], and [9]).

Another modified problem is proposed in [8]: to find integral point sets with the minimum sum of all distances.

For $d \geq 3$ nearly no examples of minimum integral point sets are known. Some first results can be found in [5].

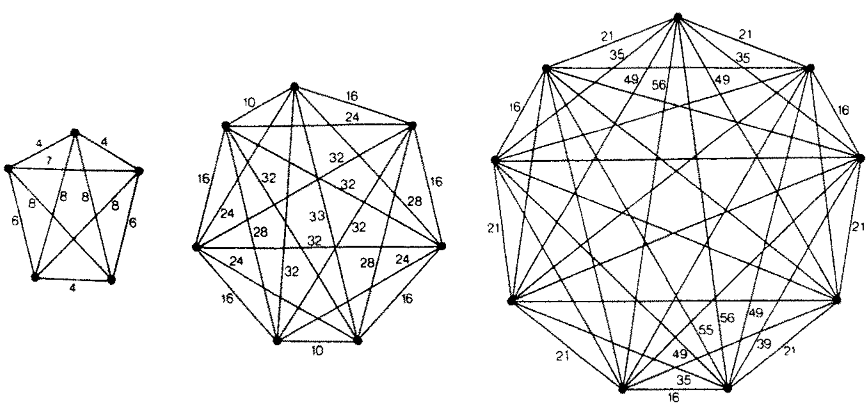

Fig. 2

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