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Analog of selfduality in dimension nine

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Abstract. We introduce a type of Riemannian geometry in nine dimensions, which can be viewed as the counterpart of selfduality in four dimensions. This geometry is related to a 9-dimensional irreducible representation of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ and it turns out to be defined by a differential 4-form. Structures admitting a metric connection with totally antisymmetric torsion and preserving the 4-form are studied in detail, producing locally homogeneous examples which can be viewed as analogs of self-dual 4-manifolds in dimension nine.

1. Introduction

The special feature of four dimensions is that the rotation group $\mathrm{SO}(4)$ is not simple but it is locally isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$, since $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$.

Given an oriented 4-dimensional Riemannian manifold (M^4, g), the Hodge-star-operator $* : \Lambda^2 \to \Lambda^2$ satisfies $*^2 = \mathrm{id}$ and the bundle of 2-forms Λ^2 splits as

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-, \tag{1.1}$$

where Λ^2_+ is the space of self-dual forms and Λ^2_- is the space of anti-self-dual forms.

The Riemann curvature tensor defines a self-adjoint transformation $\mathcal{R} : \Lambda^2 \to \Lambda^2$ which can be written, with respect to the decomposition (1.1), as the block matrix

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where $B \in \mathrm{Hom}(\Lambda^2_-, \Lambda^2_+)$ and $A \in \mathrm{End}\,\Lambda^2_+$, $C \in \mathrm{End}\,\Lambda^2_-$ are self-adjoint.

This decomposition of $\mathcal{R}$ gives the complete description of the Riemannian curvature tensor into irreducible components obtained in [21]:

$$\big(\mathrm{tr}\,A, B, A - \tfrac{1}{3}\,\mathrm{tr}\,A, C - \tfrac{1}{3}\,\mathrm{tr}\,C\big),$$

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where $\operatorname{tr} A = \operatorname{tr} C$ is the Ricci scalar, B is the traceless Ricci tensor, and the last two components $W_+ = A - \frac{1}{3}\operatorname{tr} A$ and $W_- = C - \frac{1}{3}\operatorname{tr} C$, together give the conformally invariant Weyl tensor $W = W_+ + W_-$. We recall from [4] that g is Einstein if and only if $B = 0$ and g is self-dual if and only if $W_- = 0$.

In terms of the Lie algebra valued 1-form $\overset{\mathrm{LC}}{\Gamma}$ of the Levi-Civita connection and of its curvature 2-form $\overset{\mathrm{LC}}{\Omega}$ we have the decompositions

$$\overset{\mathrm{LC}}{\Gamma} = \overset{+}{\Gamma} + \overset{-}{\Gamma} \quad \text{and} \quad \overset{\mathrm{LC}}{\Omega} = \overset{+}{\Omega} + \overset{-}{\Omega},$$

where $\overset{+}{\Gamma}$ and $\overset{+}{\Omega}$ are $\mathfrak{su}(2)_L$-valued, and $\overset{-}{\Gamma}$ and $\overset{-}{\Omega}$ are $\mathfrak{su}(2)_R$-valued.

Then the condition for the Riemannian metric g to be Einstein and self-dual is equivalent to $\overset{-}{\Omega} = 0$.

A natural problem is to study a geometry in higher dimensions, which can be viewed as the counterpart of selfduality in four dimensions. The Lie group $\mathrm{SO}(n)$ for $n \geq 5$ is simple and there is no splitting of $\mathfrak{so}(n)$, so an idea is to try with a Lie group of the form $H \times H$ in $\mathrm{SO}(n)$.

In this paper we will consider the case of $\mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(9)$. To this aim we need an irreducible 9-dimensional representation of $\mathrm{SO}(3) \times \mathrm{SO}(3)$, which turns out to be related to a 9-dimensional irreducible representation ρ of the Lie group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Perhaps for the first time the representation ρ was used by G. Peano [20] in his extension of the classical invariant theory to the action of the Cartesian product $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on the Cartesian product $\mathbb{R}^2 \times \mathbb{R}^2$. Similarly to the classical invariant theory [19, Chapter 10, p. 242], Peano in [20] defines irreducible representations of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ group, by considering its action on homogeneous polynomials in four variables $(\phi_1, \phi_2, \psi_1, \psi_2) = (\vec{\phi}, \vec{\psi}) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Given a defining action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2$, $(h, \vec{\phi}) \to h\vec{\phi}$, the irreducible action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^{m+1} \times \mathbb{R}^{\mu+1}$ is defined as follows.

Let $a_{l\lambda}$, $l = 0, \ldots, m$, $\lambda = 0, \ldots, \mu$, be coordinates in $\mathbb{R}^{m+1} \times \mathbb{R}^{\mu+1}$. They define a homogeneous polynomial

$$w(\vec{\phi}, \vec{\psi}) = \sum_{l=0}^{m} \sum_{\lambda=0}^{\mu} a_{l\lambda} \binom{m}{l} \binom{\mu}{\lambda} \phi_1^{m-l} \phi_2^{l} \psi_1^{\mu-\lambda} \psi_2^{\lambda}. \tag{1.2}$$

Now given $(h_L, h_R) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, we define $a_{l\lambda}^{(h_L,h_R)} \in \mathbb{R}^{m+1} \times \mathbb{R}^{\mu+1}$ via

$$\sum_{l=0}^{m} \sum_{\lambda=0}^{\mu} a_{l\lambda}^{(h_L,h_R)} \binom{m}{l} \binom{\mu}{\lambda} \phi_1^{m-l} \phi_2^{l} \psi_1^{\mu-\lambda} \psi_2^{\lambda} = w(h_L\vec{\phi}, h_R\vec{\psi}).$$

It follows that the map

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^{(m+1)(\mu+1)} \ni (h_L, h_R, a_{l\lambda}) \to (a_{l\lambda}^{(h_L,h_R)}) \in \mathbb{R}^{(m+1)(\mu+1)}$$

is an action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^{(m+1)(\mu+1)}$, and therefore it defines an $(m+1)(\mu+1)$-dimensional representation ρ of this group by $\rho(h_L, h_R)a_{l\lambda} = a_{l\lambda}^{(h_L,h_R)}$.

For each value of (m, μ) this representation is irreducible. In the paper we are interested in the case $(m, \mu) = (2, 2)$. In such case the polynomial w reads

$$\begin{aligned} w(\vec{\phi}, \vec{\psi}) = {} & a_{00}\phi_1^2\psi_1^2 + 2a_{10}\phi_1\phi_2\psi_1^2 + a_{20}\phi_2^2\psi_1^2 + 2a_{01}\phi_1^2\psi_1\psi_2 \\ & + 4a_{11}\phi_1\phi_2\psi_1\psi_2 + 2a_{21}\phi_2^2\psi_1\psi_2 + a_{02}\phi_1^2\psi_2^2 \\ & + 2a_{12}\phi_1\phi_2\psi_2^2 + a_{22}\phi_2^2\psi_2^2. \end{aligned} \tag{1.3}$$

The 9-dimensional space $\mathbb{R}^9$ consisting of vectors

$$\vec{x} = (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (a_{00}, a_{10}, a_{20}, a_{01}, a_{11}, a_{21}, a_{02}, a_{12}, a_{22}),$$

is equipped with the irreducible representation ρ of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. This representation induces the action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on homogeneous polynomials in variables x_i. Peano showed that the lowest order *invariant* polynomials under this action are

$$g = \sum_{i,j} g_{ij} x_i x_j = 2\big(x_0 x_8 + x_2 x_6 - 2x_1 x_7 - 2x_3 x_5 + 2x_4^2\big), \tag{1.4}$$

$$\begin{aligned} \Upsilon &= \sum_{i,j,k} \Upsilon_{ijk} x_i x_j x_k \\ &= 24\big(x_0 x_4 x_8 - x_0 x_5 x_7 - x_1 x_3 x_8 + x_1 x_5 x_6 + x_2 x_3 x_7 - x_2 x_4 x_6\big). \end{aligned} \tag{1.5}$$

They equip $\mathbb{R}^9$ with a metric g_{ij} of signature $(4,5)$ and a totally symmetric third rank tensor Υ_{ijk}, which turns out to be traceless, $g^{ij}\Upsilon_{ijk} = 0$.

The common stabilizer of the two tensors g and Υ, defined above, is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ in the 9-dimensional irreducible representation ρ of Peano.

This is very similar to the situation in $\mathbb{R}^5$, where we have a pair of tensors (g_{ij}, Υ_{ijk}) which reduces the $\mathrm{GL}(5,\mathbb{R})$ group to the irreducible $\mathrm{SO}(3)$ in dimension five [2,5,8]. The only difference with the 5-dimensional case considered in [5] is that there the metric g_{ij} is of *purely Riemannian* signature[1)]; see also [13, 17, 18].

The Riemannian version of tensors associated with Peano biquadrics may be obtained by making the following formal substitutions in (1.4)–(1.5):

$$\begin{aligned} &x_0 = y_1 + i y_2, && x_8 = y_1 - i y_2, && x_2 = y_3 + i y_4, \\ &x_6 = y_3 - i y_4, && x_1 = \tfrac{1}{\sqrt{2}}(y_5 + i y_6), && x_7 = -\tfrac{1}{\sqrt{2}}(y_5 - i y_6), \\ &x_3 = \tfrac{1}{\sqrt{2}}(y_7 + i y_8), && x_5 = -\tfrac{1}{\sqrt{2}}(y_7 - i y_8), && x_4 = \tfrac{1}{\sqrt{2}} y_9. \end{aligned}$$

In these formulae, the coefficients y_μ, $\mu = 1, \ldots, 9$, are *real*, and i is the imaginary unit. With these substitutions (1.4)–(1.5) become:

$$g = \sum_{i,j} g_{ij} y_i y_j = 2\big(y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2 + y_9^2\big), \tag{1.6}$$

$$\begin{aligned} \Upsilon &= \sum_{i,j,k} \Upsilon_{ijk} y_i y_j y_k \\ &= 12\big(-2y_1 y_5 y_7 - 2y_3 y_5 y_7 - 2y_2 y_6 y_7 - 2y_4 y_6 y_7 - 2y_2 y_5 y_8 + 2y_4 y_5 y_8 \\ &\quad + 2y_1 y_6 y_8 - 2y_3 y_6 y_8 + \sqrt{2} y_1^2 y_9 + \sqrt{2} y_2^2 y_9 - \sqrt{2} y_3^2 y_9 - \sqrt{2} y_4^2 y_9\big). \end{aligned} \tag{1.7}$$

This equips $\mathbb{R}^9$ parametrized by y_μ, $\mu = 1, 2, \ldots, 9$, with a pair of totally symmetric tensors (g_{ij}, Υ_{ijk}), in which g_{ij} is now a Riemannian metric.

1) This indicates that the geometry associated with tensors g and Υ as above can be related to the geometry of a certain type of systems of differential equations of finite type [12, 14]. Actually, the biquadrics (1.3) are related to the general solution of the finite type system $z_{xxx} = 0$ and $z_{yyy} = 0$ of PDEs on the plane for the unknown $z = z(x, y)$. We expect that the geometry associated with g and Υ is the geometry of generalizations of this system [9].

In Section 2 we obtain a better realization of $(\mathbb{R}^9, g, \Upsilon)$ by using the identification of $\mathbb{R}^9$ with a space $\mathbb{M}_{3\times 3}(\mathbb{R})$ of 3×3 matrices with real coefficients. This enables us to show that $\mathrm{SO}(3) \times \mathrm{SO}(3)$ is a stabilizer of a certain 4-form ω. In Section 3 irreducible representations of $\mathrm{SO}(3) \times \mathrm{SO}(3)$ are studied in detail. Following the approach presented in [5], in Section 4 we introduce the irreducible $\mathrm{SO}(3) \times \mathrm{SO}(3)$ geometry in dimension nine as the geometry of 9-dimensional manifolds M^9 equipped either with a pair of totally symmetric tensors (g, Υ) as in (1.6)–(1.7) or with the differential 4-form ω. In Section 5 we determine the conditions for Υ which will guarantee that $(M^9, g, \Upsilon, \omega)$ admits a unique metric connection Γ, with values in the symmetry algebra $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R)$ of (g, Υ) and with totally antisymmetric torsion [1,3,6,10,16]. This $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R)$-connection Γ, also called the characteristic connection, naturally splits into

$$\Gamma = \overset{+}{\Gamma} + \overset{-}{\Gamma}, \quad \text{with } \overset{+}{\Gamma} \in \mathfrak{so}(3)_L \otimes \mathbb{R}^9 \text{ and } \overset{-}{\Gamma} \in \mathfrak{so}(3)_R \otimes \mathbb{R}^9.$$

Because $\mathfrak{so}(3)_L$ commutes with $\mathfrak{so}(3)_R$, this split defines two independent $\mathfrak{so}(3)$-valued connections $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$. So an irreducible $\mathrm{SO}(3) \times \mathrm{SO}(3)$ geometry $(M^9, g, \Upsilon, \omega)$ equipped with an $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R)$ connection Γ can be Einstein in several meanings, by considering not only the Levi-Civita connection but also the connections Γ, $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$. In the last section we study irreducible $\mathrm{SO}(3) \times \mathrm{SO}(3)$ geometries $(M^9, g, \Upsilon, \omega)$ admitting a characteristic connection Γ with 'special' torsion T. In particular, we provide locally homogeneous (non-Riemannian symmetric) examples for which $T \neq 0$, $\overset{+}{\Gamma}$ has vanishing curvature and $\overset{-}{\Gamma}$ is Einstein and not flat. These examples can be viewed as analogs of self-dual structures in dimension four. It would be very interesting to find examples of such structures which are not locally homogeneous. It is an open question whether such examples are possible.

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2. Invariant SO(3) × SO(3) tensors

We identify the 9-dimensional real vector space $\mathbb{R}^9$ with a space $\mathbb{M}_{3\times 3}(\mathbb{R})$ of 3×3 matrices with real coefficients, via the map $\sigma : \mathbb{R}^9 \to \mathbb{M}_{3\times 3}(\mathbb{R})$, defined by

$$\mathbb{R}^9 \ni A = a^i \mathbf{e}_i \mapsto \sigma(A) = \begin{pmatrix} a^1 & a^2 & a^3 \\ a^4 & a^5 & a^6 \\ a^7 & a^8 & a^9 \end{pmatrix} \in \mathbb{M}_{3\times 3}(\mathbb{R}). \tag{2.1}$$

This map is obviously invertible, so we also have the inverse $\sigma^{-1} : \mathbb{M}_{3\times 3}(\mathbb{R}) \to \mathbb{R}^9$.

The unique irreducible 9-dimensional representation ρ of the group $G = \mathrm{SO}(3) \times \mathrm{SO}(3)$ in $\mathbb{R}^9$ is then defined as follows.

Let $h = (h_L, h_R)$ be the most general element of G, i.e., let h_L and h_R be two arbitrary elements of $\mathrm{SO}(3)$ in the standard representation of 3×3 real matrices. Then, for every vector A from $\mathbb{R}^9$, we have

$$\rho(h)A = \sigma^{-1}\big(h_L \sigma(A) h_R^{-1}\big). \tag{2.2}$$

In the rest of the article we adopt the convention that the symbol G is reserved to denote the group SO(3) × SO(3) in the irreducible 9-dimensional representation defined above, and that $\mathfrak{g}$ denotes its Lie algebra, $\mathfrak{g} = \mathfrak{so}(3) \times \mathfrak{so}(3)$.

Consider now $\theta = (\theta^1, \theta^2, \dots, \theta^9)$ with components θ^i being covectors in $\mathbb{R}^9$. This means that θ is a vector-valued 1-form, $\theta \in \mathbb{R}^9 \otimes (\mathbb{R}^9)^*$. We identify it with the matrix-valued 1-form $\sigma(\theta) \in \mathbb{M}_{3\times 3}((\mathbb{R}^9)^*)$.

The group G acts on forms θ via $\theta \mapsto \theta' = \rho(h)\theta$. Its action is then extended to all tensors T of the form

$$T = T_{i_1 i_2 \dots i_r} \theta^{i_1} \otimes \theta^{i_2} \otimes \cdots \otimes \theta^{i_r}$$

via $T \mapsto T' = T_{i_1 i_2 \dots i_r} \theta'^{i_1} \otimes \theta'^{i_2} \otimes \cdots \otimes \theta'^{i_r}$.

We say that the tensor T is G-invariant iff $T' = T$.

An example of a G-invariant tensor is obtained by considering the determinant

$$\det(\sigma(A)) = \tfrac{1}{6} \Upsilon_{ijk} a^i a^j a^k$$

and its corresponding *symmetric* tensor

$$\Upsilon := \tfrac{1}{6} \Upsilon_{ijk} \theta^i \odot \theta^j \odot \theta^k. \tag{2.3}$$

This is obviously G-invariant by the properties of the determinant, and by the fact that $\det(h) = 1$, for every element of SO(3).

Thus we have at least one G-invariant tensor Υ.

To create others we note the G-invariance of the expressions

$$\mathrm{Tr}(\sigma(\theta) \odot \sigma(\theta)^T), \quad \mathrm{Tr}(\sigma(\theta) \wedge \sigma(\theta)^T), \quad \mathrm{Tr}(\sigma(\theta) \otimes \sigma(\theta)^T). \tag{2.4}$$

Here, the product sign under the trace is considered as the usual row-by-columns product of 3×3 matrices, but with the product between the matrix elements in each sum being the respective tensor products $\odot$, $\wedge$ and $\otimes$. The G-invariance of these three expressions is an immediate consequence of the defining property of the elements of SO(3), namely $h^T h = h h^T = \mathrm{id}$. Having observed this, we now see that any function F, multilinear in expressions (2.4), also defines a G-invariant tensor.

This enables us to define a new SO(3) × SO(3)-invariant tensor:

$$g = \mathrm{Tr}(\sigma(\theta) \odot \sigma(\theta)^T) = g_{ij} \theta^i \theta^j. \tag{2.5}$$

This tensor is symmetric, rank $\binom{0}{2}$ and non-degenerate. It defines a Riemannian metric g on $\mathbb{R}^9$.

Another set of G-invariant tensors is given by the $2k$-forms

$$\mathrm{Tr}\big(\sigma(\theta) \wedge \sigma(\theta)^T \wedge \sigma(\theta) \wedge \sigma(\theta)^T \wedge \dots \wedge \sigma(\theta) \wedge \sigma(\theta)^T\big). \tag{2.6}$$

One would expect that these identically vanish, but surprisingly, we have the following proposition.

Proposition 2.1. *The* 4*-form*

$$\omega = \tfrac{1}{4} \mathrm{Tr}\big(\sigma(\theta) \wedge \sigma(\theta)^T \wedge \sigma(\theta) \wedge \sigma(\theta)^T\big) = \tfrac{1}{4!} \omega_{ijkl} \theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^l \tag{2.7}$$

does not *vanish,* $\omega \neq 0$. *In the remaining cases, when* $k = 1, 3, 4$, *the forms* (2.6) *are identically equal to zero.*

We have the following formulae for the three G-invariant objects defined above:

$$\Upsilon = -\theta^3\theta^5\theta^7 + \theta^2\theta^6\theta^7 + \theta^3\theta^4\theta^8 - \theta^1\theta^6\theta^8 - \theta^2\theta^4\theta^9 + \theta^1\theta^5\theta^9, \tag{2.8}$$

$$g = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 + (\theta^4)^2 + (\theta^5)^2 + (\theta^6)^2 + (\theta^7)^2 + (\theta^8)^2 + (\theta^9)^2, \tag{2.9}$$

$$\begin{aligned}\omega = {} & \theta^1\wedge\theta^2\wedge\theta^4\wedge\theta^5 + \theta^1\wedge\theta^2\wedge\theta^7\wedge\theta^8 + \theta^1\wedge\theta^3\wedge\theta^4\wedge\theta^6 \\ & + \theta^1\wedge\theta^3\wedge\theta^7\wedge\theta^9 + \theta^2\wedge\theta^3\wedge\theta^5\wedge\theta^6 + \theta^2\wedge\theta^3\wedge\theta^8\wedge\theta^9 \\ & + \theta^4\wedge\theta^5\wedge\theta^7\wedge\theta^8 + \theta^4\wedge\theta^6\wedge\theta^7\wedge\theta^9 + \theta^5\wedge\theta^6\wedge\theta^8\wedge\theta^9.\end{aligned} \tag{2.10}$$

Here, to simplify the notation, we abbreviated expressions like $\theta^3 \odot \theta^5 \odot \theta^7$ or $\theta^1 \odot \theta^1$ to $\theta^3\theta^5\theta^7$ and $(\theta^1)^2$, respectively.

Proposition 2.2. (1) *The simultaneous stabilizer in* $\mathrm{GL}(9,\mathbb{R})$ *of the tensors* g *and* Υ *defined respectively in* (2.3) *and* (2.5) *is* $G = \mathrm{SO}(3)\times\mathrm{SO}(3)$ *in the irreducible 9-dimensional representation* ρ.

(2) *The stabilizer in* $\mathrm{GL}(9,\mathbb{R})$ *of the 4-form* ω *defined in* (2.7) *is also* $G = \mathrm{SO}(3)\times\mathrm{SO}(3)$ *in the irreducible 9-dimensional representation* ρ.

Proof. We know from the considerations preceding the proposition that the stabilizers contain G. To show that they are actually equal to G we do as follows:

A stabilizer G' of g and Υ consists of those elements h in GL(9, R) for which

$$g(hX, hY) = g(X,Y) \quad \text{and} \quad \Upsilon(hX,hY,hZ) = \Upsilon(X,Y,Z). \tag{2.11}$$

We find the Lie algebra of G'. Taking h in the form $h = \exp(sX)$ and taking $\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}$ of the equations (2.11), we see that the matrices $X = (X^i{}_j)$ representing the elements of the Lie algebra $\mathfrak{g}'$ of G' must satisfy

$$g_{lj}X^l{}_i + g_{il}X^l{}_j = 0 \tag{2.12}$$

and

$$\Upsilon_{ljk}X^l{}_i + \Upsilon_{ilk}X^l{}_j + \Upsilon_{ijl}X^l{}_k = 0. \tag{2.13}$$

The first of the above equations tells that the matrices X must be antisymmetric, i.e., it reduces 81 components of a matrix X to 36. The second equation gives another 30 independent conditions restricting the number of free components of X to 6. Explicitly the matrix X solving (2.12)–(2.13) is of the form

$$X = X^1e_1 + X^2e_2 + X^3e_3 + X^{1'}e_{1'} + X^{2'}e_{2'} + X^{3'}e_{3'}, \tag{2.14}$$

where

$$e_1 = \begin{pmatrix} 0&0&0&-1&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}, \quad e_{1'} = \begin{pmatrix} 0&-1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}, \tag{2.15}$$

$$e_2 = \begin{pmatrix} 0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0 \end{pmatrix}, \quad e_{2'} = \begin{pmatrix} 0&0&-1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0 \end{pmatrix}, \tag{2.16}$$

$$e_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0 \end{pmatrix}, \quad e_{3'} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}. \tag{2.17}$$

It is easy to check that the matrices e satisfy the following commutation relations: $[e_1, e_2] = e_3$, $[e_3, e_1] = e_2$, $[e_2, e_3] = e_1$, $[e_{1'}, e_{2'}] = e_{3'}$, $[e_{3'}, e_{1'}] = e_{2'}$, $[e_{2'}, e_{3'}] = e_{1'}$, with all the other commutators being zero modulo the antisymmetry. Thus the system $(e_A, e_{A'})$, $A = 1, 2, 3$, spans the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$, confirming that the Lie algebra $\mathfrak{g}'$ of the stabilizer G' of tensors (2.3) and (2.5) is $\mathfrak{g}' = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. In an analogous way we find the Lie algebra $\mathfrak{g}''$ of the stabilizer G'' of ω. This stabilizer consists of those elements h in $\mathrm{GL}(9, \mathbb{R})$ for which

$$\omega(hX, hY, hZ) = \omega(X, Y, Z). \tag{2.18}$$

Taking h in the form $h = \exp(sX)$ and taking $\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0}$ of the equations (2.18), we see that the matrices $X = (X^i{}_j)$ representing the elements of the Lie algebra $\mathfrak{g}''$ of G'' must satisfy

$$\omega_{ljkm} X^l{}_i + \omega_{ilkm} X^l{}_j + \omega_{ijlm} X^l{}_k + \omega_{ijkl} X^l{}_m = 0. \tag{2.19}$$

A short algebra shows that this imposes 75 independent conditions on the 81 components of X, and that the most general solution to this equation is given by (2.14) with the generators $(e_A, e_{A'})$ as in (2.15)–(2.17). Thus $\mathfrak{g}' = \mathfrak{g}'' = \mathfrak{so}(3) \oplus \mathfrak{so}(3) := \mathfrak{g}$.

As a consequence $G' = G'' = \mathrm{SO}(3) \times \mathrm{SO}(3)$, since $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is a maximal Lie subalgebra of $\mathfrak{so}(9)$. □

Remark 2.3. Note that the form ω *alone* is enough to reduce $\mathrm{GL}(9, \mathbb{R})$ to G. One does not need the metric g for this reduction! On the other hand, the tensor Υ alone is not enough to reduce the $\mathrm{GL}(9, \mathbb{R})$ to G. The equation (2.13) imposes only 65 independent conditions on the matrix X. Thus it reduces $\mathfrak{gl}(9, \mathbb{R})$ to a Lie algebra of dimension 16. Since 16 is the dimension of $\mathfrak{sl}(3, \mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R})$, and Υ is clearly $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$-invariant, the stabilizer of the tensor Υ alone is $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$. To reduce it further to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ one needs to preserve g. If in addition to Υ we preserve g we get, via the equation (2.12), the remaining 10 conditions.

Remark 2.4. For the geometric relevance of the form ω see Remark 4.5 suggested by Robert Bryant [7], see also [15] for the details.

Remark 2.5. We remark that in addition to the 4-form ω we have also the 5-form $*\omega$ (Hodge-dual of ω) which is G-invariant. One can say that given only ω in $\mathbb{R}^9$ we do not have any metric structure on it. But ω defines the reduction of the Lie algebra of $\mathrm{GL}(9, \mathbb{R})$ to

$\mathfrak{g} = \mathfrak{so}(3) \times \mathfrak{so}(3)$. In particular it defines the explicit representation of $\mathfrak{g}$ given by (2.14) with the explicit form of the generators $(e_A, e_{A'})$ given by (2.15)–(2.17). Thus, given ω we have explicitly X as in (2.14).

Now we define the metric g_{ij} as a $\binom{0}{2}$-tensor such that (2.12) holds. It is a matter of checking that given X as in (2.14) with $(e_A, e_{A'})$ as in (2.15)–(2.17) the only metric g_{ij} satisfying (2.12) (miraculously!) is $g_{ij} = \text{const} \times \delta_{ij}$. Thus the 4-form ω defines the metric g up to a scale, and this in turn defines the unique (up to a scale) 5-form $*\omega$, being its standard Hodge-star with respect to the metric g.

Another way of defining the 5-form $*\omega$, which provides the explicit relation between (g, Υ) and ω, is given by Proposition 2.6 below. To formulate it we consider a coframe θ^i and the corresponding components Υ_{ijk} of the tensor Υ as in (2.3). Using them we define a (9×9)-matrix-valued 1-form $\Upsilon(\theta) = (\Upsilon(\theta)^i_{\ j})$ with matrix elements $\Upsilon(\theta)^i_{\ j} = g^{il}\Upsilon_{ljk}\theta^k$.

Here (g^{ij}) is the matrix inverse of (g_{ij}), i.e., $g^{ik}g_{kj} = g_{jk}g^{ki} = \delta^i{}_j$. Having the matrix $\Upsilon(\theta)$, we consider traces of the skew symmetric powers of it,

$$\mathrm{Tr}(\Upsilon(\theta)^{\wedge k}) = \mathrm{Tr}\big(\Upsilon(\theta) \wedge \Upsilon(\theta) \wedge \ldots \wedge \Upsilon(\theta)\big),$$

where again the expressions like $\Upsilon(\theta) \wedge \Upsilon(\theta)$ denote the usual row-by-columns multiplication of 9×9 matrices, with the multiplication between the matrix elements being the wedge product $\wedge$.

Proposition 2.6. *If $k \neq 5$ and $k \in \{1, 2, \ldots, 9\}$, then* $\mathrm{Tr}(\Upsilon(\theta)^{\wedge k}) = 0$.
If $k = 5$, then the 5-form $\mathrm{Tr}(\Upsilon(\theta)^{\wedge 5})$ *does not vanish,*

$$\mathrm{Tr}(\Upsilon(\theta)^{\wedge 5}) = \mathrm{Tr}\big(\Upsilon(\theta) \wedge \Upsilon(\theta) \wedge \Upsilon(\theta) \wedge \Upsilon(\theta) \wedge \Upsilon(\theta)\big) \neq 0.$$

Up to a scale this form is equal to the G-invariant 5-form $*\omega$. In turn, the relation between the form ω and tensors (g, Υ) is given by

$$\omega = * \,\mathrm{Tr}\big(\Upsilon(\theta) \wedge \Upsilon(\theta) \wedge \Upsilon(\theta) \wedge \Upsilon(\theta) \wedge \Upsilon(\theta)\big).$$

We proved this proposition by a brute force, using (2.8)–(2.10), and calculating the expression of $\mathrm{Tr}(\Upsilon(\theta)^{\wedge k})$ for each value of $k = 1, 2, \ldots, 9$. It would be interesting to get a 'pure thought' proof of it.

Remark 2.7. The situation with G-invariant totally *antisymmetric* p-forms is clear: there are only (up to a scale) one 0- and one 9-form (a constant and its Hodge dual), and there are only (up to a scale) one 4- and one 5-form (the 4-form ω and its Hodge dual). All the other G-invariant p-forms are equal to zero.

Remark 2.8. The situation with G-invariant totally *symmetric* p-forms is more complex because of the infinite dimension of $\bigoplus_{k=0}^{\infty} \bigodot^k \mathbb{R}^9$: up to a scale there is only one totally symmetric G-invariant 0-form; totally symmetric G-invariant 1-forms are all equal to zero; there is only one totally symmetric G-invariant 2-form (the metric g), and only one totally symmetric G-invariant 3-form (the tensor Υ). Continuing this one gets that, in particular, there is only a 2-real-parameter family of totally symmetric G-invariant 4-forms: the family is spanned by $g_{(ij}g_{kl)}$ and by a tensor $\Xi_{ijkl} = \Xi_{(ijkl)}$, which in our coframe θ is expressed by

$$\begin{aligned}\Xi &= \tfrac{1}{24}\Xi_{ijkl}\theta^i\theta^j\theta^k\theta^l\\ &= 2(\theta^1)^4 + 4(\theta^1)^2(\theta^2)^2 + 2(\theta^2)^4 + 4(\theta^1)^2(\theta^3)^2 + 4(\theta^2)^2(\theta^3)^2 + 2(\theta^3)^4\\ &\quad + 4(\theta^1)^2(\theta^4)^2 - 7(\theta^2)^2(\theta^4)^2 - 7(\theta^3)^2(\theta^4)^2 + 2(\theta^4)^4 + 22\theta^1\theta^2\theta^4\theta^5\\ &\quad - 7(\theta^1)^2(\theta^5)^2 + 4(\theta^2)^2(\theta^5)^2 - 7(\theta^3)^2(\theta^5)^2 + 4(\theta^4)^2(\theta^5)^2 + 2(\theta^5)^4\\ &\quad + 22\theta^1\theta^3\theta^4\theta^6 + 22\theta^2\theta^3\theta^5\theta^6 - 7(\theta^1)^2(\theta^6)^2 - 7(\theta^2)^2(\theta^6)^2 + 4(\theta^3)^2(\theta^6)^2\\ &\quad + 4(\theta^4)^2(\theta^6)^2 + 4(\theta^5)^2(\theta^6)^2 + 2(\theta^6)^4 + 4(\theta^1)^2(\theta^7)^2 - 7(\theta^2)^2(\theta^7)^2\\ &\quad - 7(\theta^3)^2(\theta^7)^2 + 4(\theta^4)^2(\theta^7)^2 - 7(\theta^5)^2(\theta^7)^2 - 7(\theta^6)^2(\theta^7)^2 + 2(\theta^7)^4\\ &\quad + 22\theta^1\theta^2\theta^7\theta^8 + 22\theta^4\theta^5\theta^7\theta^8 - 7(\theta^1)^2(\theta^8)^2 + 4(\theta^2)^2(\theta^8)^2\\ &\quad - 7(\theta^3)^2(\theta^8)^2 - 7(\theta^4)^2(\theta^8)^2 + 4(\theta^5)^2(\theta^8)^2 - 7(\theta^6)^2(\theta^8)^2 + 4(\theta^7)^2(\theta^8)^2\\ &\quad + 2(\theta^8)^4 + 22\theta^1\theta^3\theta^7\theta^9 + 22\theta^4\theta^6\theta^7\theta^9 + 22\theta^2\theta^3\theta^8\theta^9 + 22\theta^5\theta^6\theta^8\theta^9\\ &\quad - 7(\theta^1)^2(\theta^9)^2 - 7(\theta^2)^2(\theta^9)^2 + 4(\theta^3)^2(\theta^9)^2 - 7(\theta^4)^2(\theta^9)^2 - 7(\theta^5)^2(\theta^9)^2\\ &\quad + 4(\theta^6)^2(\theta^9)^2 + 4(\theta^7)^2(\theta^9)^2 + 4(\theta^8)^2(\theta^9)^2 + 2(\theta^9)^4.\end{aligned}$$

The G-invariant tensor Ξ_{ijkl} defined above may be characterized as the unique (up to a scale) G-invariant totally symmetric $\binom{0}{4}$ tensor which has vanishing trace, $g^{ij}\,\Xi_{ijkl} = 0$.

3. Irreducible representations of SO(3) × SO(3)

As it is well known all finite dimensional real irreducible representations of SO(3) have dimensions $d_k = 2k+1$, $k = 0, 1, 2, 3, \ldots$, and are enumerated by the weight vectors $[2k]$. The representations with the weight vectors $[m] = [2k]$ and $[\mu] = [2l]$ are equivalent[2)] iff $k = l$. We denote the vector spaces of these representations by $V_{[2k]}$. Consequently, all pairwise inequivalent finite dimensional real irreducible representations of SO(3) × SO(3) are given by tensor products

$$V_{[2k]} \otimes V_{[2l]} := V_{[2k,2l]}, \quad \text{with } k, l = 0, 1, 2, 3, \ldots,$$

and have the respective dimensions $d_{[2k,2l]} = (2k+1)(2l+1)$.

In particular, for each number $d_{[2k,2l]}$, with $k \neq l$, there are two non-equivalent irreducible representations of SO(3) × SO(3) with the respective carrier spaces $V_{[2k,2l]}$ and $V_{[2l,2k]}$.

In the following we will need decompositions of various tensor products of spaces $V_{[2k,2l]}$ into irreducible components with respect to the action of SO(3)×SO(3). These are summarized in the next proposition.

2) Note that m and μ here are related to the order of the Peano polynomials in (1.2).

Proposition 3.1.

$$\bigwedge^2 V_{[2,2]} = V_{[0,2]} \oplus V_{[2,0]} \oplus V_{[2,4]} \oplus V_{[4,2]},$$
$$\bigwedge^3 V_{[2,2]} = V_{[0,2]} \oplus V_{[2,0]} \oplus V_{[0,6]} \oplus V_{[6,0]} \oplus V_{[2,4]} \oplus V_{[4,2]} \oplus V_{[2,2]} \oplus V_{[4,4]},$$
$$\bigwedge^4 V_{[2,2]} = V_{[0,0]} \oplus V_{[0,4]} \oplus V_{[4,0]} \oplus 2V_{[2,2]} \oplus V_{[2,4]} \oplus V_{[4,2]} \oplus V_{[2,6]} \oplus V_{[6,2]} \oplus V_{[4,4]},$$
$$\bigodot^2 V_{[2,2]} = V_{[0,0]} \oplus V_{[0,4]} \oplus V_{[4,0]} \oplus V_{[2,2]} \oplus V_{[4,4]},$$
$$\bigodot^3 V_{[2,2]} = V_{[0,0]} \oplus 2V_{[2,2]} \oplus V_{[2,4]} \oplus V_{[4,2]} \oplus V_{[2,6]} \oplus V_{[6,2]} \oplus V_{[4,4]} \oplus V_{[6,6]},$$
$$\bigodot^4 V_{[2,2]} = 2V_{[0,0]} \oplus 2V_{[0,4]} \oplus 2V_{[4,0]} \oplus 2V_{[2,2]} \oplus V_{[2,4]} \oplus V_{[4,2]} \oplus V_{[0,8]} \oplus V_{[8,0]}$$
$$\oplus V_{[2,6]} \oplus V_{[6,2]} \oplus 3V_{[4,4]} \oplus V_{[4,6]} \oplus V_{[6,4]} \oplus V_{[4,8]} \oplus V_{[8,4]} \oplus V_{[6,6]} \oplus V_{[8,8]}.$$

In addition we have the following identifications:

$$\text{'left'} \quad \mathfrak{so}(3) = V_{[0,2]}, \qquad \text{'right'} \quad \mathfrak{so}(3) = V_{[2,0]},$$
$$\mathbb{R}^9 = V_{[2,2]}, \qquad \mathfrak{so}(9) = \bigwedge^2 \mathbb{R}^9 = \bigwedge^2 V_{[2,2]}.$$

In the following we will conveniently denote the $\mathfrak{so}(3)$ Lie algebra corresponding to $V_{[0,2]}$ by $\mathfrak{so}(3)_L$ and the $\mathfrak{so}(3)$ Lie algebra corresponding to $V_{[2,0]}$ by $\mathfrak{so}(3)_R$, i.e.,

$$V_{[0,2]} = \mathfrak{so}(3)_L \quad \text{and} \quad V_{[2,0]} = \mathfrak{so}(3)_R.$$

Using these identifications and the decompositions from Proposition 3.1, we obtain:

Proposition 3.2.

$$\begin{aligned}
\mathfrak{so}(9) \otimes \mathbb{R}^9 &= 2V_{[0,2]} \oplus 2V_{[2,0]} \oplus V_{[0,4]} \oplus V_{[4,0]} \oplus V_{[0,6]} \oplus V_{[6,0]} \\
&\quad \oplus 3V_{[2,4]} \oplus 3V_{[4,2]} \oplus V_{[2,6]} \oplus V_{[6,2]} \oplus 4V_{[2,2]} \oplus 2V_{[4,4]} \\
&\quad \oplus V_{[4,6]} \oplus V_{[6,4]}, \\
\mathfrak{so}(3)_L \otimes \mathbb{R}^9 &= V_{[2,0]} \oplus V_{[2,2]} \oplus V_{[2,4]}, \\
\mathfrak{so}(3)_R \otimes \mathbb{R}^9 &= V_{[0,2]} \oplus V_{[2,2]} \oplus V_{[4,2]}, \\
\big(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R\big) \otimes \mathbb{R}^9 &= V_{[0,2]} \oplus V_{[2,0]} \oplus 2V_{[2,2]} \oplus V_{[2,4]} \oplus V_{[4,2]}, \\
\mathfrak{so}(3)_L \otimes \bigwedge^2 \mathbb{R}^9 &= V_{[0,0]} \oplus V_{[0,2]} \oplus V_{[2,6]} \oplus V_{[0,4]} \oplus V_{[4,0]} \oplus V_{[2,4]} \oplus V_{[4,2]} \\
&\quad \oplus 2V_{[2,2]} \oplus V_{[4,4]}, \\
\mathfrak{so}(3)_R \otimes \bigwedge^2 \mathbb{R}^9 &= V_{[0,0]} \oplus V_{[2,0]} \oplus V_{[6,2]} \oplus V_{[0,4]} \oplus V_{[4,0]} \oplus V_{[2,4]} \oplus V_{[4,2]} \\
&\quad \oplus 2V_{[2,2]} \oplus V_{[4,4]},
\end{aligned}$$

$$\big(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R\big) \otimes \bigwedge^2 \mathbb{R}^9 = 2V_{[0,0]} \oplus V_{[0,2]} \oplus V_{[2,0]} \oplus 2V_{[0,4]} \oplus 2V_{[4,0]} \oplus 2V_{[2,4]} \oplus 2V_{[4,2]} \oplus V_{[2,6]} \oplus V_{[6,2]} \oplus 4V_{[2,2]} \oplus 2V_{[4,4]}.$$

The proofs of the above propositions can be obtained by the standard representation theory methods using weights. Instead of presenting them we identify various useful components of the decompositions mentioned in the propositions as eigenspaces of certain $\mathrm{SO}(3) \times \mathrm{SO}(3)$ invariant operators.

For example the four irreducible components in the decomposition of $\bigwedge^2 \mathbb{R}^9$ in Proposition 3.1 can be distinguished by means of the action of the endomorphism of $\bigotimes^2 \mathbb{R}^9$ defined by the structural 4-form ω. Indeed the 4-form $\omega = \frac{1}{24}\omega_{ijkl}\theta^i \wedge \theta^j \wedge \theta^k \wedge \theta^l$, as in (2.8)–(2.10), defines a linear map $\omega : \bigotimes^2 \mathbb{R}^9 \to \bigotimes^2 \mathbb{R}^9$ given by

$$\bigotimes^2 \mathbb{R}^9 \ni t_{ij} \stackrel{\omega}{\longmapsto} \omega(t)_{kl} = \omega^{ij}{}_{kl} t_{ij} \in \bigotimes^2 \mathbb{R}^9.$$

Here and in the following, we raise the indices by means of the inverse g^{ij} of the metric $g = g_{ij}\theta^i\theta^j$ given by (2.10). In particular

$$\omega^{ij}{}_{kl} = g^{ip} g^{jq} \omega_{pqkl}.$$

The eigenspaces of this endomorphism give the desired decomposition of $\bigwedge^2 \mathbb{R}^9$. We have the following proposition.

Proposition 3.3. *The 45-dimensional vector space* $\bigodot^2 \mathbb{R}^9$ *is an* $\mathrm{SO}(3) \times \mathrm{SO}(3)$ *invariant subspace in* $\bigotimes^2 \mathbb{R}^9$ *which corresponds to the eigenvalue* 0 *of the operator* $\omega : \bigotimes^2 \mathbb{R}^9 \to \bigotimes^2 \mathbb{R}^9$. *The decomposition* $\bigwedge^2 \mathbb{R}^9 = V_{[2,0]} \oplus V_{[0,2]} \oplus V_{[2,4]} \oplus V_{[4,2]}$ *is given by*

$$V_{[0,2]} = \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : \omega(F)_{ij} = -4F_{ij} \Big\} = \mathfrak{so}(3)_L,$$

$$V_{[2,0]} = \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : \omega(F)_{ij} = 4F_{ij} \Big\} = \mathfrak{so}(3)_R,$$

$$V_{[2,4]} = \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : \omega(F)_{ij} = 2F_{ij} \Big\},$$

$$V_{[4,2]} = \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : \omega(F)_{ij} = -2F_{ij} \Big\}.$$

The respective dimensions are

$$\dim V_{[2,0]} = \dim V_{[0,2]} = 3, \quad \dim V_{[4,2]} = \dim V_{[2,4]} = 15.$$

Remark 3.4. Convenient bases for the 2-forms spanning $V_{[0,2]}$ and $V_{[2,0]}$ are

$$\kappa_0^A = \tfrac{1}{2} e_{Aij}\theta^i \wedge \theta^j, \quad \text{and} \quad \kappa_0^{A'} = \tfrac{1}{2} e_{A'ij}\theta^i \wedge \theta^j.$$

Here e_{Aij} and $e_{A'ij}$ are the matrix elements of the bases (e_A) and $(e_{A'})$ of $\mathfrak{so}(3)_L$ and $\mathfrak{so}(3)_R$ as given in (2.15)–(2.17). Explicitly:

$$\left\{\begin{aligned} -\kappa^1_0 &= \theta^1\wedge\theta^4+\theta^2\wedge\theta^5+\theta^3\wedge\theta^6,\\ -\kappa^{1'}_0 &= \theta^1\wedge\theta^2+\theta^4\wedge\theta^5+\theta^7\wedge\theta^8,\\ -\kappa^2_0 &= \theta^1\wedge\theta^7+\theta^2\wedge\theta^8+\theta^3\wedge\theta^9,\\ -\kappa^{2'}_0 &= \theta^1\wedge\theta^3+\theta^4\wedge\theta^6+\theta^7\wedge\theta^9,\\ -\kappa^3_0 &= \theta^4\wedge\theta^7+\theta^5\wedge\theta^8+\theta^6\wedge\theta^9,\\ -\kappa^{3'}_0 &= \theta^2\wedge\theta^3+\theta^5\wedge\theta^6+\theta^8\wedge\theta^9.\end{aligned}\right.\tag{3.1}$$

Thus we have $\mathrm{Span}_{\mathbb{R}}(\kappa^1_0,\kappa^2_0,\kappa^3_0)=\mathfrak{so}(3)_L$ and $\mathrm{Span}_{\mathbb{R}}(\kappa^{1'}_0,\kappa^{2'}_0,\kappa^{3'}_0)=\mathfrak{so}(3)_R$.

A convenient basis for the space $V_{[2,4]}$ is given by

$$\left\{\begin{aligned} \lambda^1_0 &= \theta^1\wedge\theta^4-\theta^3\wedge\theta^6, & \lambda^2_0 &= \theta^1\wedge\theta^5+\theta^2\wedge\theta^4,\\ \lambda^3_0 &= \theta^1\wedge\theta^6+\theta^3\wedge\theta^4, & \lambda^4_0 &= \theta^1\wedge\theta^7-\theta^3\wedge\theta^9,\\ \lambda^5_0 &= \theta^1\wedge\theta^8+\theta^2\wedge\theta^7, & \lambda^6_0 &= \theta^1\wedge\theta^9+\theta^3\wedge\theta^7,\\ \lambda^7_0 &= \theta^2\wedge\theta^5-\theta^3\wedge\theta^6, & \lambda^8_0 &= \theta^2\wedge\theta^6+\theta^3\wedge\theta^5,\\ \lambda^9_0 &= \theta^2\wedge\theta^8-\theta^3\wedge\theta^9, & \lambda^{10}_0 &= \theta^2\wedge\theta^9+\theta^3\wedge\theta^8,\\ \lambda^{11}_0 &= \theta^4\wedge\theta^7-\theta^6\wedge\theta^9, & \lambda^{12}_0 &= \theta^4\wedge\theta^8+\theta^5\wedge\theta^7,\\ \lambda^{13}_0 &= \theta^4\wedge\theta^9+\theta^6\wedge\theta^7, & \lambda^{14}_0 &= \theta^5\wedge\theta^8-\theta^6\wedge\theta^9,\\ \lambda^{15}_0 &= \theta^5\wedge\theta^9+\theta^6\wedge\theta^8. \end{aligned}\right.\tag{3.2}$$

Similarly, a basis for $V_{[4,2]}$ is

$$\left\{\begin{aligned} \lambda^{1'}_0 &= \theta^1\wedge\theta^2-\theta^7\wedge\theta^8, & \lambda^{2'}_0 &= \theta^1\wedge\theta^3-\theta^7\wedge\theta^9,\\ \lambda^{3'}_0 &= \theta^2\wedge\theta^3-\theta^8\wedge\theta^9, & \lambda^{4'}_0 &= \theta^1\wedge\theta^5-\theta^2\wedge\theta^4,\\ \lambda^{5'}_0 &= \theta^1\wedge\theta^6-\theta^3\wedge\theta^4, & \lambda^{6'}_0 &= \theta^2\wedge\theta^6-\theta^3\wedge\theta^5,\\ \lambda^{7'}_0 &= \theta^1\wedge\theta^8-\theta^2\wedge\theta^7, & \lambda^{8'}_0 &= \theta^1\wedge\theta^9-\theta^3\wedge\theta^7,\\ \lambda^{9'}_0 &= \theta^2\wedge\theta^9-\theta^3\wedge\theta^8, & \lambda^{10'}_0 &= \theta^4\wedge\theta^5-\theta^7\wedge\theta^8,\\ \lambda^{11'}_0 &= \theta^4\wedge\theta^6-\theta^7\wedge\theta^9, & \lambda^{12'}_0 &= \theta^5\wedge\theta^6-\theta^8\wedge\theta^9,\\ \lambda^{13'}_0 &= \theta^4\wedge\theta^8-\theta^5\wedge\theta^7, & \lambda^{14'}_0 &= \theta^4\wedge\theta^9-\theta^6\wedge\theta^7,\\ \lambda^{15'}_0 &= \theta^5\wedge\theta^9-\theta^6\wedge\theta^8. \end{aligned}\right.\tag{3.3}$$

A partial decomposition of $\bigodot^2\mathbb{R}^9$ can be obtained by means of the Casimir operator $C^{ij}{}_{kl}$ for the tensorial representation $\otimes^2\rho$ of the irreducible representation of $\mathfrak{so}(3)_L\oplus\mathfrak{so}(3)_R$ defined in (2.15)–(2.17). To get an explicit formula for the operator $C^{ij}{}_{kl}$ we introduce a collective index $\mu=1,2,3,4,5,6$, so that the six vectors $(e_\mu)=(e_A,e_{A'})$ are the basis of the Lie algebra $\mathfrak{so}(3)_L\oplus\mathfrak{so}(3)_R$. Using this basis one easily calculates the Killing form k for $\mathfrak{so}(3)_L\oplus\mathfrak{so}(3)_R$. We have

$$k(e_\mu,\varepsilon_\nu)=k_{\mu\nu}=-2\delta_{\mu\nu}.$$

The inverse of the Killing form has components $k^{\mu\nu} = -\frac{1}{2}\delta^{\mu\nu}$. Then, modulo the terms proportional to the identity, the Casimir operator $C^{ij}{}_{kl}$ reads

$$C^{ij}{}_{kl} = k^{\mu\nu}(e_\mu{}^i{}_k e_\nu{}^j{}_l + e_\nu{}^i{}_k e_\mu{}^j{}_l).$$

Here $e_\mu{}^i{}_k$ denotes the matrix element from the ith raw and kth column of the Lie algebra matrix e_μ given by (2.15)–(2.17). This defines an endomorphism

$$C : \bigotimes^2 \mathbb{R}^9 \to \bigotimes^2 \mathbb{R}^9$$

given by

$$\bigotimes^2 \mathbb{R}^9 \ni t_{ij} \overset{C}{\longmapsto} C(t)_{kl} = C^{ij}{}_{kl} t_{ij} \in \bigotimes^2 \mathbb{R}^9.$$

We have the following proposition.

Proposition 3.5. *The Casimir operator C decomposes $\bigotimes^2 \mathbb{R}^9$ so that*

$$\bigotimes^2 \mathbb{R}^9 = V_{[0,0]} \oplus V_{[2,2]} \oplus V_{[4,4]} \oplus W_6 \oplus W_{10} \oplus W_{30},$$

where

$$\begin{aligned}
V_{[0,0]} &= \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : C(F)_{ij} = -4F_{ij} \Big\},\\
V_{[2,2]} &= \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : C(F)_{ij} = -2F_{ij} \Big\},\\
V_{[4,4]} &= \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : C(F)_{ij} = 2F_{ij} \Big\},\\
W_6 &= \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : C(F)_{ij} = -3F_{ij} \Big\} = V_{[2,0]} \oplus V_{[0,2]},\\
W_{30} &= \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : C(F)_{ij} = 0 \Big\} = V_{[2,4]} \oplus V_{[4,2]},\\
W_{10} &= \Big\{ \bigotimes^2 \mathbb{R}^9 \ni F_{ij} : C(F)_{ij} = -F_{ij} \Big\}.
\end{aligned}$$

We further have

$$\bigwedge^2 \mathbb{R}^9 = W_6 \oplus W_{30} \quad \textit{and} \quad \bigodot^2 \mathbb{R}^9 = V_{[0,0]} \oplus V_{[2,2]} \oplus V_{[4,4]} \oplus W_{10}.$$

The dimensions of the carrier spaces W_6, W_{10} and W_{30} are $6, 10, 30$*, respectively. The spaces $V_{[0,0]}$, $V_{[2,2]}$ and $V_{[4,4]}$ have the respective dimensions* $1, 9, 25$.

The symmetric representation W_{10} further decomposes into 5-dimensional $\mathrm{SO}(3)\times\mathrm{SO}(3)$ irreducible and non-equivalent bits: $W_{10} = V_{[4,0]} \oplus V_{[0,4]}$.

One can use the Casimir operator C to decompose the higher rank tensors as well. In particular, the third rank tensors, $t_{ijk}\in\bigotimes^3\mathbb{R}^9$, can be decomposed using the operator

$$\tilde{C}^{ijk}{}_{pqr}=C^{ij}{}_{pq}\delta^k{}_r+C^{ik}{}_{pr}\delta^j{}_q+C^{jk}{}_{qr}\delta^i{}_p.$$

This defines an endomorphism $\tilde{C}:\bigotimes^3\mathbb{R}^9\to\bigotimes^3\mathbb{R}^9$ given by

$$\bigotimes^3\mathbb{R}^9\ni t_{ijk}\stackrel{\tilde{C}}{\longmapsto}\tilde{C}(t)_{lmn}=\tilde{C}^{ijk}{}_{lmn}t_{ijk}\in\bigotimes^3\mathbb{R}^9.$$

Applying it to $\bigwedge^3\mathbb{R}^9$ we get:

Proposition 3.6. *The eigendecomposition of $\bigwedge^3\mathbb{R}^9$ by the operator $\tilde{C}$ is given by*

$$\bigwedge^3\mathbb{R}^9=Z_6\oplus Z_9\oplus Z_{30}\oplus Z_{39},$$

where

$$Z_6=\Big\{\bigwedge^3\mathbb{R}^9\ni H_{ijk}:\tilde{C}(H)_{ijk}=-5H_{ijk}\Big\}=V_{[2,0]}\oplus V_{[0,2]},$$

$$Z_9=\Big\{\bigwedge^3\mathbb{R}^9\ni H_{ijk}:\tilde{C}(H)_{ijk}=-4H_{ijk}\Big\}=V_{[2,2]},$$

$$Z_{30}=\Big\{\bigwedge^3\mathbb{R}^9\ni H_{ijk}:\tilde{C}(H)_{ijk}=-2H_{ijk}\Big\}=V_{[2,4]}\oplus V_{[4,2]},$$

$$Z_{39}=\Big\{\bigwedge^3\mathbb{R}^9\ni H_{ijk}:\tilde{C}(H)_{ijk}=0\Big\}=V_{[4,4]}\oplus V_{[0,6]}\oplus V_{[6,0]}.$$

A more refined decomposition of $\bigwedge^3\mathbb{R}^9$ is obtained by using the structural 4-form ω. It produces an endomorphism $\tilde{\omega}:\bigwedge^3\mathbb{R}^9\to\bigwedge^3\mathbb{R}^9$ given by

$$\bigwedge^3\mathbb{R}^9\ni t_{ijk}\stackrel{\tilde{\omega}}{\longmapsto}\tilde{\omega}(t)_{ijk}=3\omega^{lm}{}_{[ij}t_{k]lm}\in\bigwedge^3\mathbb{R}^9.$$

Proposition 3.7. *The eigendecomposition of $\bigwedge^3\mathbb{R}^9$ by the operator $\tilde{\omega}$ is given by*

$$\bigwedge^3\mathbb{R}^9=V_{[6,0]}\oplus V_{[0,6]}\oplus Z_{18}\oplus Z_{18'}\oplus Z_{34},$$

where

$$V_{[0,6]}=\Big\{\bigwedge^3\mathbb{R}^9\ni H_{ijk}:\tilde{\omega}(H)_{ijk}=-6H_{ijk}\Big\},$$

$$V_{[6,0]}=\Big\{\bigwedge^3\mathbb{R}^9\ni H_{ijk}:\tilde{\omega}(H)_{ijk}=6H_{ijk}\Big\},$$

$$Z_{18}=\Big\{\bigwedge^3\mathbb{R}^9\ni H_{ijk}:\tilde{\omega}(H)_{ijk}=4H_{ijk}\Big\}=V_{[2,4]}\oplus V_{[0,2]},$$

$$Z_{18'} = \Big\{ \bigwedge^3 \mathbb{R}^9 \ni H_{ijk} : \tilde{\omega}(H)_{ijk} = -4H_{ijk} \Big\} = V_{[4,2]} \oplus V_{[2,0]},$$
$$Z_{34} = \Big\{ \bigwedge^3 \mathbb{R}^9 \ni H_{ijk} : \tilde{\omega}(H)_{ijk} = 0 \Big\} = V_{[2,2]} \oplus V_{[4,4]}.$$

Using Propositions 3.6 and 3.7, we identify all the irreducible components of the $\mathrm{SO}(3) \times \mathrm{SO}(3)$ decomposition of $\bigwedge^3 \mathbb{R}^9$, e.g., $V_{[2,0]} = Z_6 \cap Z_{18'}$, $V_{[4,4]} = Z_{39} \cap Z_{34}$.

4. Irreducible SO(3) × SO(3) geometry in dimension nine

We are now prepared to define the basic object of our studies in this article.

Definition 4.1. The irreducible $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometry in dimension nine (M^9, g, Υ) is a 9-dimensional manifold M^9, equipped with totally symmetric tensor fields (g, Υ) of the respective ranks $\binom{0}{2}$ and $\binom{0}{3}$, which at each point $x \in M^9$, reduce the structure group $\mathrm{GL}(9, \mathbb{R})$ of the tangent space $\mathrm{T}_x M$ to the irreducible submodule $(\mathrm{SO}(3)\times\mathrm{SO}(3)) \subset \mathrm{SO}(9) \subset \mathrm{GL}(9, \mathbb{R})$.

Alternatively, the irreducible $\mathrm{SO}(3) \times \mathrm{SO}(3)$ geometry in dimension nine is a 9-dimensional manifold M^9, equipped with a differential 4-form ω which, at each point $x \in M^9$, reduces the structure group $\mathrm{GL}(9, \mathbb{R})$ of the tangent space $\mathrm{T}_x M$ to the irreducible submodule $(\mathrm{SO}(3) \times \mathrm{SO}(3)) \subset \mathrm{SO}(9) \subset \mathrm{GL}(9, \mathbb{R})$.

Definition 4.2. Let (M^9, g, Υ) be an irreducible $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometry in dimension nine. A diffeomorphism $\phi : M^9 \to M^9$ such that $\phi^* g = g$ and $\phi^*\Upsilon = \Upsilon$ is called a symmetry of (M^9, g, Υ). An infinitesimal symmetry of (M^9, g, Υ) is a vector field X on M^9 such that $\mathcal{L}_X g = 0$ and $\mathcal{L}_X \Upsilon = 0$.

Symmetries of (M^9, g, Υ) form a Lie group of symmetries, and infinitesimal symmetries form a Lie algebra of symmetries.

4.1. $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ connection. We want to analyze the properties of the irreducible $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometries in dimension nine by means of an $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$-valued *connection*. Since $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ seats naturally in $\mathfrak{so}(9)$, such connection is automatically *metric*. It also preserves Υ and ω.

For the purpose of this paper it is convenient to think about a connection as a Lie-algebra-valued 1-form Γ on M^9. Thus, the 1-form Γ of the connection we are going to define for geometries $(M^9, g, \Upsilon, \omega)$ has values in $\mathfrak{g} = \mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R \subset \mathfrak{so}(9)$, i.e., in the Lie algebra defined by (2.14)–(2.17).

For further use we need the following notion:

Definition 4.3. Given an irreducible $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometry $(M^9, g, \Upsilon, \omega)$, a coframe $\theta = (\theta^1, \theta^2, \theta^3, \theta^4, \theta^5, \theta^6, \theta^7, \theta^8, \theta^9)$ on M^9 is called adapted to it iff the structural tensors g, Υ and ω assume the form (2.8)–(2.10) in it.

Since the manifold $(M^9, g, \Upsilon, \omega)$ is equipped with a Riemannian metric g it carries the Levi-Civita connection $\overset{\text{LC}}{\Gamma}$ of g. This can be split into

$$\overset{\text{LC}}{\Gamma} = \Gamma + \text{‘the rest’}. \tag{4.1}$$

The only requirement that Γ has values in $\mathfrak{g}$ is to weak to make the above split unique. In order to achieve the uniqueness one has to impose some (e.g., algebraic) restrictions on ‘the rest’. The strongest of such restrictions is that ‘the rest’ $\equiv 0$. In Section 5 we will provide another much weaker condition that makes the split (4.1) unique. Here we do some preparatory steps to this.

Given the geometry $(M^9, g, \Upsilon, \omega)$ we use a coframe θ adapted to it and write down the structure equations as

$$\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^j = T^i, \quad \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = K^i{}_j. \tag{4.2}$$

Here the matrices $\Gamma = (\Gamma^i{}_j)$ have values in the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R \subset \mathfrak{so}(9)$ and therefore can be written as

$$\Gamma^i{}_j = \gamma^A e_A{}^i{}_j + \gamma^{A'} e_{A'}{}^i{}_j, \tag{4.3}$$

where $(\gamma^A, \gamma^{A'})$ are 1-forms on M^9, and the matrices $e_A = (e_A{}^i{}_j)$ and $e_{A'} = (e_{A'}{}^i{}_j)$ are given by (2.15)–(2.17).

The vector-valued 2-forms $T^i = \frac{1}{2} T^i{}_{jk} \theta^j \wedge \theta^k$ represent the ‘torsion’ of connection Γ. The ‘a priori’ $\mathfrak{so}(9)$-valued 2-forms $K^i{}_j = \frac{1}{2} K^i{}_{jkl} \theta^k \wedge \theta^l$ are actually $\mathfrak{g}$-valued. Hence they can also be written as

$$K^i{}_j = \kappa^A e_A{}^i{}_j + \kappa^{A'} e_{A'}{}^i{}_j,$$

where

$$\kappa^A = \tfrac{1}{2} \kappa^A{}_{ij} \theta^i \wedge \theta^j \quad \text{and} \quad \kappa^{A'} = \tfrac{1}{2} \kappa^{A'}{}_{ij} \theta^i \wedge \theta^j$$

are 2-forms on M^9. They describe the ‘curvature’ of the connection Γ.

We want that the first of the structure equations (4.2), which defines the torsion T of the $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ connection Γ, be nothing else but a reinterpretation of the ‘zero’-torsion equation

$$\mathrm{d}\theta^i + \overset{\text{LC}}{\Gamma}{}^i{}_j \wedge \theta^j = 0 \tag{4.4}$$

for the Levi-Civita connection $\overset{\text{LC}}{\Gamma}$. For this we need that

$$\overset{\text{LC}}{\Gamma}{}_{ijk} = \Gamma_{ijk} + \tfrac{1}{2}(T_{ijk} - T_{jik} - T_{kij}),$$

or, what is the same,

$$\overset{\text{LC}}{\Gamma}{}^i{}_j = \Gamma^i{}_j + \tfrac{1}{2} T^i{}_j - \tfrac{1}{2}(T_j{}^i{}_k + T_k{}^i{}_j)\theta^k. \tag{4.5}$$

Indeed, inserting (4.5) into (4.4), because of the symmetry of the last two terms in indices $\{jk\}$, we get precisely the first of the structure equations (4.2).

The structure equations (4.2) when written explicitly in terms of $(\theta^i, \gamma^A, \gamma^{A'})$ read

$$
(4.6)\quad \begin{cases}
d\theta^1 = \gamma^1 \wedge \theta^4 + \gamma^2 \wedge \theta^7 + \gamma^{1'} \wedge \theta^2 + \gamma^{2'} \wedge \theta^3 + T^1, \\
d\theta^2 = \gamma^1 \wedge \theta^5 + \gamma^2 \wedge \theta^8 - \gamma^{1'} \wedge \theta^1 + \gamma^{3'} \wedge \theta^3 + T^2, \\
d\theta^3 = \gamma^1 \wedge \theta^6 + \gamma^2 \wedge \theta^9 - \gamma^{2'} \wedge \theta^1 - \gamma^{3'} \wedge \theta^2 + T^3, \\
d\theta^4 = -\gamma^1 \wedge \theta^1 + \gamma^3 \wedge \theta^7 + \gamma^{1'} \wedge \theta^5 + \gamma^{2'} \wedge \theta^6 + T^4, \\
d\theta^5 = -\gamma^1 \wedge \theta^2 + \gamma^3 \wedge \theta^8 - \gamma^{1'} \wedge \theta^4 + \gamma^{3'} \wedge \theta^6 + T^5, \\
d\theta^6 = -\gamma^1 \wedge \theta^3 + \gamma^3 \wedge \theta^9 - \gamma^{2'} \wedge \theta^4 - \gamma^{3'} \wedge \theta^5 + T^6, \\
d\theta^7 = -\gamma^2 \wedge \theta^1 - \gamma^3 \wedge \theta^4 + \gamma^{1'} \wedge \theta^8 + \gamma^{2'} \wedge \theta^9 + T^7, \\
d\theta^8 = -\gamma^2 \wedge \theta^2 - \gamma^3 \wedge \theta^5 - \gamma^{1'} \wedge \theta^7 + \gamma^{3'} \wedge \theta^9 + T^8, \\
d\theta^9 = -\gamma^2 \wedge \theta^3 - \gamma^3 \wedge \theta^6 - \gamma^{2'} \wedge \theta^7 - \gamma^{3'} \wedge \theta^8 + T^9,
\end{cases}
$$

$$
(4.7)\quad \begin{cases}
d\gamma^1 = -\gamma^2 \wedge \gamma^3 + \kappa^1, \quad d\gamma^{1'} = -\gamma^{2'} \wedge \gamma^{3'} + \kappa^{1'}, \\
d\gamma^2 = -\gamma^3 \wedge \gamma^1 + \kappa^2, \quad d\gamma^{2'} = -\gamma^{3'} \wedge \gamma^{1'} + \kappa^{2'}, \\
d\gamma^3 = -\gamma^1 \wedge \gamma^2 + \kappa^3, \quad d\gamma^{3'} = -\gamma^{1'} \wedge \gamma^{2'} + \kappa^{3'}.
\end{cases}
$$

The equations (4.6)–(4.7), together with their integrability conditions implied by $d^2 \equiv 0$, encode all the geometric information about the most general irreducible SO(3) × SO(3) geometry in dimension nine.

4.2. $\mathfrak{so}(6)$ Cartan connection. The standard point of view on the structure equations (4.2) is that the equations are written just on M^9. This point of view was assumed when we have introduced (4.6)–(4.7) above.

The less standard point of view is in the spirit of E. Cartan: One considers equations (4.6)–(4.7) as written on the principal fiber bundle

$$\mathrm{SO}(3) \times \mathrm{SO}(3) \to P \to M^9,$$

with the structure group G. This is the Cartan bundle for the geometry $(M^9, g, \Upsilon, \omega)$. In this point of view the $(9 + 3 + 3) = 15$ one-forms $(\theta^i, \gamma^A, \gamma^{A'})$ are considered to live on P, rather than on M^9. They are linearly independent at each point of P defining a preferred coframe there.

The system may be ultimately interpreted as a system for the curvature of an $\mathfrak{so}(6)$-valued Cartan connection on P. This connection is defined in terms of the preferred coframe $(\theta^i, \gamma^A, \gamma^{A'})$ on P as follows. We define a 6×6 real *antisymmetric* matrix

$$
\Gamma_{\text{Cartan}} = \begin{pmatrix}
0 & -\gamma^1 & -\gamma^2 & | & \theta^1 & \theta^2 & \theta^3 \\
\gamma^1 & 0 & -\gamma^3 & | & \theta^4 & \theta^5 & \theta^6 \\
\gamma^2 & \gamma^3 & 0 & | & \theta^7 & \theta^8 & \theta^9 \\
- & - & - & - & - & - & - \\
-\theta^1 & -\theta^4 & -\theta^7 & | & 0 & -\gamma^{1'} & -\gamma^{2'} \\
-\theta^2 & -\theta^5 & -\theta^8 & | & \gamma^{1'} & 0 & -\gamma^{3'} \\
-\theta^3 & -\theta^6 & -\theta^9 & | & \gamma^{2'} & \gamma^{3'} & 0
\end{pmatrix}
$$

of 1-forms, and a 9×9 matrix of 2-forms K_0 given by $K_0 = \kappa_0^A e_A + \kappa_0^{A'} e_{A'}$.

The forms $(\kappa_0^A, \kappa_0^{A'})$ are the respective basis of $\mathfrak{so}(3)_R$ and $\mathfrak{so}(3)_L$ as defined in Remark 3.4. The matrix Γ_{Cartan} of 1-forms on P, being antisymmetric, has values in the Lie algebra $\mathfrak{so}(6)$, $\Gamma_{\text{Cartan}} \in \mathfrak{so}(6) \otimes \bigwedge^1(P)$. It defines an $\mathfrak{so}(6)$-valued *Cartan connection* on P. Due to the equations (4.6)–(4.7) its *curvature*,

$$\tilde{R} = \mathrm{d}\Gamma_{\text{Cartan}} + \Gamma_{\text{Cartan}} \wedge \Gamma_{\text{Cartan}},$$

has the form

$$\tilde{R} = \begin{pmatrix} 0 & -R^1 & -R^2 & \mid & T^1 & T^2 & T^3 \\ R^1 & 0 & -R^3 & \mid & T^4 & T^5 & T^6 \\ R^2 & R^3 & 0 & \mid & T^7 & T^8 & T^9 \\ - & - & - & - & - & - & - \\ -T^1 & -T^4 & -T^7 & \mid & 0 & -R^{1'} & -R^{2'} \\ -T^2 & -T^5 & -T^8 & \mid & R^{1'} & 0 & -R^{3'} \\ -T^3 & -T^6 & -T^9 & \mid & R^{2'} & R^{3'} & 0 \end{pmatrix},$$

where

$$R^A = \kappa^A - \kappa_0^A, \quad R^{A'} = \kappa^{A'} - \kappa_0^{A'}, \quad A, A' = 1, 2, 3.$$

Thus the curvature of the $\mathfrak{so}(6)$-Cartan connection keeps track of both the curvature K and the torsion T of the $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ connection Γ. In particular the connection Γ_{Cartan} is *flat* iff $T \equiv 0$, $R \equiv 0$, i.e., iff the connection Γ has *vanishing* torsion, $T \equiv 0$, and has *constant positive curvature*, $K = K_0$.

4.3. No torsion. It is very easy to find all 9-dimensional irreducible SO(3) × SO(3) geometries with vanishing torsion. It follows that the system (4.2), or equivalently (4.6)–(4.7), with $T^i \equiv 0$, $i = 1, 2, \ldots, 9$, is so rigid on P that it admits only a 1-parameter family of solutions. More specifically, the first Bianchi identities, $\mathrm{d}(\mathrm{d}\theta^i) \equiv 0$, $i = 1, 2, \ldots, 9$, applied to the equations (4.6), with $T^i \equiv 0$, very quickly show that the curvatures κ^A and $\kappa^{A'}$ must be of the form $\kappa^A = s\kappa_0^A$ and $\kappa^{A'} = s\kappa_0^{A'}$, where s is a real function on P. Then, the second Bianchi identities, $\mathrm{d}(\mathrm{d}\gamma^A) \equiv 0 \equiv \mathrm{d}(\mathrm{d}\gamma^{A'})$, applied to (4.7) with the κ's as above, show that $\mathrm{d}s \equiv 0$, i.e., that the function s is constant on P. This proves the following proposition, which also follows from Berger's classification.

Proposition 4.4. *All irreducible* SO(3) × SO(3) *geometries* $(M^9, g, \Upsilon, \omega)$ *with vanishing torsion are locally isometric to one of the symmetric spaces*

$$M^9 = \mathcal{G}/(\mathrm{SO}(3) \times \mathrm{SO}(3)),$$

where $\mathcal{G} = \mathrm{SO}(6), \mathrm{SO}(3,3)$*, or* $(\mathrm{SO}(3) \times \mathrm{SO}(3)) \rtimes_\rho \mathbb{R}^9$.

The Riemannian metric g, the tensor Υ, and the 4*-form ω defining the* SO(3) × SO(3) *structure are defined in terms of the left invariant* 1*-forms* $(\theta^1, \theta^2, \ldots, \theta^9)$*, which on* $P = \mathcal{G}$ *satisfy equations* (4.6)–(4.7) *and* $T^i \equiv 0$*. These forms, via* (2.8)–(2.10)*, define objects* g, Υ *and ω on P, which descend to a well defined Riemannian metric g, the symmetric tensor Υ and the* 4*-form ω on* $M^9 = \mathcal{G}/(\mathrm{SO}(3) \times \mathrm{SO}(3))$*. The Levi-Civita connection of the metric g has Einstein Ricci tensor on M^9,*

$$\overset{\mathrm{LC}}{\mathrm{Ric}}(g) = 4sg,$$

and has holonomy reduced to $SO(3) \times SO(3)$. *The metric* g *is flat if and only if* $s = 0$. *Otherwise it is not conformally flat. The Cartan* $\mathfrak{so}(6)$ *connection for these structures has constant curvature,*

$$\tilde{R} = (s-1)\begin{pmatrix} 0 & -\kappa_0^1 & -\kappa_0^2 & | & 0 & 0 & 0 \\ \kappa_0^1 & 0 & -\kappa_0^3 & | & 0 & 0 & 0 \\ \kappa_0^2 & \kappa_0^3 & 0 & | & 0 & 0 & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & | & 0 & -\kappa_0^{1'} & -\kappa_0^{2'} \\ 0 & 0 & 0 & | & \kappa_0^{1'} & 0 & -\kappa_0^{3'} \\ 0 & 0 & 0 & | & \kappa_0^{2'} & \kappa_0^{3'} & 0 \end{pmatrix},$$

and is flat if and only if $s = 1$. *The symmetry group of these structures is* $\mathcal{G} = SO(6)$ *for* $s > 0$, $SO(3,3)$ *for* $s < 0$ *and* $(SO(3) \times SO(3)) \rtimes_\rho \mathbb{R}^9$ *for* $s = 0$.

Remark 4.5. The space $SO(6)/(SO(3) \times SO(3))$ appearing in this proposition is just the Grassmannian $Gr(3,6)$ of oriented 3-planes in 6-space and the 4-form ω coincides (up to a multiple) with the first Pontryagin class of the canonical 3-plane bundle over $Gr(3,6)$ (see [7, 15]) and the 5-form $*\omega$ is its dual. Indeed, ω is induced by the first Pontryagin class of the canonical 3-plane bundle over the Grassmannian $Gr(3,7)$. In his Ph.D. thesis C. Michael [15] showed that $*\omega$ calibrates the special Lagrangian Grassmannian $SU(3)/SO(3) \subset Gr(3,6)$ and its congruent submanifolds (and nothing else). Moreover, he classified also the 8-dimensional submanifolds of $Gr(3,7)$ that are calibrated by the dual of the first Pontryagin class of the canonical 3-plane bundle [11].

4.4. Spin connections. Denote by $\mathcal{C}_9$ the real Clifford algebra of the positive definite quadratic form. $\mathcal{C}_9$ is generated by the vectors of $\mathbb{R}^9$ and the relation

$$v \cdot w + w \cdot v = 2\langle v, w\rangle, \quad v, w \in \mathbb{R}^9,$$

holds. The spin representation of the group Spin(9) is a faithful real representation in the 16-dimensional space Δ_9 of real spinors and it is the unique irreducible representation of the group Spin(9) in dimension 16. With respect to this representation the orthonormal vectors $(\mathbf{e}_1, \ldots, \mathbf{e}_9)$ may be represented by the matrices

$$\mathbf{e}_1 = \sum_{k=0}^{15} M_{16-k,k+1}, \quad \mathbf{e}_2 = i\sum_{k=0}^{15}(-1)^k M_{16-k,k+1},$$

$$\mathbf{e}_3 = \sum_{k=0}^{7}(M_{15-2k,2k+1} - M_{16-2k,2k+2}),$$

$$\mathbf{e}_4 = i\sum_{k=0}^{7}(-1)^k(M_{15-2k,2k+1} + M_{16-2k,2k+2}),$$

$$\mathbf{e}_5 = \sum_{k=0}^{3}(M_{13-4k,4k+1} + M_{14-4k,4k+2} - M_{15-4k,4k+3} - M_{16-4k,4k+4}),$$

$$\mathbf{e}_6 = i\sum_{k=0}^{3}(-1)^k(M_{13-4k,4k+1} + M_{14-4k,4k+2} + M_{15-4k,4k+3} + M_{16-4k,4k+4}),$$
$$\mathbf{e}_7 = \sum_{k=0}^{3}(M_{9+k,k+1} - M_{13+k,k+5} + M_{1+k,k+9} - M_{5+k,k+13}),$$
$$\mathbf{e}_8 = i\sum_{k=0}^{7}(M_{9+k,k+1} - M_{1+k,k+9}), \quad \mathbf{e}_9 = \sum_{k=0}^{7}(M_{k+1,k+1} - M_{k+9,k+9}),$$

where by $M_{i,j}$ we denote the 16×16-matrix having value 1 at its entry (i, j) and value 0 in all the remaining entries. In particular we have

$$\mathbf{e}_i^2 = 1, \quad \mathbf{e}_i \cdot \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{e}_i = 0, \quad \text{for all } i, j = 1, 2, \ldots, 9.$$

The double covering homomorphism $\mathrm{Spin}(9) \to \mathrm{SO}(9)$ induces the isomorphism of Lie algebras $\mathfrak{spin}(9) \to \mathfrak{so}(9)$. By means of this isomorphism the basis of the Lie algebra $\mathfrak{spin}(3)_L \oplus \mathfrak{spin}(3)_L$ corresponding to the basis $(e_1, e_2, e_3, e'_1, e'_2, e'_3)$ of $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ is

$$\mathbf{E}_1 = -\frac{1}{2}(\mathbf{e}_1\cdot\mathbf{e}_4 + \mathbf{e}_2\cdot\mathbf{e}_5 + \mathbf{e}_3\cdot\mathbf{e}_6), \quad \mathbf{E}'_1 = -\frac{1}{2}(\mathbf{e}_1\cdot\mathbf{e}_2 + \mathbf{e}_4\cdot\mathbf{e}_5 + \mathbf{e}_6\cdot\mathbf{e}_8),$$
$$\mathbf{E}_2 = -\frac{1}{2}(\mathbf{e}_1\cdot\mathbf{e}_7 + \mathbf{e}_2\cdot\mathbf{e}_8 + \mathbf{e}_3\cdot\mathbf{e}_9), \quad \mathbf{E}'_2 = -\frac{1}{2}(\mathbf{e}_1\cdot\mathbf{e}_3 + \mathbf{e}_4\cdot\mathbf{e}_6 + \mathbf{e}_7\cdot\mathbf{e}_3),$$
$$\mathbf{E}_3 = -\frac{1}{2}(\mathbf{e}_4\cdot\mathbf{e}_7 + \mathbf{e}_5\cdot\mathbf{e}_8 + \mathbf{e}_6\cdot\mathbf{e}_9), \quad \mathbf{E}'_3 = -\frac{1}{2}(\mathbf{e}_2\cdot\mathbf{e}_3 + \mathbf{e}_5\cdot\mathbf{e}_6 + \mathbf{e}_8\cdot\mathbf{e}_9).$$

Thus, in this spinorial 16-dimensional representation, we have

$$\begin{aligned}\mathfrak{spin}(3)_L \oplus \mathfrak{spin}(3)_L &= \mathrm{Span}(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3) \oplus \mathrm{Span}(\mathbf{E}'_1, \mathbf{E}'_2, \mathbf{E}'_3)\\ &\subset \mathfrak{spin}(9) = \mathrm{Span}\big(\tfrac{1}{2}\mathbf{e}_i\mathbf{e}_j, i < j = 1, 2, \ldots, 9\big).\end{aligned}$$

Now given an $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$-valued connection $\Gamma = \gamma^A e_A + \gamma^{A'} e_{A'}$ as in (4.3), we define a spin connection

$$\Gamma_{\mathfrak{spin}} = \gamma^A \mathbf{E}_A + \gamma^{A'}\mathbf{E}_{A'} \in (\mathfrak{spin}(3)_L \oplus \mathfrak{spin}(3)_R) \otimes \mathbb{R}^9.$$

4.5. $\mathfrak{so}(3)_L$ and $\mathfrak{so}(3)_R$ connections. Since every $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R)$-connection Γ, as defined in Section 4.1, has values in the *direct sum* of Lie algebras $\mathfrak{so}(3)_L$ and $\mathfrak{so}(3)_R$, it naturally splits into

$$\Gamma = \overset{+}{\Gamma} + \overset{-}{\Gamma}, \quad \text{with } \overset{+}{\Gamma} \in \mathfrak{so}(3)_L \otimes \mathbb{R}^9 \text{ and } \overset{-}{\Gamma} \in \mathfrak{so}(3)_R \otimes \mathbb{R}^9.$$

Because $\mathfrak{so}(3)_L$ commutes with $\mathfrak{so}(3)_R$, this split defines two *independent* $\mathfrak{so}(3)$-valued connections $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$. The two independent curvatures of these connections

$$\overset{+}{\Omega}{}^i{}_j = \mathrm{d}\overset{+}{\Gamma}{}^i{}_j + \overset{+}{\Gamma}{}^i{}_k \wedge \overset{+}{\Gamma}{}^k{}_j = \tfrac{1}{2}\overset{+}{R}{}^i{}_{jkl}\theta^k \wedge \theta^l$$

and

$$\overset{-}{\Omega}{}^i{}_j = \mathrm{d}\overset{-}{\Gamma}{}^i{}_j + \overset{-}{\Gamma}{}^i{}_k \wedge \overset{-}{\Gamma}{}^k{}_j = \tfrac{1}{2}\overset{-}{R}{}^i{}_{jkl}\theta^k \wedge \theta^l$$

are equal to the respective $\mathfrak{so}(3)_L$ and $\mathfrak{so}(3)_R$ parts of the curvature of Γ:

$$\Omega^i{}_j = \mathrm{d}\Gamma^i{}_j + \Gamma^i{}_k \wedge \Gamma^k{}_j = \overset{+}{\Omega}{}^i{}_j + \overset{-}{\Omega}{}^i{}_j.$$

Moreover, since, via the identifications $\mathfrak{so}(3)_L = \mathfrak{so}(3)_L \oplus 0$ and $\mathfrak{so}(3)_R = 0 \oplus \mathfrak{so}(3)_R$, both $\mathfrak{so}(3)_L$ and $\mathfrak{so}(3)_R$ are naturally included in $\mathfrak{so}(9)$, we can define not only the Ricci tensor of Γ: $R_{ij} = R^k{}_{ikj}$, but also the corresponding Ricci tensors of $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$:

$$\overset{+}{R}_{ij} = \overset{+}{R}{}^k{}_{ikj}, \quad \overset{-}{R}_{ij} = \overset{-}{R}{}^k{}_{ikj}.$$

Thus an irreducible $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometry $(M^9, g, \Upsilon, \omega)$ equipped with a $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R)$ connection Γ can be Einstein in several meanings:

(1) with respect to the Levi-Civita connection $\overset{\mathrm{LC}}{\Gamma}$, i.e., $\overset{\mathrm{LC}}{\mathrm{Ric}}_{ij} = \lambda g_{ij}$;

(2) with respect to the $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R)$ connection Γ, i.e., $R_{ij} = \lambda g_{ij}$;

(3) with respect to the $\mathfrak{so}(3)_L$ connection $\overset{+}{\Gamma}$, i.e., $\overset{+}{R}_{ij} = \lambda g_{ij}$;

(4) with respect to the $\mathfrak{so}(3)_R$ connection $\overset{-}{\Gamma}$, i.e., $\overset{-}{R}_{ij} = \lambda g_{ij}$.

Of course the functions λ appearing in the four above formulae do not need to be the same.

Calculating the Ricci curvature R_{ij} for the 'no-torsion' examples from Section 4.3, obviously yields

$$\overset{\mathrm{LC}}{\mathrm{Ric}}_{ij} = R_{ij} = 4s g_{ij},$$

since the connections $\overset{\mathrm{LC}}{\Gamma}$ and Γ coincide. But it follows that in these examples also the connections $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$ are Einstein. Actually we have

$$\overset{+}{R}_{ij} = \overset{-}{R}_{ij} = 2s g_{ij}$$

for all the examples in Section 4.3.

Similar considerations as for connections Γ, $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$, can be performed for the spin connection $\Gamma_{\mathfrak{spin}}$. Here we have

$$\Gamma_{\mathfrak{spin}} = \overset{+}{\Gamma}_{\mathfrak{spin}} + \overset{-}{\Gamma}_{\mathfrak{spin}},$$

with $\overset{+}{\Gamma} \in \mathfrak{spin}(3)_L \otimes \mathbb{R}^9$ and $\overset{-}{\Gamma}_{\mathfrak{spin}} \in \mathfrak{spin}(3)_R \otimes \mathbb{R}^9$. Since $\mathfrak{spin}(3)_L$ commutes with $\mathfrak{spin}(3)_R$ we again have two independent connections $\overset{+}{\Gamma}_{\mathfrak{spin}}$ and $\overset{-}{\Gamma}_{\mathfrak{spin}}$. Since they yield essentially the same information as $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$ we will not comment about them any further.

5. Nearly integrable SO(3) × SO(3) geometries

In the previous section we discussed general $\mathrm{SO}(3) \times \mathrm{SO}(3)$ geometries in dimension nine, and general $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ connections Γ, which were obtained from the Levi-Civita connection $\overset{\mathrm{LC}}{\Gamma}$ via the split (4.5). The problem with such connections is that in general they are *not* unique. In this section we will restrict ourselves to a subclass of irreducible $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometries in dimension nine for which the connection Γ appearing in the formula (4.5) will be uniquely defined. This class is distinguished by the following definition.

Definition 5.1. An irreducible SO(3) × SO(3) geometry $(M^9, g, \Upsilon, \omega)$ is called nearly integrable iff its structural tensor Υ is a Killing tensor with respect to the Levi-Civita connection, i.e., iff

$$\overset{\text{LC}}{\nabla}_X \Upsilon(X, X, X) = 0, \quad \text{for all } X \in \mathrm{T}M. \tag{5.1}$$

We first write the condition (5.1) in a coframe θ adapted to $(M^9, g, \Upsilon, \omega)$. In such a coframe we define the Levi-Civita connection coefficients $\overset{\text{LC}}{\Gamma}{}^j{}_{ki}$ to be given by

$$\overset{\text{LC}}{\nabla}_{X_i}\theta^j = -\overset{\text{LC}}{\Gamma}{}^j{}_{ki}\theta^k,$$

where X_i are the vector fields X_i dual on M^9 to the 1-forms θ^i,

$$X_i \lrcorner\, \theta^j = \delta^j{}_i.$$

The coefficients $\overset{\text{LC}}{\Gamma}{}^j{}_{ki}$ are related to the Levi-Civita connection 1-form $\overset{\text{LC}}{\Gamma} = (\overset{\text{LC}}{\Gamma}{}^i{}_j)$ via

$$\overset{\text{LC}}{\Gamma}{}^i{}_j = \overset{\text{LC}}{\Gamma}{}^i{}_{jk}\theta^k.$$

In this setting the condition (5.1) reads

$$\overset{\text{LC}}{\Gamma}{}^m{}_{(ji}\Upsilon_{kl)m} \equiv 0. \tag{5.2}$$

This motivates an introduction of the map $\Upsilon' : \bigwedge^2 \mathbb{R}^9 \otimes \mathbb{R}^9 \mapsto \bigodot^4 \mathbb{R}^9$ such that

$$\begin{aligned} \Upsilon'(\overset{\text{LC}}{\Gamma})_{ijkl} &= 12\overset{\text{LC}}{\Gamma}{}^p{}_{(ji}\Upsilon_{kl)p} \\ &= \overset{\text{LC}}{\Gamma}{}^p{}_{ji}\Upsilon_{pkl} + \overset{\text{LC}}{\Gamma}{}^p{}_{ki}\Upsilon_{jpl} + \overset{\text{LC}}{\Gamma}{}^p{}_{li}\Upsilon_{jkp} \\ &\quad + \overset{\text{LC}}{\Gamma}{}^p{}_{ij}\Upsilon_{pkl} + \overset{\text{LC}}{\Gamma}{}^p{}_{kj}\Upsilon_{ipl} + \overset{\text{LC}}{\Gamma}{}^p{}_{lj}\Upsilon_{ikp} \\ &\quad + \overset{\text{LC}}{\Gamma}{}^p{}_{ik}\Upsilon_{pjl} + \overset{\text{LC}}{\Gamma}{}^p{}_{jk}\Upsilon_{ipl} + \overset{\text{LC}}{\Gamma}{}^p{}_{lk}\Upsilon_{ijp} \\ &\quad + \overset{\text{LC}}{\Gamma}{}^p{}_{il}\Upsilon_{pjk} + \overset{\text{LC}}{\Gamma}{}^p{}_{jl}\Upsilon_{ipk} + \overset{\text{LC}}{\Gamma}{}^p{}_{kl}\Upsilon_{ijp}. \end{aligned} \tag{5.3}$$

Comparing this with (5.2) we have the following proposition.

Proposition 5.2. *An irreducible* SO(3) × SO(3) *geometry* $(M^9, g, \Upsilon, \omega)$ *is nearly integrable if and only if its Levi-Civita connection* $\overset{\text{LC}}{\Gamma}$ *is in* $\ker \Upsilon'$.

It is worth noting that each of the last four rows of (5.3) resembles the left-hand side of the equality

$$X^p{}_j\Upsilon_{pkl} + X^p{}_k\Upsilon_{jpl} + X^p{}_l\Upsilon_{jkp} = 0$$

satisfied by every matrix $X \in \mathfrak{g} = \mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$. Thus, $\mathfrak{g} \otimes \mathbb{R}^9 \subset \ker \Upsilon'$. Now let us consider tensors $T^i{}_{jk}$, such that $T_{ijk} = g_{il}T^l{}_{jk}$ is totally antisymmetric, $T_{ijk} = T_{[ijk]} \in \bigwedge^3 \mathbb{R}^9$. Via g we identify the space of the considered tensors $T^i{}_{jk}$ with $\bigwedge^3 \mathbb{R}^9$.

Because of the antisymmetry in the last pair of indices, and due to the first equality in (5.3), every such $T^i{}_{jk}$ also belongs to $\ker \Upsilon'$. Since $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R) \otimes \mathbb{R}^9 \subset \ker \Upsilon'$ and $\bigwedge^3 \mathbb{R}^9 \subset \ker \Upsilon'$, this proves the following lemma.

Lemma 5.3. $\big(\big[(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R) \otimes \mathbb{R}^9\big] + \bigwedge^3 \mathbb{R}^9\big) \subset \ker \Upsilon'$.

It is now crucial to calculate the dimension of $\ker \Upsilon'$. We did it using the symbolic algebra calculation softwares Mathematica and, independently, Maple, by solving equations (5.2) for the most general

$$\overset{\scriptscriptstyle LC}{\Gamma}{}^i{}_{jk} \in \mathfrak{so}(9) \otimes \mathbb{R}^9.$$

It follows that the equations impose the number 186 of independent conditions on the $\frac{9\times 8}{2} \times 9 = 324$ free coefficients $\overset{\scriptscriptstyle LC}{\Gamma}{}^i{}_{jk}$. Thus we have

Lemma 5.4. $\dim \ker \Upsilon' = 324 - 186 = 138$.

Again with the help of the Mathematica/Maple softwares we calculated the intersection of $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R) \otimes \mathbb{R}^9$ with $\bigwedge^3 \mathbb{R}^9$. In this way we obtained

Lemma 5.5. $\big((\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R) \otimes \mathbb{R}^9\big) \cap \bigwedge^3 \mathbb{R}^9 = \{0\}$.

Comparing the dimension of $(\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R) \otimes \mathbb{R}^9$, which is 54, with the dimension of $\bigwedge^3 \mathbb{R}^9$, which is 84, and $\dim \ker \Upsilon' = 138$ and using the above lemmas, we get

Proposition 5.6. $\ker \Upsilon' = \big((\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R) \otimes \mathbb{R}^9\big) \oplus \bigwedge^3 \mathbb{R}^9$.

This leads to the following theorem.

Theorem 5.7. *Every nearly integrable irreducible geometry $(M^9, g, \Upsilon, \omega)$ defines an $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$-valued connection, whose torsion is totally antisymmetric. This connection is unique and defined in an adapted coframe θ via the formula*

$$\overset{\scriptscriptstyle LC}{\Gamma}{}^i{}_{jk} = \Gamma^i{}_{jk} + \tfrac{1}{2} T^i{}_{jk}, \tag{5.4}$$

where $\overset{\scriptscriptstyle LC}{\Gamma}{}^i{}_{jk}$ are the Levi-Civita connection coefficients in the coframe θ, $\Gamma = (\Gamma^i{}_j) = (\Gamma^i{}_{jk}\theta^k)$ is a 1-form on M^9 with values in $\mathfrak{g} = \mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$, and $T_{ijk} = g_{il} T^l{}_{jk}$ is totally antisymmetric, i.e., $T_{ijk} = T_{[ijk]}$.

Conversely, every irreducible $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ geometry in dimension nine admitting a unique $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$ connection with totally skew symmetric torsion is nearly integrable.

Proof. See Propositions 5.6 and 5.2. □

Definition 5.8. The unique $\mathfrak{so}(3)_L \oplus \mathfrak{so}(3)_R$-valued connection Γ of a nearly integrable $SO(3) \times SO(3)$ geometry $(M^9, g, \Upsilon, \omega)$ as described in Theorem 5.7 is called *characteristic connection* for the geometry $(M^9, g, \Upsilon, \omega)$.

We close this section with a proposition, which relates the torsion of the characteristic connection of a nearly integrable structure $(M^9, g, \Upsilon, \omega)$, and the exterior derivatives $d\omega$ and $d * \omega$.

Proposition 5.9. *The derivatives* $\mathrm{d}\omega$ *and* $\mathrm{d}*\omega$ *of the structural* 4*-forms* ω *and* $*\omega$ *of a nearly integrable geometry* $(M^9, g, \Upsilon, \omega)$ *decompose as*

$$\mathrm{d}\omega \in V_{[2,2]} \oplus V_{[2,4]} \oplus V_{[4,2]} \oplus V_{[4,4]} \tag{5.5}$$

and

$$\mathrm{d}*\omega \in V_{[0,2]} \oplus V_{[2,0]} \oplus V_{[0,6]} \oplus V_{[6,0]} \oplus V_{[2,4]} \oplus V_{[4,2]}. \tag{5.6}$$

In particular, the torsion $T \in \bigwedge^3 \mathbb{R}^9$ *of the characteristic connection is related to these decompositions via*

$$\mathrm{d}\omega \equiv 0 \Longleftrightarrow \Big(T \in V_{[0,2]} \oplus V_{[2,0]} \oplus V_{[0,6]} \oplus V_{[6,0]} \subset \bigwedge^3 \mathbb{R}^9\Big)$$

and

$$\mathrm{d}*\omega \equiv 0 \Longleftrightarrow \Big(T \in V_{[2,2]} \oplus V_{[4,4]} \subset \bigwedge^3 \mathbb{R}^9\Big).$$

Proof. It follows from the first structure equations (4.6) that the derivatives $\mathrm{d}\omega$ and $\mathrm{d}*\omega$ are totally expressible in terms of the torsion components T_{ijk} of the characteristic connection. It is also clear that the relations between $\mathrm{d}\omega$ and $\mathrm{d}*\omega$ and the torsion is algebraic, and linear in the components of T. Thus each of $\mathrm{d}\omega$ and $\mathrm{d}*\omega$ must be contained in an 84-dimensional $\mathrm{SO}(3)\times\mathrm{SO}(3)$-invariant submodule of the respective modules

$$\bigwedge^5 \mathbb{R}^9 \simeq \bigwedge^4 V_{[2,2]} \quad \text{and} \quad \bigwedge^6 \mathbb{R}^9 \simeq \bigwedge^3 V_{[2,2]}.$$

Now a quick calculation using Maple/Mathematica shows that the equation $\mathrm{d}\omega \equiv 0$ imposes 64 conditions on the 84 components of the torsion. Similarly, one can check that the equation $\mathrm{d}*\omega \equiv 0$ imposes 50 conditions on the torsion. Thus $\mathrm{d}\omega$ has 64 independent components, and $\mathrm{d}*\omega$ has 50 independent components.

Comparison of these numbers with the $\mathrm{SO}(3)\times\mathrm{SO}(3)$ decompositions of $\bigwedge^4 V_{[2,2]}$ and $\bigwedge^3 V_{[2,2]}$ given in Proposition 3.1 quickly yields the conclusion that $\mathrm{d}\omega$ and $\mathrm{d}*\omega$ must be in the submodules of $\bigwedge^5 \mathbb{R}^9$ and $\bigwedge^6 \mathbb{R}^9$ indicated in the proposition. To get the decompositions (5.5)–(5.6) explicitly, dualize the forms $\mathrm{d}\omega$ and $\mathrm{d}*\omega$, i.e., calculate $*\mathrm{d}\omega$ and $*\mathrm{d}*\omega$, and use the respective operators defined in Section 3. □

Note that it follows from this proposition that if the torsion T of the characteristic connection has a component in $V_{[2,4]}$, or in $V_{[4,2]}$, then the forms $\mathrm{d}\omega$ and $\mathrm{d}*\omega$ are both non-vanishing.

6. Examples of nearly integrable SO(3) × SO(3) geometries

We begin this section by considering the most general situation of a *nearly integrable* irreducible geometry $(M^9, g, \Upsilon, \omega)$. Thus, its characteristic connection has a general torsion in $\bigwedge^3 \mathbb{R}^9$.

The group SO(3) × SO(3) acts on the torsion space $\bigwedge^3 \mathbb{R}^9$ in the following way. One of the SO(3) groups in SO(3) × SO(3) is just $\exp(\mathfrak{so}(3)_L)$. The other is $\exp(\mathfrak{so}(3)_R)$. Thus we have

$$\mathrm{SO}(3) \times \mathrm{SO}(3) = \mathrm{SO}(3)_L \times \mathrm{SO}(3)_R$$

with

$$\mathrm{SO}(3)_L = \exp(\mathfrak{so}(3)_L), \quad \mathrm{SO}(3)_R = \exp(\mathfrak{so}(3)_R).$$

The 9×9 matrices $h \in \mathrm{SO}(3)_L$ and $h' \in \mathrm{SO}(3)_R$ act on the torsion coefficients T_{ijk} via

$$T_{ijk} \stackrel{h}{\mapsto} (hT)_{ijk} = h^p{}_i h^q{}_j h^r{}_k T_{pqr}, \tag{6.1}$$

$$T_{ijk} \stackrel{h'}{\mapsto} (h'T)_{ijk} = h'^p{}_i h'^q{}_j h'^r{}_k T_{pqr}. \tag{6.2}$$

There is an obvious invariant of both of these actions. It is the square of the torsion:

$$\|T\|^2 = T_{ijk} T_{pqr} g^{ip} g^{jq} g^{kr}. \tag{6.3}$$

Thus the 84-dimensional space $\bigwedge^3 \mathbb{R}^9$ is foliated by the SO(3) × SO(3)-invariant 83-dimensional spheres

$$\mathbb{S}_T = \Big\{ T_{ijk} \in \bigwedge^3 \mathbb{R}^9 : T_{ijk} T_{pqr} g^{ip} g^{jq} g^{kr} = r^2 \Big\},$$

parametrized by the real parameter $r > 0$. The group SO(3) × SO(3) preserves these spheres. But, for the dimensional reasons, its action is not transitive on them. Note that if one restricts the torsion, forcing it to lie in an SO(3) × SO(3)-invariant submodule of $\bigwedge^3 \mathbb{R}^9$, then the restrictions of the spheres $\mathbb{S}_T$ to this submodule will be still invariant with respect to both actions, but the quadrics obtained by this restriction will have smaller dimension than 83.

For example when the torsion T_{ijk} is in the invariant module $\mathfrak{so}(3)_L \subset \bigwedge^3 \mathbb{R}^9$, the spheres $\mathbb{S}_T$ restrict to 2-dimensional spheres. In such case the 3-dimensional torsion space $\mathfrak{so}(3)_L \simeq \mathbb{R}^3$ is foliated by 2-dimensional spheres with radius r and center at the origin – the zero torsion. The orbit space of the action of the groups $\mathrm{SO}(3)_L$ and $\mathrm{SO}(3)_R$ on these spheres will be discussed in the next subsection.

6.1. Torsion in $V_{[0,2]} = \mathfrak{so}(3)_L$. The aim of this section is to find all *nearly integrable* irreducible geometries $(M^9, g, \Upsilon, \omega)$, whose characteristic connection Γ has totally skew symmetric torsion T in the irreducible representation $\mathfrak{so}(3)_L$. Thus

$$T \in \mathfrak{so}(3)_L \subset \bigwedge^3 \mathbb{R}^9$$

in this subsection.

The assumption that $T \in \mathfrak{so}(3)_L \subset \bigwedge^3 \mathbb{R}^9$ is equivalent to the requirement that, in a coframe θ^i adapted to $(M^9, g, \Upsilon, \omega)$, we have

$$T^i = \tfrac{1}{2} g^{ij} T_{jkl} \theta^k \wedge \theta^l, \quad T_{ijk} = T_{[ijk]}, \quad \tilde{C}(T)_{ijk} = -5T_{ijk}, \quad \tilde{\omega}(T)_{ijk} = 4T_{ijk}.$$

The last two conditions mean that, in accordance with the results of Section 3, the torsion is in the intersection $Z_6 \cap Z_{18}$. These algebraic conditions for T_{ijk} can be easily solved. The result is summarized in the following proposition.

Proposition 6.1. *In an adapted coframe (θ^i) the $\mathfrak{so}(3)_L$ torsion of the characteristic connection of a nearly integrable geometry $(M^9, g, \Upsilon, \omega)$ reads*

$$
(6.4)\quad \begin{cases}
T^1 = -3t_3\theta^2\wedge\theta^3 + t_2\theta^2\wedge\theta^6 - t_1\theta^2\wedge\theta^9 - t_2\theta^3\wedge\theta^5 + t_1\theta^3\wedge\theta^8 \\
\qquad - t_3\theta^5\wedge\theta^6 - t_3\theta^8\wedge\theta^9, \\
T^2 = 3t_3\theta^1\wedge\theta^3 - t_2\theta^1\wedge\theta^6 + t_1\theta^1\wedge\theta^9 + t_2\theta^3\wedge\theta^4 - t_1\theta^3\wedge\theta^7 \\
\qquad + t_3\theta^4\wedge\theta^6 + t_3\theta^7\wedge\theta^9, \\
T^3 = -3t_3\theta^1\wedge\theta^2 + t_2\theta^1\wedge\theta^5 - t_1\theta^1\wedge\theta^8 - t_2\theta^2\wedge\theta^4 + t_1\theta^2\wedge\theta^7 \\
\qquad - t_3\theta^4\wedge\theta^5 - t_3\theta^7\wedge\theta^8, \\
T^4 = t_2\theta^2\wedge\theta^3 - t_3\theta^2\wedge\theta^6 + t_3\theta^3\wedge\theta^5 + 3t_2\theta^5\wedge\theta^6 - t_1\theta^5\wedge\theta^9 \\
\qquad + t_1\theta^6\wedge\theta^8 + t_2\theta^8\wedge\theta^9, \\
T^5 = -t_2\theta^1\wedge\theta^3 + t_3\theta^1\wedge\theta^6 - t_3\theta^3\wedge\theta^4 - 3t_2\theta^4\wedge\theta^6 + t_1\theta^4\wedge\theta^9 \\
\qquad - t_1\theta^6\wedge\theta^7 - t_2\theta^7\wedge\theta^9, \\
T^6 = t_2\theta^1\wedge\theta^2 - t_3\theta^1\wedge\theta^5 + t_3\theta^2\wedge\theta^4 + 3t_2\theta^4\wedge\theta^5 - t_1\theta^4\wedge\theta^8 \\
\qquad + t_1\theta^5\wedge\theta^7 + t_2\theta^7\wedge\theta^8, \\
T^7 = -t_1\theta^2\wedge\theta^3 - t_3\theta^2\wedge\theta^9 + t_3\theta^3\wedge\theta^8 - t_1\theta^5\wedge\theta^6 + t_2\theta^5\wedge\theta^9 \\
\qquad - t_2\theta^6\wedge\theta^8 - 3t_1\theta^8\wedge\theta^9, \\
T^8 = t_1\theta^1\wedge\theta^3 + t_3\theta^1\wedge\theta^9 - t_3\theta^3\wedge\theta^7 + t_1\theta^4\wedge\theta^6 - t_2\theta^4\wedge\theta^9 \\
\qquad + t_2\theta^6\wedge\theta^7 + 3t_1\theta^7\wedge\theta^9, \\
T^9 = -t_1\theta^1\wedge\theta^2 - t_3\theta^1\wedge\theta^8 + t_3\theta^2\wedge\theta^7 - t_1\theta^4\wedge\theta^5 + t_2\theta^4\wedge\theta^8 \\
\qquad - t_2\theta^5\wedge\theta^7 - 3t_1\theta^7\wedge\theta^8.
\end{cases}
$$

Here (t_1, t_2, t_3) are the three independent components of the torsion T.

Remark 6.2. Rewriting the equations in (6.4) in terms of the basis of 2-forms $(\kappa^A_0, \kappa^{A'}_0, \lambda^\mu_0, \lambda^{\mu'}_0)$, as in Remark 3.4, one can see that only the primed 2-forms appear above. Explicitly:

$$
(6.5)\quad \begin{cases}
T^1 = -t_1\lambda^{9'}_0 + t_2\lambda^{6'}_0 + \tfrac{1}{3}t_3(5\kappa^{3'}_0 - 4\lambda^{3'}_0 + 2\lambda^{12'}_0), \\
T^2 = t_1\lambda^{8'}_0 - t_2\lambda^{5'}_0 + \tfrac{1}{3}t_3(-5\kappa^{2'}_0 + 4\lambda^{2'}_0 - 2\lambda^{11'}_0), \\
T^3 = -t_1\lambda^{7'}_0 + t_2\lambda^{4'}_0 + \tfrac{1}{3}t_3(5\kappa^{1'}_0 - 4\lambda^{1'}_0 + 2\lambda^{10'}_0), \\
T^4 = -t_1\lambda^{15'}_0 + \tfrac{1}{3}t_2(-5\kappa^{3'}_0 - 2\lambda^{3'}_0 + 4\lambda^{12'}_0) - t_3\lambda^{6'}_0, \\
T^5 = t_1\lambda^{14'}_0 + \tfrac{1}{3}t_2(5\kappa^{2'}_0 + 2\lambda^{2'}_0 - 4\lambda^{11'}_0) + t_3\lambda^{5'}_0, \\
T^6 = -t_1\lambda^{13'}_0 + \tfrac{1}{3}t_2(-5\kappa^{1'}_0 - 2\lambda^{1'}_0 + 4\lambda^{10'}_0) - t_3\lambda^{4'}_0, \\
T^7 = \tfrac{1}{3}t_1(5\kappa^{3'}_0 + 2\lambda^{3'}_0 + 2\lambda^{12'}_0) + t_2\lambda^{15'}_0 - t_3\lambda^{9'}_0, \\
T^8 = -\tfrac{1}{3}t_1(5\kappa^{2'}_0 + 2\lambda^{2'}_0 + 2\lambda^{11'}_0) - t_2\lambda^{14'}_0 + t_3\lambda^{8'}_0, \\
T^9 = \tfrac{1}{3}t_1(5\kappa^{1'}_0 + 2\lambda^{1'}_0 + 2\lambda^{10'}_0) + t_2\lambda^{13'}_0 - t_3\lambda^{7'}_0.
\end{cases}
$$

Once the torsion in $\mathfrak{so}(3)_L \simeq \mathbb{R}^3$ is totally determined and parametrized as above by a 'vector' $\mathbf{t} = (t_1, t_2, t_3)$, we can check what are the orbits of the action of the groups $\mathrm{SO}(3)_L$ and $\mathrm{SO}(3)_R$ on the torsion space $\mathfrak{so}(3)_L \simeq \mathbb{R}^3$. A direct calculation yields that the action of $\mathrm{SO}(3)_R$ on $V_{[0,2]} = \mathfrak{so}(3)_L$, as defined in (6.1)–(6.2), is trivial and that the group $\mathrm{SO}(3)_L$ acts transitively on each of the 2-spheres $\mathbb{S}_T \subset \mathfrak{so}(3)_L$. The orbit space of the action of $\mathrm{SO}(3)_L$ on $\mathfrak{so}(3) \simeq \mathbb{R}^3$ is $\mathbb{R}_+ \cup \{0\}$ and is parametrized by the radius r of these spheres. Thus the orbit structure of this action is represented by $\mathfrak{so}(3)_L = \mathbb{S}^2 \times \mathbb{R}_+ \cup \{0\}$.

Now we analyze the differential consequences of the structure equations (4.6)–(4.7) with torsion T^i as in (6.4). We consider the equations (4.6)–(4.7) *on the bundle*

$$\mathrm{SO}(3) \times \mathrm{SO}(3) \to P \to M.$$

Thus the 15 forms $(\theta^i, \gamma^A, \gamma^{A'})$ appearing in these equations are considered to be linearly independent. Also the unknown torsions (t_1, t_2, t_3), as well as the curvatures $K^i{}_{jkl}$ are considered to be functions on P.

A piece of terminology is useful here: whenever we make an analysis of a system of equations like the one given by (4.6)–(4.7), (6.4), we will say that we analyze an *exterior differential system* (EDS).

Although we have proven above that we can always gauge the 3-dimensional torsion (t_1, t_2, t_3) of our EDS in such a way that $t_2 \equiv t_3 \equiv 0$, we will not use this gauge yet. This is because the use of this gauge would imply the restriction of the EDS from 15-dimensional bundle P to its 13-dimensional section P^{13}. Since the analysis of the system is more convenient on P, rather than on P^{13} (because only from there the system nicely generalizes to torsions more general than those in $\mathfrak{so}(3)_L$), we will make the gauge $t_2 \equiv t_3 \equiv 0$ only after extracting the information from the first *Bianchi identities* of our EDS on P.

The first Bianchi identities are obtained by applying the exterior derivative on both sides of equations (4.6). Their consequences are summarized in the following proposition.

Proposition 6.3. *The first Bianchi identities imply that*

$$\mathrm{d}t_1 = t_2\gamma^3 - t_3\gamma^2, \quad \mathrm{d}t_2 = t_3\gamma^1 - t_1\gamma^3, \quad \mathrm{d}t_3 = t_1\gamma^2 - t_2\gamma^1, \tag{6.6}$$

and that the curvatures $(\kappa^A, \kappa^{A'})$*, as defined in* (4.7)*, read*

$$\begin{cases} \kappa^1 = k\kappa_0^1 + t_1t_2\kappa_0^2 + t_1t_3\kappa_0^3, \\ \kappa^2 = t_1t_2\kappa_0^1 + (k - t_1^2 + t_2^2)\kappa_0^2 + t_2t_3\kappa_0^3, \\ \kappa^3 = t_1t_3\kappa_0^1 + t_2t_3\kappa_0^2 + (k - t_1^2 + t_3^2)\kappa_0^3, \\ \kappa^{1'} = (k + t_1^2 + 2t_2^2 + 2t_3^2)\kappa_0^{1'}, \\ \kappa^{2'} = (k + t_1^2 + 2t_2^2 + 2t_3^2)\kappa_0^{2'}, \\ \kappa^{3'} = (k + t_1^2 + 2t_2^2 + 2t_3^2)\kappa_0^{3'}. \end{cases} \tag{6.7}$$

Here k is an unknown function on P, and the forms $(\kappa_0^A, \kappa_0^{A'})$ are defined in (3.1).

Thus, the first Bianchi identities show that the curvature of the characteristic connection is totally determined by the torsion (t_1, t_2, t_3) and an unknown function k.

Proof. To apply the first Bianchi identities, one needs the derivatives of the torsions t_i. So we assume the most general form for these:

$$\mathrm{d}t_\mu = t_{\mu j}\theta^j + t_{\mu A}\gamma^A + t_{\mu A'}\gamma^{A'}, \quad \mu = 1,2,3. \tag{6.8}$$

Here $t_{\mu j}, t_{\mu A}, t_{\mu A'}$ are $(3*9+3*3+3*3) = 45$ functions on P, which we hope to determine by means of the first Bianchi identities $\mathrm{d}^2\theta^i \equiv 0$, $i = 1,2,\dots,9$. Note that if one applies the exterior differential to the equations (4.6), the d of the right-hand sides must be zero, d(right-hand side) $\equiv 0$. Inserting our definitions (6.8) in these identities, we obtain nine identities each of which is a 3-form on P. Decomposing these nine 3-forms into the basis of 3-forms on P, which consists of the primitive forms $\theta^i\wedge\theta^j\wedge\theta^k$, $\theta^i\wedge\theta^j\wedge\gamma^{A/A'}$, $\theta^i\wedge\gamma^{A/A'}\wedge\gamma^{B/B'}$, and $\gamma^{A/A'}\wedge\gamma^{B/B'}\wedge\gamma^{C/C'}$, one gets relations on the unknown functions $t_{\mu j}, t_{\mu A}, t_{\mu A'}$ and the curvatures $K^i{}_{jkl}$.

Analyzing these relations step by step we get the following:

First, we consider terms at the basis forms $\theta^i\wedge\theta^j\wedge\gamma^{A/A'}$. This gives 18 conditions determining all the functions $t_{\mu A}$ and $t_{\mu A'}$ in terms of (t_1,t_2,t_3). After solving these 18 conditions we get

$$\begin{aligned}\mathrm{d}t_1 &= t_2\gamma^3 - t_3\gamma^2 + t_{1j}\theta^j,\\ \mathrm{d}t_2 &= t_3\gamma^1 - t_1\gamma^3 + t_{2j}\theta^j,\\ \mathrm{d}t_3 &= t_1\gamma^2 - t_2\gamma^1 + t_{3j}\theta^j.\end{aligned}$$

Second, the terms at the basis forms $\theta^i\wedge\theta^j\wedge\theta^k$, when equated to zero, can be split into two types of equations. The first type is obtained by eliminating the curvatures $K^i{}_{jkl}$ from the full set. This yields a system of linear equations for the unknowns $t_{\mu j}$, whose only solution is $t_{\mu j} = 0$. After these conditions are imposed, the second type of equations involves the curvatures $K^i{}_{jkl}$ only in a linear fashion. It has a unique solution for the curvatures, which is explicitly given by (6.7).

Third, after imposing the conditions described above, all the other terms in $\mathrm{d}^2\theta^i$ are automatically zero.

This proves the proposition, and also shows that the conditions (6.6)–(6.7) on the curvature and the derivatives of the torsion are equivalent to the first Bianchi identities of the system in consideration. □

Now we are in a position to impose the gauge

$$t_2 \equiv t_3 \equiv 0. \tag{6.9}$$

Since the action of $\mathrm{SO}(3)_L$ is transitive, every nearly integrable $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometry with torsion in $\mathfrak{so}(3)_L$ admits an adapted frame in which the conditions (6.9) hold. But the assumption of the gauge (6.9) reduces the degrees of freedom by 2, from 15 to 13. This means that we reduce the equation of our EDS (4.6)–(4.7), (6.4) from dimension 15 to dimension 13. Also the differential consequences (6.6)–(6.7) of this EDS must be reduced to dimension 13. This in particular means that the fifteen 1-forms $(\theta^i,\gamma^A,\gamma^{A'})$ can no longer be linearly independent. This obvious observation finds its confirmation in the integrability conditions (6.6)–(6.7).

Indeed, assuming $t_2 \equiv t_3 \equiv 0$, and comparing it with the last two integrability conditions in (6.6) yields

$$t_1\gamma^3 \equiv 0 \quad \text{and} \quad t_1\gamma^2 \equiv 0.$$

These, when confronted with the assumption that the torsion T^i is not vanishing in a neighborhood, imply that

$$\gamma^2 \equiv 0 \quad \text{and} \quad \gamma^3 \equiv 0. \tag{6.10}$$

Thus the EDS (4.6)–(4.7), (6.4) naturally reduces to 13 dimensions, and has now thirteen 1-forms $(\theta^i, \gamma^1, \gamma^{A'})$ linearly independent at each point of the 13-dimensional manifold, which we previously called P^{13}.

The relations (6.10) have further consequences, for if we compare them with the second and the third equation in (4.7), we see that $\kappa^2 \equiv 0$ and $\kappa^3 \equiv 0$.

If we now compare these with (6.10), and the second and the third integrability condition in (6.7), we get $(k - t_1^2)\kappa_0^2 \equiv 0$ and $(k - t_1^2)\kappa_0^3 \equiv 0$.

These hold iff $k \equiv t_1^2$, which we have to accept from now on. Note that this totally determines the function k, which was a mysterious unknown in Proposition 6.3.

Finally, if we insert $t_2 \equiv t_3 \equiv 0$ in the first of the integrability conditions (6.6), we get also that $\mathrm{d}t_1 \equiv 0$, i.e., that the function t_1 must be *constant* on the 13-dimensional reduced manifold P^{13} on which our EDS lives.

These considerations, when compared with the rest of the integrability conditions (6.7), prove the following proposition.

Proposition 6.4. *Every nearly integrable* $\mathrm{SO}(3)\times\mathrm{SO}(3)$ *geometry* $(M^9, g, \Upsilon, \omega)$ *with a non-vanishing torsion* T *of the characteristic connection lying in* $\mathfrak{so}(3)_L = V_{[0,2]}$, $T \in \mathfrak{so}(3)_L$, *can be described in terms of thirteen linearly independent* 1*-forms* $(\theta^i, \gamma^1, \gamma^{A'})$, $i = 1, 2, \ldots, 9$, $A' = 1, 2, 3$, *satisfying*

$$\begin{cases}
\mathrm{d}\theta^1 = \gamma^1\wedge\theta^4 + \gamma^{1'}\wedge\theta^2 + \gamma^{2'}\wedge\theta^3 + t(-\theta^2\wedge\theta^9 + \theta^3\wedge\theta^8),\\
\mathrm{d}\theta^2 = \gamma^1\wedge\theta^5 - \gamma^{1'}\wedge\theta^1 + \gamma^{3'}\wedge\theta^3 + t(\theta^1\wedge\theta^9 - \theta^3\wedge\theta^7),\\
\mathrm{d}\theta^3 = \gamma^1\wedge\theta^6 - \gamma^{2'}\wedge\theta^1 - \gamma^{3'}\wedge\theta^2 + t(-\theta^1\wedge\theta^8 + \theta^2\wedge\theta^7),\\
\mathrm{d}\theta^4 = -\gamma^1\wedge\theta^1 + \gamma^{1'}\wedge\theta^5 + \gamma^{2'}\wedge\theta^6 + t(-\theta^5\wedge\theta^9 + \theta^6\wedge\theta^8),\\
\mathrm{d}\theta^5 = -\gamma^1\wedge\theta^2 - \gamma^{1'}\wedge\theta^4 + \gamma^{3'}\wedge\theta^6 + t(\theta^4\wedge\theta^9 - \theta^6\wedge\theta^7),\\
\mathrm{d}\theta^6 = -\gamma^1\wedge\theta^3 - \gamma^{2'}\wedge\theta^4 - \gamma^{3'}\wedge\theta^5 + t(-\theta^4\wedge\theta^8 + \theta^5\wedge\theta^7),\\
\mathrm{d}\theta^7 = \gamma^{1'}\wedge\theta^8 + \gamma^{2'}\wedge\theta^9 - t(\theta^2\wedge\theta^3 + \theta^5\wedge\theta^6 + 3\theta^8\wedge\theta^9),\\
\mathrm{d}\theta^8 = -\gamma^{1'}\wedge\theta^7 + \gamma^{3'}\wedge\theta^9 + t(\theta^1\wedge\theta^3 + \theta^4\wedge\theta^6 + 3\theta^7\wedge\theta^9),\\
\mathrm{d}\theta^9 = -\gamma^{2'}\wedge\theta^7 - \gamma^{3'}\wedge\theta^8 - t(\theta^1\wedge\theta^2 + \theta^4\wedge\theta^5 + 3\theta^7\wedge\theta^8),
\end{cases} \tag{6.11}$$

$$\begin{cases}
\mathrm{d}\gamma^1 = t^2(\theta^1\wedge\theta^4 + \theta^2\wedge\theta^5 + \theta^3\wedge\theta^6),\\
\mathrm{d}\gamma^{1'} = -\gamma^{2'}\wedge\gamma^{3'} + 2t^2(\theta^1\wedge\theta^2 + \theta^4\wedge\theta^5 + \theta^7\wedge\theta^8),\\
\mathrm{d}\gamma^{2'} = -\gamma^{3'}\wedge\gamma^{1'} + 2t^2(\theta^1\wedge\theta^3 + \theta^4\wedge\theta^6 + \theta^7\wedge\theta^9),\\
\mathrm{d}\gamma^{3'} = -\gamma^{1'}\wedge\gamma^{2'} + 2t^2(\theta^2\wedge\theta^3 + \theta^5\wedge\theta^6 + \theta^8\wedge\theta^9).
\end{cases} \tag{6.12}$$

Here $\mathrm{d}t \equiv 0$, *i.e., the function* t *is constant.*

Note that the system (6.11)–(6.12) involves only constant coefficients on the right-hand sides. Thus the manifold P^{13} is a Lie group $P^{13} = \mathcal{G}^{13}$, with the forms $(\theta^i, \gamma^1, \gamma^{A'})$ constituting a basis of its left invariant forms. A calculation of the Killing form for $\mathcal{G}^{13}$, by using the structure constants red off from (6.11)–(6.12), shows that this group is semisimple, unless the torsion $t \equiv 0$. The group $\mathcal{G}^{13}$ is a transitive group of symmetries of the underlying nearly integrable geometry $(M^9, g, \Upsilon, \omega)$. The 9-dimensional manifold M^9 is a homogeneous space $M^9 = \mathcal{G}^{13}/H$, where H is a certain 4-dimensional subgroup of $\mathcal{G}^{13}$. The structural tensors g, Υ and ω of the corresponding SO(3) $\times$ SO(3) structure are obtained, via formulae (2.8)–(2.10), from the 1-forms (θ^i) solving (6.11)–(6.12). The system (6.11)–(6.12) guarantees that although tensors g, Υ, ω defined in this way live on $\mathcal{G}^{13}$, they actually descend to tensors g, Υ, ω on the manifold $M^9 = \mathcal{G}^{13}/H$, defining a homogeneous nearly integrable geometry $(M^9, g, \Upsilon, \omega)$ with 13-dimensional group of symmetries $\mathcal{G}^{13}$ there.

For $t = 0$ the Lie group $\mathcal{G}^{13}$ is just a semidirect product $(\mathrm{SO}(3) \times \mathrm{SO}(2)) \ltimes \mathbb{R}^9$. For $t \neq 0$, by considering the new basis of 1-forms

$$\begin{gathered}
\tilde{\theta}^i = t\theta^i, \quad i = 1, \dots, 6, \\
\tilde{\gamma}^1 = \gamma'^1 + t\theta^9, \quad \tilde{\gamma}^2 = \gamma'^2 - t\theta^8, \quad \tilde{\gamma}^3 = \gamma'^3 + t\theta^7, \\
\tilde{\theta}^7 = \gamma'^3 + 2t\theta^7, \quad \tilde{\theta}^8 = \gamma'^2 - 2t\theta^8, \quad \tilde{\theta}^9 = \gamma'^1 + 2t\theta^9,
\end{gathered}$$

one sees that for any $t \neq 0$ the Lie group $\mathcal{G}^{13}$ is the product SO(3) $\times$ K^{10} with structure equations

$$\begin{aligned}
d\tilde{\theta}^1 &= \gamma^1 \wedge \tilde{\theta}^4 + \tilde{\gamma}^1 \wedge \tilde{\theta}^2 + \tilde{\gamma}^2 \wedge \tilde{\theta}^3, \\
d\tilde{\theta}^2 &= \gamma^1 \wedge \tilde{\theta}^5 - \tilde{\gamma}^1 \wedge \tilde{\theta}^1 + \tilde{\gamma}^3 \wedge \tilde{\theta}^3, \\
d\tilde{\theta}^3 &= \gamma^1 \wedge \tilde{\theta}^6 - \tilde{\gamma}^2 \wedge \tilde{\theta}^1 + \tilde{\gamma}^2 \wedge \tilde{\theta}^2, \\
d\tilde{\theta}^4 &= -\gamma^1 \wedge \tilde{\theta}^1 + \tilde{\gamma}^1 \wedge \tilde{\theta}^5 + \tilde{\gamma}^2 \wedge \tilde{\theta}^6, \\
d\tilde{\theta}^5 &= -\gamma^1 \wedge \tilde{\theta}^2 - \tilde{\gamma}^1 \wedge \tilde{\theta}^4 + \tilde{\gamma}^3 \wedge \tilde{\theta}^6, \\
d\tilde{\theta}^6 &= -\gamma^1 \wedge \tilde{\theta}^3 - \tilde{\gamma}^2 \wedge \tilde{\theta}^4 - \tilde{\gamma}^3 \wedge \tilde{\theta}^5, \\
d\gamma^1 &= \tilde{\theta}^1 \wedge \tilde{\theta}^4 + \tilde{\theta}^2 \wedge \tilde{\theta}^5 + \tilde{\theta}^3 \wedge \tilde{\theta}^6, \\
d\tilde{\gamma}^1 &= -\tilde{\gamma}^2 \wedge \tilde{\gamma}^3 + \tilde{\theta}^1 \wedge \tilde{\theta}^2 + \tilde{\theta}^4 \wedge \tilde{\theta}^5, \\
d\tilde{\gamma}^2 &= -\tilde{\gamma}^3 \wedge \tilde{\gamma}^1 + \tilde{\theta}^1 \wedge \tilde{\theta}^3 + \tilde{\theta}^4 \wedge \tilde{\theta}^6, \\
d\tilde{\gamma}^3 &= -\tilde{\gamma}^1 \wedge \tilde{\gamma}^2 + \tilde{\theta}^2 \wedge \tilde{\theta}^3 + \tilde{\theta}^5 \wedge \tilde{\theta}^6, \\
d\tilde{\theta}^7 &= \tilde{\theta}^8 \wedge \tilde{\theta}^9, \\
d\tilde{\theta}^8 &= \tilde{\theta}^9 \wedge \tilde{\theta}^7, \\
d\tilde{\theta}^9 &= \tilde{\theta}^7 \wedge \tilde{\theta}^8.
\end{aligned}$$

To say what K^{10} is, we calculate the Killing forms. In the basis

$$\left(\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3, \tilde{\theta}^4, \tilde{\theta}^5, \tilde{\theta}^6, \gamma^1, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3, \tilde{\theta}^7, \tilde{\theta}^8, \tilde{\theta}^9\right)$$

the Killing form of $\mathcal{G}^{13}$ reads

$$\mathrm{Kil}_{13} = \mathrm{diag}(6, 6, 6, 6, 6, 6, -6, -6, -6, -6, -2, -2, -2).$$

The Lie algebra of K^{10} is spanned by $(\tilde\theta^1, \tilde\theta^2, \tilde\theta^3, \tilde\theta^4, \tilde\theta^5, \tilde\theta^6, \gamma^1, \tilde\gamma^1, \tilde\gamma^2, \tilde\gamma^3)$. Its Killing form in this basis is

$$\mathrm{Kil}_{10} = \mathrm{diag}(6, 6, 6, 6, 6, 6, -6, -6, -6, -6),$$

showing that K^{10} is semisimple, and as such, having dimension 10, it must be locally isomorphic to a non-compact real form of $\mathrm{SO}(5, \mathbb{C})$. Comparison of Killing forms for $\mathrm{SO}(1, 4)$ and $\mathrm{SO}(2, 3)$ shows that K^{10} is locally $\mathrm{SO}(2, 3)$.

In both cases ($t = 0$ and $t \neq 0$) the Lie algebra of the group $H = \mathrm{SO}(3) \times \mathrm{SO}(2)$ is given by the annihilator of the 1-forms θ^i, $i = 1, 2, \ldots, 9$.

After calculating the curvatures of the various connections associated with this geometry we get the following theorem.

Theorem 6.5. *Any nearly integrable irreducible* $\mathrm{SO}(3)\times\mathrm{SO}(3)$ *geometry* $(M^9, g, \Upsilon, \omega)$ *with torsion of the characteristic connection* Γ *in* $V_{[0,2]} = \mathfrak{so}(3)_L$ *is locally a homogeneous space* $\mathcal{G}^{13}/H$*. It has a transitive symmetry group* $\mathcal{G}^{13}$ *of dimension* 13*. For* $t = 0$ *the Lie group* $\mathcal{G}^{13}$ *is a semidirect product* $(\mathrm{SO}(3) \times \mathrm{SO}(2)) \ltimes \mathbb{R}^9$ *and for* $t \neq 0$ *it is a direct product* $\mathrm{SO}(3) \times \mathrm{SO}(2, 3)$.

The metric g *is conformally* non-flat *and* not *locally symmetric. The Ricci tensors of the Levi-Civita connection* $\overset{\mathrm{LC}}{\Gamma}$*, of the characteristic connection* Γ*, and of the* $\mathfrak{so}(3)_L$ *part* $\overset{+}{\Gamma}$ *of the characteristic connection have all two distinct eigenvalues.*

The $\mathfrak{so}(3)_R$ *part* $\overset{-}{\Gamma}$ *of the characteristic connection is* Einstein.

Explicitly, in the adapted coframe (θ^i) in which the structure equations read as in (6.11) and in which the structural tensors g, Υ, ω are given by (2.8)–(2.10), we have the following:

- The Cartan connection Γ_{Cartan} has the curvature given by

$$\tilde R = \begin{pmatrix} 0 & (1+t^2)\kappa_0^1 & \kappa_0^2 & | & T^1 & T^2 & T^3 \\ -(1+t^2)\kappa_0^1 & 0 & \kappa_0^3 & | & T^4 & T^5 & T^6 \\ -\kappa_0^2 & -\kappa_0^3 & 0 & | & T^7 & T^8 & T^9 \\ - & - & - & - & - & - & - \\ -T^1 & -T^4 & -T^7 & | & 0 & (1+2t^2)\kappa_0^{1'} & (1+2t^2)\kappa_0^{2'} \\ -T^2 & -T^5 & -T^8 & | & -(1+2t^2)\kappa_0^{1'} & 0 & (1+2t^2)\kappa_0^{3'} \\ -T^3 & -T^6 & -T^9 & | & -(1+2t^2)\kappa_0^{2'} & -(1+2t^2)\kappa_0^{3'} & 0 \end{pmatrix},$$

  where the torsions T^i are given by (6.5) with $t_1 = t = \mathrm{const}$ and $t_2 = t_3 = 0$.

- The Levi-Civita connection Ricci tensor reads

$$\overset{\mathrm{LC}}{\mathrm{Ric}} = \mathrm{diag}\big(-4t^2, -4t^2, -4t^2, -4t^2, -4t^2, -4t^2, \tfrac{3}{2}t^2, \tfrac{3}{2}t^2, \tfrac{3}{2}t^2\big)$$

  and has the Ricci scalar equal to $-\frac{39}{2}t^2$.

- The $\mathfrak{so}(3)_L$ part $\overset{+}{\Gamma}$ of the characteristic connection has the curvature

$$\overset{+}{\Omega} = -t^2\kappa_0^1 e_1,$$

where the matrix $e_1 = (e_1{}^i{}_j)$ is given by (2.15). It has the Ricci tensor $\overset{+}{R}_{ij}$ given by

$$\overset{+}{R}_{ij} = \mathrm{diag}\big(-t^2, -t^2, -t^2, -t^2, -t^2, -t^2, 0, 0, 0\big),$$

with the Ricci scalar equal to $-6t^2$.

- The $\mathfrak{so}(3)_R$ part $\overset{-}{\Gamma}$ of the characteristic connection has the curvature

$$\overset{-}{\Omega} = -2t^2\kappa_0^{A'} e_{A'},$$

where the matrices $e_{A'} = (e_{A'}{}^i{}_j)$ are given by (2.15)–(2.17). Its Ricci tensor is *Einstein*,

$$\overset{-}{R}_{ij} = -4t^2 g_{ij},$$

and has Ricci scalar equal to $-36t^2$.

- The characteristic connection $\Gamma = \overset{+}{\Gamma} + \overset{-}{\Gamma}$ has curvature

$$\Omega = \overset{+}{\Omega} + \overset{-}{\Omega} = -t^2\kappa_0^1 e_1 - 2t^2\kappa_0^{A'} e_{A'}$$

and the Ricci tensor

$$R_{ij} = \mathrm{diag}\big(-5t^2, -5t^2, -5t^2, -5t^2, -5t^2, -5t^2, -4t^2, -4t^2, -4t^2\big).$$

6.2. Torsion in $V_{[0,6]}$. In this section, we will find examples of *nearly integrable* geometries $(M^9, g, \Upsilon, \omega)$ in dimension nine, whose characteristic connection Γ has totally skew symmetric torsion T in the irreducible representation $V_{[0,6]}$. Thus $T \in V_{[0,6]} \subset \bigwedge^3 \mathbb{R}^9$ in this subsection.

The assumption that $T \in V_{[0,6]} \subset \bigwedge^3 \mathbb{R}^9$ is equivalent to the requirement that, in a coframe θ^i adapted to $(M^9, g, \Upsilon, \omega)$, we have

$$T^i = \tfrac{1}{2} g^{ij} T_{jkl}\theta^k \wedge \theta^l, \quad T_{ijk} = T_{[ijk]}, \quad \text{and} \quad \tilde{\omega}(T)_{ijk} = -6T_{ijk}.$$

Solving these algebraic conditions for T_{ijk} we get the following proposition.

Proposition 6.6. *In an adapted coframe (θ^i) the $V_{[0,6]}$ torsion of a characteristic connection of a nearly integrable geometry $(M^9, g, \Upsilon, \omega)$ reads*

$$(6.13)\quad \begin{cases} T^1 = u_1(-\lambda_0^{3'} + \lambda_0^{12'}) - u_2\lambda_0^{15'} - u_3\lambda_0^{3'} - u_4\lambda_0^{6'} - u_5\lambda_0^{9'} - u_6\lambda_0^{6'} - u_7\lambda_0^{9'}, \\ T^2 = u_1(\lambda_0^{2'} - \lambda_0^{11'}) + u_2\lambda_0^{14'} + u_3\lambda_0^{2'} + u_4\lambda_0^{5'} + u_5\lambda_0^{8'} + u_6\lambda_0^{5'} + u_7\lambda_0^{8'}, \\ T^3 = u_1(-\lambda_0^{1'} + \lambda_0^{10'}) - u_2\lambda_0^{13'} - u_3\lambda_0^{1'} - u_4\lambda_0^{4'} - u_5\lambda_0^{7'} - u_6\lambda_0^{4'} - u_7\lambda_0^{7'}, \\ T^4 = u_1\lambda_0^{6'} - u_2\lambda_0^{9'} + u_4(-\lambda_0^{3'} + \lambda_0^{12'}) + u_5\lambda_0^{15'} - u_6\lambda_0^{3'}, \\ T^5 = -u_1\lambda_0^{5'} + u_2\lambda_0^{8'} + u_4(\lambda_0^{2'} - \lambda_0^{11'}) - u_5\lambda_0^{14'} + u_6\lambda_0^{2'}, \\ T^6 = u_1\lambda_0^{4'} - u_2\lambda_0^{7'} + u_4(-\lambda_0^{1'} + \lambda_0^{10'}) + u_5\lambda_0^{13'} - u_6\lambda_0^{1'}, \\ T^7 = -u_2\lambda_0^{6'} + u_3\lambda_0^{9'} + u_5(-\lambda_0^{3'} + \lambda_0^{12'}) + u_6\lambda_0^{15'} - u_7\lambda_0^{3'}, \\ T^8 = u_2\lambda_0^{5'} - u_3\lambda_0^{8'} + u_5(\lambda_0^{2'} - \lambda_0^{11'}) - u_6\lambda_0^{14'} + u_7\lambda_0^{2'}, \\ T^9 = -u_2\lambda_0^{4'} + u_3\lambda_0^{7'} + u_5(-\lambda_0^{1'} + \lambda_0^{10'}) + u_6\lambda_0^{13'} - u_7\lambda_0^{1'}, \end{cases}$$

where $(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ are the seven independent components of the torsion T, and $(\lambda_0^{\mu'})$, $\mu' = 1, 2, \ldots, 15$, is a basis of 2-forms in $V_{[4,2]}$ as defined in (3.3).

Proposition 6.7. *The action of* $\mathrm{SO}(3)_R$ *on* $V_{[0,6]}$*, as defined in* (6.1)–(6.2)*, is trivial, i.e.,* $(h'T)_{ijk} = T_{ijk}$ *for all* $h' \in \mathrm{SO}(3)_R$ *and for all* $T_{ijk} \in V_{[0,6]}$.

The 'left' SO(3) acts non-trivially on $V_{[0,6]}$. It has a 4-parameter family of generic orbits in this 7-dimensional space. As in the $V_{[0,2]}$ case, instead of restricting ourselves to the representatives of these orbits, we will analyze the EDS (4.6)–(4.7) for the torsion in $V_{[0,6]}$, with general torsions $(u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ as in (6.13). Thus the EDS (4.6)–(4.7), (6.13) we consider, lives on the Cartan bundle $\mathrm{SO}(3)_L \times \mathrm{SO}(3)_R \to P \to M$, where the 15 forms $(\theta^i, \gamma^A, \gamma^{A'})$ are linearly independent at each point.

Now the $V_{[0,6]}$ analog of Proposition 6.3 is as follows.

Proposition 6.8. *The first Bianchi identities* $\mathrm{d}^2\theta^i \equiv 0$*, for the EDS* (4.6)–(4.7), (6.13) *imply that*

$$\begin{cases} \mathrm{d}u_1 = (3u_4 + 2u_6)\gamma^1 + u_5\gamma^2 - 2u_2\gamma^3, \\ \mathrm{d}u_2 = -(2u_5 + u_7)\gamma^1 - (u_4 + 2u_6)\gamma^2 + (u_1 - u_3)\gamma^3, \\ \mathrm{d}u_3 = u_6\gamma^1 + (2u_5 + 3u_7)\gamma^2 + 2u_2\gamma^3, \\ \mathrm{d}u_4 = -3u_1\gamma^1 + 3u_5\gamma^3, \\ \mathrm{d}u_5 = 2u_2\gamma^1 - u_1\gamma^2 + (2u_6 - u_4)\gamma^3, \\ \mathrm{d}u_6 = -u_3\gamma^1 + 2u_2\gamma^2 + (u_7 - 2u_5)\gamma^3, \\ \mathrm{d}u_7 = -3u_3\gamma^2 - 3u_6\gamma^3, \end{cases} \tag{6.14}$$

and that the curvatures $(\kappa^A, \kappa^{A'})$*, as defined in* (4.7)*, are*

$$\begin{cases} \kappa^1 = k_1\kappa^1_0 + k_2\kappa^2_0 + k_3\kappa^3_0, \\ \kappa^2 = k_2\kappa^1_0 + k_4\kappa^2_0 + k_5\kappa^3_0, \\ \kappa^3 = k_3\kappa^1_0 + k_5\kappa^2_0 + k_6\kappa^3_0, \\ \kappa^{1'} = k_7\kappa^{1'}_0, \\ \kappa^{2'} = k_7\kappa^{2'}_0, \\ \kappa^{3'} = k_7\kappa^{3'}_0, \end{cases} \tag{6.15}$$

where

$$\begin{cases} k_2 = 2(u_1 + u_3)u_2 - (2u_4 + 3u_6)u_5 - (u_4 + 2u_6)u_7, \\ k_3 = 2u_2u_4 + (2u_1 - u_3)u_5 + u_1u_7, \\ k_4 = k_1 + 2u_1^2 - 2u_3^2 + 2u_4^2 + 2u_4u_6 - 2u_5u_7 - 2u_7^2, \\ k_5 = -u_3u_4 + (u_1 - 2u_3)u_6 - 2u_2u_7, \\ k_6 = k_1 + 2u_1^2 + 2u_1u_3 + 2u_4^2 + 4u_4u_6 + 2u_5u_7, \\ k_7 = k_1 + 2u_1^2 + u_2^2 + u_1u_3 + 2u_4^2 + u_5^2 + 3u_4u_6 + u_6^2 + u_5u_7. \end{cases} \tag{6.16}$$

Here k_1 *is an unknown function and* $(\kappa^A_0, \kappa^{A'}_0)$ *are given by* (3.1).

Proof. The proof here is very similar to the proof of Proposition 6.3. So we first assume the most general form for the derivatives of the torsions u_μ:

$$\mathrm{d}u_\mu = u_{\mu j}\theta^j + u_{\mu A}\gamma^A + u_{\mu A'}\gamma^{A'}, \quad \mu = 1, 2, \ldots, 7. \tag{6.17}$$

Here $u_{\mu j}, u_{\mu A}, u_{\mu A'}$ are $(7 * 9 + 7 * 3 + 7 * 3) = 105$ functions on P, which we will determine by means of the first Bianchi identities $\mathrm{d}^2\theta^i \equiv 0$, $i = 1, 2, \ldots, 9$. Inserting our definitions (6.17) in these identities, we obtain nine identities each of which is a 3-form on P. We decompose these nine 3-forms into the basis of 3-forms on P, $\theta^i \wedge \theta^j \wedge \theta^k$, $\theta^i \wedge \theta^j \wedge \gamma^{A/A'}$, $\theta^i \wedge \gamma^{A/A'} \wedge \gamma^{B/B'}$, and $\gamma^{A/A'} \wedge \gamma^{B/B'} \wedge \gamma^{C/C'}$. This brings the relations between the unknown functions $u_{\mu j}, t_{\mu A}, t_{\mu A'}$ and the curvatures $K^i{}_{jkl}$.

Analyzing these relations step by step we get the following:

First, we consider terms at the basis forms $\theta^i \wedge \theta^j \wedge \gamma^{A/A'}$. This gives 42 conditions determining all the functions $u_{\mu A}$ and $u_{\mu A'}$ in terms of (u_μ). After solving these 42 conditions we get

$$\begin{aligned}
\mathrm{d}u_1 &= (3u_4 + 2u_6)\gamma^1 + u_5\gamma^2 - 2u_2\gamma^3 + u_{1j}\theta^j,\\
\mathrm{d}u_2 &= -(2u_5 + u_7)\gamma^1 - (u_4 + 2u_6)\gamma^2 + (u_1 - u_3)\gamma^3 + u_{2j}\theta^j,\\
\mathrm{d}u_3 &= u_6\gamma^1 + (2u_5 + 3u_7)\gamma^2 + 2u_2\gamma^3 + u_{3j}\theta^j,\\
\mathrm{d}u_4 &= -3u_1\gamma^1 + 3u_5\gamma^3 + u_{4j}\theta^j,\\
\mathrm{d}u_5 &= 2u_2\gamma^1 - u_1\gamma^2 + (2u_6 - u_4)\gamma^3 + u_{5j}\theta^j,\\
\mathrm{d}u_6 &= -u_3\gamma^1 + 2u_2\gamma^2 + (u_7 - 2u_5)\gamma^3 + u_{6j}\theta^j,\\
\mathrm{d}u_7 &= -3u_3\gamma^2 - 3u_6\gamma^3 + u_{7j}\theta^j.
\end{aligned}$$

Second, the terms at the basis forms $\theta^i \wedge \theta^j \wedge \theta^k$, when equated to zero, can be split into two types of equations. The first type is obtained by eliminating the curvatures $K^i{}_{jkl}$ from the full set. This yields a system of linear equations for the unknowns $u_{\mu j}$, whose only solution is $u_{\mu j} = 0$. After these conditions are imposed, the second type of equations involves the curvatures $K^i{}_{jkl}$ only in a linear fashion. It has a unique solution for the curvatures, which is explicitly given by (6.15)–(6.16).

Third, after imposing the conditions described above, all the other terms in $\mathrm{d}^2\theta^i$ are automatically zero.

This proves the proposition. □

The next proposition determines the derivatives of the unknown k_1.

Proposition 6.9. *The second Bianchi identities* $\mathrm{d}^2\gamma^A \equiv 0 \equiv \mathrm{d}^2\gamma^{A'}$, $A, A' = 1, 2, 3$, *imply that*

$$\mathrm{d}k_1 = -2k_3\gamma^2 + 2k_2\gamma^3. \tag{6.18}$$

Proof. To prove this we write $\mathrm{d}k_1$ in the most general form

$$\mathrm{d}k_1 = k_{1i}\theta^i + k_{1A}\gamma^A + k_{1A'}\gamma^{A'},$$

and consider the terms $\theta^i \wedge \theta^j \wedge \gamma^{A/A'}$ in the decomposition of $\mathrm{d}^2\gamma^{A/A'}$. This immediately yields $k_{1A'} = 0$ for all $A' = 1, 2, 3$, and $k_{11} = 0, k_{12} = -2k_3$, and $k_{13} = 2k_2$.

Eliminating u_μs from the equations implied by equating to zero the coefficients at the terms $\theta^i \wedge \theta^j \wedge \theta^k$ in $\mathrm{d}^2\gamma^{A/A'} \equiv 0$, shows that all the remaining coefficients k_{1i} in $\mathrm{d}k_1$ must also vanish, i.e., $k_{1i} = 0$, for all $i = 1, 2, \ldots, 9$. This finishes the proof. □

The lack of the θ^i terms on the right-hand sides of equations (6.14) and (6.18) proves that the functions u_μ and k_1, and as a consequence the functions $k_2, \ldots, k_7$, are *constant* along the base manifold M. They depend only on the fiber coordinates. Moreover, since only γ^As appear on the right-hand sides of these equations, they only depend on the fiber coordinates associated with $\mathrm{SO}(3)_L$. This means that there exists an $\mathrm{SO}(3)_L$ gauge in which all the functions u_μ, $k_1, \ldots, k_7$ are constant. This is the same as to say that there exists a subbundle $\mathcal{G}$ of P, with fibers at least as large as $\mathrm{SO}(3)_R$, on which we have $\mathrm{d}u_\mu = 0 = \mathrm{d}k_1 = \cdots = \mathrm{d}k_7$.

To see the examples of such solutions we look at the fourth and the seventh equation in (6.14). Since we want $\mathrm{d}u_4 = \mathrm{d}u_7 = 0$, we obtain that

$$u_1\gamma^1 = u_5\gamma^3 \quad \text{and} \quad u_3\gamma^2 = -u_6\gamma^3.$$

Now, assuming that $u_1 \neq 0 \neq u_3$, we solve it for γ^1 and γ^2, obtaining

$$\gamma^1 = \frac{u_5}{u_1}\gamma^3 \quad \text{and} \quad \gamma^2 = -\frac{u_6}{u_3}\gamma^3. \tag{6.19}$$

Thus these equations show that we have reduced our original manifold P to its 13-dimensional submanifold $\mathcal{G}$ on which the forms γ^1 and γ^2 become dependent on γ^3. On this manifold we further want that $\mathrm{d}u_\mu = 0$ for all $\mu = 1, 2, \ldots, 7$. Inserting (6.19) into the right-hand sides of equations (6.14) for $\mathrm{d}u_1$, $\mathrm{d}u_2$, $\mathrm{d}u_3$, $\mathrm{d}u_5$, $\mathrm{d}u_6$, and equating the result to zero, we obtain the five equations

$$\begin{aligned}
&2u_1u_2u_3 - 3u_3u_4u_5 + u_1u_5u_6 - 2u_3u_5u_6 = 0,\\
&u_1^2u_3 - u_1u_3^2 - 2u_3u_5^2 + u_1u_4u_6 + 2u_1u_6^2 - u_3u_5u_7 = 0,\\
&2u_1u_2u_3 - 2u_1u_5u_6 + u_3u_5u_6 - 3u_1u_6u_7 = 0,\\
&u_1u_3u_4 - 2u_2u_3u_5 - u_1^2u_6 - 2u_1u_3u_6 = 0,\\
&2u_1u_3u_5 + u_3^2u_5 + 2u_1u_2u_6 - u_1u_3u_7 = 0.
\end{aligned}$$

A particular solution is given by

$$\begin{cases}
u_2 = \dfrac{u_6\sqrt{4u_1 + u_3}\sqrt{u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2}}{3u_6^2 - u_3^2},\\[2ex]
u_4 = \dfrac{u_6(u_1u_6^2 - 3u_1u_3^2 - 2u_3u_6^2)}{u_3(3u_6^2 - u_3^2)},\\[2ex]
u_5 = -\dfrac{u_1\sqrt{u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2}}{u_3\sqrt{4u_1 + u_3}},\\[2ex]
u_7 = \dfrac{(2u_1u_3^3 + u_3^3 + 2u_1u_6^2 - u_3u_6^2)\sqrt{u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2}}{u_3(3u_6^2 - u_3^2)\sqrt{4u_1 + u_3}}.
\end{cases} \tag{6.20}$$

Of course we restrict the range of the free real torsion parameters u_1, u_3 and u_6, so that u_2, u_4, u_5 and u_7 are real and finite! This happens, e.g., for

$$-1 < \frac{4u_3}{u_1} < 4, \quad u_6 \neq \pm\sqrt{\tfrac{1}{3}}u_3 \neq 0.$$

This solution is compatible with the structure equations

$$\mathrm{d}\gamma^1 = -\gamma^2 \wedge \gamma^3 + \kappa^1, \quad \mathrm{d}\gamma^2 = -\gamma^3 \wedge \gamma^1 + \kappa^2$$

having κ^1, κ^2 and κ^3 as in (6.15), and with $\mathrm{d}k_1 = 0$ if and only if

$$k_1 = \frac{4(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2(u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2)}{u_3^2(4u_1 + u_3)(u_3^2 - 3u_6^2)^2}. \tag{6.21}$$

This leads to the following proposition.

Proposition 6.10. *Assume that the forms* $(\theta^i, \gamma^3, \gamma^{A'})$ *satisfy the equations for* $\mathrm{d}\theta^i$, $\mathrm{d}\gamma^3$*, and* $\mathrm{d}\gamma^{A'}$ *as in the system* (4.6)–(4.7)*,* (6.13) *with*

- *the forms* γ^1 *and* γ^2 *given by* (6.19)*,*
- *the coefficients* u_1*,* u_3 *and* u_6 *being constants,*
- *the coefficients* u_2*,* u_4*,* u_5 *and* u_7 *given by* (6.20)*,*
- *the curvatures* κ^1*,* $\kappa^{A'}$ *given by* (6.15)–(6.16) *and* (6.21)*.*

Then

- *the equations for* $\mathrm{d}\gamma^1$ *and* $\mathrm{d}\gamma^2$ *in the system* (4.6)–(4.7)*,* (6.13) *are automatically satisfied, and*
- *the Bianchi identities* $\mathrm{d}^2\theta^i = \mathrm{d}^2\gamma^3 = \mathrm{d}^2\gamma^{A'} = 0$ *are also automatically satisfied.*

In such a case the manifold on which the forms $(\theta^i, \gamma^3, \gamma^{A'})$ *are defined becomes a* 13*-dimensional Lie group* $\mathcal{G}^{13}$*, with the forms* $(\theta^i, \gamma^3, \gamma^{A'})$ *being its Maurer-Cartan forms. The Lie group* $\mathcal{G}^{13}$ *is a subbundle of the bundle* $\mathrm{SO}(3) \times \mathrm{SO}(3) \to P \to M^9$*, so that the manifold* M^9 *is a homogeneous space* $M^9 = \mathcal{G}^{13}/H$*, with* H *being a certain* 4*-dimensional subgroup of* $\mathcal{G}^{13}$ *containing* $\mathrm{SO}(3)_R$*. The nearly integrable* $\mathrm{SO}(3) \times \mathrm{SO}(3)$ *structure* (g, Υ, ω) *on* M^9 *is given by the forms* θ^i *and the formulae* (2.8)–(2.10)*.*

For all of these geometries the metric g is conformally non-flat *and* not *locally symmetric. The Ricci tensors of the Levi-Civita connection* $\overset{\mathrm{LC}}{\Gamma}$*, of the characteristic connection* Γ*, and of the* $\mathfrak{so}(3)_L$ *part* $\overset{+}{\Gamma}$ *of* Γ *have all two distinct eigenvalues.*

The $\mathfrak{so}(3)_R$ *part* $\overset{-}{\Gamma}$ *of the characteristic connection* Γ *is* Einstein.

Explicitly, in the adapted coframe (θ^i) in which the structure equations read as in (6.11) and in which the structural tensors g, Υ, ω are given by (2.8)–(2.10), we have the following:

- The eigenvalues of the Levi-Civita connection Ricci tensor read

$$(45s, 45s, 45s, 55s, 55s, 55s, 55s, 55s, 55s),$$

where

$$s = \frac{(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^3}{u_3^2(4u_1 + u_3)(u_3^2 - 3u_6^2)^2}.$$

The Ricci scalar is equal to $465s$. The Levi-Civita connection is never Ricci flat, because the equation $u_1u_3^2 + u_1u_6^2 + u_3u_6^2 = 0$ contradicts the reality of u_2, u_5 and u_7.

- The $\mathfrak{so}(3)_L$ part $\overset{+}{\Gamma}$ of Γ has the curvature $\overset{+}{\Omega} = \kappa^A e_A$, with

$$\begin{aligned}
\kappa^1 = {} & \frac{4(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2(u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2)}{u_3^2(4u_1 + u_3)(u_3^2 - 3u_6^2)^2}\kappa_0^1 \\
& + \frac{4u_6(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2\sqrt{u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2}}{u_3^2\sqrt{4u_1 + u_3}(u_3^2 - 3u_6^2)^2}\kappa_0^2 \\
& - \frac{4(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2\sqrt{u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2}}{u_3\sqrt{4u_1 + u_3}(u_3^2 - 3u_6^2)^2}\kappa_0^3, \\
\kappa^2 = {} & \frac{4u_6(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2\sqrt{u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2}}{u_3^2\sqrt{4u_1 + u_3}(u_3^2 - 3u_6^2)^2}\kappa_0^1 \\
& + \frac{4u_6^2(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2}{u_3^2(u_3^2 - 3u_6^2)^2}\kappa_0^2 - \frac{4u_6(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2}{u_3(u_3^2 - 3u_6^2)^2}\kappa_0^3, \\
\kappa^3 = {} & -\frac{4(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2\sqrt{u_1u_3^2 - u_3^3 + u_1u_6^2 + 4u_3u_6^2}}{u_3\sqrt{4u_1 + u_3}(u_3^2 - 3u_6^2)^2}\kappa_0^1 \\
& - \frac{4u_6(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2}{u_3(u_3^2 - 3u_6^2)^2}\kappa_0^2 + \frac{4(u_1u_3^2 + u_1u_6^2 + u_3u_6^2)^2}{(u_3^2 - 3u_6^2)^2}\kappa_0^3
\end{aligned}$$

and the matrices $e_A = (e_A{}^i{}_j)$ given by (2.15)–(2.17). It has the Ricci tensor $\overset{+}{R}_{ij}$ with two different eigenvalues

$$(0, 0, 0, 20s, 20s, 20s, 20s, 20s, 20s),$$

with the Ricci scalar equal to $120s$.

- The $\mathfrak{so}(3)_R$ part $\overset{-}{\Gamma}$ of Γ has the curvature $\overset{-}{\Omega} = 15s\kappa_0^{A'}e_{A'}$, where as before the matrices $e_{A'} = (e_{A'}{}^i{}_j)$ are given by (2.15)–(2.17). Its Ricci tensor is *Einstein*,

$$\overset{-}{R}_{ij} = 30sg_{ij},$$

and has Ricci scalar equal to $270s$.

- The characteristic connection $\Gamma = \overset{+}{\Gamma} + \overset{-}{\Gamma}$ has curvature

$$\Omega = \overset{+}{\Omega} + \overset{-}{\Omega} = \kappa^A e_A + 15s\kappa_0^{A'}e_{A'}$$

and the Ricci tensor with eigenvalues

$$(30s, 30s, 30s, 50s, 50s, 50s, 50s, 50s, 50s).$$

The examples of nearly integrable $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometries with torsion of the characteristic connection in $V_{[0,6]}$ described by this proposition have quite similar features to the nearly integrable $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometries with torsion in $V_{[0,2]}$. In particular, if any of these geometries has curvature $\overset{+}{\Omega}\equiv 0$, then it must be flat and torsion free.

It turns out however that there is another branch of nearly integrable $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometries with torsion of their characteristic connections in $V_{[0,6]}$ for which $\overset{+}{\Omega}\equiv 0$ neither implies vanishing torsion nor vanishing of $\overset{-}{\Omega}$. Below we present these examples.

Assuming that $\overset{+}{\Omega}\equiv 0$ is the same as to assume that

$$k_1=k_2=k_3=k_4=k_5=k_6=0.$$

(Compare with the first three equations in (6.15)). But since $\overset{+}{\Omega}\equiv 0$ is the condition for the connection $\overset{+}{\Gamma}$ to be flat, in such a situation we can use a gauge in which $\overset{+}{\Gamma}\equiv 0$. This condition means that the system (4.6)–(4.7), (6.13) reduces from P to a 12-dimensional $\mathcal{G}^{12}$ manifold on which $\gamma^1\equiv\gamma^2\equiv\gamma^3\equiv 0$.

Having these conditions and the requirement that $T\in V_{[0,6]}$, we see, via (6.14), that all u_μ are *constants*. The rest of the equations $\mathrm{d}^2\theta^i\equiv 0$ finally imply that

$$\begin{aligned}
&2u_2u_4+2u_1u_5-u_3u_5+u_1u_7=0,\\
&u_3u_4-u_1u_6+2u_3u_6+2u_2u_7=0,\\
&2u_1u_2+2u_2u_3-2u_4u_5-3u_5u_6-u_4u_7-2u_6u_7=0,\\
&2u_1^2+2u_1u_3+2u_4^2+4u_4u_6+2u_5u_7=0,\\
&2u_1u_3+2u_3^2+2u_4u_6+4u_5u_7+2u_7^2=0,\\
&2u_1^2-2u_3^2+2u_4^2+2u_4u_6-2u_5u_7-2u_7^2=0.
\end{aligned}$$

We have found six different particular solutions to these equations. These are

$$\text{(6.22)}\qquad \begin{cases} u_2=\dfrac{(u_1-2u_3)u_7^2-(2u_1-u_3)\big((u_1-2u_3)(u_1+u_3)+u_4^2\big)}{6u_4u_7},\\[2mm] u_5=\dfrac{(u_1-2u_3)(u_1+u_3)+u_4^2-2u_7^2}{3u_7},\\[2mm] u_6=\dfrac{-(2u_1-u_3)(u_1+u_3)-2u_4^2+u_7^2}{3u_4};\end{cases}$$

$$\text{(6.23)}\qquad \begin{cases} u_2=\mp\dfrac{u_5\sqrt{9u_3^2-4u_4^2}}{2u_4},\quad u_6=\mp\dfrac{u_3\big(\pm 3u_3+\sqrt{9u_3^2-4u_4^2}\big)}{2u_4},\\[2mm] u_1=\tfrac12\big(u_3\pm\sqrt{9u_3^2-4u_4^2}\big),\quad u_7=0;\end{cases}$$

$$\text{(6.24)}\qquad \begin{cases} u_2=\mp\dfrac{u_6\big(\pm 9u_3^2+\sqrt{9u_3^2+8u_7^2}\big)}{8u_7},\\[2mm] u_5=\dfrac{-4u_7^2\pm u_3\big(\mp 3u_3+\sqrt{9u_3^2+8u_7^2}\big)}{8u_7},\\[2mm] u_1=\tfrac14\big(-u_3\mp\sqrt{9u_3^2+8u_7^2}\big),\quad u_4=0;\end{cases}$$

(6.25) $u_1 = u_3 = u_4 = u_5 = u_7 = 0;$

(6.26) $u_1 = -u_3, \quad u_4 = u_5 = u_6 = u_7 = 0;$

(6.27) $u_1 = u_3 = u_4 = u_6 = u_7 = 0.$

It follows that for all of these six solutions we have $d^2\theta^i \equiv 0$ and $d^2\gamma^{A'} \equiv 0$, automatically for all $i = 1, 2, \dots, 9$ and for all $A' = 1, 2, 3$. Thus each of these six solutions defines a nearly integrable $SO(3) \times SO(3)$ geometry $(M^9, g, \Upsilon, \omega)$ having the torsion of the characteristic connection in $V_{[0,6]}$ and the vanishing curvature $\overset{+}{\Omega}$ of $\overset{+}{\Gamma}$. It turns out that all the six solutions have the same qualitative behavior of the curvatures of $\overset{\text{LC}}{\Gamma}$, Γ, $\overset{+}{\Gamma}$ and $\overset{-}{\Gamma}$. The properties of the curvatures of the geometries corresponding to these six solutions are summarized in the theorem below.

Theorem 6.11. *All nearly integrable* $SO(3) \times SO(3)$ *geometries* $(M^9, g, \Upsilon, \omega)$ *corresponding to any solution* (6.22)–(6.27) *above have*

- *torsion of the characteristic connection* Γ *in* $V_{[0,6]} \subset \bigwedge^3 \mathbb{R}^9$,
- *vanishing curvature* $\overset{+}{\Omega}$ *of the* $\mathfrak{so}(3)_L$ *part of* Γ, *i.e.,* $\overset{+}{\Omega} \equiv 0$,
- *the curvature* Ω *of the characteristic connection* Γ *equal to*

$$\Omega \equiv \overset{-}{\Omega} = \tfrac{1}{36}\|T\|^2 \kappa_0^{A'} e_{A'},$$

where $\|T\|^2$ *is the square norm of the torsion* T *of* Γ*:*

$$\begin{aligned}\|T\|^2 &= T_{ijk}T^{ijk}\\ &= 36k_7 = 36\big(2u_1^2 + u_2^2 + u_1u_3 + 2u_4^2 + u_5^2 + 3u_4u_6 + u_6^2 + u_5u_7\big)\end{aligned}$$

with u_μ *being constants and satisfying one of* (6.22)–(6.27).

All these geometries $(M^9, g, \Upsilon, \omega)$ *are locally homogeneous spaces* $M^9 = \mathcal{G}^{12}/H$, *where* $\mathcal{G}^{12}$ *is a* 12*-dimensional symmetry group of* $(M^9, g, \Upsilon, \omega)$ *and* H *is its* 3*-dimensional subgroup isomorphic to* $SO(3)$, *i.e.,* $H = SO(3)_R$. *The metric* g, *the tensor* Υ *and the form* ω *defining a nearly integrable* $SO(3) \times SO(3)$ *geometry on* M^9 *are given by formulae* (2.8)–(2.10), *in terms of the forms* $(\theta^i, \gamma^A \equiv 0, \gamma^{A'})$ *satisfying* (4.6)–(4.7), (6.13), (6.15)–(6.16), *and one of* (6.22)–(6.27), *with* u_μ *being constants.*

- *In the basis* $(\theta^i, \gamma^{A'})$ *the Killing form for the group* $\mathcal{G}^{12}$ *reads*

$$\mathrm{Kil} = -8\,\mathrm{diag}(k_7, k_7, k_7, k_7, k_7, k_7, k_7, k_7, k_7, 1, 1, 1).$$

- *If* $k_7 \neq 0$ *the Riemannian manifold* $(M^9 = \mathcal{G}^{12}/SO(3)_R, g)$ *is not locally symmetric. If* $k_7 = 0$ *the solutions have flat characteristic connection,* $\Omega \equiv 0$, *and in such a case* $(M^9 = \mathcal{G}^{12}/SO(3)_R, g)$ *is a locally symmetric Riemannian manifold.*
- *For every value of* k_7 *the metric is Einstein,* $\overset{\text{LC}}{\mathrm{Ric}} = 3k_7 g$. *It is not conformally flat unless the torsion is zero,* $(u_1, u_2, \dots, u_7) = 0$.
- *Also the* $SO(3)_R$ *part* $\overset{-}{\Gamma}$ *of the characteristic connection is always Einstein,*

$$\overset{-}{R}_{ij} = 2k_7 g_{ij}.$$

It is flat, $\overset{-}{\Omega} \equiv 0$, *if and only if* $k_7 = 0$.

It is a remarkable fact that both the Levi-Civita connection $\overset{\mathrm{LC}}{\Gamma}$ and the characteristic connection Γ are Einstein and (generically) Ricci non-flat for all the geometries $(M^9, g, \Upsilon, \omega)$ described by the theorem. Moreover although the metric g is not conformally flat, the $\mathrm{SO}(3)_L$ part $\overset{+}{\Gamma}$ of Γ is flat. This makes these geometries similar to the selfdual Riemannian geometries in dimension four.

6.3. Analogs of selfduality; examples with torsion in $V_{[0,2]} \oplus V_{[0,6]}$. The examples described by the Theorem 6.11 raise the question if there are other nearly integrable $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometries $(M^9, g, \Upsilon, \omega)$ in dimension nine for which the $\mathfrak{so}(3)_L$ part $\overset{+}{\Gamma}$ of the characteristic connection Γ is flat, $\overset{+}{\Omega} \equiv 0$, and for which the $\mathfrak{so}(3)_R$ part $\overset{-}{\Gamma}$ is not flat, $\overset{-}{\Omega} \neq 0$.

In the following the nearly integrable $\mathrm{SO}(3)\times\mathrm{SO}(3)$ geometries $(M^9, g, \Upsilon, \omega)$ with these two properties, $\overset{+}{\Omega} \equiv 0$ and $\overset{-}{\Omega} \neq 0$, will be called *analogs of selfduality*.

The problem of finding all such structures is a difficult one. To generalize solutions of Theorem 6.11, on top of the analogs of selfduality conditions, we will assume in addition that the torsion T of the characteristic connection Γ is restricted from $\bigwedge^3 \mathbb{R}^9$ to $V_{[0,2]} \oplus V_{[0,6]}$. In this section we will find all such structures.

We first have an analog of Proposition 6.6 and Remark 6.2:

Proposition 6.12. *In an adapted coframe (θ^i) the $V_{[0,2]} \oplus V_{[0,6]}$ torsion of a characteristic connection of a nearly integrable geometry $(M^9, g, \Upsilon, \omega)$ reads*

$$
(6.28)\quad
\begin{cases}
T^1 = -t_1\lambda_0^{9'} + t_2\lambda_0^{6'} + \frac{1}{3}t_3(5\kappa_0^{3'} - 4\lambda_0^{3'} + 2\lambda_0^{12'}) + u_1(-\lambda_0^{3'} + \lambda_0^{12'}) \\
\qquad - u_2\lambda_0^{15'} - u_3\lambda_0^{3'} - u_4\lambda_0^{6'} - u_5\lambda_0^{9'} - u_6\lambda_0^{6'} - u_7\lambda_0^{9'}, \\
T^2 = t_1\lambda_0^{8'} - t_2\lambda_0^{5'} + \frac{1}{3}t_3(-5\kappa_0^{2'} + 4\lambda_0^{2'} - 2\lambda_0^{11'}) + u_1(\lambda_0^{2'} - \lambda_0^{11'}) \\
\qquad + u_2\lambda_0^{14'} + u_3\lambda_0^{2'} + u_4\lambda_0^{5'} + u_5\lambda_0^{8'} + u_6\lambda_0^{5'} + u_7\lambda_0^{8'}, \\
T^3 = -t_1\lambda_0^{7'} + t_2\lambda_0^{4'} + \frac{1}{3}t_3(5\kappa_0^{1'} - 4\lambda_0^{1'} + 2\lambda_0^{10'}) + u_1(-\lambda_0^{1'} + \lambda_0^{10'}) \\
\qquad - u_2\lambda_0^{13'} - u_3\lambda_0^{1'} - u_4\lambda_0^{4'} - u_5\lambda_0^{7'} - u_6\lambda_0^{4'} - u_7\lambda_0^{7'}, \\
T^4 = -t_1\lambda_0^{15'} + \frac{1}{3}t_2(-5\kappa_0^{3'} - 2\lambda_0^{3'} + 4\lambda_0^{12'}) - t_3\lambda_0^{6'} \\
\qquad + u_1\lambda_0^{6'} - u_2\lambda_0^{9'} + u_4(-\lambda_0^{3'} + \lambda_0^{12'}) + u_5\lambda_0^{15'} - u_6\lambda_0^{3'}, \\
T^5 = t_1\lambda_0^{14'} + \frac{1}{3}t_2(5\kappa_0^{2'} + 2\lambda_0^{2'} - 4\lambda_0^{11'}) + t_3\lambda_0^{5'} \\
\qquad - u_1\lambda_0^{5'} + u_2\lambda_0^{8'} + u_4(\lambda_0^{2'} - \lambda_0^{11'}) - u_5\lambda_0^{14'} + u_6\lambda_0^{2'}, \\
T^6 = -t_1\lambda_0^{13'} + \frac{1}{3}t_2(-5\kappa_0^{1'} - 2\lambda_0^{1'} + 4\lambda_0^{10'}) - t_3\lambda_0^{4'} \\
\qquad + u_1\lambda_0^{4'} - u_2\lambda_0^{7'} + u_4(-\lambda_0^{1'} + \lambda_0^{10'}) + u_5\lambda_0^{13'} - u_6\lambda_0^{1'}, \\
T^7 = \frac{1}{3}t_1(5\kappa_0^{3'} + 2\lambda_0^{3'} + 2\lambda_0^{12'}) + t_2\lambda_0^{15'} - t_3\lambda_0^{9'} \\
\qquad - u_2\lambda_0^{6'} + u_3\lambda_0^{9'} + u_5(-\lambda_0^{3'} + \lambda_0^{12'}) + u_6\lambda_0^{15'} - u_7\lambda_0^{3'}, \\
T^8 = -\frac{1}{3}t_1(5\kappa_0^{2'} + 2\lambda_0^{2'} + 2\lambda_0^{11'}) - t_2\lambda_0^{14'} + t_3\lambda_0^{8'} \\
\qquad + u_2\lambda_0^{5'} - u_3\lambda_0^{8'} + u_5(\lambda_0^{2'} - \lambda_0^{11'}) - u_6\lambda_0^{14'} + u_7\lambda_0^{2'}, \\
T^9 = \frac{1}{3}t_1(5\kappa_0^{1'} + 2\lambda_0^{1'} + 2\lambda_0^{10'}) + t_2\lambda_0^{13'} - t_3\lambda_0^{7'} \\
\qquad - u_2\lambda_0^{4'} + u_3\lambda_0^{7'} + u_5(-\lambda_0^{1'} + \lambda_0^{10'}) + u_6\lambda_0^{13'} - u_7\lambda_0^{1'},
\end{cases}
$$

where $(t_1, t_2, t_3, u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ *are the ten independent components of the torsion* T, *and* $(\lambda_0^{\mu'})$, $\mu' = 1, 2, \ldots, 15$, *is a basis of* 2*-forms in* $V_{[4,2]}$ *as defined in* (3.3). *Note that if all* u_μ *are equal to zero, then* $T \in V_{[0,2]}$, *and if all* t_A *are equal to zero, then* $T \in V_{[0,6]}$.

We want to construct nearly integrable $\mathrm{SO}(3) \times \mathrm{SO}(3)$ structures with torsion in $V_{[0,2]} \oplus V_{[0,6]}$, and with $\overset{+}{\Omega} \equiv 0$. All of them, in an adapted coframe, are therefore described by the system (4.6)–(4.7), (6.28), with $\kappa^A \equiv 0$. This enables us to reduce the system from $P \to M^9$ to a 12-dimensional subbundle of P on which

$$\gamma^1 \equiv \gamma^2 \equiv \gamma^3 \equiv 0.$$

The procedure of analyzing such a reduced system is completely the same as the procedure leading to solutions described by Theorem 6.11. We therefore only state the result.

Theorem 6.13. *All nearly integrable* $\mathrm{SO}(3) \times \mathrm{SO}(3)$ *geometries* $(M^9, g, \Upsilon, \omega)$, *which have torsion* T *of the characteristic connection* Γ *in* $V_{[0,2]} \oplus V_{[0,6]}$, *and the curvature* $\overset{+}{\Omega}$ *of the* $\mathfrak{so}(3)_L$*-part* $\overset{+}{\Gamma}$ *of* Γ *vanishing,* $\overset{+}{\Omega} \equiv 0$, *correspond to the system* (4.6)–(4.7), (6.28), *with* $\gamma^1 \equiv \gamma^2 \equiv \gamma^3 \equiv 0$ *and* $\kappa^A \equiv 0$, *and* constant *torsion coefficients*

$$(t_1, t_2, t_3, u_1, u_2, u_3, u_4, u_5, u_6, u_7)$$

satisfying the following algebraic equations:

$$\begin{cases} 2u_2u_4 + 2u_1u_5 - u_3u_5 + u_1u_7 \\ \quad + t_2u_2 + t_1u_3 - t_3u_5 - t_3u_7 - t_1t_3 = 0, \\ u_3u_4 - u_1u_6 + 2u_3u_6 + 2u_2u_7 \\ \quad - t_2u_1 - t_1u_2 - t_3u_4 - t_3u_6 + t_2t_3 = 0, \\ 2u_1u_2 + 2u_2u_3 - 2u_4u_5 - 3u_5u_6 - u_4u_7 - 2u_6u_7 \\ \quad + t_3u_2 + t_2u_5 - t_1u_6 - t_1t_2 = 0, \\ 2u_1^2 + 2u_1u_3 + 2u_4^2 + 4u_4u_6 + 2u_5u_7 \\ \quad - 2t_1u_7 - t_3u_1 - 2t_3u_3 + t_2u_4 - t_1u_5 + 2t_2u_6 + t_1^2 - t_3^2 = 0, \\ 2u_1u_3 + 2u_3^2 + 2u_4u_6 + 4u_5u_7 + 2u_7^2 \\ \quad - 2t_3u_1 - t_3u_3 + 2t_2u_4 - 2t_1u_5 + t_2u_6 - t_1u_7 + t_2^2 - t_3^2 = 0, \\ 2u_1^2 - 2u_3^2 + 2u_4^2 + 2u_4u_6 - 2u_5u_7 - 2u_7^2 \\ \quad + t_3u_1 - t_3u_3 - t_2u_4 + t_1u_5 + t_2u_6 - t_1u_7 + t_1^2 - t_2^2 = 0. \end{cases} \tag{6.29}$$

If these equations are satisfied, the metric g, *the tensor* Υ *and the* 4*-form* ω *are obtained in terms of the forms* (θ^i) *via formulae* (2.8)–(2.10). *They descend from the* 12*-dimensional subbundle* $P^{12} \to M^9$ *of the fiber bundle* $\mathrm{SO}(3) \times \mathrm{SO}(3) \to P \to M^9$ *to* M^9 *due to the structure equations* (4.6).

If the equations (6.29) *for the constants* $(t_1, t_2, t_3, u_1, u_2, u_3, u_4, u_5, u_6, u_7)$ *are satisfied, then all the integrability conditions* $\mathrm{d}^2\theta^i \equiv 0$ *and* $\mathrm{d}^2\gamma^{A'} \equiv 0$, *for all* θ^i *and all* $\gamma^{A'}$ *appearing in the system* (4.6)–(4.7), (6.28) *are automatically satisfied.*

The manifold P^{12} *is locally a* 12*-dimensional symmetry group* $P^{12} = \mathcal{G}^{12}$ *of the so obtained* $(M^9, g, \Upsilon, \omega)$, *and* M^9 *is a homogeneous space* $M^9 = \mathcal{G}^{12}/H$, *where* $H = \mathrm{SO}(3)_R$ *is a subgroup of* $\mathcal{G}^{12}$.

The curvatures $\kappa^{A'}$ are given by

$$\kappa^{A'} = \big(\tfrac{1}{36}\|T\|^2 - \tfrac{25}{6}(t_1^2+t_2^2+t_3^2)\big)\kappa_0^{A'}, \quad A'=1,2,3,$$

where

$$\|T\|^2 = 6\big(4u_1^2+6u_2^2+2u_1u_3+4u_3^2+4u_4^2+6u_5^2+6u_4u_6+6u_6^2+6u_5u_7+4u_7^2\big) + 90\big(t_1^2+t_2^2+t_3^2\big),$$

with constants $(t_1,t_2,t_3,u_1,u_2,u_3,u_4,u_5,u_6,u_7)$ fulfilling equations (6.29).

The torsion T of the characteristic connection generically seats in $V_{[02,]}\oplus V_{[0,6]}$. It is in $V_{[0,2]}$ if and only if $(u_1,u_2,u_3,u_4,u_5,u_6,u_7)=0$, and in $V_{[0,6]}$ if and only if $(t_1,t_2,t_3)=0$. The square of the torsion is $\|T\|^2$ as above.

The curvature Ω of Γ has vanishing $\mathfrak{so}(3)_L$ part, $\overset{+}{\Omega}\equiv 0$, and is equal to

$$\Omega \equiv \overset{-}{\Omega} = \big(\tfrac{1}{36}\|T\|^2 - \tfrac{25}{6}(t_1^2+t_2^2+t_3^2)\big)\kappa_0^{A'}e_{A'}.$$

The Ricci tensor of the curvature Ω of the characteristic connection, and what is the same, the Ricci tensor of the curvature $\overset{-}{\Omega}$ of its $\mathfrak{so}(3)_R$-part is Einstein,

$$\overset{-}{R}_{ij} = 2\big(\tfrac{1}{36}\|T\|^2 - \tfrac{25}{6}(t_1^2+t_2^2+t_3^2)\big)g_{ij}.$$

The metric g is Einstein if and only if $t_1=t_2=t_3=0$. In such a case the nearly integrable structures coincide with those described in Theorem 6.11.

Generically the solutions described by this theorem have $\overset{-}{\Omega}\neq 0$, and as such constitute analogs of selfduality.

Remark 6.14. Note that although $(t_1,t_2,t_3)=0$ gives all the solutions described in Theorem 6.11, the assumption $(u_1,u_2,u_3,u_4,u_5,u_6,u_7)=0$ does not recover all the solutions with $T\in V_{[0,2]}$. The reason for this is that here we additionally assumed that $\overset{+}{\Omega}\equiv 0$, and such solutions are possible for $T\in V_{[0,2]}$ only if $T=0$. Nonetheless the solutions in this section are non-trivial generalizations to $T\in V_{[0,2]}\oplus V_{[0,6]}$ of solutions from Theorems 6.5 and 6.11.

Remark 6.15. We emphasize that the system of equations (6.29) for the constants $(t_1,t_2,t_3,u_1,u_2,u_3,u_4,u_5,u_6,u_7)$ can be solved explicitly to the very end. For example, an application of a Mathematica command `Solve[]` to the system (6.29), immediately gives thirteen different solutions of these equations. The obtained formulae are not particularly illuminating. For example a generalization to the case of $T\in V_{[0,2]}\oplus V_{[0,6]}$ of the solution (6.22) from Section 6.2 is given by

$$\begin{aligned} u_2 = \Big(&\big(2(t_3+u_1-2u_3)u_7^2 \\ &+ (t_3-2u_1+u_3)\big(-2t_2^2+(t_3+u_1-2u_3)(t_3+2(u_1+u_3))-3t_2u_4+2u_4^2\big) \\ &+ 3t_1(t_3+u_1-2u_3)u_7-2t_1^2(t_3+u_1-2u_3)\big)\Big)\times\big(3(t_2+2u_4)(2u_7-t_1)\big)^{-1}, \end{aligned}$$

$$u_5 = \frac{(t_3 + u_1 - 2u_3)(t_3 + 2(u_1 + u_3)) - (2t_2 - u_4)(t_2 + 2u_4) - 4u_7^2 + t_1^2}{3(2u_7 - t_1)},$$
$$u_6 = \frac{2u_3(u_3 - u_1) - 4(u_1^2 + u_4^2) - (t_1 - 2u_7)(2t_1 + u_7) + t_2^2 + t_3^2 + 3t_3u_3}{3(2u_4 + t_2)}.$$

It is a matter of checking that this coincides with solution (6.22) from Section 6.2 in the limit $t_1 \to 0$, $t_2 \to 0$, $t_3 \to 0$.

A solution of (6.29) which has *no* limit when $t_1 \to 0$, $t_2 \to 0$, $t_3 \to 0$ is given below:

$$u_2 = \frac{3t_1t_2 - 8t_2u_5 + 8t_1u_6 + 12u_5u_6}{20t_3}, \quad u_1 = u_3 = t_3, \quad u_4 = -\tfrac{1}{2}t_2, \quad u_7 = \tfrac{1}{2}t_1.$$

Remark 6.16. It is remarkable that we have obtained analogs of selfduality *with high number of symmetries*. We did not *assume* any symmetry conditions. The homogeneity of the structures obtained were implied by the merely requirements that

$$\overset{+}{\Omega} \equiv 0 \quad \text{and} \quad T \in V_{[0,2]} \oplus V_{[0,6]}.$$

It would be very interesting to find analogs of selfduality which are not locally homogeneous. It is an open question whether such solutions exist.

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