

Analogue gravity from Bose–Einstein condensates

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Abstract

We analyse prospects for the use of Bose–Einstein condensates as condensed-matter systems suitable for generating a generic ‘effective metric’, and for mimicking kinematic aspects of general relativity. We extend the analysis due to Garay *et al* (2000 *Phys. Rev. Lett.* **85** 4643, 2001 *Phys. Rev. A* **63** 023611). Taking a long-term view, we ask what the ultimate limits of such a system might be. To this end, we consider a very general version of the nonlinear Schrödinger equation (with a 3-tensor position-dependent mass and arbitrary nonlinearity). Such equations can be used, for example, in discussing Bose–Einstein condensates in heterogeneous and highly nonlinear systems. We demonstrate that at low momenta linearized excitations of the phase of the condensate wavefunction obey a (3 + 1)-dimensional d’Alembertian equation coupling to a (3 + 1)-dimensional Lorentzian-signature ‘effective metric’ that is generic, and depends algebraically on the background field. Thus at low momenta this system serves as an analogue for the curved spacetime of general relativity. In contrast, at high momenta we demonstrate how one can use the eikonal approximation to extract a well controlled Bogoliubov-like dispersion relation, and (perhaps unexpectedly) recover non-relativistic Newtonian physics at high momenta. Bose–Einstein condensates appear to be an extremely promising analogue system for probing kinematic aspects of general relativity.

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1. Introduction

Progress in understanding classical general relativity (GR) and quantum field theory in curved spacetime (not to mention quantum gravity itself), suffers greatly from a lack of experimental feedback⁴. Moreover, direct experimental probes of many important aspects of these theories

⁴ See [1], however, for a description of some actual and proposed experiments probing aspects of the behaviour of quantum systems in GR.

(such as, for instance, Hawking radiation from a black hole) appear to be very far from current (and even foreseeable) technologies. In this regard, the possibility of using condensed matter systems (such as, for example, Bose–Einstein condensates [2, 3]) to mimic certain aspects of GR could prove to be very important. The basic idea of a condensed matter analogue model of GR is that the modifications to the propagation of a field/wave due to curved spacetime can be reproduced (at least partially) by an analogue field/wave propagating in some material background with space- and time-dependent properties.

The concept of a condensed matter analogue model was first explored in a little-known paper by Gordon, where he worked within the context of the optical properties of dielectrics [4]. After lying fallow for a considerable period, over the last decade or so this idea has been revived and elaborated on, mainly in the context of considering the propagation of sound waves in a moving fluid [5–14]. Several other analogue systems have also been analysed (see for example [15, 16], and also the mini-review by Jacobson [17]). The search for new analogue models, and the exploration of known analogue models, is ongoing [18–21]. Among these models, a particularly promising one that will be the focus of this paper is the recent proposal of Garay, Anglin, Cirac and Zoller based on Bose–Einstein condensates (BEC) [2, 3].

Over the last few years a remarkable series of experiments on vapours of rubidium [22], lithium [23] and sodium [24] have led to a renewed interest in the phenomenon of Bose–Einstein condensation [25, 26]. In these experiments gases of alkali atoms were confined in magnetic traps and cooled down to extremely low temperatures (of the order of fractions of microkelvins). In order to observe the BEC the whole gas was allowed to expand, by switching off the trapping potential, and monitored via time-of-flight measurements made with optical methods. The signature of the BEC was a sharp peak in the velocity distribution for temperatures below some critical value (see [27] for an extensive review on the subject).

As Garay *et al* have shown, perturbations in the phase of the condensate wavefunction satisfy, in the low-momentum regime, an equation equivalent to that of a massless scalar field in a curved spacetime (the d’Alembertian equation $\Delta\phi = 0$), but with the spacetime metric being replaced by an effective metric that depends on the characteristics of the background condensate. Present-day experimental achievements, and the rapid development in magnetic trapping techniques, seem to illuminate a viable path to experimentally reproducing important general relativistic features such as ergoregions and event horizons [2, 3].

In this paper we wish to explore the Bose–Einstein system in more detail, formally extending the analysis of Garay *et al* as much as possible. We will analyse a number of physically conceivable extensions of the usual Gross–Pitaevskii approximation. In particular, we will consider arbitrary nonlinear interactions and anisotropic mass-tensors, both depending explicitly on (space and time) position. In this scenario we will investigate how close we can come to mimicking a ‘generic’ gravitational field. (The simple analogue models based on ordinary fluid dynamics are somewhat limited in this regard because the spatial slices are always conformally flat [5, 9, 12, 14]). After including these generalizations the only significant constraint left on the effective metric is due to the irrotational nature of the condensate 3-velocity. Within this scenario we will then show how various quantum corrections come into play in the whole analysis, distinguishing three useful regimes: the quasi-classical, low-momentum and high-momentum regimes.

By using eikonal techniques we shall investigate the high-momentum dispersion relation of the collective excitations of the condensate. In particular, we shall study it with regard to the possibility of analysing the ‘Hawking radiation’ that may be emitted from ‘horizons’ that form in this system. We find a Bogoliubov-like dispersion relation of the superluminal (more properly, supersonic) type. For situations with both an outer and an inner horizon, this dispersion relation seems to lead to exponential amplification of the radiation flux coming from

the outer horizon [28] (a so-called black hole laser), in agreement with the unstable behaviour of some of the solutions found by Garay *et al.* (They also found some metastable solutions whose origin might be related to the specific boundary conditions of the configurations considered.) Finally, we discuss the way in which in the high-momentum limit one recovers the underlying non-relativistic Newtonian physics.

2. Bose–Einstein condensates

In a quantum system of N interacting bosons the crucial feature of a Bose–Einstein condensate is that it corresponds to a configuration in which most of the bosons lie in the same single-particle quantum state. This system can be suitably described in a second quantization framework by a many-body Hamiltonian of the form [27]

$$H = \int d\mathbf{x} \widehat{\Psi}^\dagger(t, \mathbf{x}) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{x}) \right] \widehat{\Psi}(t, \mathbf{x}) + \frac{1}{2} \int d\mathbf{x} d\mathbf{x}' \widehat{\Psi}^\dagger(t, \mathbf{x}) \widehat{\Psi}^\dagger(t, \mathbf{x}') V(\mathbf{x} - \mathbf{x}') \widehat{\Psi}(t, \mathbf{x}') \widehat{\Psi}(t, \mathbf{x}). \quad (1)$$

Here $V_{\text{ext}}(\mathbf{x})$ is some confining external potential, $V(\mathbf{x} - \mathbf{x}')$ is the interatomic two-body potential (other possible multi-body interactions are neglected at this stage), and m is the mass of the bosons undergoing condensation (in current experiments these bosons are actually alkali atoms). Finally, $\widehat{\Psi}(t, \mathbf{x})$ is the boson field operator.

Although the quantum state for the Bose–Einstein condensate, as well as its thermodynamic properties, can, in principle, be computed exactly from (1), it is clear that for large ensemble of atoms this approach can become impractical. It was Bogoliubov [29, 30] who first recognized that a natural ansatz for studying such a system is the mean-field approach, which consists of separating the bosonic field operator $\widehat{\Psi}(t, \mathbf{x})$ into a classical condensate contribution $\psi(t, \mathbf{x})$ plus excitations $\widehat{\varphi}(t, \mathbf{x})$:

$$\widehat{\Psi}(t, \mathbf{x}) = \psi(t, \mathbf{x}) + \widehat{\varphi}(t, \mathbf{x}). \quad (2)$$

Here $\psi(t, \mathbf{x})$ is defined as the expectation value of the field $\psi \equiv \langle \widehat{\Psi}(t, \mathbf{x}) \rangle$. It is sometimes referred in the literature as the ‘wavefunction of the Bose–Einstein condensate’. Its modulus fixes the particle density of the condensate, $\rho = N/V$, in such a way that $|\psi(t, \mathbf{x})|^2 = \rho(t, \mathbf{x})$.

The Bogoliubov decomposition (2) is particularly useful when the number of atoms which do not lie in the ground state of the condensate is small. In this case it is, in fact, possible to consider the ‘zeroth-order approximation’ to the Heisenberg equation given by the many-body Hamiltonian (1) by just replacing $\widehat{\Psi}(t, \mathbf{x})$ with the condensate wavefunction

$$i\hbar \frac{\partial}{\partial t} \psi(t, \mathbf{x}) = [\psi, H] = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{x}) + \int d\mathbf{x}' \psi^\dagger(\mathbf{x}', t) V(\mathbf{x}' - \mathbf{x}) \psi(t, \mathbf{x}') \right] \psi(t, \mathbf{x}). \quad (3)$$

The next approximation is to assume that the intermolecular interaction term $V(\mathbf{x}' - \mathbf{x})$ represents a short-range interaction. (In helium-based Bose–Einstein systems this is *not* the case and a considerably more complicated analysis is required.) Part of the theoretical interest in the previously cited heavy-alkali-atom Bose–Einstein condensates is that these are extremely dilute condensates; systems in which at low energy only binary collisions are relevant. This permits one to model the interaction with a short-distance delta-like term times a unique self-coupling constant λ which is determined by the s-wave scattering length a and the atomic

mass m

$$V(\mathbf{x}' - \mathbf{x}) \approx \lambda \delta(\mathbf{x}' - \mathbf{x}), \quad (4)$$

$$\lambda = \frac{4\pi a \hbar^2}{m}. \quad (5)$$

The use of the approximate potential (4) in equation (3) leads to a standard closed-form equation for the weakly interacting boson condensate

$$i\hbar \frac{\partial}{\partial t} \psi(t, \mathbf{x}) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{x}) + \lambda |\psi(t, \mathbf{x})|^2 \right) \psi(t, \mathbf{x}). \quad (6)$$

This equation, most commonly known as the Gross–Pitaevskii (GP) equation (sometimes called the nonlinear Schrödinger equation, or even the time-dependent Landau–Ginzburg equation), can be associated with an effective action of the form

$$\mathcal{S} = \int dt d^3x \left\{ \psi^* \left(i\hbar \partial_t + \frac{\hbar^2}{2m} \nabla^2 - V_{\text{ext}}(\mathbf{x}) \right) \psi - \frac{1}{2} \lambda |\psi(t, \mathbf{x})|^4 \right\}, \quad (7)$$

which is called the time-dependent Landau–Ginzburg action⁵. (There is some disagreement on terminology here: some authors prefer to use the phrase ‘time-dependent Landau–Ginzburg equation’ for equation (6) only after formally discarding the i . This has the net effect of replacing the Schrödinger differential operator by a diffusion operator. We always keep the i ; all our physics is oscillatory rather than diffusion-based.)

The Gross–Pitaevskii equation describes, in a simple and compact form, the relevant phenomena associated with BEC. In particular, it reproduces typical properties exhibited by superfluid systems, such as the propagation of collective excitations and the interference effects originating from the phase of the condensate wavefunction. It is precisely from this (non-relativistic) theory that we shall now see how an analogue of GR can be constructed, and possibly used for experimental purposes.

3. Generalized Gross–Pitaevskii equation

The aim of our investigation is to explore the ultimate extent of the ability of BEC systems to mimic general relativistic ones. In order to do this in the most general way we shall now introduce a generalization of the nonlinear Schrödinger equation (6). In particular, we shall formally consider a series of generalizations each of which is, in principle, allowed for systems undergoing Bose–Einstein condensation, though they are not (yet) commonly encountered in the experimental literature.

- (a) The first generalization we will make is to replace the quartic $\frac{1}{2} \lambda |\psi(t, \mathbf{x})|^4$ by an arbitrary nonlinearity $\pi(\psi^* \psi) = \pi(|\psi|^2)$. We note, in particular, that for two-dimensional systems Kolomeisky *et al* have argued [34] that in many experimentally interesting cases the nonlinearity will be cubic or even logarithmic in $|\psi|^2$. (Although in two-dimensional systems standard Bose–Einstein condensation does not occur it is experimentally almost certain that ‘quasi-condensates’ exist, and possess collective excitations which are treatable in an analogous way [35].)

⁵ The same line of reasoning we have followed with this non-relativistic boson system can be repeated for relativistic bosons, arriving to the generalized nonlinear Klein–Gordon equation [31, 32] (from which the nonlinear Schrödinger equation arises as the non-relativistic limit). When coupled to gravity, this equation serves as the basis for the description of boson stars [33], that is, an astrophysical BEC.

- (b) The second generalization is that we will also permit the nonlinearity function to be explicitly space and time dependent: $\pi \rightarrow \pi(x, [\psi^* \psi])$ with $x \equiv (t, \mathbf{x})$. While intrinsic properties of the bosons that we are trying to condense (such as the scattering length and mass) will typically not change from point to point, we can certainly envisage a more general situation in which the condensate is either selectively ‘doped’, or perhaps (as suggested in Garay *et al* [2, 3]) physically constrained to move through a narrow tube of varying cross section, thereby altering its effective properties in a position-dependent way. (We do not mean to imply that this would be easy, or that such techniques are ‘just around the corner’; instead we are interested in seeing how far we might ultimately be able to push this system.)
- (c) The third generalization we will make is to permit the mass to be a 3-tensor: $m \rightarrow m_{ij}$. Such anisotropic masses are well known from condensed matter physics where they are most typically encountered in effective mass calculations for electrons immersed in a band structure (see, for example, [36]). They have also been discussed for the case of excitons (electron–hole pairs held together by Coulomb attraction) in BEC for semiconductors. The doping structure of the semiconductor and its anisotropies would give place to an effective mass matrix for the paraexcitons (singlet excitons) at least in the low-momentum approximation [37, 38]. Formally, we shall consider the possibility of anisotropic masses in the more general context of the nonlinear Schrödinger equation, regardless of whether or not we are dealing with a band structure or even a Bose–Einstein condensate. (If you wish, you might like to think of this as a liquid-crystal BEC.)
- (d) The fourth generalization we will make is to also permit the 3-tensor mass to depend on position (both time and space). Again we might like to think of a doping gradient or similar situation. Mathematically, this has the effect that the mass matrix must be viewed as a ‘metric’ on a curved three-dimensional space. It is convenient to introduce an arbitrary but fixed (time- and space-independent) scale μ with the dimensions of mass and then write

$$m_{ij} = \mu {}^{(3)}h_{ij} \quad (8)$$

with ${}^{(3)}h_{ij}$ being a properly dimensionless 3-metric. Note that the introduction of μ is a *mathematical convenience*, not a *physical necessity*, and that all properly formulated physical questions will be independent of μ .

- (e) The fifth and last generalization will be to allow a time dependence for the confining potential $V_{\text{ext}} = V_{\text{ext}}(t, \mathbf{x})$. (This is already implicit in the analysis of Garay *et al*, but is somewhat non-standard from the usual condensed-matter viewpoint.)

We do not claim that any of these generalizations will be easy to achieve experimentally, it is sufficient for our purposes that they are at least physically conceivable. After this series of generalizations we obtain an action which now reads

$$\mathcal{S} = \int dt d^3x \sqrt{\det [{}^{(3)}h]} \left\{ \psi^* \left(i\hbar \partial_t + \frac{\hbar^2}{2\mu} \Delta_h + \frac{\xi \hbar^2}{2\mu} {}^{(3)}R(h) - V_{\text{ext}}(t, \mathbf{x}) \right) \psi - \pi(x, |\psi|^2) \right\}. \quad (9)$$

Here Δ_h is the three-dimensional Laplacian defined by

$$\Delta_h \psi = \frac{1}{\sqrt{\det [{}^{(3)}h]}} \nabla_i \left(\sqrt{\det [{}^{(3)}h]} [{}^{(3)}h^{-1}]^{ij} \nabla_j \psi \right), \quad (10)$$

where $[(^3)h^{-1}]^{ij}$ is the inverse of the 3-metric $(^3)h_{ij}$. Additionally, note the presence of the DeWitt term

$$\frac{\xi \hbar^2}{2\mu} (^3)R(h) \quad (11)$$

involving the dimensionless parameter ξ and the three-dimensional Ricci scalar—this term arises from operator-ordering ambiguities in going from the ‘flat space’ metric to ‘curved space’ (going from position-independent m to a position-varying effective mass) [39, 40]. We include the DeWitt term here for completeness, and because it should be included as a matter of principle, but note that it is unlikely to lead to experimentally measurable effects.

Varying this generalized action with respect to ψ^* now gives the ‘generalized’ nonlinear Schrödinger equation that will be of central interest in this paper:

$$i\hbar \frac{\partial}{\partial t} \psi(t, \mathbf{x}) = -\frac{\hbar^2}{2\mu} \Delta_h \psi(t, \mathbf{x}) - \frac{\xi \hbar^2}{2\mu} (^3)R(h) \psi(t, \mathbf{x}) \\ + V_{\text{ext}}(t, \mathbf{x}) \psi(t, \mathbf{x}) + \pi'(\psi^* \psi) \psi(t, \mathbf{x}). \quad (12)$$

We will now demonstrate that this non-relativistic generalized nonlinear Schrödinger equation has a $(3+1)$ -dimensional ‘effective’ Lorentzian spacetime metric hiding inside it.

4. Madelung representation and the hydrodynamic limit

The so-called Madelung representation [41–45] of a Schrödinger wavefunction consists of writing

$$\psi(t, \mathbf{x}) = \sqrt{\rho(t, \mathbf{x})} \exp[-i\theta(t, \mathbf{x})/\hbar]. \quad (13)$$

The factor of \hbar is introduced for future convenience (it suppresses \hbar as much as possible in the following equations). This implies that θ has the dimensions of an action. The Madelung representation is well known in the context of the ordinary linear Schrödinger equation, and generalizes to the present situation without difficulty.

Garay *et al* [2, 3] substitute the Madelung representation into the Gross–Pitaevskii equation, and take the real and imaginary parts. We could do the same thing here with the generalized nonlinear Schrödinger equation. Alternatively, we could insert the Madelung representation directly into the action, and vary with respect to θ and ρ to deduce Euler–Lagrange equations. Either way, you will obtain two equations:

(a) Continuity:

$$\partial_t \rho + \frac{1}{\mu} \nabla \cdot (\rho \nabla \theta) = 0. \quad (14)$$

Here and hereafter ∇ denotes a *covariant* derivative with respect to the 3-metric $(^3)h_{ij}$. If we define a ‘velocity’ (and at this stage this is a purely formal step)

$$(v)^i \equiv \frac{\hbar^{ij} \nabla_j \theta}{\mu} \equiv [m^{-1}]^{ij} \nabla_j \theta, \quad (15)$$

then this ‘velocity’ is actually independent of μ , and equation (14) above is formally equivalent to the usual equation of continuity in a curved 3-space

$$\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0. \quad (16)$$

(b) The quantum analogue of the Hamilton–Jacobi equation:

$$\frac{\partial}{\partial t}\theta + \frac{1}{2\mu}(\nabla\theta)^2 + V_{\text{ext}} + \pi' - \frac{\hbar^2}{2\mu} \left(\frac{\Delta_h\sqrt{\rho}}{\sqrt{\rho}} + \xi {}^{(3)}R(h) \right) = 0. \quad (17)$$

Here and hereafter $(\nabla\theta)^2$ denotes the 3-metric scalar inner product ${}^{(3)}[h^{-1}]^{ij} \partial_i\theta \partial_j\theta$.

The quantity

$$V_Q(\rho, \xi) \equiv -\frac{\hbar^2}{2\mu} \left(\frac{\Delta_h\sqrt{\rho}}{\sqrt{\rho}} + \xi {}^{(3)}R(h) \right) \quad (18)$$

is a generalization (because it now includes the three-dimensional Ricci scalar term) of what is often called the ‘quantum potential’ [46, 47]. (Note that the quantum potential is actually independent of μ because rescaling μ simultaneously rescales h_{ij} in a compensating manner.) With our conventions this is now the only place where \hbar appears.

In terms of the velocity field (15) we can write the Hamilton–Jacobi equation in the form

$$\frac{\partial}{\partial t} [m_{ij}(\mathbf{v})^j] + \nabla_i \left(\frac{m_{pq}(\mathbf{v})^p(\mathbf{v})^q}{2} + V_{\text{ext}} + \pi' + V_Q \right) = 0. \quad (19)$$

(Note that this form makes the μ independence of the physics manifest.) The two real equations (14) and (17) (or alternatively equations (16) and (19), subject to the definition (15)) are completely equivalent to the generalized nonlinear Schrödinger equation (12).

An interesting physical regime is that in which one can safely neglect the quantum potential term. This approximation can be justified either as the classical limit of the theory (it corresponds to neglecting all terms with powers of \hbar) or as the regime of strong repulsive interaction among atoms. In the latter case the density profile can be safely considered smooth and hence it is reasonable to neglect the kinetic-pressure term $\Delta_h\sqrt{\rho}/\sqrt{\rho}$. This, together with the smallness of the DeWitt term, permits one to discard V_Q .

In any case one can see that on neglecting V_Q , equations (16) and (19) have the form typical of those for superfluids in the $T \rightarrow 0$ limit. In particular, we can see, by the absence of any term proportional to $\mathbf{v} \times (\nabla \times \mathbf{v})$ in equation (19), that the equations we are working with are automatically vorticity free. This is generally a necessary assumption in order to obtain tractable equations in the analogue of GR from standard hydrodynamics [9, 12]; here it is a free byproduct of the GP equation (and this conclusion does not depend on the assumption of neglecting V_Q).

The hydrodynamical form of equations (16) and (19) allows one to describe the Bose–Einstein condensate as a gas whose pressure and density are related by the barotropic equation of state

$$P(\rho) = \pi'(\rho) \rho - \pi(\rho). \quad (20)$$

Therefore, it is also possible to *formally* define a local speed of sound by varying this pressure with respect to the mass density of the condensate ‘fluid’ $\varrho = \mu\rho$

$$c^2 = \frac{\partial P}{\partial \varrho} = \frac{\pi''\rho}{\mu}. \quad (21)$$

We say formally because for the general anisotropic case this ‘velocity of sound’ is not physical. Physically, there will be three principal sound velocities in three orthogonal directions, and, as we have mentioned before and will see again, the formally convenient parameter μ does not appear in any true physical result. This does not diminish the convenience of introducing the parameter μ for intermediate stages of the calculation.

It is also interesting to give a more quantitative estimate of the magnitude of this sound velocity for the Bose–Einstein systems based on alkali atoms. In particular, we can consider the specific case of the rubidium gas with trivial mass tensor ($\mu = m$) and standard interaction term $\pi(t, \mathbf{x}, |\psi|^2) = \pi(t, \mathbf{x}, \rho) = \frac{1}{2}\lambda\rho^2$, so that $\pi'(t, \mathbf{x}, |\psi|^2) = \lambda|\psi(t, \mathbf{x})|^2 = \lambda\rho$. (Here λ is given by equation (5).) In this case the equation of state becomes $P = \frac{1}{2}\rho^2$ and the speed of sound takes the well known form

$$c = \sqrt{\frac{\lambda\rho}{m}} = \frac{2\hbar}{m} \sqrt{a\rho\pi}. \quad (22)$$

For the rubidium gas one has $a(^{87}\text{Rb}) \approx 5.77$ nm, $m(^{87}\text{Rb}) \approx 86.9$ u and in standard BEC experiments $\rho \approx 10^{15}$ cm⁻³ [27]. These numbers lead to a value of the speed of sound $c \approx 6.2 \times 10^{-3}$ m s⁻¹ ≈ 6 mm s⁻¹. This is indeed one of the lowest speeds of sound one can obtain experimentally. We shall see in what follows how this number can play an important role in the simulation of gravitational phenomena in BEC systems.

5. Fluctuations

Now that we have seen how it is possible, at least in some appropriate regime, to introduce a hydrodynamical interpretation of the condensate equations, and how a speed of sound can be meaningfully introduced, we are naturally lead to investigate the propagation of fluctuations in the condensate.

In order to pursue such an investigation we shall linearize the equations of motion (14) and (17) around some assumed background (ρ_0, θ_0) . In particular, we shall set $\rho = \rho_0 + \epsilon\rho_1 + O(\epsilon^2)$ and $\theta = \theta_0 + \epsilon\theta_1 + O(\epsilon^2)$. Then, we will be left with two equations for the background configuration plus two more (often called Bogoliubov equations) for the linearized quantities. Linearizing the continuity equation results in the pair of equations

$$\partial_t \rho_0 + \frac{1}{\mu} \nabla \cdot (\rho_0 \nabla \theta_0) = 0, \quad (23)$$

$$\partial_t \rho_1 + \frac{1}{\mu} \nabla \cdot (\rho_1 \nabla \theta_0 + \rho_0 \nabla \theta_1) = 0. \quad (24)$$

Here and hereafter all inner products $(a \cdot b)$ are calculated using the 3-metric $(h_{ij} a^i b^j)$.

Linearizing the Hamilton–Jacobi equation we obtain the pair

$$\partial_t \theta_0 + \frac{1}{2\mu} (\nabla \theta_0)^2 + V_{\text{ext}} + \pi'(\rho_0) + V_Q(\rho_0) = 0, \quad (25)$$

$$\partial_t \theta_1 + \frac{1}{\mu} \nabla \theta_0 \cdot \nabla \theta_1 + \pi''(\rho_0) \rho_1 - \frac{\hbar^2}{2\mu} D_2 \rho_1 = 0. \quad (26)$$

Here D_2 represents a relatively messy second-order differential operator obtained from linearizing the quantum potential, explicitly

$$D_2 \rho_1 \equiv -\frac{1}{2} \rho_0^{-3/2} [\Delta_h(\rho_0^{+1/2})] \rho_1 + \frac{1}{2} \rho_0^{-1/2} \Delta_h(\rho_0^{-1/2} \rho_1). \quad (27)$$

The linearized Hamilton–Jacobi equation may be rearranged to yield

$$\rho_1 = - \left[\pi''(\rho_0) - \frac{\hbar^2}{2\mu} D_2 \right]^{-1} \left(\partial_t \theta_1 + \frac{1}{\mu} \nabla \theta_0 \cdot \nabla \theta_1 \right). \quad (28)$$

Now substitute this consequence of the linearized Hamilton–Jacobi equation back into the linearized equation of continuity. We obtain, up to an overall sign, the wave-like equation:

$$\begin{aligned}
 & -\partial_t \left\{ \left[\pi''(\rho_0) - \frac{\hbar^2}{2\mu} D_2 \right]^{-1} \left(\partial_t \theta_1 + \frac{1}{\mu} \nabla \theta_0 \cdot \nabla \theta_1 \right) \right\} \\
 & + \frac{1}{\mu} \nabla \cdot \left(\rho_0 \nabla \theta_1 - \nabla \theta_0 \left\{ \left[\pi''(\rho_0) - \frac{\hbar^2}{2\mu} D_2 \right]^{-1} \right. \right. \\
 & \left. \left. \times \left(\partial_t \theta_1 + \frac{1}{\mu} \nabla \theta_0 \cdot \nabla \theta_1 \right) \right\} \right) = 0.
 \end{aligned} \tag{29}$$

This wave-like equation describes the propagation of the linearized Schrödinger phase θ_1 . (The coefficients of this wave-like equation depend on the background field (ρ_0, θ_0) that you are linearizing around.) Once θ_1 is determined, then equation (28) determines ρ_1 . Thus this wave equation completely determines the propagation of linearized disturbances. The background fields ρ_0 and θ_0 , which appear as time-dependent and position-dependent coefficients in this wave equation, are constrained to solve our generalized nonlinear Schrödinger equation. Apart from this constraint, they are otherwise permitted to have *arbitrary* temporal and spatial dependences. To simplify things construct the symmetric 4×4 matrix

$$f^{\mu\nu}(t, \mathbf{x}) \equiv \begin{bmatrix} f^{00} & \vdots & f^{0j} \\ \dots\dots & \cdot & \dots\dots\dots \\ f^{i0} & \vdots & f^{ij} \end{bmatrix}. \tag{30}$$

(Greek indices run from 0–3, while Roman indices run from 1–3.) Then, introducing $(3 + 1)$ -dimensional spacetime coordinates $(x^\mu \equiv (t; x^i))$ the above wave equation (29) is easily rewritten as

$$\partial_\mu (f^{\mu\nu} \partial_\nu \theta_1) = 0. \tag{31}$$

Here

$$f^{00} = - \left[\pi''(\rho_0) - \frac{\hbar^2}{2\mu} D_2 \right]^{-1} \tag{32}$$

$$f^{0j} = - \left[\pi''(\rho_0) - \frac{\hbar^2}{2\mu} D_2 \right]^{-1} \frac{h^{jk} \nabla_k \theta_0}{\mu} \tag{33}$$

$$f^{i0} = - \frac{h^{ik} \nabla_k \theta_0}{\mu} \left[\pi''(\rho_0) - \frac{\hbar^2}{2\mu} D_2 \right]^{-1} \tag{34}$$

$$f^{ij} = \frac{\rho_0 {}^{(3)}h^{ij}}{\mu} - \frac{h^{ik} \nabla_k \theta_0}{\mu} \left[\pi''(\rho_0) - \frac{\hbar^2}{2\mu} D_2 \right]^{-1} \frac{h^{jl} \nabla_l \theta_0}{\mu}. \tag{35}$$

Thus $f^{\mu\nu}$ is a 4×4 matrix of *differential operators* (the differential operators in question consistently operating on *everything* to the right). Note that the precise placement of h^{ij} above is immaterial since the operator D_2 is built using ∇ , the covariant derivative associated with h^{ij} . This remarkably compact formulation (31) is completely equivalent to equation (29) and is a much more promising stepping stone for further manipulations.

The major obstruction to interpreting this wave equation in terms of Lorentzian geometry is the fact that $f^{\mu\nu}$ is itself still a matrix of differential operators, not functions. We now consider several different approximations (valid in different regimes) which have the effect of replacing these differential operators by functions. After making those approximations, the remaining steps are a straightforward application of the techniques of curved space $(3 + 1)$ -dimensional Lorentzian geometry (see, for instance [9, 12]).

6. Quasi-classical approximation

The most straightforward approximation we could make is to simply neglect D_2 completely: this is actually what is done in the analysis of Garay *et al* [2, 3]. We (and they) justify this approximation by pointing out that D_2 is always multiplied by \hbar^2 and in this sense is suitably ‘small’. We shall call this the quasi-classical approximation. The wave equation then simplifies to

$$-\partial_t \left(\frac{(\partial_t \theta_1 + [1/\mu] \nabla \theta_0 \cdot \nabla \theta_1)}{\pi''} \right) + \frac{1}{\mu} \nabla \cdot \left(\frac{\rho_0 \nabla \theta_1 - \nabla \theta_0 (\partial_t \theta_1 + [1/\mu] \nabla \theta_0 \cdot \nabla \theta_1)}{\pi''} \right) = 0. \quad (36)$$

To simplify things algebraically, we can define, in analogy with equation (21), a speed c as

$$c^2 \equiv \frac{\pi'' \rho_0}{\mu}, \quad (37)$$

and the background ‘velocity’ v_0 by

$$(v_0)^i = \frac{h^{ij} \nabla_j \theta_0}{\mu} = [m^{-1}]^{ij} \nabla_j \theta_0. \quad (38)$$

(Remember that c is not necessarily the physical speed of sound; it is simply a convenient parametrization; in contrast v_0 really is μ -independent and physical.) Now construct the symmetric 4×4 matrix

$$f^{\mu\nu}(t, \mathbf{x}) \equiv \frac{1}{\pi''} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \dots & \cdot & \dots \\ -v_0^i & \vdots & (c^2 h^{ij} - v_0^i v_0^j) \end{bmatrix}. \quad (39)$$

This is now just a 4×4 matrix of numbers.

In any Lorentzian (that is, pseudo-Riemannian) manifold the curved space scalar d’Alembertian is given in terms of the $(3 + 1)$ -metric $g_{\mu\nu}(t, \mathbf{x})$ by

$$\Delta \theta \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \theta). \quad (40)$$

The $(3 + 1)$ -dimensional inverse metric, $g^{\mu\nu}(t, \mathbf{x})$, is pointwise the matrix inverse of $g_{\mu\nu}(t, \mathbf{x})$, while $g \equiv \det(g_{\mu\nu})$. Thus one can rewrite the physically derived wave equation (36) in terms of the d’Alembertian provided one identifies

$$\sqrt{-g} g^{\mu\nu} = f^{\mu\nu}. \quad (41)$$

This implies, on the one hand

$$\det(f^{\mu\nu}) = (\sqrt{-g})^4 g^{-1} = g. \quad (42)$$

On the other hand, from the explicit expression (30), expanding the determinant in minors

$$\det(f^{\mu\nu}) = (\pi'')^{-4} [(-1) \cdot (c^2 - v_0^2) - (-v_0)^2] \cdot [c^2] \cdot [c^2] = -c^6 / (\pi'')^4. \quad (43)$$

Thus

$$g = -c^6 / (\pi'')^4; \quad \sqrt{-g} = c^3 / (\pi'')^2. \quad (44)$$

We can therefore pick off the coefficients of the inverse ‘condensate metric’

$$g^{\mu\nu}(t, \mathbf{x}) \equiv \frac{\pi''}{c^3} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \dots & \cdot & \dots \\ -v_0^i & \vdots & (c^2 h^{ij} - v_0^i v_0^j) \end{bmatrix} = \frac{\mu}{\rho_0 c} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \dots & \cdot & \dots \\ -v_0^i & \vdots & (c^2 h^{ij} - v_0^i v_0^j) \end{bmatrix}. \quad (45)$$

This is the effective Lorentzian metric seen by the perturbations of the phase of the condensate wavefunction. At this point let us compare this metric with the acoustic metric of [5–13]

- (a) The two metrics (acoustic and condensate) possess the same conformal factor (up to a physically irrelevant constant rescaling). In view of this we will just call it the acoustic metric from now on, keeping in the back of our minds that the relevant ‘acoustics’ is now the propagation of oscillations in the phase of the condensate wavefunction.
- (b) Note that Garay *et al* did not keep track of the conformal factor, as it was not needed for the points they wanted to make.
- (c) There are now slightly different physical interpretations for c and v_0 .
- (d) There is already a non-flat 3-metric h_{ij} present in the analysis, even before the linearization procedure is carried out. It is this feature that departs furthest from the previous implementations of the notion of an ‘acoustic metric’.

We could now determine the metric itself simply by inverting this 4×4 matrix. On the other hand, as is by now standard, it is even easier to recognize that one has in front of one an example of the Arnowitt–Deser–Misner split of a $(3 + 1)$ -dimensional Lorentzian spacetime metric into space + time, more commonly used in discussing initial-value data in Einstein’s theory of gravity (general relativity). The (direct) acoustic metric is easily read off as

$$g_{\mu\nu}(t, \mathbf{x}) \equiv \frac{c}{\pi''} \begin{bmatrix} -(c^2 - v_0^2) & \vdots & -v_0^k h_{kj} \\ \dots & \cdot & \dots \\ -v_0^k h_{ki} & \vdots & h_{ij} \end{bmatrix} = \frac{\rho_0}{c\mu} \begin{bmatrix} -(c^2 - v_0^2) & \vdots & -v_0^k h_{kj} \\ \dots & \cdot & \dots \\ -v_0^k h_{ki} & \vdots & h_{ij} \end{bmatrix}. \quad (46)$$

Equivalently, the ‘acoustic interval’ can be expressed as

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = \frac{\rho_0}{c\mu} \left[-c^2 dt^2 + h_{ij} (dx^i - v_0^i dt) (dx^j - v_0^j dt) \right]. \quad (47)$$

A few brief comments should be made before proceeding.

- Observe that the signature of this metric is indeed $(-, +, +, +)$, as it should be to be regarded as Lorentzian. There is an interesting physical subtlety here: some alkali atomic gases have a negative scattering length, that is, there are attractive forces between atoms physically leading to the collapse of the BEC. A negative scattering length is formally equivalent to an imaginary speed of sound, and in terms of the effective metric is equivalent to a Euclidean-signature metric. That is, negative scattering length corresponds to an elliptic differential operator, instead of the more usual hyperbolic differential operator. In terms of general relativity, manipulating the sign of the scattering length corresponds to building an analogue for a signature-changing spacetime.
- It should be emphasized that there are (at least) *three* distinct metrics that are relevant to the current discussion:

* The *physical spacetime metric* is just the usual flat metric of Minkowski space

$$\eta_{\mu\nu} \equiv (\text{diag}[-c_{\text{light}}^2, 1, 1, 1])_{\mu\nu}. \quad (48)$$

(Here c_{light} = speed of light.) The quantum field couples only to the physical metric $\eta_{\mu\nu}$. In fact, the quantum field is completely non-relativistic, $\|v_0\| \ll c_{\text{light}}$.

- * The ‘mass metric’ $m_{ij} \equiv \mu h_{ij}$ describing the position-dependent effective mass for the fundamental particles described by the generalized nonlinear Schrödinger equation.
- * Fluctuations in the condensate field on the other hand, do not ‘see’ the physical metric at all. Perturbations couple only to the *acoustic metric* $g_{\mu\nu}$.
- * As is common to all 2-metric theories, one could also construct a distinct ‘associated metric’ by the prescription

$$[g_{\text{associated}}]_{\mu\nu} \equiv \eta_{\mu\sigma} [g^{-1}]^{\sigma\rho} \eta_{\rho\nu}. \quad (49)$$

There seems to not be any clean physical interpretation for this object.

- The geometry determined by the acoustic metric inherits some key properties (such as, for example, stable causality) from the existence of the underlying flat physical metric (see [9, 12]).
- This acoustic metric is now sufficiently general to be able to mimic a wide class of ‘generic’ gravitational fields—the presence of the position-dependent 3-tensor mass matrix is crucial to this observation. In Garay *et al* [2, 3] the mass was both isotropic and position-independent, consequently, the spatial slices of their analogue spacetimes were always conformally flat (the same phenomenon occurs in the acoustic geometries of [5–14]). This is the fundamental reason we have gone to the technical trouble of adding a position-dependent 3-tensor mass. The only significant restriction on our version of the effective metric is that the 3-velocity is irrotational (zero vorticity, curl-free).
- The major differences with the analysis of Garay *et al* is that their nonlinearity was strictly quartic, their mass both isotropic and position-independent, and that they further approximated the current quasi-classical approximation by going to a ‘geometrical optics’ version thereof that permitted them to also neglect the conformal factor.
- As we have already commented, one might be a little worried that the ‘speed’ c depends on the arbitrary but fixed scale μ . However, the 3-metric h_{ij} also depends on μ and

$$c^2 h^{ij} = \pi'' \rho_0 [m^{-1}]^{ij} \quad (50)$$

is μ -independent. Similarly, $(v_0)^i$ is independent of μ . If you ask physical questions like (for instance) ‘what are the null curves of the metric $g_{\mu\nu}$?’ they are determined by the equation

$$h_{ij} \left(\frac{dx^i}{dt} - v_0^i \right) \left(\frac{dx^j}{dt} - v_0^j \right) = c^2. \quad (51)$$

And this equation is completely independent of the arbitrary fixed scale μ . Equivalently,

$$m_{ij} \left(\frac{dx^i}{dt} - v_0^i \right) \left(\frac{dx^j}{dt} - v_0^j \right) = \pi'' \rho_0. \quad (52)$$

- We should add that in Einstein gravity the spacetime metric is related to the distribution of matter by the nonlinear Einstein–Hilbert differential equations. In contrast, in the present context, the acoustic metric is related to the background wavefunction in a simple algebraic fashion. There are certainly constraints on the acoustic metric, but they arise from the generalized nonlinear Schrödinger equation, not from the Einstein equations of general relativity. (We belabour this trivial point because we have seen it lead to considerable confusion.)

- Finally, we reiterate the main reason the relativity community is interested in these systems: regions where the speed of the condensate flow exceeds the speed of sound very closely mimic the key kinematic features of black hole physics. We will not repeat the relevant details as they are more than adequately dealt with elsewhere in the literature (see, for example, [2, 3, 5–14]).

Many features of this acoustic metric survive beyond the current quasi-classical approximation, and we now initiate a systematic analysis of how much further the model can be pushed.

7. Low-momentum approximation

The low-momentum approximation is subtly different from the naive quasi-classical approximation. We shall now retain the $\hbar^2 D_2$ term, but take the approximation that within this quantum potential term the gradients of the background are more important than gradients of the fluctuation. (We justify this with the observation that gradients of the fluctuation are doubly small, being suppressed by both a factor of \hbar and a factor of the linearization parameter ϵ). Specifically, we take

$$D_2 \rho_1 \equiv -\frac{1}{2} \rho_0^{-3/2} [\Delta_h (\rho_0^{+1/2})] \rho_1 + \frac{1}{2} \rho_0^{-1/2} \Delta_h (\rho_0^{-1/2} \rho_1) \quad (53)$$

$$\approx \left\{ -\frac{1}{2} \rho_0^{-3/2} [\Delta_h (\rho_0^{+1/2})] + \frac{1}{2} \rho_0^{-1/2} [\Delta_h (\rho_0^{-1/2})] \right\} \rho_1 \quad (54)$$

$$= -\frac{1}{2} \left\{ \frac{\Delta_h \rho_0}{\rho_0^2} - \frac{(\nabla \rho_0)^2}{\rho_0^3} \right\} \rho_1. \quad (55)$$

That is, under this low-momentum approximation we can simply replace the *operator* D_2 by the *function*

$$D_2 \rightarrow -\frac{1}{2} \left\{ \frac{\Delta_h \rho_0}{\rho_0^2} - \frac{(\nabla \rho_0)^2}{\rho_0^3} \right\}. \quad (56)$$

The net result of this approximation is that wherever the quantity π'' appears in the naive quasi-classical analysis it should be replaced by

$$\pi'' \rightarrow \pi'' + \frac{\hbar^2}{4\mu} \left\{ \frac{\Delta_h \rho_0}{\rho_0^2} - \frac{(\nabla \rho_0)^2}{\rho_0^3} \right\}. \quad (57)$$

This does not affect the background flow velocity v_0 , but it does modify the propagation speed so that now

$$c^2 \rightarrow \frac{\rho_0}{\mu} \left[\pi'' + \frac{\hbar^2}{4\mu} \left\{ \frac{\Delta_h \rho_0}{\rho_0^2} - \frac{(\nabla \rho_0)^2}{\rho_0^3} \right\} \right] = c_{\text{quasiclassical}}^2 \left[1 + \frac{\hbar^2}{4\mu\pi''} \left\{ \frac{\Delta_h \rho_0}{\rho_0^2} - \frac{(\nabla \rho_0)^2}{\rho_0^3} \right\} \right]. \quad (58)$$

This is a new higher-order correction to the quasi-classical speed of sound we have previously introduced in equation (21); in this sense is a generalization of the previously known results regarding the propagation of collective excitations in BEC. This effect was not contemplated in the analysis of Garay *et al.* (Justifiably so, since they were only interested in the geometrical ‘optics’ approximation within the quasi-classical limit.) If, using the speed of sound, we introduce a notion of ‘acoustic Compton wavelength’

$$\lambda_c \equiv \frac{h}{\mu c}, \quad (59)$$

then

$$\lambda_c^2 = \frac{\hbar^2}{\mu\rho_0\pi''}. \quad (60)$$

So this modification to the speed of sound is seen to be the first-order term in a gradient expansion governed by the dimensionless parameter

$$\lambda_c \frac{\|\nabla\rho_0\|}{4\pi\rho_0}. \quad (61)$$

8. Eikonal approximation

In contrast to the low-momentum approximation, the *eikonal* approximation is a high-momentum approximation where the phase fluctuation θ_1 is itself treated as a slowly varying amplitude times a rapidly varying phase. This phase will be taken to be the same for both ρ_1 and θ_1 fluctuations. In fact, if one discards the unphysical possibility that the respective phases differ by a time-varying quantity, any time-constant difference can be safely reabsorbed in the definition of the (complex) amplitudes.

Specifically, we shall write

$$\theta_1(t, \mathbf{x}) = \text{Re} \{ \mathcal{A}_\theta \exp(-i\phi) \}, \quad (62)$$

$$\rho_1(t, \mathbf{x}) = \text{Re} \{ \mathcal{A}_\rho \exp(-i\phi) \}. \quad (63)$$

As a consequence of our starting assumptions, gradients of the amplitude, and gradients of the background fields, are systematically ignored relative to gradients of ϕ . (Warning: what we are doing here is not quite a ‘standard’ eikonal approximation, in the sense that it is not applied directly on the fluctuations of the field $\psi(t, \mathbf{x})$ but separately on their amplitudes and phases ρ_1 and ϕ_1 .) We adopt the notation

$$\omega = \frac{\partial\phi}{\partial t}; \quad k_i = \nabla_i\phi. \quad (64)$$

Then the operator D_2 can be approximated as

$$D_2\rho_1 \equiv -\frac{1}{2}\rho_0^{-3/2}[\Delta_h(\rho_0^{+1/2})]\rho_1 + \frac{1}{2}\rho_0^{-1/2}\Delta_h(\rho_0^{-1/2}\rho_1) \quad (65)$$

$$\approx +\frac{1}{2}\rho_0^{-1}[\Delta_h\rho_1] \quad (66)$$

$$= -\frac{1}{2}\rho_0^{-1}k^2\rho_1. \quad (67)$$

A similar result holds for D_2 acting on θ_1 . That is, under the eikonal approximation we effectively replace the *operator* D_2 by the *function*

$$D_2 \rightarrow -\frac{1}{2}\rho_0^{-1}k^2. \quad (68)$$

For the matrix $f^{\mu\nu}$ this effectively results in the replacement

$$f^{00} \rightarrow -\left[\pi''(\rho_0) + \frac{\hbar^2 k^2}{4\mu\rho_0}\right]^{-1} \quad (69)$$

$$f^{0j} \rightarrow -\left[\pi''(\rho_0) + \frac{\hbar^2 k^2}{4\mu\rho_0}\right]^{-1} \frac{\hbar^{jk}\nabla_k\theta_0}{\mu} \quad (70)$$

$$f^{i0} \rightarrow -\frac{h^{ik}\nabla_k\theta_0}{\mu} \left[\pi''(\rho_0) + \frac{\hbar^2 k^2}{4\mu\rho_0} \right]^{-1} \quad (71)$$

$$f^{ij} \rightarrow \frac{\rho_0^{(3)}h^{ij}}{\mu} - \frac{h^{ik}\nabla_k\theta_0}{\mu} \left[\pi''(\rho_0) + \frac{\hbar^2 k^2}{4\mu\rho_0} \right]^{-1} \frac{h^{jl}\nabla_l\theta_0}{\mu} \quad (72)$$

(as desired, this has the net effect of making $f^{\mu\nu}$ a matrix of numbers, not operators).

The physical wave equation (29) now becomes a nonlinear dispersion relation

$$f^{00}\omega^2 + (f^{0i} + f^{i0})\omega k_i + f^{ij}k_i k_j = 0. \quad (73)$$

After substituting the approximate D_2 into this dispersion relation and rearranging, we see (remember $k^2 = \|k\|^2 = [h^{-1}]^{ij}k_i k_j$)

$$-\omega^2 + 2v_0^i\omega k_i + \frac{\rho_0 k^2}{\mu} \left[\pi'' + \frac{\hbar^2}{4\mu\rho_0} k^2 \right] - (v_0^i k_i)^2 = 0. \quad (74)$$

That is

$$(\omega - v_0^i k_i)^2 = \frac{\rho_0 k^2}{\mu} \left[\pi'' + \frac{\hbar^2}{4\mu\rho_0} k^2 \right]. \quad (75)$$

Alternatively,

$$\omega = v_0^i k_i \pm \sqrt{\frac{\rho_0 k^2}{\mu} \left[\pi'' + \frac{\hbar^2}{4\mu\rho_0} k^2 \right]} = v_0^i k_i \pm \sqrt{\rho_0 \pi'' (k_i [m^{-1}]^{ij} k_j) + \left(\frac{1}{2} \hbar (k_i [m^{-1}]^{ij} k_j) \right)^2}. \quad (76)$$

In the case of an isotropic mass matrix, $[m^{-1}]^{ij} \rightarrow 1/m$. It also makes sense to set $\mu \rightarrow m$ and $h_{ij} \rightarrow \delta_{ij}$, in which case c really does represent the physical speed of sound. Then we can write the dispersion relation in a more illuminating form

$$\omega = v_0^i k_i \pm \sqrt{c^2 k^2 + \left(\frac{\hbar}{2m} k^2 \right)^2}. \quad (77)$$

Note how the previous anisotropic dispersion relation differs from this isotropic one: in that case there are three different physical sound velocities in three orthogonal principal directions.

At this stage some observations are in order

- (a) It is interesting to recognize that the dispersion relation (77) is exactly in agreement with that found in 1947 by Bogoliubov [29, 30] for the collective excitations of a homogeneous Bose gas in the limit $T \rightarrow 0$ (almost complete condensation). In his derivation Bogoliubov applied a diagonalization procedure for the Hamiltonian describing the system of bosons.
- (b) It is easy to see that (76), and its isotropic partner (77) actually interpolates between two different regimes depending on the value of the wavelength $\lambda = 2\pi/\|k\|$ with respect to the ‘acoustic Compton wavelength’ $\lambda_c = h/(\mu c)$. (Remember that c is the speed of sound; this is not a standard particle physics Compton wavelength. Furthermore, $\|k\|$ is defined using the inverse 3-metric h^{ij} .) In particular, if we assume $v_0 = 0$ (no background velocity), then for large wavelengths $\lambda \gg \lambda_c$ one obtains a standard phonon dispersion relation $\omega \approx c\|k\|$. As stressed by Braaten [48] this can be related to the fact that the quantum theory we are working with has a $U(1)$ symmetry which is spontaneously broken. At low momenta we are just seeing the dispersion relation of the corresponding Goldstone mode. For wavelengths $\lambda \ll \lambda_c$ the quasi-particle energy tends to the kinetic energy of an individual gas particle and, in fact, $\omega \approx \hbar^2 k^2 / (2m)$.

- (c) The dispersion relation (77) exhibits a contribution due to the background flow $v_0^i k_i$, plus a quartic dispersion at high momenta. The group velocity is

$$v_g^i = \frac{\partial \omega}{\partial k_i} = v_0^i \pm \frac{\left(c^2 + \frac{\hbar^2}{2m^2} k^2\right)}{\sqrt{c^2 k^2 + \left(\frac{\hbar}{2m} k^2\right)^2}} k^i. \quad (78)$$

Dispersion relations of this type (but in most cases with the sign of the quartic term reversed) have been used by Corley and Jacobson in analysing the issue of trans-Planckian modes in the Hawking radiation from general relativistic black holes [6, 8, 21]. In their analysis the group velocity reverses its sign for large momenta. (Unruh's analysis of this problem used a slightly different toy model in which the dispersion relation is saturated at high momentum [10].) In our case, however, the group velocity grows without bound, allowing high-momentum modes to escape from behind the 'horizon'. (Thus the acoustic horizon is not 'absolute' in these models, but is instead frequency dependent, a phenomenon that is common once non-trivial dispersion is included.)

This type of 'superluminal'⁶ dispersion relation has also been analysed by Corley and Jacobson [28]. They found that this escape of modes from behind the horizon often leads to self-amplified instabilities in systems possessing both an inner horizon as well as an outer horizon, possibly causing them to disappear in an explosion of phonons. This is also in partial agreement with the stability analysis performed by Garay *et al* using the whole Bogoliubov equations. They found unstable solutions with the kind of behaviour just mentioned, but they also find stability regions (depending on the value of certain configuration parameters). The existence of these stable configurations might be related to the specific boundary conditions imposed in their configurations.

Indeed, with hindsight the fact that the group velocity goes to infinity for large k was pre-ordained: after all, we started from the generalized nonlinear Schrödinger equation, and we know what its characteristic curves are. Like the diffusion equation the characteristic curves of the Schrödinger equation (linear or nonlinear) move at infinite speed. If we then approximate this generalized nonlinear Schrödinger equation in any manner, for instance by linearization, we cannot change the characteristic curves: for any well behaved approximation technique, at high frequency and momentum we should recover the characteristic curves of the system we started with. However, what we certainly do see in this analysis is a suitably large region of momentum space for which the concept of the effective metric both makes sense, and leads to a finite propagation speed for medium-frequency oscillations.

- (d) There is an amusing feature to the (generalized) Bogoliubov dispersion relation which it may be worth making explicit: consider the dispersion relation

$$\omega(k) = \sqrt{m_0^2 + k^2 + \left(\frac{k^2}{2m_\infty}\right)^2} \quad (79)$$

(BEC condensates correspond to $m_0 = 0$, we retain this term here for generality. We have made $c = \hbar = 1$). At low momenta ($k \ll m_0$) this dispersion relation has the usual non-relativistic limit

$$\omega(k) = m_0 + \frac{k^2}{2m_0} + O(k^4). \quad (80)$$

⁶ The term 'superluminal' refers here to real physical propagation beyond c , in our case the velocity of propagation for phonons in the condensate, and has nothing to do with the 'apparent' superluminal behaviour found for wavepackets in certain tunnelling situations [49, 50].

At intermediate momenta ($m_0 \ll k \ll m_\infty$) this dispersion relation has an approximately relativistic form. Finally, at large momenta ($k \gg m_\infty$) the dispersion relation again (perhaps surprisingly) recovers a non-relativistic form

$$\omega(k) = \frac{k^2}{2m_\infty} + m_\infty + \mathcal{O}(k^{-2}). \quad (81)$$

This serves to drive home in a particularly simple way the point that observing a Lorentz invariant spectrum does not guarantee that the underlying physics is Lorentz invariant. Indeed, the entire programme of searching for analogue models of general relativity generically seeks to take some simple (physically accessible and typically non-relativistic) model and determine whether it nevertheless hides within it some useful approximation to Lorentzian geometry.

9. Summary and discussion

In seeking to see how far we might be able to push the BEC system as an analogue model for general relativity we have encountered a number of intriguing features. First, we have shown that the existence of a regime in which phase perturbations of the wavefunction of the BEC (or quasi-BEC) behave as though coupled to an ‘effective Lorentzian metric’ is a generic feature, independent of the explicit form of the nonlinear terms in the Schrödinger equation. The only exception comes about when the forces exerted between atoms are attractive. In this case the equation of motion for the phase perturbations are no longer hyperbolic, so the whole notion of a wave disappears for these systems. (In GR language, this corresponds to a Euclidean-signature metric.)

Second, we have seen that in contrast to the isotropic acoustic systems considered to date, mimicking a generic gravitational field is not *a priori* implausible though it would technically be very challenging, relying as it does on the direct introduction of anisotropies into the generalized nonlinear Schrödinger equation via a position-dependent 3-tensor effective mass.

We have also explicitly seen how the whole notion of an ‘effective metric’ in these BEC systems is intrinsically an approximation which is valid for certain ranges of frequency and wavenumber—this should not really be a surprise, since even for normal acoustic systems eventually the atomic nature of matter provides a natural cut-off. In the BEC system we have seen that the acoustic Compton wavelength plays a similar role: wavelengths long compared with the acoustic Compton wavelength see a Lorentzian ‘effective metric’, while wavelengths short compared with the acoustic Compton wavelength probe the ‘high-energy’ physics (which in this situation is the non-relativistic Schrödinger equation).

Finally, we mention that from the sound velocities typically encountered in BEC systems we can make a crude estimate of the Hawking temperature to be expected in these systems. Using the standard estimate based on dimensional analysis [5, 12, 13]

$$T \approx \frac{\hbar}{2\pi k_B} \frac{c}{R}, \quad (82)$$

and choosing a value of $R \approx 10\mu$ for the size of the acoustic black hole, we would have $T \approx 10^{-9}$ K. While extremely small this temperature is only three orders of magnitude less than that of the BEC itself. Furthermore, as argued in [13] this order of magnitude estimate for the Hawking temperature is often misleadingly low—and this fact, in addition to the intrinsically interesting features of the effective geometry approach, makes further investigation of these BEC systems worthwhile—both for the condensed matter and relativity communities.

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References

- [1] Hehl F W, Lemke J and Mielke E W 1991 Two lectures on fermions and gravity *Geometry and Theoretical Physics (Bad Honnef Lectures, 12–16 February 1990)* ed J Debrus and A C Hirshfeld (Berlin: Springer)
- [2] Garay L J, Anglin J R, Cirac J I and Zoller P 2000 Black holes in Bose–Einstein condensates *Phys. Rev. Lett.* **85** 4643
(Garay L J, Anglin J R, Cirac J I and Zoller P 2000 *Preprint* gr-qc/0002015)
- [3] Garay L J, Anglin J R, Cirac J I and Zoller P 2001 Sonic black holes in dilute Bose–Einstein condensates *Phys. Rev. A* **63** 023611
(Garay L J, Anglin J R, Cirac J I and Zoller P 2000 *Preprint* gr-qc/0005131)
- [4] Gordon W 1923 Zur Lichtfortpflanzung nach der Relativitätstheorie *Ann. Phys., Lpz.* **72** 421
- [5] Unruh W G 1981 Experimental black hole evaporation? *Phys. Rev. Lett.* **46** 1351
- [6] Jacobson T 1991 Black hole evaporation and ultrashort distances *Phys. Rev. D* **44** 1731
- [7] Comer G 1992 Superfluid analog of the Davies–Unruh effect (unpublished)
- [8] Jacobson T 1993 Black hole radiation in the presence of a short distance cutoff *Phys. Rev. D* **48** 728
(Jacobson T 1993 *Preprint* hep-th/9303103)
- [9] Visser M 1993 Acoustic propagation in fluids: an unexpected example of Lorentzian geometry *Preprint* gr-qc/9311028
- [10] Unruh W G 1995 Sonic analogues of black holes and the effects of high frequencies on black hole evaporation *Phys. Rev. D* **51** 2827
(Unruh W G 1994 Dumb holes and the effects of high frequencies on black hole evaporation *Preprint* gr-qc/9409008)
- [11] Hochberg D 1997 Evaporating black holes and collapsing bubbles in fluids (unpublished)
- [12] Visser M 1998 Acoustic black holes: Horizons, ergospheres and Hawking radiation *Class. Quantum Grav.* **15** 1767
(Visser M 1997 *Preprint* gr-qc/9712010)
- [13] Liberati S, Sonogo S and Visser M 2000 Unexpectedly large surface gravities for acoustic horizons? *Class. Quantum Grav.* **17** 2903
(Liberati S, Sonogo S and Visser M 2000 *Preprint* gr-qc/0003105)
- [14] Liberati S 2000 Quantum vacuum effects in gravitational fields: theory and detectability *Preprint* gr-qc/0009050
- [15] Volovik G E 1998 Simulation of quantum field theory and gravity in superfluid ^3He *Low Temp. Phys.* **24** 127
(Volovik G E 1997 *Preprint* cond-mat/9706172)
Kopnin N B and Volovik G E 1998 Critical velocity and event horizon in pair-correlated systems with relativistic fermionic quasiparticles *Pis. Zh. Eksp. Teor. Fiz.* **67** 124
(Kopnin N B and Volovik G E 1997 *Preprint* cond-mat/9712187)
Volovik G E 1998 Gravity of monopole and string and gravitational constant in $^3\text{He-A}$ *Pis. Zh. Eksp. Teor. Fiz.* **67** 666
Volovik G E 1998 *JETP Lett.* **67** 698
(Volovik G E 1998 *Preprint* cond-mat/9804078)
Volovik G E 2000 Links between gravity and dynamics of quantum liquids *Preprint* gr-qc/0004049
- [16] Leonhardt U and Piwnicki P 1999 Optics of nonuniformly moving media *Phys. Rev. A* **60** 4301
(Leonhardt U and Piwnicki P 1999 *Preprint* physics/9906038)
Leonhardt U and Piwnicki P 2000 Relativistic effects of light in moving media with extremely low group velocity *Phys. Rev. Lett.* **84** 822
(Leonhardt U and Piwnicki P 1999 *Preprint* cond-mat/9906332)
Visser M 2000 Comment on relativistic effects of light in moving media with extremely low group velocity *Phys. Rev. Lett.* **85** 5252

- (Visser M 2000 *Preprint* gr-qc/0002011)
 Leonhardt U and Piwnicki P 2000 Reply to the comment on relativistic effects of light in moving media with extremely low group velocity *Phys. Rev. Lett.* **85** 5253
 (Leonhardt U and Piwnicki P 2000 *Preprint* gr-qc/0003016)
- [17] Jacobson T 1999 Trans-Planckian redshifts and the substance of the space-time river *Prog. Theor. Phys. Suppl.* **136** 1
 (Jacobson T 2000 *Preprint* hep-th/0001085)
- [18] Reznik B 1997 Trans-Planckian tail in a theory with a cutoff *Phys. Rev. D* **55** 2152
 (Reznik B 1996 *Preprint* gr-qc/9606083)
 Reznik B 2000 Origin of the thermal radiation in a solid-state analog of a black hole *Phys. Rev. D* **62** 0440441
 (Reznik B 1997 *Preprint* gr-qc/9703076)
- [19] Dittrich W and Gies H 1998 Light propagation in nontrivial QED vacua *Phys. Rev. D* **58** 025004
 (Dittrich W and Gies H 1998 *Preprint* hep-ph/9804375)
 Novello M, de Lorenci V A, Salim J M and Klippert R 2000 Geometrical aspects of light propagation in nonlinear electrodynamics *Phys. Rev. D* **61** 045001
 (Novello M, de Lorenci V A, Salim J M and Klippert R 1999 *Preprint* gr-qc/9911085)
 de Lorenci V A, Klippert R, Novello M and Salim J M 2000 Light propagation in non-linear electrodynamics *Phys. Lett. B* **482** 134
 (de Lorenci V A, Klippert R, Novello M and Salim J M 2000 *Preprint* gr-qc/0005049)
- [20] Liberati S, Sonogo S and Visser M 2001 Scharnhorst effect at oblique incidence *Phys. Rev. D* at press
 (Liberati S, Sonogo S and Visser M 2000 *Preprint* quant-ph/0010055)
- [21] Corley S and Jacobson T 1996 Hawking spectrum and high frequency dispersion *Phys. Rev. D* **54** 1568
 (Corley S and Jacobson T 1996 *Preprint* hep-th/9601073)
 Corley S 1997 Particle creation via high frequency dispersion *Phys. Rev. D* **55** 6155
 Corley S 1998 Computing the spectrum of black hole radiation in the presence of high frequency dispersion: an analytical approach *Phys. Rev. D* **57** 6280
 (Corley S 1997 *Preprint* hep-th/9710075)
 Jacobson T and Mattingly D 2000 Spontaneously broken Lorentz symmetry and gravity *Preprint* gr-qc/0007031
 Jacobson T and Mattingly D 2001 Generally covariant model of a scalar field with high frequency dispersion and the cosmological horizon problem *Phys. Rev. D* **63** 041502
 (Jacobson T and Mattingly D 2000 *Preprint* hep-th/0009052)
- [22] Anderson M H, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1995 Observation of Bose–Einstein condensation in a dilute atomic vapor *Science* **269** 198
- [23] Bradley C C, Sackett C A, Tollett J J and Hulet R G 1995 Evidence of Bose–Einstein condensation in an atomic gas with attractive interactions *Phys. Rev. Lett.* **75** 1687
- [24] Davis K B, Mewes M-O, Andrews M R, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 Bose–Einstein condensation in a gas of sodium atoms *Phys. Rev. Lett.* **75** 3969
- [25] Bose S N 1924 Planck’s Gesetz und Lichtquantenhypothese *Z. Phys.* **26** 178
- [26] Einstein A 1924 *Sitzber. Kgl. Preuss. Akad. Wiss.* 261
- [27] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 Theory of Bose–Einstein condensation in trapped gases *Rev. Mod. Phys.* **71** 463
- [28] Corley S and Jacobson T 1999 Black hole lasers *Phys. Rev. D* **59** 124011
 (Corley S and Jacobson T 1998 *Preprint* hep-th/9806203)
- [29] Bogoliubov N 1947 *J. Phys. (USSR)* **11** 23
- [30] Lifshitz E M and Pitaevskii L P 1984 *Statistical Physics, Part 2* (Oxford: Pergamon)
- [31] Ruffini R and Bonazzola S 1969 Systems of self-gravitating particles in general relativity and the concept of an equation of state *Phys. Rev.* **187** 1767
- [32] Mielke E W and Schunck F E 2000 Boson stars: alternatives to primordial black holes? *Nucl. Phys. B* **564** 185
- [33] Mielke E W and Scherzer R 1981 Geon-type solutions of the nonlinear Heisenberg–Klein–Gordon equation *Phys. Rev. D* **24** 2111
 Schunck F E and Mielke E W 1998 Rotating boson star as an effective mass torus in general relativity *Phys. Lett. A* **249** 389
 Schunck F E and Mielke E W 1999 Boson stars: rotation, formation and evolution *Gen. Rel. Grav.* **31** 787
- [34] Kolomeisky E B, Newman T J, Straley J P and Qui X 2000 Low-dimensional Bose liquids: beyond the Gross–Pitaevskii approximation *Phys. Rev. Lett.* **85** 1146
- [35] Mullin W J 1997 Bose–Einstein condensation in a harmonic potential *J. Low Temp. Phys.* **106** 615
- [36] Ziman J M 1972 *Principles of the Theory of Solids* (Cambridge: Cambridge University Press)
- [37] Wolfe J P, Lin J L and Snoke D W 1995 Bose–Einstein condensation of a nearly ideal gas: excitons in Cu₂O *Bose–*

- Einstein Condensation* ed A Griffin, D W Snoke and S Stringari (Cambridge: Cambridge University Press)
- [38] Moskaleiko S A and Snoke D W 2000 *Bose–Einstein Condensation of Excitons and Biexcitons* (Cambridge: Cambridge University Press)
- [39] DeWitt B S 1957 Dynamical theory in curved spaces I. A review of classical and quantum action principles *Rev. Mod. Phys.* **29** 377
- [40] Schulman L 1981 *Techniques and Applications of Path Integration* (New York: Wiley) (see especially ch 24)
- [41] Madelung E 1926 Quantentheorie in hydrodynamischer Form *Z. Phys.* **38** 322
- [42] Takabayasi T 1953 Remarks on the formulation of quantum mechanics with classical pictures and on relations between linear scalar fields and hydrodynamical fields *Prog. Theor. Phys.* **9** 187
- [43] Wong C Y 1976 On the Schrödinger equation in fluid-dynamical form *J. Math. Phys.* **17** 1008
- [44] Ghosh S K and Deb B M 1982 Densities, density-functionals and electron fluids *Phys. Rep.* **92** 1
- [45] Sonogo S 1991 Interpretation of the hydrodynamical formalism of quantum mechanics *Found. Phys.* **21** 1135
- [46] Bohm D 1952 A suggested interpretation of the quantum theory in terms of hidden variables: I *Phys. Rev.* **85** 166
- Bohm D 1952 A suggested interpretation of the quantum theory in terms of hidden variables: II *Phys. Rev.* **85** 180
- [47] Holland P R 1993 *The Quantum Theory of Motion: an Account of the de Broglie–Bohm Causal Interpretation of Quantum Mechanics* (Cambridge: Cambridge University Press)
- [48] Braaten E 1998 Bose–Einstein condensation of atoms and thermal field theory *Preprint* hep-ph/9809405
- [49] Nimtz G 1999 The special features of superluminal evanescent mode propagation *Gen. Rel. Grav.* **31** 737
- [50] Wang L J, Kuzmich A and Dogariu A 2000 Gain-assisted superluminal light propagation *Nature* **406** 277