Analog-to-Information Conversion via Random Demodulation

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***Abstract*— Many problems in radar and communication signal processing involve radio frequency (RF) signals of very high bandwidth. This presents a serious challenge to systems that might attempt to use a high-rate analog-to-digital converter (ADC) to sample these signals, as prescribed by the Shannon/Nyquist sampling theorem. In these situations, however, the information level of the signal is often far lower than the actual bandwidth, which prompts the question of whether more efficient schemes can be developed for measuring such signals. In this paper we propose a system that uses modulation, filtering, and sampling to produce a low-rate set of digital measurements. Our "analog-to-information converter" (AIC) is inspired by the recent theory of Compressive Sensing (CS), which states that a discrete signal having a sparse representation in some dictionary can be recovered from a small number of linear projections of that signal. We generalize the CS theory to continuous-time sparse signals, explain our proposed AIC system in the CS context, and discuss practical issues regarding implementation.**

I. Introduction

The power, stability, and low cost of digital signal processing (DSP) have pushed the *analog-to-digital converter* (ADC) increasingly close to the front-end of many important sensing, imaging, and communication systems. Unfortunately, many systems, especially those operating in the radio frequency (RF) bands, severely stress current ADC technologies. For example, some important radar and communications applications would be best served by an ADC sampling over 5 GSample/s and resolution of over 20 bits, a combination that greatly exceeds current capabilities.

It could be decades before ADCs based on current technology will be fast and precise enough for these applications. And even after better ADCs become available, the deluge of data will swamp back-end DSP algorithms. For example, sampling a 1GHz band using 2 GSample/s at 16 bits-per-sample generates data at a rate of 4GB/s, enough to fill a modern hard disk in roughly one minute. In a typical application, only a tiny fraction of this information is actually relevant; the wideband signals in many RF applications often have a large bandwidth but a small "information rate" [1].

Fortunately, recent developments in mathematics and signal processing have uncovered a promising approach to the ADC bottleneck that enables sensing at a rate comparable to the signal's information rate. A new field, known as *Compressive Sensing* (CS) [2], [3], establishes mathematically that a relatively small number of non-adaptive, linear measurements can harvest all of the information necessary to faithfully reconstruct sparse or compressible signals. An intriguing aspect of the theory is the central role played by randomization.

CS suggests a new framework for *analog-to-information conversion* (AIC) as an alternative to conventional ADC. A typical system is illustrated in Figure 1. The information extraction denoted by the operation $\mathbf{\Phi}$ replaces conventional sampling. Back-end DSP reconstructs the signal, approximates the signal, computes key statistics, or produces other information. For sparse input signals, AIC promises greatly reduced digital data rates (matching the information rate of the signal), and it offers the ability to focus only on the relevant information.

In this paper, we develop a practical AIC architecture based on a wideband pseudorandom demodulator and a low-rate sampler that can efficiently acquire a large class of compressible signals. The remainder of the paper is organized as follows. In Section II, we explain the traditional discrete-time CS problem, discuss methods for extending the basic theory to continuous-time signals, and present a system-level AIC design for low-rate sampling of continuous-time signals having a low information rate. In Section III, we discuss practical issues surrounding the implementation of such a system. Section IV conducts a series of simulation experiments to validate the design. We conclude in Section V.

II. Compressive Sensing for Analog Systems

A. Compressive sensing background

CS deals with the problem of acquiring an $N \times 1$ discrete-time signal vector x that is K*-sparse* or *compressible* in some *sparsity basis* matrix $\mathbf{\Psi}$ (where each column is a basis vector ψ_i). By K-sparse we mean that only $K \ll N$ of the expansion coefficients α representing $x = \mathbf{\Psi}\alpha$ are nonzero. By compressible we mean that the entries of α, when sorted from largest to smallest, decay rapidly to zero; such a signal is well approximated using a K-term representation.

The theory of CS as introduced by Candès, Romberg, and Tao [2] and Donoho [3] demonstrates that a signal that is K-sparse or compressible in one basis $\mathbf{\Psi}$ can be recovered

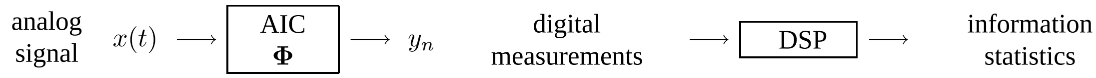

Fig. 1. *Analog-to-information converter (AIC). The operator $\mathbf{\Phi}$ takes nonadaptive linear measurements of the analog signal $x(t)$ to create the digital sequence y_n that preserves its salient information. Back-end DSP produces the desired output, from signal reconstruction to signal detection.*

from $M = cK$ *nonadaptive* linear projections onto a second basis $\mathbf{\Phi}$ that is *incoherent* with the first, where c is a small *overmeasuring* constant. By incoherent we mean that the rows ϕ_j of the matrix $\mathbf{\Phi}$ cannot sparsely represent the elements of the sparsity-inducing basis ψ_i, and vice versa [2], [3]. Thus, rather than measuring the N-point signal x directly, we acquire the $M \ll N$ linear projections $y = \mathbf{\Phi}x = \mathbf{\Phi\Psi}\alpha$. Define the $M \times N$ matrix $\mathbf{V} = \mathbf{\Phi\Psi}$.

Since $M < N$, recovery of the signal x from the measurements y is ill-posed; however the additional assumption of signal *sparsity* in the basis $\mathbf{\Psi}$ makes recovery both possible and practical. The recovery of the sparse set of significant coefficients α can be achieved using *optimization* by searching for the signal with ℓ_0-sparsest[1] coefficients α that agrees with the M observed measurements in y. While solving this ℓ_0 optimization problem is prohibitively complex (it is believed to be NP-hard [4]), if we use $M = O(K \log(N/K))$ measurements, then we need only solve for the ℓ_1-sparsest coefficients α that agree with the measurements y [2], [3]

$$\widehat{\alpha} = \arg\min \|\alpha\|_1 \quad \text{s.t. } y = \mathbf{\Phi\Psi}\alpha. \tag{1}$$

This optimization problem, also known as *Basis Pursuit* [5] can be solved with traditional linear programming techniques whose computational complexities are polynomial in N. At the expense of slightly more measurements, iterative greedy algorithms like Orthogonal Matching Pursuit (OMP) [6] can also be applied to the recovery problem.

In its present form, CS is only applicable to discrete signals. Below we extend the framework to continuous signals in order to build new kinds of samplers. Developing a framework for continuous CS will require defining new analog signal models for sparse signals and constructing an analog system that has CS-compatible properties.

B. Analog-to-information conversion: signal processing issues

1) Analog signal model: Supposing our analog signal has finite information rate, then it is reasonable to assume that it can be represented using a finite number of parameters per unit time in some continuous basis. More concretely, let the analog signal $x(t)$ be composed of a *discrete*, *finite* number of weighted continuous basis or dictionary components

$$x(t) = \sum_{n=1}^{N} \alpha_n \, \psi_n(t), \tag{2}$$

[1]The ℓ_0 "norm" $\|\alpha\|_0$ merely counts the number of nonzero entries in the vector α.

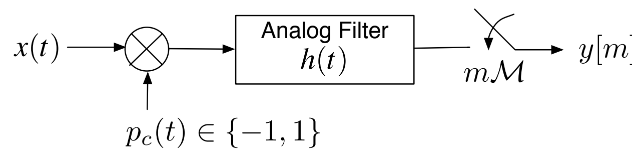

Fig. 2. Pseudo-random demodulation scheme for AIC.

with $t, \alpha_n \in \mathbb{R}$. In cases where there are a small number of nonzero entries in α, we may again say that the signal x is sparse. Although each of the dictionary elements ψ_n may have high bandwidth, the signal itself has relatively few degrees of freedom. Ideally, we would like to sample the signal at some multiple of the sparsity level, rather than at twice the bandwidth as demanded by the Shannon/Nyquist sampling theorem.

2) Analog processing: Our signal acquisition system consists of three main components; demodulation, filtering, and uniform sampling. As seen in Figure 2, the signal is modulated by a psuedo-random maximal-length PN sequence of ± 1's. We call this the *chipping sequence* $p_c(t)$, and it must alternate between values at or faster than the Nyquist frequency of the input signal. The purpose of the demodulation is to spread the frequency content of the signal so that it is not destroyed by the second stage of the system, a low-pass filter with impulse response $h(t)$. Finally, the signal is sampled at rate $\mathcal{M}$ using a traditional ADC.

3) Analog system as a CS matrix: Although our system involves the sampling of continuous-time signals, the discrete measurement vector y can be characterized as a linear transformation of the discrete coefficient vector α. As in the discrete CS framework, we can express this transformation as an $M \times N$ matrix $\mathbf{V}$ that combines two operators: $\mathbf{\Psi}$, which maps the discrete coefficient vector α to an analog signal x, and $\mathbf{\Phi}$, which maps the analog signal x to the discrete set of measurements y.

To find the matrix $\mathbf{V}$ we start by looking at the output $y[m]$, which is a result of convolution and demodulation followed by sampling at rate $\mathcal{M}$

$$y[m] = \int_{-\infty}^{\infty} x(\tau) \, p_c(\tau) \, h(t - \tau) \, d\tau \Big|_{t = m\mathcal{M}}. \tag{3}$$

Our analog input signal (2) is composed of a finite and discrete number of components of $\mathbf{\Psi}$, and so we can expand (3) to

$$y[m] = \sum_{n=1}^{N} \alpha_n \int_{-\infty}^{\infty} \psi_n(\tau) \, p_c(\tau) \, h(m\mathcal{M} - \tau) \, d\tau. \tag{4}$$

Fig. 3. Image depicting the magnitude of one realization of the $M \times N$ complex matrix $\mathbf{V}$ for acquiring Fourier-sparse signals.

It is now clear that we can separate out an expression for each element $v_{m,n} \in \mathbf{V}$ for row m and column n

$$v_{m,n} = \int_{-\infty}^{\infty} \psi_n(\tau)\, p_c(\tau)\, h(m\mathcal{M} - \tau)\, d\tau. \tag{5}$$

Figure 3 displays an image of the magnitude of a realization of such a $\mathbf{V}$ (assuming that $\mathbf{\Psi}$ is the FFT).

4) Idealized simulations: Consider a smooth signal consisting of the sum of 10 sine waves; this corresponds to 10 spikes in the Fourier domain. We operated on the sparse coefficients using the matrix $\mathbf{V}$ constructed via Equation (5) and illustrated in Figure 3. We perform several tests; for clarity, the following figures show the results in the Fourier domain. Figure 4 (a) shows the original signal, and Figure 4 (b) shows a reconstruction of the signal from a measurement at 20% of the Nyquist rate. The recovery is correct to within machine precision (mean squared error is 2.22×10^{-15}). We next apply noise to the sparse vector (see Figure 4 (c)). Figures 4 (d) and (e) show reconstruction results from measurement rates of 20% and 40% of Nyquist. In the noisy situation, 20% of the Nyquist rate is still enough to reconstruct several of the sinusoids, however the noise floor (maximum noise value) decreases from Figures 4 (d) to (e) with increased measurements. This demonstrates that the system still performs reasonably well in substantial amounts of additive noise, but more measurements may be required to produce a higher quality result.

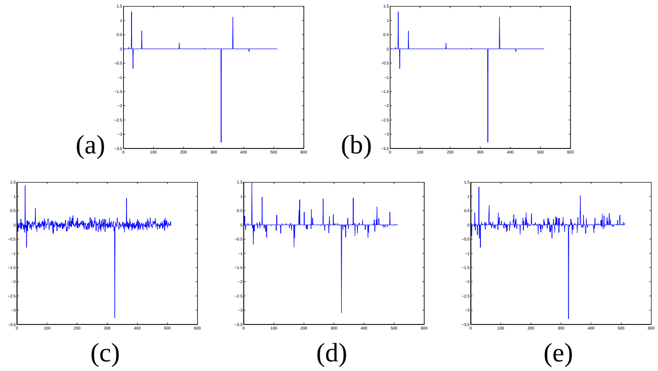

Fig. 4. Idealized AIC simulations. (a) Original sparse vector α. (b) Reconstructed sparse vector from measurements at 20% of the Nyquist rate. (c) Noisy sparse vector with additive Gaussian noise. (d),(e) Reconstructed sparse vector from measurements at 20% and 40% of Nyquist rate.

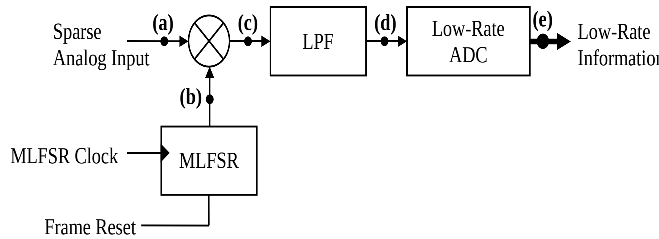

Fig. 5. Architecture of the random demodulation AIC.

III. AIC System Implementation

In order to verify the feasibility of our proposed AIC system, we examine the system implementation shown in Figure 5. The multiplier modulates the input signal with a ± 1 sequence coming from a pseudo-random number generator. The random number generator is implemented using a 10-bit Maximal-Length Linear Feedback Shift Register (MLFSR). The MLFSR has the benefit of providing a random sequence of ± 1 with zero average, while offering the possibility of regenerating the same sequence again given the initial seed. This feature allows the decoder to re-generate the pseudo-random sequence in the reconstruction algorithm. The MLFSR is reset to its initial state every time frame, which is the period of time that is captured from the simulations and fed to the frame-based reconstruction algorithm. The time-frame based operation imposes synchronization between the encoder and the decoder for proper signal reconstruction. To identify the beginning of each frame, header bits can be added in the beginning of each data frame in order to synchronize the decoder; the overhead in the number of data bits is much smaller than the data rate compression of the decoder.

Column n of the transfer function of the system $\mathbf{V}$ for use in the reconstruction algorithm can be extracted as the output of the AIC when we input the analog signal ψ_n. However, the system is time-varying because the random number generator has different values at different time steps. Therefore, we must input all N of the ψ_n in order to account for the corresponding N elements in the pseudo-random number sequence. The resultant system impulse response can then be reshaped to form the $\mathbf{V}$ matrix. Alternatively, we can input impulses in order to extract the columns of the operator $\mathbf{\Phi}$ and then determine $\mathbf{V}$ via (5) using, for example, numerical integration.

IV. AIC System Simulations

Figure 6(a) illustrates an example analog input composed of two sinusoid tones located at 10 MHz and 20 MHz. The clock frequency of the random number generator is 100 MHz. (The MLFSR frequency must be at least $2\times$ higher than the maximum analog input frequency in order to provide the necessary randomization.) The output of the demodulator is low-pass filtered as shown in Figure 6(d), then its output is sampled with a low-rate ADC. In Figure 6(e), the output

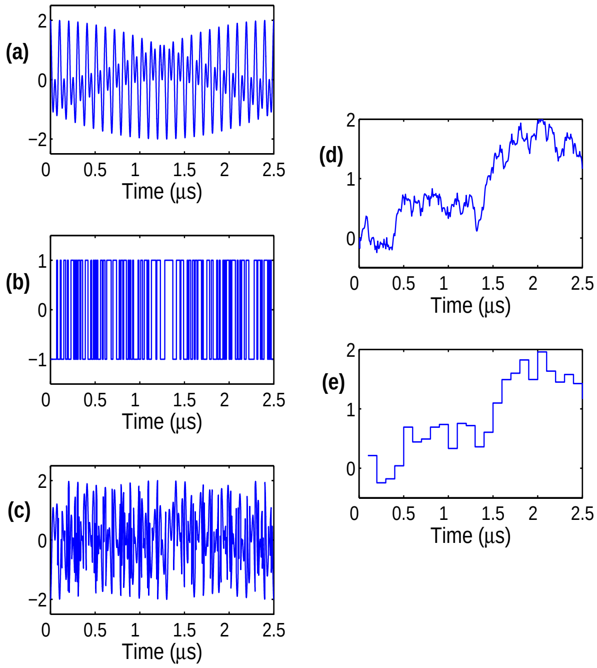

Fig. 6. Time signals inside the AIC of Figure 5: (a) input signal, (b) pseudo-random chipping sequence, (c) signal after demodulation, (d) signal after the low-pass filter, (e) quantized, low-rate final output.

sampling rate is 10 MSample/s, which is 1/4 the traditional Nyquist rate.

In order to quantify the performance of the AIC in term of the probability of success in recognizing the sparse components in the original signal without adding unnecessary spikes in other frequency locations, we measure the performance in terms of the Spurious Free Dynamic Range (SFDR) as shown in Figure 7. The SFDR is the difference between the original signal amplitude and the highest spurs. For this example at 1/4 Nyquist sampling, the SFDR was measured as 80 dB as shown in Figure 7. Higher SFDR values can be obtained by increasing the sampling frequency. Figure 8 presents another example with the sampling frequency further decreased to 5 MSample/s. This frequency is 1/8 of the Nyquist rate; the SFDR is reduced to 29 dB as expected.

V. Conclusions

In this paper, we have developed a novel analog-to-information converter (AIC) architecture. Our design is based on simple off-the-shelf components – a wideband pseudorandom demodulator, a low-pass filter, and a low-rate ADC – yet we demonstrated promising reconstruction results despite sampling well below the Nyquist rate. These concepts could give rise to new generation of AICs for applications where the bandwidth significantly exceeds the "information rate."

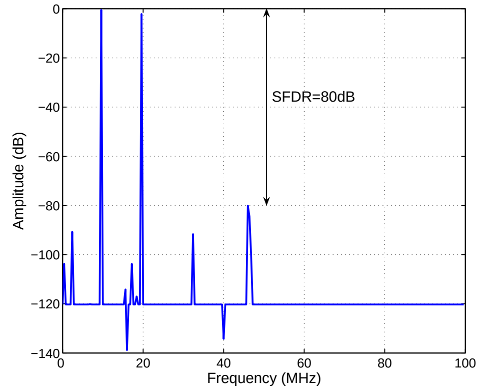

Fig. 7. SFDR for a dual tone signal (10 MHz and 20 MHz) AIC'ed at 10 MSample/s.

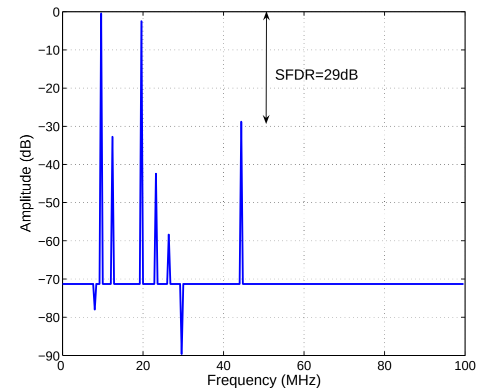

Fig. 8. SFDR for a dual tone signal (10 MHz and 20 MHz) AIC'ed at 5 MSample/s.

Acknowledgments

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