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Analogue of Inverse Scattering Theory for the Discrete Hill's Equation and Exact Solutions for the Periodic Toda Lattice

Etsuro DATE and Shunichi TANAKA

Department of Mathematics, Osaka University, Toyonaka, Osaka 560

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An analogue of the inverse scattering theory for the discrete Hill's equation is developed. A method consists of the introduction of abelian integrals and the solution of Jacobi's inversion problem. The equation of motion of the periodic Toda lattice is integrated exactly by means of Riemann theta functions.

§ 1. Introduction

The purpose of the present paper is an integration of the equation of motion of the periodic Toda lattice.[1)]

Simultaneously with the invention of his exponential lattice, Toda has constructed both of the infinite and periodic traveling wave solutions (soliton). The latter is expressed by elliptic function. Solutions showing the interaction of two solitons are also found by Toda[2)] for the infinite case. Hirota[3)] has found the explicit form of N-solitons for the infinite case. Flaschka[4)] has found Lax representation[5)] for the Toda lattice and applied the inverse scattering theory to the discrete Sturm-Liouville equation with similar conclusion to the one obtained by Gardner, Greene, Kruskal and Miura[6)] on the Korteweg-de Vries (KdV) equation. These developments of Toda lattice theory is summarized in Toda's review article.[7)] Kac and Moerbeke[8)] studied a variant of the Toda lattice. They also consider the periodic problem (especially soliton solution) from the standpoint of the spectral theory of the discrete Sturm-Liouville equation.

Recently Dubrovin[9),10)] and Its-Matveev[11)] developed an analogue of the inverse scattering theory for Hill's equation (Sturm-Liouville equation with periodic coefficients) and have given an explicit form of the periodic potentials with the finite number of gaps in the spectrum. The KdV equation is exactly solved in that class of potentials leading to the effective construction of the periodic N-solitons.[10),11)]

Main part of the present paper is devoted to the inverse scattering theory of the discrete Hill's equation which is also of its own interest. Principal method of Dubrovin, Its-Matveev and also of this paper is the introduction of abelian integrals and solution of Jacobi's inversion problem. This idea has been introduced into the study of Hill's equation and its discrete version by Akhiezer.[12),13)]

In § 2 we describe the generalities on the spectral properties of the discrete

Hill's equation.

In § 3 hyperelliptic Riemann surface is introduced so that the Bloch eigenfunction is single-valued meromorphic function on the surface. Abelian integrals and Jacobi's inversion problem for them are described and the potentials are expressed explicitly by theta functions.

Section 4 concerns exact solutions for the equation of motion of the periodic Toda lattice.

§ 2. Spectral properties of the discrete Hill's equation

Consider the discrete Hill's equation

$$Lu=\lambda u\,,$$
$$(Lu)(n)=a_n u(n+1)+b_n u(n)+a_{n-1}u(n-1),$$
$$a_n>0;\ a_{n+N}=a_n,\ b_{n+N}=b_n\,. \tag{2·1}$$

We define a fundamental system of solutions of (2·1), $y(n)$, $z(n)$, by the initial conditions

$$y(0)=1\,,\qquad y(1)=a_0^{-1}(\lambda-b_0);$$
$$z(0)=0\,,\qquad z(1)=a_0^{-1}\,. \tag{2·2}$$

Then for $n\geq 0$, $y(n)$ is a polynomial of the form

$$y(n)=(\prod_{j=0}^{n-1}a_j)^{-1}\{\lambda^n-(\sum_{j=0}^{n-1}b_j)\lambda^{n-1}+(\sum_{0\leq j<k\leq n-1}b_jb_k-\sum_{j=0}^{n-2}a_j^2)\lambda^{n-2}+\cdots\} \tag{2·3}$$

and $z(n)$ is a polynomial of the form

$$z(n)=(\prod_{j=0}^{n-1}a_j)^{-1}\{\lambda^{n-1}-(\sum_{j=1}^{n-1}b_j)\lambda^{n-2}+(\sum_{1\leq j<k\leq n-1}b_jb_k-\sum_{j=1}^{n-2}a_j^2)\lambda^{n-3}+\cdots\}. \tag{2·4}$$

For $n<0$, similar expressions hold.

By the periodicity of a_n, b_n, we have

$$\begin{bmatrix}y(n+N)\\ z(n+N)\end{bmatrix}=M(\lambda)\begin{bmatrix}y(n)\\ z(n)\end{bmatrix},\qquad M(\lambda)=\begin{bmatrix}y(N) & -a_{N-1}y(N-1)\\ z(N) & -a_{N-1}z(N-1)\end{bmatrix}. \tag{2·5}$$

Noting that $\det M$ is an analogue of Wronskian, we have $\det M=1$

Put

$$\Delta(\lambda)=y(N)-a_{N-1}z(N-1).$$

The roots of the equation

$$\Delta(\lambda)^2-4=0 \tag{2·6}$$

are all real and are ordered as

$$\lambda_1<\lambda_2\leq\lambda_3<\cdots<\lambda_{2j}\leq\lambda_{2j+1}<\cdots<\lambda_{2N-2}\leq\lambda_{2N-1}<\lambda_{2N}\,.$$

The roots of the equation

$$z(N)=0$$

are real and distinct. If we order them by

$$\mu_1<\mu_2<\cdots<\mu_{N-1},$$

we have $\mu_j\in[\lambda_{2j},\lambda_{2j+1}]$. These properties are shown by the analogous arguments as in the continuous case.[14),15),16)]

Denote by $y(k,n)$, $z(k,n)$ the solutions of $(2\cdot1)$, $(2\cdot2)$ in which the coefficients a_n, b_n are replaced by a_{n+k}, b_{n+k}. Then the relations

$$y(k,n)=a_{k-1}y(k-1)z(k+n)-a_{k-1}z(k-1)y(k+n),$$

$$z(k,n)=y(k)z(k+n)-z(k)y(k+n) \tag{2·7}$$

hold.
Using these relations we have

$$y(N)-a_{N-1}z(N-1)=y(k,N)-a_{k-1}z(k,N-1).$$

Therefore if we denote the roots of the equation

$$z(k,N)=0$$

by $\mu_1(k)<\cdots<\mu_{N-1}(k)$, $\mu_j(k)$ belongs to $[\lambda_{2j},\lambda_{2j+1}]$. Accordingly if the interval to which $\mu_j(0)=\mu_j$ belongs degenerates to a single point, we have $\mu_j(k)=\mu_j(0)$ for all k.

The coefficients of λ^{N-2} and λ^{N-3} of $z(k,N)$ are calculated in two ways leading to

$$\sum_{j=0}^{N-1}b_j-b_k=\sum_{j=1}^{N-1}\mu_j(k), \tag{2·8}$$

$$\sum_{0\leq i<j\leq N-1;\,i,\,j\neq k}b_ib_j-\sum_{j=0}^{N-1}a_j^2+a_{k-1}^2+a_k^2=\sum_{1\leq i<j\leq N-1}\mu_i(k)\,\mu_j(k). \tag{2·9}$$

Introducing the notations

$$\Lambda_1=2^{-1}\sum_{j=1}^{2N}\lambda_j\,,\qquad \Lambda_2=\sum_{1\leq i<j\leq 2N}\lambda_i\lambda_j$$

and applying the analogous argument to $\Delta(\lambda)^2-4$, we have

$$\sum_{j=0}^{N-1}b_j=\Lambda_1\,, \tag{2·10}$$

$$\sum_{0\leq i<j\leq N-1}b_ib_j-\sum_{j=0}^{N-1}a_j^2=2^{-1}\Lambda_2-2^{-1}\Lambda_1^2\,. \tag{2·11}$$

We define the Bloch eigenfunction of $(2\cdot1)$ by

$$x_\pm(n)=y(n)+\frac{-\{a_{N-1}z(N-1)+y(N)\}\pm(\Delta(\lambda)^2-4)^{1/2}}{2z(N)}z(n).$$

By the direct calculation using $(2\cdot5)$, $(2\cdot7)$, we have

$$x_+(k)x_-(k)=z(k,N)/z(N). \tag{2·12}$$

§ 3. Hyperelliptic abelian integrals and the solution of Jacobi's inversion problem

In what follows, among roots of (2·6) simple roots play important roles. Assuming their number to be $2g+2$ and changing the numbering, we denote simple roots by

$$\lambda_1<\lambda_2<\cdots<\lambda_{2g+2}$$

and double roots by $\lambda_{2j+1}=\lambda_{2j+2}$ $(j=g+1,\cdots,N-1)$. We also change the numbering for $\mu_j(k)$ so that the relations

$$\lambda_{2j}\leq\mu_j(k)\leq\lambda_{2j+1}, \qquad j=1,\cdots,g,$$
$$\mu_j(k)=\lambda_{2j+1}=\lambda_{2j+2}, \qquad j=g+1,\cdots,N-1$$

hold.

We introduce the Riemann surface of the hyperelliptic curve[17)]

$$\mu^2=R(\lambda)=\prod_{j=1}^{2g+2}(\lambda-\lambda_j).$$

This Riemann surface is realized by cross-connecting two copies of the λ-planes which are cut along the intervals $(\lambda_{2j-1},\lambda_{2j})$, $j=1,\cdots,g+1$. On this surface S we take a system of canonical cuts α_j,β_j, $j=1,\cdots,g$. For α_j we take a closed contour which starts at λ_2, goes on the upper sheet as far as λ_{2j+1}, crosses to the lower sheet and ends at λ_2. For β_j we take a closed contour which surrounds the cut $(\lambda_{2j+1},\lambda_{2j+2})$, $j=1,\cdots,g$ on the upper sheet. For the point $\lambda\in S$ the corresponding point on the other sheet is denoted by λ'. We mean by the upper sheet the sheet on which $R(\lambda)^{1/2}$ is positive on (λ_{2g+2},∞).

Bloch eigenfunctions $x_+(k)$ and $x_-(k)$ can be regarded as the branch of the single-valued function $x(k)$ on S. For sufficiently large λ, we have

$$x_+(k)=y(k)-z(N)^{-1}a_{N-1}z(k)z(N-1)+\cdots,$$
$$x_-(k)=z(N)^{-1}z(k,N-k)+\cdots$$

up to the terms of lower order in λ. Therefore by the above expressions and (2·12), $x(k)$ is a meromorphic function on S, has simple zeros at $\mu_j(k)$, has simple poles at $\mu_j(0)$, has zero of k-th order at ∞' (on the lower sheet) and has pole of k-th order at ∞ (on the upper sheet), where $\lambda_{2j}\leq\mu_j(0)$, $\mu_j(k)\leq\lambda_{2j+1}$, $j=1,\cdots,g$.

We want to know the dependence of $\mu_j(k)$ on k. For this purpose we use the idea of Akhiezer.[12),13)] We denote a base of the abelian differentials[17),18)] of the first kind by

$$\omega_m=\sum_{j=0}^{g-1}c_{mj}\lambda^jR(\lambda)^{-1/2}d\lambda, \qquad m=1,\cdots,g,$$

normalized by

$$\int_{\beta_j}\omega_m=-\pi i\delta_{jm}\,,\qquad j,\ m=1,\cdots,g\,.$$

Put

$$\int_{\alpha_j}\omega_m=t_{jm}\,,\qquad\qquad j,\ m=1,\cdots,g\,.$$

Then $c_{m,j}$ are real and the matrix (t_{jm}) is a real symmetric negative-definite matrix.

Next we take the abelian differential of the third kind whose residues at $\mu_j(k)$ and $\mu_j(0)$ are 1 and -1 respectively:

$$\omega(\mu_j(k),\mu_j(0))=2^{-1}[(\lambda-\mu_j(k))^{-1}\{R(\lambda)^{1/2}+R(\mu_j(k))^{1/2}\}$$
$$-(\lambda-\mu_j(0))^{-1}\{R(\lambda)^{1/2}+R(\mu_j(0))^{1/2}\}+p_j(\lambda)]R(\lambda)^{-1/2}d\lambda\,,$$

where $p_j(\lambda)$ is a certain polynomial of at most $(g-1)$-th degree. Finally we take the abelian differential of the third kind whose residues at ∞ and ∞' are -1 and 1 respectively:

$$\omega(\infty,\infty')=(\lambda^g+p(\lambda))R(\lambda)^{-1/2}d\lambda\,,$$

where $p(\lambda)$ is a certain polynomial of at most $(g-1)$-th degree. These differentials are normalized so that all of β_j periods are zero. These conditions determine $p_j(\lambda)$ and $p(\lambda)$ uniquely.

Consider

$$\omega(k)=x(k)^{-1}(dx(k)/d\lambda)\,d\lambda\,.$$

This is the abelian differential which has poles at $\mu_j(k)$, $\mu_j(0)$, ∞,∞' and has residues 1, -1, $-k, k$ respectively. Then $\omega(k)$ is expressed as

$$\omega(k)=k\omega(\infty,\infty')+\sum_{j=1}^{g}\omega(\mu_j(k),\mu_j(0))+\sum_{j=1}^{g}c_j\omega_j\,,$$

c_j being some complex numbers. Since $x(k)$ is single-valued on S the relations

$$\int_{\alpha_l}\omega(k)=2\pi m_l i\,,\qquad \int_{\beta_l}\omega(k)=2\pi n_l i\,,\qquad l=1,\cdots,g$$

must hold for some integers m_l, n_l. Then from the second relation we have $c_j=-2n_j$, and from the first relation

$$k\int_{\alpha_l}\omega(\infty,\infty')+\sum_{j=1}^{g}\int_{\alpha_l}\omega(\mu_j(k),\mu_j(0))-2\sum_{j=1}^{g}n_j\int_{\alpha_l}\omega_j=2\pi m_l i\,.$$

Using the relations among the periods of the normalized differentials, we have

$$-2k\int_{\infty'}^{\infty}\omega_l+2\sum_{j=1}^{g}\int_{\mu_j(0)}^{\mu_j(k)}\omega_l-2\sum_{j=1}^{g}n_jt_{lj}=2\pi m_l i\,.$$

We rewrite this as

$$\sum_{j=1}^{g}\int_{\mu_0}^{\mu_j(k)}\omega_l=k\int_{\infty'}^{\infty}\omega_l+\sum_{j=1}^{g}\int_{\mu_0}^{\mu_j(0)}\omega_l+\sum_{j=1}^{g}n_jt_{lj}+m_l\pi i\,, \tag{3·1}$$

where μ_0 is a fixed point on S.

We introduce Riemann theta function defined by

$$\theta(u)=\sum_{m_1,\cdots,m_g=-\infty}^{\infty}\exp[2\sum_{j=1}^{g}m_ju_j+\sum_{j,k=1}^{g}t_{jk}m_jm_k],$$
$$u=(u_1,\cdots,u_g)\in\boldsymbol{C}^g\,.$$

Solution of Jacobi's inversion problem[17),18)] permits us to express symmetric polynomials of $\mu_j(k)$ by the right-hand side of (3·1) as rational function of theta functions. Following Its-Matveev[11)] we write first of them as

$$\sum_{j=1}^{g}\mu_j(k)=\pi^{-1}i\sum_{j=1}^{g}\int_{\beta_j}\lambda\omega_j+\sum_{j=1}^{g}c_{j,g-1}D_j\log\theta(u(\infty)+kc+d)$$
$$-\sum_{j=1}^{g}c_{j,g-1}D_j\log\theta(u(\infty)+(k+1)c+d), \tag{3·2}$$

where

$$u(\lambda)=\left(\int_{\mu_0}^{\lambda}\omega_1,\cdots,\int_{\mu_0}^{\lambda}\omega_g\right),$$
$$c=\left(\int_{\infty}^{\infty'}\omega_1,\cdots,\int_{\infty}^{\infty'}\omega_g\right),$$
$$d=(d_1,\cdots,d_g),$$
$$d_j=-\sum_{k=1}^{g}\int_{\mu_0}^{\mu_k(0)}\omega_j+2^{-1}j\pi i-2^{-1}\sum_{k=1}^{g}t_{kj}\,,$$

and D_j denotes the partial differentiation with respect to the j-th variable.

By (2·8) and (2·10), we have

$$b_k=\Lambda_1^*-\sum_{j=1}^{g}\mu_j(k),$$
$$\Lambda_1^*=2^{-1}\sum_{j=1}^{2g+2}\lambda_j\,. \tag{3·3}$$

With (3·2), this expresses b_k by Riemann theta functions.

By (2·9) and (2·11), $a_{k-1}^2+a_k^2$ and, if N (the period) is odd, even a_k^2 can be expressed (in a complicated way) by theta functions. Because we do not need these formulas for a_k^2 to integrate Toda lattice, we do not write them.

§ 4. Integration of the periodic Toda lattice

The equation of motion of Toda lattice has the form

$$\dot{Q}_n=P_n\,,\qquad \dot{P}_n=-\{\exp(-(Q_{n+1}-Q_n))-\exp(-(Q_n-Q_{n-1}))\},$$

where the dot denotes the differentiation with respect to the time variable t.

By putting

$$a_n = 2^{-1}\exp\{-(Q_n - Q_{n-1})/2\}, \qquad b_n = -2^{-1}P_{n-1},$$

these equations take the following forms:

$$\dot{a}_n = a_n(b_{n+1} - b_n),$$
$$\dot{b}_n = 2(a_n{}^2 - a_{n-1}^2).$$

These equations are equivalent to the evolution equation of linear operators[4)]

$$\dot{L} = [B, L] = BL - LB,$$

where

$$(Bu)(n) = a_n u(n+1) - a_{n-1}u(n-1).$$

Uisng these expressions we have

$$\dot{y}(n) = a_n y(n+1) + (b_0 - \lambda)y(n) - a_{n-1}y(n-1) + 2a_{N-1}^2 z(n),$$
$$\dot{z}(n) = -2y(n) + a_n z(n+1) + (\lambda - b_0)z(n) - a_{n-1}z(n-1). \qquad (4\cdot1)$$

From these formulas, we have

$$\dot{\Delta}(\lambda) = 0,$$

i.e., λ_j are independent of t. So the Riemann surface and the normalized differentials on it introduced in § 3 are also independent of t (namely, they are determined by the initial conditions). In the construction of § 3 dependency on t comes only through $\mu_j(0)$. We also note that

$$A = \prod_{j=0}^{N-1} a_j$$

is independent of t.

From (4·1) we have

$$\dot{z}(N) = -2y(N) - 2(b_0 - \lambda)z(N) - 2a_{N-1}z(N-1). \qquad (4\cdot2)$$

Differentiating the relation

$$z(N) = A^{-1}\sum_{j=1}^{N-1}(\lambda - \mu_j(0))$$

with respect to t, we have

$$\dot{z}(N) = -A^{-1}\sum_{j=1}^{g}\dot{\mu}_j(0)\prod_{k=1,\neq j}^{N-1}(\lambda - \mu_k(0)).$$

Putting $\lambda = \mu_j(0)$ $(j = 1, \cdots, g)$ in (4·2), we have

$$-A^{-1}\dot{\mu}_j(0)\prod_{k=1,\neq j}^{N-1}(\mu_j(0) - \mu_k(0)) = -2\{y(N) + a_{N-1}z(N-1)\}|_{\lambda=\mu_j(0)}.$$

If we consider $\mu_j(0)$ not as a number but as the point on S where Bloch eigenfunction

$$x(n)=y(n)+\frac{-\{a_{N-1}z(N-1)+y(N)\}+(\Delta(\lambda)^2-4)^{1/2}}{2z(N)}z(n)$$

has pole, it does not have pole at $\mu_j(0)'$. Therefore we have

$$\begin{aligned}-\{a_{N-1}z(N-1)+y(N)\}|_{\lambda=\mu_j(0)}&=-\{\Delta(\mu_j(0)')^2-4\}^{1/2}\\&=\{\Delta(\mu_j(0))^2-4\}^{1/2}.\end{aligned}$$

Thus we have

$$\dot{\mu}_j(0)=-2R(\mu_j(0))^{1/2}\prod_{k=1,\neq j}^{g}(\mu_j(0)-\mu_k(0))^{-1}. \tag{4·3}$$

The corresponding differential equations for the case of the KdV equation have been derived by Marchenko,[19] Dubrovin-Novikov[10] and Its-Matveev.[11]

Put

$$\xi_n(t)=\sum_{j=1}^{g}\int_{\mu_0}^{\mu_j(0,t)}\omega_n.$$

Differentiating with respect to t, we have

$$\dot{\xi}_n(t)=\sum_{j=1}^{g}\dot{\mu}_j(0,t)R(\mu_j(0,t))^{-1/2}\sum_{l=0}^{g-1}c_{nl}\mu_j^{\,l}(0,t).$$

Inserting (4·3), we have

$$\dot{\xi}_n(t)=-2\sum_{j=1}^{g}\prod_{k=1,\neq j}^{g}(\mu_j(0,t)-\mu_k(0,t))^{-1}\sum_{l=0}^{g-1}c_{nl}\mu_j^{\,l}(0,t).$$

The right-hand side is simplified to $-2c_{n,g-1}$ by the Lagrange interpolation formula. Therefore in (3·2), d_j is replaced by

$$d_j(t)=d_j(0)+2c_{j,g-1}t.$$

By (3·2) and (3·3), we have

$$b_n(t)=A_1{}^*-\pi^{-1}i\sum_{j=1}^{g}\int_{\beta_j}\lambda\omega_j-2^{-1}(d/dt)\log\frac{\theta(u(\infty)+nc+d(t))}{\theta(u(\infty)+(n+1)c+d(t))},$$

the right-hand side being determined by the initial conditions. Formulas for $a_n(t)$, $P_n(t)$ and $Q_n(t)$ are direct consequences of this formula.

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Note added in proof: The authors became aware that Manakov[20] also obtained results similar to those of Ref. 4).