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ANALOGUE OF DINI-RIEMANN THEOREM FOR NON-ABSOLUTELY CONVERGENT INTEGRALS

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An analogue of classical Dini-Riemann theorem related to non-absolutely convergent series of real number is proved for the Lebesgue improper integral.

The classical Dini-Riemann theorem (see [3]) states that if a series of real numbers is non-absolutely convergent, then it can be so rearranged, that the new series converges to an arbitrary assigned sum. If one want to obtain an analog of this theorem for the non-absolute convergent integral it is natural to use a measure preserving mapping instead of permutation. It is important to note that this analog is not true for some non-absolute integrals. For example the Kolmogorov A-integral (see [1] and [2]) being non-absolute is known to be invariant under measure preserving mapping. So the problem arise for which non-absolute integrals this analogue is true.

In this paper we prove the analogue of Dini-Riemann theorem for the Lebesgue improper integral. We do it by presenting the direct construction of measure preserving mapping that change the value of the integral.

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Theorem 0.1. *Let a measurable function* $f : [0,1] \to \mathbb{R}$ *be Lebesgue integrable on* $[0,\eta]$ *for any* η, $0 < \eta < 1$ *and let* $\lim_{\eta\to 1}\int_0^\eta$ *exist and be finite, with*

$$(L)\int_0^1 |f| d\mu = +\infty \tag{1}$$

Then for any $\xi \in \mathbb{R}$ *there exists a measure preserving mapping* $\psi_\xi : [0,1] \to [0,1]$ *such that* $f(\psi_\xi(x))$ *is also* L*-integrable on* $[0,\eta]$ *for any* η , $0 < \eta < 1$ *and* $\lim_{\eta\to 1}\int_0^\eta f(\psi_\xi) d\mu = \xi$

Proof. Define $A^+ = \{x \in [0,1] : f(x) \geq 0\}$ and $A^- = \{x \in [0,1] : f(x) < 0\}$. It follows from (1) and from existence of improper integral that

$$(L)\int_{A^+} f d\mu = (L)\int_{A^-} (-f) d\mu = +\infty. \tag{2}$$

Choose a series $\sum c_n = \xi$, with $c_n \in \mathbb{R}$ such that $\sum |c_n| = +\infty$. We split the sequence $\{c_n\}$ in two subsequences $\{c_{n_j}\}$ and $\{c_{m_i}\}$ where $c_{n_j} > 0$ and $c_{m_i} < 0$. We construct by induction two increasing sequences $\{\alpha_j\}$ and $\{\beta_i\}$ convergent to 1.

We start with defining $\alpha_1 = \inf\{\alpha > 0 : \int_{(0,\alpha)\cap A^+} f d\mu = c_{n_1}\}$, then if α_{j-1} is already defined we put $\alpha_j = \inf\{\alpha > \alpha_{j-1} : \int_{(\alpha_{j-1},\alpha)\cap A^+} f d\mu = c_{n_j}\}$; similarly we define $\beta_1 = inf\{\beta > 0 : \int_{(0,\beta)\cap A^-} f d\mu\} = c_{m_1}$ and assuming that β_{i-1} are already defined we put $\beta_i = \inf\{\beta > \beta_{i-1} : \int_{(\beta_{i-1},\beta)\cap A^-} f d\mu = c_{n_i}\}$.

We introduce the following numbers: $k^-(j) = |\{i : m_i < n_j\}|$ and $k^+(i) = |\{j : n_j < m_i\}|$ where $|M|$ denote the cardinality of the set M.

We note that by the previous notation we can get the position of the element c_{n_j} in the sequence $\{c_n\}$. In fact $n_j = j + k^-(j)$ and in the same way for the element c_{m_i} we have $m_i = i + k^+(i)$.

Now let E^+ (respectively E^-) be the subset of all the points of A^+ (or A^-) which are of density 1 of A^+ (respectively A^-). To simplify the notation we denote $E^+ \cap (\alpha_{j-1}, \alpha_j) = E_j^+$ and $E^- \cap (\beta_{i-1}, \beta_i) = E_i^-$. On the set $E^+ \cup E^-$ we define the following function φ in this way:

$$\varphi(x) = \begin{cases} \mu(E^+ \cap (0,x)) + \mu(E^- \cap (0, \beta_{k^-(j)})) & \text{if} \quad x \in E_j^+, \\ \mu(E^- \cap (0,x)) + \mu(E^+ \cap (0, \alpha_{k^+(j)})) & \text{if} \quad x \in E_i^-. \end{cases}$$

In other words if $x \in E_j^+$ then $\varphi(x) = \sum_{l=1}^{j-1} \mu(E_j^+) + \sum_{p=1}^{k^-(j)} \mu(E_p^-) +$

$\mu(E_j^+ \cap (\alpha_{j-1}, x))$ and similarly if $x \in E_i^-$, then $\varphi(x) = \sum_{l=1}^{i-1} \mu(E_i^-) + \sum_{p=1}^{k^+(i)} \mu(E_p^+) + \mu(E_i^- \cap (\beta_{i-1}, x))$.

On each set E_j^+ (respectively E_i^-) the function φ is strictly increasing because from the definition of density point it follows that if x_1, x_2 are different points of E_j^+ (or E_i^-) then $\mu((x_1, x_2) \cap E^+) > 0$ (and $\mu((x_1, x_2) \cap E^-) > 0$.

Let $t_0 = 0$,

$$t_{n_j} = \lim_{x \to \alpha_j - 0, x \in E^+} \varphi(x) \quad \text{and} \quad t_{m_i} = \lim_{x \to \beta_i - 0, x \in E^-} \varphi(x).$$

Note that $t_{n_j} = \sum_{l=1}^{j} \mu(E_l^+) + \sum_{p=1}^{k^-(j)} \mu(E_p^-)$ and $t_{m_i} = \sum_{r=1}^{i} \mu(E_r^-) + \sum_{s=1}^{k^+(i)} \mu(E_s^+)$. Therefore t_{n_j+1} could be either $t_{n_{j+1}}$ or $t_{m^-(j)+1}$ but in both cases, because of above notations, the value of t_{n_j+1} is strictly greater than t_{n_j}. The same is true if we compare t_{m_i+1} with t_{m_i}.

So the sequence $\{t_n\}$ is strictly increasing with $\varphi(E_j^+) \subset [t_{n_j-1}, t_{n_j}]$ and $\varphi(E_i^-) \subset [t_{m_i-1}, t_{m_i}]$. So, the images of E_j^+ for different j as well as E_i^- are non-overlapping. Therefore φ, is one-to-one, as a mapping from $E^+ \cup E^-$ onto the set $\varphi(E^+ \cup E^-)$.

Now we show that φ is measure preserving mapping. Because of σ-additivity of the measure and because the sets E_j^+, $j = 1, 2, ...,$, E_i^-, $i = 1, 2, ...$ and also their images are mutually disjoints, it is enough to prove that φ is measure preserving mapping on each E_j^+ and each E_i^-. We prove it for the first set, the proof for the other one is the same.

We shall use the following known estimate (see [4], ch. VII, theorem 6.5):

If a measurable function F is differentiable on a measurable set A then

$$\mu(F(A)) \leq \int_A |F'(x)| d\mu \tag{3}$$

We apply (3) for a function φ_1 defined on $x \in (\alpha_{j-1}, \alpha_j)$ by

$$\varphi_1(x) = \int_0^{\beta_{k^-(j)}} \chi_{E^-} d\mu + \int_0^x \chi_{E^+} d\mu.$$

We have by the above definition

$$\varphi_1(\alpha_j) - \varphi_1(\alpha_{j-1}) = \mu(E_j^+). \tag{4}$$

We note that $\varphi_1(x) = \varphi(x)$ for $x \in E_j^+$, i.e., $\varphi = \varphi_1|_{E_j^+}$. As $\varphi_1'(x) = 1$ if $x \in E_j^+$, we get for any set $M \subset E_j^+$

$$\mu(\varphi(M)) = \mu(\varphi_1(M)) \leq \int_M \chi_{E^+} d\mu = \mu(M). \tag{5}$$

In particular we have

$$\mu(\varphi(E_j^+)) \leq \mu(E_j^+). \tag{6}$$

Let $S_j = \{x \in [\alpha_{j+1}, \alpha_j] : \varphi_1'(x) = 0\}$ and

$$P_j = \{x \in [\alpha_{j+1}, \alpha_j] : 0 < \varphi_1'(x) < 1 \text{ or } \varphi_1'(x) \quad \text{does not exists}\}.$$

The Lebesgue density theorem implies that

$$\mu(S_j) = \mu([\alpha_{j+1}, \alpha_j] \setminus E_j^+) \text{ and } \mu(P_j) = 0$$

Applying (3) to the function φ_1 and the set S_j we get

$$\mu(\varphi_1(S_j)) = 0. \tag{7}$$

The function φ_1 being the indefinite Lebesgue integral is absolutely continuous and so has Lusin (N)-property hence

$$\mu(\varphi_1(P_j)) = 0. \tag{8}$$

Now combining the (6), (7) and (9) we obtain

$$\mu(\varphi_1([\alpha_{j+1}, \alpha_j])) \leq \mu(\varphi_1(E_j^+)) + \mu(\varphi_1(P_j)) + \mu(\varphi_1(S_j)) = \tag{9}$$

$$= \mu(\varphi(E_j^+)) \leq \mu(E_j^+)$$

The function φ_1 is clearly monotonic and continuous on (α_{j-1}, α_j), so $\mu(\varphi_1(\alpha_{j-1}, \alpha_j)) = \varphi_1(\alpha_j) - \varphi_1(\alpha_{j-1})$. Combining this with (4) and (9) we get

$$\mu(E_j^+) \leq \mu(\varphi(E_j^+)) \leq \mu(E_j^+).$$

Therefore we finally obtain

$$\mu(\varphi(E_j^+)) = \mu(E_j^+). \tag{10}$$

To get the same equality for any $M \subset E^+$ we rewrite (5) for $E_j^+ \setminus M$ obtaining $\mu(\varphi(E_j^+ \setminus M)) \leq \mu(E_j^+ \setminus M)$. Comparing this with (5) and (10) we get

$$\mu(\varphi(M)) = \mu(\varphi(E_j^+)) - \mu(\varphi(E_j^+ \setminus M)) \geq \mu(E_j^+) - \mu(E_j^+ \setminus M) = \mu(M).$$

So we have proved that $\mu(\varphi(M)) = \mu(M)$ for any $M \subset E_j^+$ and as we have observed above, this implies that φ is measure preserving mapping on $E^+ \cup E^-$.

In this way we obtain that $\varphi(E^+ \cup E^-)$ is a set of full measure on $[0,1]$, so the inverse function φ^{-1} is defined almost everywhere on $[0,1]$. Using notation for t_n we also have $\mu(\varphi(E_j^+)) = \mu(E_j^+) = t_{n_j} - t_{n_j-1}$ and $\mu(\varphi(E_i^-)) = \mu(E_i^-) = t_{m_i} - t_{m_i-1}$.

We show now that φ^{-1} can be take as ψ_ξ we are looking for.

The function $f(\varphi^{-1}(y))$ is defined almost everywhere on $[0,1]$. As the Lebesgue integral is invariant under measure preserving mapping we get

$$\int_{t_{n_j-1}}^{t_{n_j}} f(\varphi^{-1}(y))d\mu = \int_{\varphi(E_j^+)} f(\varphi^{-1}(y))d\mu = \int_{E_j^+} f d\mu =$$

$$= \int_{(\alpha_{j-1},\alpha_j)\cup A^+} f d\mu = c_{n_j}$$

and

$$\int_{t_{m_i}-1}^{t_{m_i}} f(\varphi^{-1}(y))d\mu = \int_{\varphi(E_i^-)} f(\varphi^{-1}(y))d\mu = \int_{E_i^-} f d\mu =$$

$$= \int_{(\beta_{i-1},\beta_i)\cup A^-} f d\mu = c_{m_i}$$

Therefore we get $\int_0^{t_n} f(\varphi^{-1}(y))d\mu = \sum_{k=1}^{n} c_n$ and so having in mind that $\sum_{n=1}^{+\infty} c_n = \xi$ we obtain

$$\lim_{n\longrightarrow\infty} \int_0^{t_n} f(\varphi^{-1}(y))d\mu = \lim_{n\longrightarrow\infty} \sum_{n=1}^{+\infty} c_n = \xi.$$

Considering now any t, there exists n such that $t_{n-1} < t < t_n$ and the interval (t_{n-1}, t_n) is the image of either E_j^+ or E_i^-, up to a set of measure zero. So the value of $\int_0^t f(\varphi^{-1}(y))d\mu$ is between the values $\int_0^{t_{n-1}} f(\varphi^{-1}(y))d\mu$ and $\int_0^{t_n} f(\varphi^{-1}(y))d\mu$, and we conclude

$$\lim_{n\longrightarrow\infty} \int_0^{t} f(\varphi^{-1}(y))d\mu = \xi$$

completing the proof. □

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