AN UPPER BOUND TO THE SPECTRUM OF $\varDelta$ ON A MANIFOLD OF NEGATIVE CURVATURE

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1. Introduction

The spectrum of the standard Laplace operator $\varDelta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ acting in $L^2 (R^2, dxdy)$ is the whole left half-line $(-\infty, 0]$. By contrast, the spectrum of the Laplace operator $\varDelta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ for the hyperbolic plane $H = R^1 \times (0, \infty)$, acting in the appropriate space $L^2(H, y^{-2}dxdy)$ reaches only up to $-1/4$.

The proof is easy. To see that the spectrum lies to the left of $-1/4$, take a compact function $f \in C^\infty(0, \infty)$, and conclude from

$$\frac{1}{4} \left(\int_0^\infty f^2 y^{-2} dy \right)^2 = \frac{1}{4} \left(\int_0^\infty y^{-1} df^2 \right)^2 = \left(\int_0^\infty y^{-1} ff' dy \right)^2$$

$$\leq \int_0^\infty y^{-2} f^2 dy \int_0^\infty (f')^2 dy$$

that

$$\frac{1}{4} \int_0^\infty f^2 y^{-2} dy \leq \int_0^\infty (f')^2 dy = -\int_0^\infty f^2 f'' y^{-2} dy \ .$$

This bound is applied to a compact function $f \in C^\infty(H)$ viewed as a function of $y > 0$ for fixed $x \in R^1$, and the result is integrated from $-\infty$ to ∞ with regard to x. This gives

$$\frac{1}{4} \int_H f^2 y^{-2} dxdy \leq -\int_H f\varDelta f y^{-2} dxdy \ ,$$

proving that the spectrum of $\varDelta$ lies to the left of $-1/4$.

To see that the spectrum fills up the half-line $(\infty, -1/4]$, we can use the fact that if l is hyperbolic distance from $\sqrt{-1}$, then the so-called conical function

$$f_{-1/2 + \sqrt{-1}c}(\text{ch } l) = \int_0^1 (\text{ch } l + \text{sh } l \sin 2\pi x)^{-1/2 + \sqrt{-1}c} dx$$

is an eigenfunction of Δ with eigenvalue $-(1/4 + c^2)$ for any $c \geq 0$, and

$$f_{ab} = \int_a^b f_{-1/2 + \sqrt{-1}c}\, dc$$

lies in $L^2(H, y^{-2}dxdy)$ for any $0 < a < b < \infty$. The fact that $-1/4$ is a sharp bound can also be read off the formula for the fundamental solution of $\partial p/\partial t = \Delta p$ with pole at $\sqrt{-1}$:

$$p(t, z) = (4\pi t)^{-3/2} e^{-t/4} \sqrt{2} \int_l^\infty \frac{x e^{-x^2/4t}\, dx}{\sqrt{\operatorname{ch} x - \operatorname{ch} l}},$$

l being the hyperbolic distance from z to $\sqrt{-1}$. The point is that

$$\lim_{t \uparrow \infty} t^{-1} \ln p = -1/4 .$$

The aim of this paper is to extend the above to simply-connected manifolds of negative curvature. The exact statement is the following.

Consider a smooth, n-dimensional, simply-connected Riemannian manifold M with negative sectional curvatures K bounded away from 0: specifically, suppose $K \leq k$ for some constant $k < 0$. Then the spectrum of the corresponding Laplace-Beltrami operator Δ acting in $L^2(M, \sqrt{g}\, dx)$ is also bounded from 0: specifically, the top of the spectrum lies to the left of $(n - 1)^2 k/4 < 0$, and this bound is sharp.

The proof is carried out in §§ 2, 3, and 4 below. The principal tool is the so-called index form of Morse theory, connecting the second fundamental form and the curvature, with which the author became acquainted through the kindness of J. Cheeger and J. Simons. An alternative proof for surfaces ($n = 2$) is presented in § 5.

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2. Reducing the problem

The first point to be clarified is the meaning of Δ. Denote the first fundamental form of M by (g_{ij}), and let g be its determinant and (g^{ij}) its inverse. Then

$$\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i} g^{ij} \sqrt{g} \frac{\partial}{\partial x_j}$$

is a negative symmetric operator on $C^\infty(M)$ which extends to a negative self-adjoint operator on $L^2(M, \sqrt{g}\, dx)$. The top of the spectrum of this operator

is the supremum

$$\gamma = \sup \int f \Delta f \sqrt{g}\, dx : f \text{ compact and smooth with } \int f^2 \sqrt{g}\, dx = 1 \ .$$

Recall that a simply-connected, negatively-curved, n-dimensional Riemannian manifold is homeomorphic to R^n. This homeomorphism may be carried out by fixing a point $0 \in M$ and mapping the corresponding tangent space (R^n) onto M via the exponential map. This provides M with a global system of (polar) coordinates $x_1 = r \geq 0$ and $(x_2, \cdots, x_n) \in S^{n-1}$, in which

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}.$$

This system of coordinates is preferred from now on.

To prove the stated bound for the spectrum of Δ [$\gamma \leq (n-1)^2 k/4$], it suffices to check that

$$(1) \qquad \frac{1}{4}(n-1)^2 |k| \int f^2 \sqrt{g}\, dx \leq -\int f \Delta f \sqrt{g}\, dx = +\int |df|^2 \sqrt{g}\, dx$$

for any compact $f \in C^\infty(M)$. By the special form of (g_{ij}), it is clear that (1) would follow from

$$(2) \qquad \frac{1}{4}(n-1)^2 |k| \int_0^\infty f^2 \sqrt{g}\, dr \leq \int_0^\infty \left(\frac{\partial f}{\partial r}\right)^2 \sqrt{g}\, dr \ .$$

As before, f is a compact function from $C^\infty(M)$, but now it (and $\sqrt{g}$ also) is viewed as a function of $0 \leq r < \infty$ alone. (1) is obtained from (2) simply by integrating both sides over $(x_2, \cdots, x_n)$ and noticing that $(\partial f / \partial r)^2 \leq |df|^2$.

The proof of (2) is broken into two steps. The first is to prove a purely 1-dimensional fact:

$$(3) \qquad \int_0^\infty (f')^2 \sqrt{g}\, dr \Big/ \int_0^\infty f^2 \sqrt{g}\, dr \geq \frac{1}{4} \inf (\sqrt{g}'')/\sqrt{g} \ .$$

The second step is to estimate this below by $(n-1)^2 |k|/4$. This follows by a consideration of the index form, as will be seen in § 4. But first the proof of (3).

3. A lemma

The problem is to show that *if $\sqrt{g} > 0$ is a smooth function of $r \geq 0$ and if*

$$(4) \qquad K \equiv (\sqrt{g}'')/\sqrt{g} \text{ is bounded below by a constant } k > 0 ,$$

then the infemum of the left-hand side of (3) *is bounded below by* $k/4$.

This infemum is the limit as $R \uparrow \infty$ of the smallest eigenvalue α of the problem

(5)
$$(f'\sqrt{g})' + \alpha f \sqrt{g} = 0 \, ,$$
$$f(0) = f(R) = 0 \, ,$$

so it is enough to prove that $\alpha \geq k/4$. Define a new scale so that $f'^{\text{new}} = \sqrt{g} \, f'^{\text{old}}$. (5) becomes

(6)
$$f'' + \alpha f g = 0 \, ,$$
$$f(0) = f(R) = 0 \, ,$$

and (4) takes the form

(7)
$$K = g^{-1}(\ln \sqrt{g})'' \geq k \, .$$

To estimate α, perform a smooth change in g so as make $K = g^{-1}(\ln \sqrt{g})''$ decrease to the constant value k: Specifically, make $g = e^h$ depend smoothly upon the parameter $0 \leq t \leq 1$, keeping $g(0)$ and $g(R)$ fixed, and make

(8)
$$K = (1/2)e^{-h}h''$$

decrease to k as $t \uparrow 1$. By standard perturbation arguments, the principal eigenvalue of problem (5) is a (smooth) function of the parameter, as is the associated eigenfunction f; to prove this, one uses the fact that this principal eigenvalue is *simple*. By (8),

(9)
$$0 \geq K^{\cdot} = \tfrac{1}{2}e^{-h}(h''' - h''h^{\cdot}) \, ,$$
$$h^{\cdot}(0) = h^{\cdot}(R) = 0 \, ,$$

and so $h^{\cdot} \geq 0$, where a spot on the shoulder stands for differentiation with regard to the parameter $0 \leq t \leq 1$. By (6),

(10)
$$f''' + \dot\alpha f e^h + \alpha f^{\cdot} e^h + \alpha f e^h h^{\cdot} = 0 \, ,$$
$$f^{\cdot}(0) = f^{\cdot}(R) = 0 \, .$$

This formula is multiplied by f and integrated to obtain

(11)
$$0 = \dot\alpha \int f^2 e^h + \alpha \int f^2 e^h h^{\cdot} \, ,$$

and $\dot\alpha \leq 0$ follows. Therefore, α is diminished by pushing H down to its lower bound k and the rest of the proof is to check that $\alpha \geq k/4$ in case $K \equiv k$.

For this purpose, it is convenient to abandon (6) and (7) in favor of (4) and (5). Then $(\sqrt{g''}) = k\sqrt{g}$, and up to translation, reflection, and magnification, three cases arise according as

$$\sqrt{g} = \exp(\sqrt{k}\, r)\,, \quad \text{or} \quad \mathrm{sh}\,(\sqrt{k}\, r)\,, \quad \text{or} \quad \mathrm{ch}\,(\sqrt{k}\, r)\,.$$

The second case is typical: $\alpha \geq k/4$ follows from

$$\frac{k}{4}\left(\int f^2\sqrt{g}\,dr\right)^2 \leq \frac{k}{4}\left(\int f^2\,\mathrm{ch}\,(\sqrt{k}\,r)dr\right)^2 = \frac{1}{4}\left(\int f^2 d\,\mathrm{sh}\,(\sqrt{k}\,r)\right)^2$$

$$= \left(\int \mathrm{sh}\,(\sqrt{k}\,r)ff'dr\right)^2 \leq \int f^2\,\mathrm{sh}\,(\sqrt{k}\,r)dr \int (f')^2\,\mathrm{sh}\,(\sqrt{k}\,r)dr$$

$$= \int f^2\sqrt{g}\,dr \int (f')^2\sqrt{g}\,dr = -\int f^2\sqrt{g}\,dr \int f(f'\sqrt{g})'dr$$

$$= \alpha\left(\int f^2\sqrt{g}\,dr\right)^2\,.$$

4. Application of the index form

The second step of the proof is to check the underestimate

$$(12) \qquad\qquad \frac{1}{\sqrt{g}}\frac{\partial^2\sqrt{g}}{\partial r^2} \geq (n-1)^2|k|$$

under the assumption that the sectional curvatures of M are bounded above by the constant $k < 0$.

Recall the first fundamental form $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$ and bring in the second fundamental form defined by

$$f_{ij} = \tfrac{1}{2}g^{il}\,\partial g_{lj}/\partial r \qquad (i,j \geq 2)\,.$$

An easy computation verifies that for $2 \leq j \leq n$ the sectional curvature of the $1j$ submanifold can be expressed as the jj entry of $-\partial f/\partial r - f^2$:

$$(13) \qquad\qquad -R^{1j}{}_{1j} = -\partial f_{jj}/\partial r - (f^2)_{jj} \qquad (j \geq 2)\,.$$

By assumption, each of these numbers is $\leq k < 0$. Besides,

$$(14) \qquad \frac{1}{\sqrt{g}}\frac{\partial^2\sqrt{g}}{\partial r^2} = \frac{\partial^2 \ln\sqrt{g}}{\partial r^2} + \left(\frac{\partial \ln\sqrt{g}}{\partial r}\right)^2 = \frac{\partial}{\partial r}\,\mathrm{sp}\,f + (\mathrm{sp}\,f)^2\,,$$
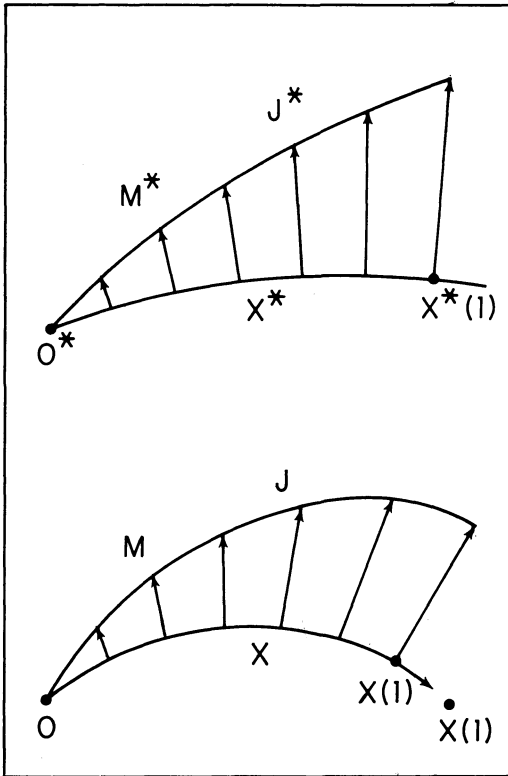
where sp means *spur* or *trace*.

By (13),

$$\frac{\partial}{\partial r} \operatorname{sp} f + \operatorname{sp} (f^2) \geq (n-1)|k| \, ,$$

so by (14),

(15) $$\frac{1}{\sqrt{g}} \frac{\partial^2 \sqrt{g}}{\partial r^2} \geq (n-1)|k| + (\operatorname{sp} f)^2 - \operatorname{sp} (f^2) \, ,$$

and if we bring f into diagonal form, we will easily see that for the proof of (12) it suffices to verify that all the eigenvalues of f are $\geq \sqrt{-k}$. This is where the index form comes in, as a means of expressing f in terms of the curvature.

Pick a point $x(1) \in M$, let $x(t)$: $0 \leq t \leq 1$ be the geodesic joining it to $x(0) = 0$, and let $J(1)$ be tangent to M at $x(1)$ and perpendicular to $\dot{x}(1)$, as in the Diagram. $J(1)$ can be extended along x to a Jacobi field $J(t)$: $0 \leq t \leq 1$ joining $J(1)$ and $J(0) = 0$. This means that among all tangent fields T along x, subject to $T(0) = 0$ and $T(1) = J(1)$, the present field J makes the index form

$$I(T) = \int_0^1 (|T|^2 - K|T|^2) dt$$

as small as possible, K being the curvature of the 2-dimensional submanifold tangent to T and $\dot{x}$. The identity of the index form $I(J)$ and the second fundamental form

$$f[J(1)] = \sum_{i,j \geq 2} f_{ij}[x(1)]J_i(1)J_j(1)$$

is a standard fact; see, for instance, [1, p. 219].

Now take a manifold M^* of constant curvature k, fix a point $0^* \in M^*$, and lift up x to a geodesic x^* via the exponential map, as in the Diagram. An isometry is now established between the tangent spaces at $x(1)$ and $x^*(1)$, mapping $x^{\cdot}(1)$ into $(x^*)^{\cdot}(1)$, and this isometry is extended along x and its lift x^*

by parallel transport. J is thereby lifted up to a (non-Jacobi) field J^* perpendicular to $x^{*\cdot}$, and with this picture in mind, the second fundamental form $f[J(1)]$ may be compared to the second fundamental form $f^*[J^*(1)]$ at $x^*(1)$ as follows:

$$f[J(1)] = I(J) = \int_0^1 (|\dot{J}|^2 - K|J|^2) dt$$

$$= \int_0^1 (|J^{*\cdot}|^{*2} - K|J^*|^{*2}) dt \geq \int_0^1 (|J^{*\cdot}|^{*2} - k|J^*|^{*2}) dt$$

$$= \int_0^1 (|J^{*\cdot}|^{*2} - K^*|J^*|^{*2}) dt = I(J^*) \geq I(J^{**}) = f^*[J^*(1)] \ ,$$

J^{**} being the Jacobi field along x^* joining $J^{**}(1) = J^*(1)$ to $J^{**}(0) = 0$.

But for the constantly curved space M^*, the second fundamental form is easily computed. A convenient model is $M^* = R^n$ provided with the first fundamental form:

$$(1 + k r^2/4)^{-2} \times \text{the identity}, \quad r^2 = x_1^2 + \cdots + x_n^2 < 4|k|^{-1} \ .$$

Take a new coordinate $dx_1^* = (1 + k r^2/4)^{-1} dr$, and let $x_2^*, \cdots, x_n^*$ be the remaining angular variables running over S^{n-1}. The first fundamental form is now

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2(1 + k r^2/4)^{-2} \times \text{identity} \end{pmatrix} \ ,$$

while the second fundamental form is the $(n-1)$-dimensional identity form multiplied by

$$\tfrac{1}{2} r^{-2}(1 + kr^2/4)^2 [r^2(1 + kr^2/4)^{-2}]' \geq r^{-1}(1 - kr^2/4)$$
$$= (r^{-1/2} - \tfrac{1}{2}\sqrt{-k}\, r^{+1/2})^2 + \sqrt{-k} \geq \sqrt{-k} \ .$$

But now, by (16),

$$f[J(1)] \geq f^*[J^*(1)] \geq \sqrt{-k}\, |J^*(1)|^{*2} = \sqrt{-k}\, |J(1)|^2 \ ,$$

and this proves the desired result that all the eigenvalues of f are $\geq \sqrt{-k}$.

5. Special proof for surfaces

A considerable simplification takes place if M is a surface ($n = 2$). In this case, any geodesic ball B of M can be covered with a single patch provided with isothermal coordinates, which means that the first fundamental form is

the identity times $\sqrt{g}$, and the curvature is expressed as

(17) $K = (-1/2)\Delta \ln \sqrt{g}$.

The proof that the spectrum of Δ lies to the left of $k/4$ can now be carried out much as in § 3.

Make the first fundamental form depend smoothly upon a parameter $0 \le t \le 1$, and let f and γ be the principal eigenfunction and eigenvalue of Δ in B: $\Delta f = \gamma f$ inside B and $f = 0$ on ∂B. Take $0 \le h \in C^\infty(B)$, $h = 0$ on ∂B, and $g_{ij}' = hg$. By a simple computation, we find

$$\dot{\gamma} \int f^2 \sqrt{g} = (-\gamma n/2) \int h f^2 \sqrt{g} + (1 - n/2) \int h |df|^2 \sqrt{g} ,$$

especially, for $n = 2$,

(18) $-\dot{\gamma}/\gamma = \int h f^2 \sqrt{g} \Big/ \int f^2 \sqrt{g}$.

By (17), $(\Delta + 2K)h = -2\dot{K}$, and so we can drive K up to its bound $k < 0$, keeping $h \ge 0$ and hence, by (18), keeping also $\dot{\gamma} \ge 0$. Accordingly, it is enough to prove the bound $\gamma \le k/4$ for surfaces of constant curvature; the proof of this was already indicated in § 1.

Bibliography

[1]   R. Bishop & R. Crittenden, *Geometry of manifolds*, Academic Press, New York, 1964.

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