Analogues of Classical Limit Theorems for the Supercritical Galton–Watson Process with Immigration

C. C. HEYDE AND E. SENETA
Australian National University
Canberra, Australia

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ABSTRACT

In [6] it was shown that under standard conditions $X_n/m^n \to V > 0$ with probability one, where $\{X_n\}$ is a Galton–Watson process with immigration, and offspring mean $m > 1$. The authors obtain convergence-rate results under additional conditions for this asymptotic behavior in the form of analogues of the central limit theorem and the law of the iterated logarithm, and similar results for the estimator X_{n+r}/X_n, which for fixed $r \geqslant 1$ approaches m^r with probability one.

INTRODUCTION

The classical central limit theorem and the law of the iterated logarithm may be regarded as convergence-rate results for the strong law of large numbers. Recently, one of the authors has adopted this interpretation with regard to the well-known relation $Z_n/m^n \overset{\text{a.s.}}{\to} W$ for the ordinary Galton–Watson process $\{Z_n\}$ with offspring mean $m > 1$, and proved analogous convergence-rate theorems for this strong convergence result [1, 2].

A similar strong convergence result is known for the supercritical Galton–Watson process with immigration $\{X_n\}$; this has been derived simply in an earlier article in this journal [6], to which we refer the reader for a detailed description, and whose notation and assumptions we adopt. (The result may also be eventually deduced from a general limit theorem on an ordinary decomposable multitype Galton–Watson process, namely, Theorem 2.1 of Kesten and Stigum [4].) Thus, our basic assumptions are that the process is initiated by a single ancestor; and that the offspring and immigration distributions $\{f_j\}$, $\{b_j\}$, generated respectively by the probability generating functions (pgf's) $F(s)$, $B(s)$, $s \in [0, 1]$ satisfy:

$$1 < F'(1-) \equiv m < \infty;\ f_j \neq 1,\ j \geqslant 0;\ 0 < \lambda \equiv B'(1-) < \infty.$$

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The theorem of [6] then implies that if $\Sigma\, j(\log j)f_j < \infty$, $\{X_n/m^n\}$ converges almost surely to a proper random variable V with finite mean EV, and such that $P(V = 0) = 0$. One of the aims of this article is to indicate that for the process with immigration also analogous convergence-rate results hold, under further conditions, to be introduced as required.

We note also that the foregoing convergence result implies that for any fixed integer $r \geqslant 1$, $X_{n+r}/X_n \to m^r$ with probability one. It will be seen from the representation (2), which follows, that this may be regarded as an analogue of the strong law of large numbers for the estimator X_{n+r}/X_n of m^r. We give a parallel set of convergence-rate results for this relationship also, which may have some statistical utility.

The overall treatment is relatively brief, because much of the approach may be taken over directly from that for the ordinary supercritical Galton–Watson case; in fact, the present situation is intrinsically simpler in that $X_n \overset{\text{a.s.}}{\to} \infty$, a property not shared in general by the ordinary process. Indeed, the main difficulty is to obtain for the present process an analogue of the Berry–Esseen bound on the deviation from normality, which is needed to prove the analogue of the iterated logarithm law. This is done via Lemma 3.1, which may be of some independent interest, and two further lemmas.

DESCRIPTION OF THE RESULTS

The keys to the limit results are representations for $X_n - m^n V$ and $X_{n+r} - m^r X_n$ as sums of independent random variables.

In the first case, we have

$$X_n - m^n V = \lim_{r\to\infty}\left(X_n - m^n \frac{X_{n+r}}{m^{n+r}}\right) \quad \text{a.s.} \tag{1}$$

Furthermore,

$$X_{n+r} = Z_r^{(1)} + \cdots + Z_r^{(X_n)} + Y_{r,n} \tag{2}$$

where

$$Y_{r,n} = U_{n+r}^{(n+1)} + \cdots + U_{n+r}^{(n+r)}.$$

Here $Z_r^{(i)}$, $i = 1, 2, \ldots, X_n$ each have the distribution of Z_r, where $\{Z_n\}$ is an ordinary Galton–Watson process generated by $F(s)$; and $U_{n+r}^{(i)}$ is the number of direct descendants in the $(n + r)$th generation from immigration at the ith. Further, the $Z_r^{(i)}$, $i = 1, 2, \ldots, X_n$ and $U_{n+r}^{(i)}$, $i = n + 1, \ldots, n + r$ are all independent, and $Y_{r,n}$ has the same distribution for each n. Moreover, it has been shown in [6] that, for fixed n, $\lim_{r\to\infty} m^{-r} Y_{r,n} = I^{(n)}$ a.s. where $I^{(n)}$ has the distribution of $I = V - W$. From (1) and (2)

$$
\begin{aligned}
X_n - m^n V &= \lim_{r\to\infty}\left[\left(1 - \frac{Z_r^{(1)}}{m^r}\right) + \cdots + \left(1 - \frac{Z_r^{(X_n)}}{m^r}\right) - \frac{Y_{r,n}}{m^r}\right] \\
&= (1 - W^{(1)}) + \cdots + (1 - W^{(X_n)}) - I^{(n)} \\
&= T_1 + \cdots + T_{X_n} - I^{(n)} \quad \text{a.s.,} \qquad (3)
\end{aligned}
$$

(say) where the components on the right-hand side are all independent and are each distributed independently of X_n.

The representation for $X_{n+r} - m^r X_n$ follows simply from (2):

$$X_{n+r} - m^r X_n = (Z_r^{(1)} - m^r) + \cdots + (Z_r^{(X_n)} - m^r) + Y_{r,n} \qquad (4)$$

where again the components on the right-hand side are all independent, and are independent of X_n; and $Y_{r,n}$ has the same distribution for each n.

Making use of (3) and (4) together with an adaptation of the techniques of [2] and [3] yields the following results.

THEOREM 1. *Let* $\operatorname{var} Z_1 = \sigma^2 < \infty$. *Then*

$$\lim_{n\to\infty} P((m^2 - m)^{1/2}\sigma^{-1}X_n^{-1/2}(m^n V - X_n) \leqslant x \mid X_n > 0) = \Phi(x)$$

and

$$\lim_{n\to\infty} P(\sigma_r^{-1}X_n^{-1/2}(X_{n+r} - m^r X_n) \leqslant x \mid X_n > 0) = \Phi(x)$$

where $\Phi(x)$ *is the distribution function of* $N(0, 1)$ *and*

$$\sigma_r^2 = \operatorname{var} Z_r = \sigma^2 m^r(m^r - 1)(m^2 - m)^{-1}.$$

THEOREM 2. *Let* $EZ_1^3 < \infty$. *Then there exists a* δ, $0 < \delta < 1$, *such that*

$$\sup_x |P((m^2 - m)^{1/2}\sigma^{-1}X_n^{-1/2}(m^n V - X_n) \leqslant x \mid X_n > 0) - \Phi(x)| = 0(\delta^n)$$

and

$$\sup_x |P(\sigma_r^{-1}X_n^{-1/2}(X_{n+r} - m^r X_n) \leqslant x \mid X_n > 0) - \Phi(x)| = 0(\delta^n)$$

as $n \to \infty$.

(Explicit forms for the remainders may be found from Lemmas 2 and 3 of the next section.)

THEOREM 3. *Let* $EZ_1^3 < \infty$. *Then with probability one*

$$\limsup_{n\to\infty} \frac{m^n V - X_n}{(2\sigma^2(m^2 - m)^{-1}X_n \log n)^{1/2}} = 1,$$

$$\liminf_{n\to\infty} \frac{m^n V - X_n}{(2\sigma^2(m^2 - m)^{-1}X_n \log n)^{1/2}} = -1,$$

$$\limsup_{n\to\infty} \frac{X_{n+r} - m^r X_n}{(2\sigma_r^2 X_n \log n)^{1/2}} = 1, \qquad \liminf_{n\to\infty} \frac{X_{n+r} - m^r X_n}{(2\sigma_r^2 X_n \log n)^{1/2}} = -1.$$

PROOFS

Proof of Theorem 1. Theorem 1 is very easily obtained from the representations (3) and (4). Since $X_n \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$, we certainly have $X_n^{-1/2} I^{(n)}$ and $X_n^{-1/2} Y_{r,n}$ converging in probability to zero on $\{X_n > 0\}$ as $n \to \infty$. Then, noting that X_n is independent of the summands in each case, an appeal to the classical central limit theorem yields the result, since from ordinary Galton–Watson theory the remaining (identically and independently distributed) summands have zero mean. ∎

In order to prove Theorem 2 we need first to extend the lemma in Section 4 of [3] so that the troublesome components $I^{(n)}$ and $Y_{r,n}$ in the representations (3) and (4) can be accounted for.

LEMMA 1. *Let ξ_i, $i = 1, 2, \ldots$, be independent and identically distributed random variables with $E\xi_1 = 0$, $\operatorname{var} \xi_1 = \alpha^2$, and $E|\xi_1|^3 < \infty$. Let η_n, with $E|\eta_n| < \infty$, be a random variable independent of $\{\xi_i\}$ and with the same distribution for each n; and N_n be a positive integer-valued random variable that is independent of the $\{\xi_i\}$ and η_n. Then for any sequence $\{\varepsilon_n\}$ of positive constants with $\varepsilon_n \to 0$ as $n \to \infty$*

$$\sup_x |P(\alpha^{-1} N_n^{-1/2}(\xi_1 + \cdots + \xi_{N_n} + \eta_n) \leqslant x) - \Phi(x)|$$
$$\leqslant CE(N_n^{-1/2}) + \alpha^{-1}\varepsilon_n^{-1} E|\eta_n| E(N_n^{-1/2}) + \frac{\varepsilon_n}{2}$$

where $C = K\alpha^{-3}E|\xi_1|^3$, K being the universal constant in the Berry–Esseen bound.

Proof. Using the Berry–Esseen bound and since η_n is independent of $\{\xi_i\}$,

$$\begin{aligned}
&-CE(N_n^{-1/2}) \\
&= -C \sum_{j=1}^{\infty} j^{-1/2} P(N_n = j) \\
&= -C \sum_{j=1}^{\infty} j^{-1/2} \left[\int_{-\infty}^{\infty} dP(\alpha^{-1} j^{-1/2} \eta_n \leqslant y)\right] P(N_n = j) \\
&\leqslant \sum_{j=1}^{\infty} \left[\int_{-\infty}^{\infty} \{P(\alpha^{-1} j^{-1/2}(\xi_1 + \cdots + \xi_j) \leqslant x - y) - \Phi(x - y)\}\right. \\
&\qquad\qquad \left. dP(\alpha^{-1} j^{-1/2} \eta_n \leqslant y)\right] P(N_n = j) \\
&\leqslant CE(N_n^{-1/2}).
\end{aligned}$$

But,

$$\sum_{j=1}^{\infty} \left[\int_{-\infty}^{\infty} \{P(\alpha^{-1} j^{-1/2}(\xi_1 + \cdots + \xi_j) \leqslant x - y) - \Phi(x - y)\}\right.$$
$$\left. dP(\alpha^{-1} j^{-1/2} \eta_n \leqslant y)\right] P(N_n = j)$$

$$= \sum_{j=1}^{\infty} [P(\alpha^{-1}j^{-1/2}(\xi_1 + \cdots + \xi_j + \eta_n) \leqslant x) \\ -P(\zeta + \alpha^{-1}j^{-1/2}\eta_n \leqslant x)]P(N_n = j)$$

$$= \sum_{j=1}^{\infty} [P(\alpha^{-1}j^{-1/2}(\xi_1 + \cdots + \xi_j + \eta_n) \leqslant x | N_n = j) \\ -P(\zeta + \alpha^{-1}j^{-1/2}\eta_n \leqslant x | N_n = j)]P(N_n = j)$$

$$= P(\alpha^{-1}N_n^{-1/2}(\xi_1 + \cdots + \xi_{N_n} + \eta_n) \leqslant x) - P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x)$$

where ζ has an $N(0, 1)$ distribution and is independent of η_n, so that

$$\sup_x |P(\alpha^{-1}N_n^{-1/2}(\xi_1 + \cdots + \xi_{N_n} + \eta_n) \leqslant x) \\ -P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x)| \leqslant CE(N_n^{-1/2}). \quad (5)$$

To this point, the argument is a straightforward generalization of the lemma of [3].

Now, for any $\varepsilon_n > 0$,

$$P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x) \\ = P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x, \alpha^{-1}N_n^{-1/2}|\eta_n| \leqslant \varepsilon_n) \\ +P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x, \alpha^{-1}N_n^{-1/2}|\eta_n| > \varepsilon_n) \\ \leqslant P(\zeta \leqslant x + \varepsilon_n) + P(\alpha^{-1}N_n^{-1/2}|\eta_n| > \varepsilon_n), \quad (6a)$$

and similarly

$$P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n > x) \leqslant P(\zeta > x - \varepsilon_n) + P(\alpha^{-1}N_n^{-1/2}|\eta_n| > \varepsilon_n),$$

or equivalently,

$$P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x) \geqslant P(\zeta \leqslant x - \varepsilon_n) - P(\alpha^{-1}N_n^{-1/2}|\eta_n| > \varepsilon_n). \quad (6b)$$

Also, it is readily checked that

$$\sup_x |P(\zeta \leqslant x - \varepsilon_n) - P(\zeta \leqslant x)| < \frac{\varepsilon_n}{2} \quad (7a)$$

from, for example, the mean value theorem, whence

$$\sup_x |P(\zeta \leqslant x) - P(\zeta \leqslant x + \varepsilon_n)| < \frac{\varepsilon_n}{2}. \quad (7b)$$

Using the double inequality for $P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x)$ provided by (6a) and (6b) in conjunction with (7a) and (7b)

$$\sup_x |P(\zeta + \alpha^{-1}N_n^{-1/2}\eta_n \leqslant x) - \Phi(x)| \\ \leqslant P(\alpha^{-1}N_n^{-1/2}|\eta_n| > \varepsilon_n) + \frac{\varepsilon_n}{2} \\ \leqslant \alpha^{-1}\varepsilon_n^{-1}E(N_n^{-1/2}|\eta_n|) + \frac{\varepsilon_n}{2}$$

by Markov's inequality;

$$= \alpha^{-1}\varepsilon_n^{-1}E(N_n^{-1/2})E|\eta_n| + \frac{\varepsilon_n}{2} \tag{8}$$

by the independence of η_n and N_n. The result of the lemma then follows from (5) and (8). ■

LEMMA 2. *Let $EZ_1^3 < \infty$. Then for any sequence $\{\varepsilon_n\}$ of positive constants such that $\varepsilon_n \to 0$ as $n \to \infty$*

$$\sup_x |P((m^2 - m)^{1/2}\sigma^{-1}X_n^{-1/2}(m^nV - X_n) \leqslant x|X_n > 0) - \Phi(x)|$$
$$\leqslant K\sigma^{-3}(m^2 - m)^{3/2}E|W - 1|^3E(X_n^{-1/2}|X_n > 0)$$
$$+\sigma^{-1}(m^2 - m)^{1/2}\varepsilon_n^{-1}E(I)E(X_n^{-1/2}|X_n > 0) + \frac{\varepsilon_n}{2} \tag{9}$$

and

$$\sup_x |P(\sigma_r^{-1}X_n^{-1/2}(X_{n+r} - m^rX_n) \leqslant x|X_n > 0) - \Phi(x)|$$
$$\leqslant K\sigma_r^{-3}E|Z_r - m^r|^3E(X_n^{-1/2}|X_n > 0)$$
$$+\sigma_r^{-1}\varepsilon_n^{-1}E(Y_{r,1})E(X_n^{-1/2}|X_n > 0) + \frac{\varepsilon_n}{2} \tag{10}$$

where $K(<0.82)$ is the universal constant in the Berry–Esseen bound.

Proof. The inequalities (9) and (10) follow respectively from the representations (3) and (4), and application of Lemma 1, having noted that all expectations occurring on the right-hand sides of (9) and (10) are finite on account of the totality of conditions being assumed to be satisfied. ■

LEMMA 3. *Let γ be any positive number satisfying*

$$1 > \gamma^2 > \max(B(q), m^{-1}).$$

Then

$$E(X_n^{-1}|X_n > 0) = 0(\gamma^n) \tag{11}$$

as $n \to \infty$. (An explicit, of somewhat inconvenient, form for the right-hand side may be deduced from the body of the proof.)

Proof. Let us denote by $P_n(x)$ the pgf of X_n, and write

$$P_n(x) = F_n(x)\tilde{P}_n(x),$$

$x \in [0, 1]$, where from [6], $\tilde{P}_n(x) \equiv \Pi_{j=0}^{n-1} B\{F_j(x)\}$, $n \geqslant 1$, denotes the pgf of the contribution to X_n resulting from immigration since generation 1. It is easily checked that

$$D_n \equiv E[X_n^{-1}|X_n > 0] = \int_0^1 \frac{P_n(x) - P_n(0)}{x\{1 - P_n(0)\}}\, dx,$$

so that we proceed to the estimation of the right-hand integral. *We assume initially that* $q > 0$, where q is the unique root in $[0, 1)$ of the equation $F(x) = x$, and decompose the integral into components.

$$\begin{aligned} D_n &= \int_0^q \frac{P_n(x) - P_n(0)}{x\{1 - P_n(0)\}}\,dx + \int_q^1 \frac{P_n(x) - P_n(0)}{x\{1 - P_n(0)\}}\,dx \\ &= D_n(1) \qquad\qquad\qquad\quad + D_n(2), \quad \text{say.} \end{aligned}$$

To estimate the first integral we note that

$$\begin{aligned} D_n(1) &= \{1 - P_n(0)\}^{-1} \int_0^q \left(\sum_{j=1}^{\infty} P(X_n = j)x^{j-1}\right) dx \\ &= \{1 - P_n(0)\}^{-1} \sum_{j=1}^{\infty} j^{-1} P(X_n = j) q^j \\ &\leqslant \{1 - P_n(0)\}^{-1} P_n(q) \\ &= \{1 - P_n(0)\}^{-1} q (B(q))^n \\ &\leqslant C_1 (B(q))^n \end{aligned} \tag{12}$$

(since $P_n(0) \to 0$ as $n \to \infty$), where C_1 is a positive constant.

To estimate $D_n(2)$ we first note that

$$D_n(2) \leqslant C_2(1 - q)^{-1} \int_q^1 P_n(x)\,dx$$

where C_2 is a constant. If we now make the change of variable $x = q + (1 - q)y$, we get

$$\begin{aligned} D_n(2) &\leqslant C_2 \int_0^1 P_n(q + (1 - q)y)\,dy \\ &\leqslant C_2 \int_0^1 \tilde{P}_n(q + (1 - q)y)\,dy. \end{aligned}$$

Now, for $j \geqslant 0$

$$B\{F_j(q + (1 - q)y)\} = B^*\{F_j^*(y)\}$$

where

$$B^*(y) = B(q + (1 - q)y) \quad \text{and} \quad F_j^*(y) = \frac{F_j(q + (1 - q)y) - q}{1 - q},$$

$F_j^*(y)$ being easily seen to be the jth functional iterate of $F_1^*(y)$, which is a pgf with mean m, and $F_1^*(0) = 0$. Thus, for $n \geqslant 1$

$$\tilde{P}_n(q+(1-q)y) \equiv \prod_{j=0}^{n-1} B^*\{F_j^*(y)\} = P_n^*(y),$$

say, where $P_n^*(y)$ may itself be construed as the pgf of the nth generation of a *supercritical* Galton–Watson process with immigration, with offspring pgf $F_1^*(y)$ and immigration pgf $B^*(y)$, and no initial ancestor. Thus

$$P_n^*(y) = P_k^*(y)P_{n-k}^*(F_k^*(y)) \tag{13}$$

for any integer k, $1 \leqslant k \leqslant n$.

Let us note at this stage, for further use, that for any fixed $y_0 \in (0, 1)$,

$$P_k^*(y_0) \leqslant \{B^*(y_0)\}^k \tag{14}$$

since in the present situation $F_1^*(y_0) < y_0$. Also, if we let $H_n(y)$, $y \in [0, 1]$, denote the inverse function of $F_n^*(y)$, $y \in [0, 1]$, then it is easily checked that $H_n(y)$ is the nth functional iterate of $H_1(y)$, and that $H_n(y_0) \uparrow 1$ as $n \to \infty$. In fact, since by the mean value theorem

$$1 - H_n(y_0) = H_1'(\xi_n)\{1 - H_{n-1}(y_0)\}$$

where $H_{n-1}(y_0) < \xi_n < 1$, and since $H'_1(y) \downarrow m^{-1}$ as $y \uparrow 1$, it follows that for any fixed $\varepsilon > 0$ satisfying $m^{-1} + \varepsilon < 1$, there exists a fixed $r_0 \equiv r_0(y_0, \varepsilon)$ such that

$$1 - H_n(y_0) \leqslant (m^{-1} + \varepsilon)^n\{(m^{-1} + \varepsilon)^{-r_0}(1 - H_{r_0}(y_0))\}$$

for $n \geqslant r_0$.

Then

$$D_n(2) \leqslant C_2 \int_0^1 P_n^*(y)\, dy \tag{15}$$

$$= C_2 \left\{\int_0^{H_k(y_0)} P_n^*(y)\, dy + \int_{H_k(y_0)}^1 P_n^*(y)\, dy\right\}$$

$$\leqslant C_2\{H_k(y_0)P_k^*(H_k(y_0))P_{n-k}^*(y_0) + 1 - H_k(y_0)\}$$

providing $1 \leqslant k \leqslant n$, by dominating by the maximum value of the integrand in each case, and in the first part using (13);

$$\leqslant C_2\{P_{n-k}^*(y_0) + 1 - H_k(y_0)\}$$

$$\leqslant C_2\{(B^*(y_0))^{n-k} + 1 - H_k(y_0)\}$$

on account of (14).

If we now choose $k = [n/2]$ and choose n so large that $[n/2] \geqslant r_0$,

$$\leqslant C_2\{(B^*(y_0))^{n/2} + \rho(\varepsilon, y_0)(m^{-1} + \varepsilon)^{n/2}\} \tag{16}$$

where $\rho > 0$ does not depend on n.

Since initially both y_0 and ε could have been chosen as close to zero as desired and $B^*(y_0) = B(q + (1 - q)y_0)$, the assertion of the lemma follows from (12) and (16).

It remains only to estimate $D_n \equiv D_n(2)$ when $q = 0$. We have then that $P_n(x) = F_n(x)\tilde{P}_n(x) = F_n(x)P_n^*(x)$, with $P_n(0) = F_n(0) = 0$, $F_n(x) < x$. Thus

$$D_n = \int_0^1 \frac{P_n(x)}{x}\,dx \leqslant \int_0^1 P_n^*(x)\,dx \tag{17}$$

whence we may proceed as before from (15). ∎

NOTES. (a) The technique for estimating the rate of geometric convergence was suggested by a method of A. V. Nagaev [5] in a similar situation in connection with the ordinary supercritical Galton–Watson process. (There appear to be some imprecise steps in his development.)

(b) The rate of convergence in (17) may also be estimated with the aid of the estimate in [3] for a similar ordinary supercritical Galton–Watson process.

(c) Other choices of k, for example, $k = [\delta n]$, for a δ fixed in (0, 1), would give similar results. The optimal choice of such k, giving optimal geometric rate at least within the context of the present proof, would appear to depend on the absolute sizes of the two values $B(q)$, $1/m$.

PROOF OF THEOREM 2. We first note that, by a standard moment inequality,

$$E(X_n^{-1/2} | X_n > 0) \leqslant E(X_n^{-1} | X_n > 0)^{1/2}.$$

The assertions of the theorem then follow from Lemmas 2 and 3 by putting $\varepsilon_n = \gamma^{n/4}$, in which case $\delta = \gamma^{1/4}$. ∎

PROOF OF THEOREM 3. This follows exactly the pattern of that of [2]. There are two main points of difference for which some modification is required:

(i) that $\sum_{n=1}^{\infty} P(A_{n+1} | X_n, \ldots, X_1) = \infty$ a.s. when

$$A_n = \{X_n - mX_{n-1} > (1 - \delta)\sigma(2X_{n-1}\log(n - 1))^{1/2}\}$$

for any δ, $0 < \delta < 1$;

(ii) that for any integer $r > 1$, $\{X_{rn}\}$, $n = 0, 1, 2, \ldots$, also describes a supercritical Galton–Watson process with immigration; and we shall confine ourselves to establishing the validity of these points.

(i) Using (4) we have, if $X_n > 0$,

$$P(A_{n+1} | X_n, \ldots, X_1)$$
$$= P((Z_1^{(1)} - m) + \cdots + (Z_1^{(X_n)} - m) + Y > (1 - \delta)\sigma(2X_n \log n)^{1/2} | X_n)$$

where Y has the immigration distribution, and the $Z_1^{(i)}$ are independent and identically distributed, each with the distribution of Z_1, and are independent of $X_1, \ldots, X_n$ and Y; and if $X_n = 0$

$$P(A_{n+1} | X_n, \ldots, X_1) = P(Y > 0).$$

In the former case

$$P((Z_1^{(1)} - m) + \cdots + (Z_1^{(X_n)} - m) + Y > (1 - \delta)\sigma(2X_n \log n)^{1/2} | X_n)$$
$$\geqslant P((Z_1^{(1)} - m) + \cdots + (Z_1^{(X_n)} - m) > (1 - \delta)\sigma(2X_n \log n)^{1/2} | X_n)$$

since $Y \geqslant 0$

$$\geqslant 1 - \Phi((1 - \delta)(2 \log n)^{1/2}) - CX_n^{-1/2}$$

using the Berry–Esseen bound. Now, with probability one, $X_n > 0$ for n sufficiently large, since $X_n \to \infty$ a.s. Hence the last inequality may be used to show divergence with probability one of $\Sigma_1^\infty P(A_{n+1} | X_n, \ldots, X_1)$ as in [2].

(ii) Consideration of the structure of the process reveals that indeed $\{X_{nr}\}$, $n \geqslant 0$, may be considered as a branching process with immigration, the offspring distribution being $F_r(x)$ and the immigration distribution $\Pi_{i=0}^{r-1} B\{F_i(x)\}$. This last may also be easily confirmed from the fact that X_{nr} has the pgf $P_{nr}(x)$ and from the fundamental recursion for the Galton–Watson process with immigration $\{X_n\}$

$$P_{nr}(x) = B(x)P_{nr-1}(F(x))$$

whence, iterating back,

$$P_{nr}(x) = B(x)B(F(x)) \cdots B(F_{r-1}(x))P_{(n-1)r}(F_r(x)). \blacksquare$$

SUPPLEMENTARY REMARKS

The results given above have been proved under conditions that are convenient, but frequently more restrictive than necessary, and we now briefly note how these can be relaxed.

Reference to [7] in conjunction with [6] reveals that Theorem 1 remains valid if the condition $0 < \lambda < \infty$ on the immigration distribution is replaced by the milder assumption that $\Sigma_{j=1}^\infty b_j \log j < \infty$ with $b_0 < 1$.

In Theorems 2 and 3 the condition $EZ_1^3 < \infty$ may be replaced by the weaker one that $EZ_1^{2+\kappa} < \infty$ where $0 < \kappa \leqslant 1$, through the application of generalized forms of Berry–Esseen bounds. In passing we also note the Lemma 3 remains valid without any auxiliary moment conditions at all (even $m = \infty$ is permissible). Finally, it may be possible to relax the condition $0 < \lambda < \infty$, which with the other conditions implies $EI < \infty$ and $EY_{r,1} < \infty$ in (9) and (10), and replace EI and $EY_{r,1}$ by something correspondingly weaker. The reader wishing to pursue this interaction may begin by consulting Lemma 2.2 of [9] in conjunction with the representation (9) of [6].

Finally, we note that it is not difficult to verify the validity of the second part of Theorem 1 even when $m = 1$, where $\sigma_r^2 = \text{var}\, Z_r = r\sigma^2$, since $P(X_n = k|X_n > 0) \to 0, k = 1, 2, \ldots$, as $n \to \infty$ (see [8]).

Note added in proof: Subsequent research by C. C. Heyde and J. R. Leslie has revealed that versions of Theorems 2 and 3 hold subject to only the condition $EZ_1^2 < \infty$ (to appear in *Bull. Austral. Math. Soc.*).

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