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# **Analysing the relationship between the problem solving-related beliefs, competence and teaching of three Cypriot primary teachers**

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## **Abstract**

In this paper we analyse the problem solving-related beliefs, competence and classroom practice of three Cypriot upper-primary teachers. Data derived from semi-structured interviews focused on teachers' beliefs about the nature of mathematical problems, problem solving, and their competence as both problem solvers and teachers of problem solving; clinical interviews during which teachers solved a context-free geometrical problem; and observations of a lesson during which teachers introduced the same geometrical problem to their grade 6 students. Analyses, which were structured by a framework derived from key problem solving literature, indicated firstly, that the framework was an effective tool, being sensitive to variation both within and across the data from the three teachers, and secondly that all three teachers, in largely explicable ways, exhibited both consistency and inconsistency in the ways in which their beliefs, competence and practice interacted. Some implications for further research are discussed.

## **1. Introduction**

Problem solving is an important mathematics education research domain, not least because it is both a key mathematical proficiency and an important means of developing other mathematical competencies (Ryve 2007). And yet, despite inclusion in most systems' curricular specifications (Skott 2009), it remains an ambiguous concept that is poorly integrated into teachers' professional repertoires (Silver, Ghouseini, Gosen, Charalambous & Strawhun 2005), due to their interpreting curricular tasks according to their professional capacities and beliefs about the nature of mathematics teaching and learning (Stein & Kaufman 2010). While much research has examined the relationship between teachers' beliefs and practice, little has framed such investigations within curricular expectations of problem solving, although the Flemish study reported by Depaepe, De Corte and Verschaffel (2010) is one example, and even less has incorporated teachers' problem solving competence. In this paper, mindful of the naivety of assuming a simple causal relationship between, say, beliefs and practice (Leatham 2006; Skott 2013), we extend research in this field by means of an analysis of the mathematical problem solving-related beliefs, competence and teaching of three Cypriot primary teachers. In so doing, we exploit a framework for analysis derived from many years of problem solving-related research undertaken by Schoenfeld in the United States and De Corte, Verschaffel and others in Flanders, and, as we explain below, an inclusive definition of beliefs. This has allowed us to make several contributions to the field. Firstly, we offer new perspectives on an old problem, not least through our inclusion of teacher problem solving competence. Secondly, we situate our analysis in a cultural context that is not well studied but which is subjected to a highly prescriptive curriculum model. Thirdly, our analytical framework derived from Schoenfeld's and De Corte and Verschaffel et al.'s work is shown to facilitate the analysis of teachers' problem solving-related beliefs, competence and practice. Fourthly, our use of a broad and inclusive definition has allowed us work with beliefs in a natural and simple to interpret manner. But first, we summarise our perspectives on beliefs in educational research.

## **2. Beliefs in mathematics education research**

Much beliefs-related research has been motivated by evidence that human beings “form and hold beliefs that serve their own needs, desires and goals” (Snow, Corno & Jackson 1996, p. 292). In particular, it is widely accepted that beliefs underpin learners’ responses to opportunities to learn mathematics (Callejo & Vila 2009; De Corte, Verschaffel & Op ’t Eynde 2000; Leder & Forgasz 2002; Schoenfeld 1983, 1985a). Thus, it is not surprising that researchers see the relationship between teachers’ beliefs and their needs, desires and goals as an important research domain. Early research in the field, such as Thompson’s (1984) case study of three middle school mathematics teachers, confirmed the significance of teachers’ beliefs in their professional decision making. Moreover, initial theoretical conceptualisations, like Ernest’s (1989) distinction between espoused and enacted beliefs, argue convincingly that one’s beliefs about an object - mathematics teaching - are likely to influence how one addresses that object.

However, some researchers, for example, Skott (2009, 2013), have rejected such models on the grounds of their being too deterministic. He writes, for example, that

“this line of research was and still is based on the assumption that teachers’ beliefs are a main obstacle to educational change, and that belief research may remedy what is generally referred to as the problems of implementation” (Skott 2013, p. 548).

While there may be elements of truth underpinning his argument, and Skott supports his position through appropriate citations, Ernest’s model does not presuppose any such determinism. Ernest’s perspective is that teachers make decisions, whether in the moment or in their planning, that are informed not only by what they believe they are trying to achieve but by the context in which they operate. This implies that an apparent dissonance between espoused and enacted beliefs should not be construed as a belief inconsistency, something for which Skott (2009, 2013) criticises other researchers, but a contingent response (Rowland & Zazkis 2013). Thus, in such contexts, the notion of belief enactment seems unproblematic; beliefs should not be construed as predictors but indicators of intentions to act.

When viewed in this light, concerns that beliefs research is located in definitional inconsistency (Mason 2004; Op ’t Eynde et al. 2002; Törner 2002) seem not insurmountable. For example, students’ mathematics-related beliefs have been defined as “the implicitly or explicitly held subjective conceptions students hold to be true, that influence their mathematical learning and problem solving” (Op ’t Eynde et al. 2002, p. 16) a definition that resonates closely with Richardson’s “psychologically held understandings, premises, or propositions about the world that are felt to be true” (Richardson 1996, p.103). Such definitions, which are deliberately broad, have a strong appeal, not least because they avoid the complexity, and therefore lack of operationalisability, of the more complex and overly prescriptive definitions of, say, Goldin (2002). Moreover, they do not try to distinguish between belief and knowledge, but suggest that beliefs, which are experientially formed, incorporate knowledge even though the latter, which require consensus, hold a higher epistemological warrant than the former. They also overcome concerns raised by Pajares (1992) pertaining to the evaluative nature of beliefs, the impermanent or episodic nature of beliefs, the various functions of belief, the relationship between beliefs, values and attitudes. They can accommodate Abelson’s (1986) distinction between testable - beliefs derived from immediate experience of an object - and distal beliefs - beliefs based on a remote experience of the same object, Ernest’s (1989) distinction between conscious and unconscious beliefs, and Skott’s (2013) concerns with respect to belief stability. This does not mean that such distinctions are not important, they are, but by taking an open stance we avoid tying ourselves

in definitional knots and are freed to work focus on what teachers say they believe and the extent to those beliefs are indicators of intentions to act.

### **3. Mathematical problems and problem solving**

In general, a mathematical problem presents an objective with no immediately obvious means of achievement (Pólya 1945; Nunokawa 2005). Significantly, problem complexity is not a function of the task but the individual solver's knowledge, experience and dispositions (Schoenfeld 1985b; Carlson & Bloom 2005). After all, multiplying two two-digit numbers is genuinely problematic for a typical five year-old but not for any moderately competent adult. A related issue concerns the context in which mathematical problems are found. Blum and Niss (1991) distinguish between problems embedded in mathematics itself and those located in some sense of a real world. That is, mathematical problems can be either purely mathematical or applied (Haylock & Cockburn 2008), a distinction we regard as both logical and functional, since the same task can exist in both domains according to the manner of its presentation. Applied mathematical problems, typically construed as located in some sense of a real-world and presented in *word* or *story* form, have attracted much research attention and represent the most common context in which children are expected to apply their mathematical knowledge (Briars & Larkin 1984; Chapman 2006). However, such problems are not necessarily straightforward, as they require not only the linguistic competence to decode the text but also the ability to extract relevant data, select and implement successfully appropriate operations before interpreting the outcomes against the original context (Nesher, Hershkovitz & Novotna 2003; Jitendra, Griffin, Deatline-Buchman & Sczesniak 2007). Interestingly, problems based purely in a world of mathematics are less well researched, although this paper, which is framed by such a problem, contributes to this debate.

Problem solving “is an activity requiring the individual to engage in a variety of cognitive actions, each of which requires some knowledge and skill, and some of which are not routine” (Cai & Lester 2005, p.221). Various frameworks have been proposed for analysing and describing the components of such activity, the best known of these being Pólya’s (1945) understand the problem, devise a plan, implement the plan, and reflect. Subsequent problem solving frameworks have typically been refinements of Pólya’s work; such as Mason, Burton and Stacey’s (1982) three-phase, Kapa’s (2001) six-phase or Nunokawa’s (2005) three-phase models. Others, such as the four component - orientation, organisation, execution and verification - model proposed by Garofalo and Lester (1985), have been explicitly meta-cognitive in their descriptions of the problem solving process. However, such frameworks have been erroneously construed as linear rather than cyclical (Kelly 2006; Nunokawa 1994), although all are sufficiently general to accommodate the distinction between mathematical and application problems (Haylock & Cockburn 2008).

### **4. Teachers’ problem solving-related beliefs**

While it is desirable that those who are expected to help students become effective problem solvers should be competent problem solvers themselves (Silver 1985), it is equally important that teachers are motivated to engage in such work (Leikin & Kawass 2005). This is a not insignificant issue as studies have shown that even in Finland, where PISA has reported consistently high ‘mathematics’ achievement, teachers seem unprepared to engage their students in mathematical problem solving (Ryve, Hemmi & Börjesson 2011). Typically, the literature indicates that research on teachers’ problem solving-related beliefs, whether pre-service or in-service, has been framed by the *traditional* versus *reform* dichotomy (see, for example, Handal & Herrington 2003; Anderson, Sullivan & White 2005; Chapman 1997). Moreover, there is much research on belief modification and practice as part of, typically,

teacher education programmes (for example, Higgins 1997; Leikin & Kawass 2005; Silver et al. 2005), which is not discussed here.

Interestingly, research concerning teachers' beliefs about the nature of mathematical problems indicates a phenomenon prone to substantial international variation. For example, Cai and Nie (2007) found significant differences in the beliefs about mathematical problems of Chinese and American teachers, typically deriving from their experiences as learners of mathematics. In similar vein, Andrews' (2007) interview study found Hungarian teachers describing mathematics as a subject based on intellectually challenging problems, which was a perspective largely unrecognised by his English informants, while others have identified substantial culturally constructed differences in the problem solving-related beliefs of pre-service primary teachers in Cyprus and England (Xenofontos & Andrews 2012, 2014).

These differences are not insignificant. In Israel, Arcavi and Friedlander (2007), found teacher beliefs not only consistent within the underlying culture but consistent with the view held by the mathematics education research community more broadly that mathematical

“problems and problem solving entail a situation for which students have some entrance knowledge or tools ... to approach and to solve it, and lack others. Implicit in these views is that the situation presented in the problem is understood and makes sense to students, and that what constitutes a problem at a certain point in time may not necessarily be a problem later on” (Arcavi & Friedlander 2007, p.358).

Such beliefs contrast starkly with those expressed in a case study of ten United States grade five teachers, who construed mathematical problems as the application of computational skills, with an emphasis on students' getting the correct answer (Ford 1994). In other words, teachers' beliefs about problem and problem solving appear to be underpinned by the cultures in which they live and cannot, *ab initio*, be assumed to resonate with the views held by the international mathematics education community.

Moreover, the reality of classroom life has a moderating effect on teachers' problem solving beliefs. For some, such a reality, particularly for novice teachers, may compromise the enactment of beliefs relating to the role and primacy of problem solving in mathematics classrooms (Andrews, Ryve, Hemmi & Sayers 2014; Cooney 1985; Drake 2002). For others, attempts to enact newly acquired beliefs about problems and problem solving may fail due to mismatches between the new and prior belief sets (Cohen 1990; Raymond 1997), while, for others, teachers' beliefs about problems and problem solving may force them to abandon as inadequate existing classroom practices in favour of more open pedagogies (Chapman 1997). In sum, teachers' problem solving beliefs are deeply contextual, not least because least because cultures create mental models of teaching (Strauss, Ravid, Magen & Berliner 1998) that inform in typically hidden ways how teachers enact their beliefs (Andrews 2011; Watson & Barton 2011).

## **5. Preparing teachers for problem solving**

It is reasonable to assume that when teachers invite their students to solve a problem they have some insight into how that problem might be solved. However, when presenting problems to their students, teachers typically have only one solution in mind and may reject student suggestions that deviate from their preconceived expectations (Karp 2010; Leikin 2003). In part this problem is due to the rarity of teachers' personal engagement with problem solving and a typical expectation that problems should be curriculum-based (Leikin & Levav-Waynberg 2007).

There is some evidence that problem solving-focused pre-service and in-service education may be effective in overcoming such difficulties. Based on the principle that an immersion in problems enables teachers to develop both their problem solving and teaching skills (Taplin & Chan 2001), problem-based teacher education programmes have enabled teachers to become more proficient problem solvers and develop a more connected mathematical knowledge irrespective of prior assessments of teachers mathematical proficiency (Guberman & Leikin 2013), a process facilitated when participants work in pairs (Leikin 2003). Indeed, as teachers become more proficient problem solvers they become not only better listeners and reactors to what their students say but more likely to engage their students with more challenging problems (Karp 2010; Silver et al. 2005). On a related theme, Crespo and Sinclair (2008) found that where pre-service teachers were given time to explore a mathematical problem in depth, they were able, in comparison with a group who were given no such exploratory time, to pose to their students significantly more challenging problems, particularly from the perspective of mathematical reasoning.

Other studies have yielded less equivocal findings. Osana, Lacroix, Tucker and Desrosiers (2006) found not only that the higher the level of problem complexity the more difficulty pre-service teachers had in classifying problems but that decisions were frequently made on superficial characteristics like problem length. Perhaps most worryingly is research showing that when they undertake a problem solving task prior to teaching it, many teachers lower their expectations of their students' success, although the most successful problem solvers are more convinced that their students will succeed also (Leikin & Kawass 2005). However, in all such research, the more profound teachers' knowledge of mathematics the more likely they were to identify and respond to problem complexity (Osana et al. 2006) and plan appropriate teaching-related prompt questions (Leikin & Kawass 2005).

## **6. The study's aims**

In this paper, we present findings from a multiple case study of three Cypriot primary teachers of mathematics. Case study affords a deep understanding of the nature and the complexity of the phenomenon under study, in this case teachers' beliefs about mathematical problems and problem solving, teachers' competence as solvers of mathematical problems and their problem solving-related teaching (Yin 2009; Stake 2002). Our analysis of three components, which takes our study beyond the typical espoused/enacted dichotomy, should afford greater insights into teachers' professional decision making than would be the case with research focused on the interactions of espoused and enacted beliefs alone. To facilitate the process, as is discussed below, we exploit an analytical framework derived from Schoenfeld, which, in its entirety, is novel. Of course, there are alternative approaches. We could, for example, have exploited the quantitative approach adopted by Depaepe, De Corte and Verschaffel (2010) in their video case study of two Flemish teachers' problem solving beliefs and practice, but argue not only that qualitative analyses allow data to speak more eloquently but that to do otherwise would land us back in Skott's (2009, 2013) trap of determinism. Thus, while our ostensible aims are to examine the interactions of teachers' problem solving-related beliefs, competence and practice, an important focus lies in the extent to which the framework provides an appropriate tool kit for so doing. Such aims, we argue, are better addressed by means of a multiple case study design than a single.

### **6.1. The participants**

The three participants were primary teachers working with children aged 11-12 in different urban schools of a provincial city in Cyprus. Their professional experience and other background characteristics are presented in Table 1. For the purposes of anonymity each has

been given a pseudonym. No assumptions have been made about their being representative of Cypriot teachers.

Anna	With 21 years of teaching experience, 43 year old Anna is a graduate of the Pedagogical Academy of Cyprus. Founded before the University of Cyprus was established, this was a training college offering three-year primary teacher preparation courses. Anna is considered a capable teacher by parents and she feels quite confident in teaching, especially mathematics.
Eleni	Newly qualified, 22 year-old, Eleni was a high achiever at school and university. Self-described as having low mathematics-related self-esteem and, despite being successful as a learner of mathematics, she believes that “mathematics and I are not good friends”. At school she specialised in history and classics before studying language education at university.
Stelios	With two years’ experience, 25 year-old Stelios was a high achiever at school and graduated from the University of Cyprus as one of the highest achieving students of all departments that year. Despite his love of and confidence in mathematics teaching, he specialised in language education because “it was easier to achieve higher grades”.

Table 1: Summary data on project participants

## 6.2. Data collection

Data collection took three forms. Firstly, all three teachers participated in individual semi-structured interviews focused on their beliefs about mathematics, mathematical problem solving and mathematics teaching. The literature search identified four broad themes around which the interviews were structured; beliefs about themselves as teachers of mathematics; beliefs about the nature of problem and problem solving; beliefs about themselves as problem solvers and, finally, beliefs about the management of problem solving in classrooms. Each interview, which lasted approximately half an hour, was audio-recorded before being transcribed and examined against the four literature-derived themes.

Secondly, following their interviews, participants took part in a clinical interview focused on their solving, in a think aloud manner, a mathematical problem. These clinical interviews were audio-recorded to capture as much of the problem solving process as possible. Informants were allowed as much time as they wished, while the second author offered prompts or asked clarifying questions as appropriate. The problem was presented in written Greek in the following manner:

*“If each small square on the paper represents an area of one square unit, how many isosceles triangles can you find that satisfy all of the following criteria?”*

1. *The area must be nine square units.*
2. *One of the vertices is at the point (3, 1).*
3. *The other two vertices are also on grid points.”*

This problem, which can be described as purely mathematical (Blum & Niss 1991), yields 36 solutions in three families of 12. The condition concerning vertices being on grid points requires the solver to find rectangles of integer sides and area 18 square units. This yields six possibilities, with the first digit in each of the following ordered pairs being the base and the second the height, [18, 1], [9, 2], [6, 3], [3, 6], [2, 9] and [1, 18]. The isosceles condition requires the base of this rectangle to be even, for otherwise the vertex opposite the base would not lie on a grid point, as shown in figure 1. This reduces the six possibilities to three;

[18,1], [6,3] and [2,9]. Of course, relaxing one or more of the conditions creates new possibilities, a characteristic of a good problem.

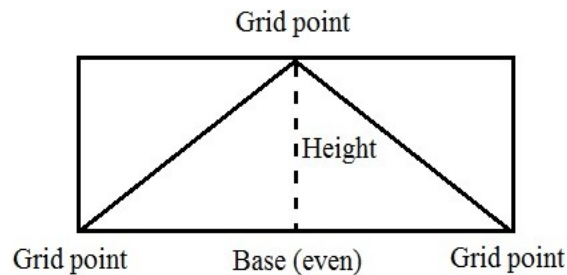


Figure 1: A hypothetical illustrative configuration

Each of these three possibilities produces a family of twelve solutions, as shown in figure 2. The crux of the argument is that each of the triangle's vertices can be placed at the fixed point (3, 1), while rotations about that point yield the remaining solutions. In sum, the solution requires that the solver understand basic notions of area, factor pairs of 18, the ways in which isosceles triangles can be configured on a Cartesian grid and, significantly, simple transformations like rotation or reflection. For more sophisticated problem solvers, the solution may include showing that there can be no isosceles triangles with the *base* lying obliquely on the grid. The fact that there are 36 solutions in a small number of distinct families creates a problem that is non-trivial but accessible. Moreover, it is a problem that facilitates both the demonstration and development of mathematical habits of mind (Cuoco, Goldenberg & Mark 1996).

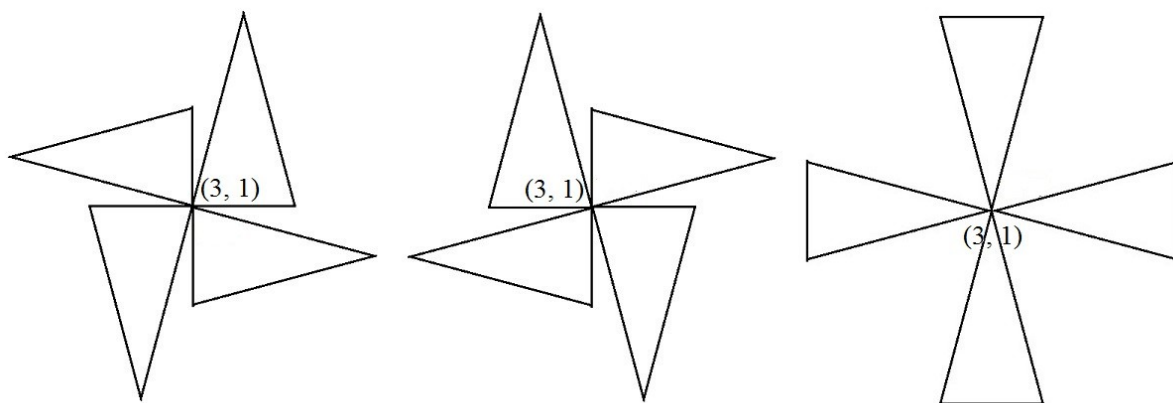


Figure 2: Three distinct sets of four triangles with same base and height

Finally, each teacher spent one forty-minute lesson working on the same problem with their students. The second author acted as a non-participant observer, making notes in addition to audio-recording the lessons. Unfortunately, due to delays in obtaining ministerial authorisation, lessons were not videotaped. The observations were structured by the teachers' actions, with particular attention paid to the ways in which they presented the problem and facilitated students' working on it.

## 7. Analysis

As indicated above, our analyses exploit a framework derived from the many years of problem solving-related research of Schoenfeld in the United States and De Corte, Verschaffel and their colleagues in Flanders, scholars who are extremely well-respected within the mathematics education community. Importantly, while there are inevitably



overlaps in their theorisations, Schoenfeld's work is generally qualitative, with a philosophical bias and an emphasis on model-building while De Corte and Verschaffel's is typically quantitative, with a psychological bias and an emphasis on model-testing. This variation strengthens the framework's intellectual warrant as an analytical tool. The framework is in two parts, which we present below.

### **7.1. Problem solving from the solver's perspective**

For several decades research undertaken by Schoenfeld in the United States and DeCorte, Verschaffel and others in Flanders has shown consistently that successful mathematical problem solving is dependent on four solver characteristics. The first reflects the solver's knowledge base (Schoenfeld 1992, 2004); what does an individual already know that would be useful in solving a particular problem, how is this knowledge organised and how is it accessed purposefully in problem solving situations? In short, successful problem solving requires a well-organised and flexible knowledge, including the rules that underpin mathematical argumentation (De Corte 1995, 2004, De Corte et al. 2000). Recently, Schoenfeld has incorporated this sense of knowledge base into a broader category of resources (Schoenfeld 2010a, 2010b). However, we do not consider this broad category as part of the problem solving process but as a characteristic of the problem solving conducive classroom, which we discuss below. The second concerns the extent to which an individual possesses a productive set of problem-solving strategies (Schoenfeld 1992, 2004) or heuristics (Schoenfeld 1985b). Does the individual have access to an organised and retrievable set of problem solving principles (Schoenfeld 1992) or strategies likely to facilitate progress on unfamiliar or atypical problems (Schoenfeld 1985b)? In other words, problem solving requires a set of problem solving strategies that, when used systematically, may not guarantee a solution but significantly enhances the likelihood of finding one argumentation (De Corte 1995, 2004, De Corte, Verschaffel, & Depaepe 2008). The third component concerns the extent to which an individual is able to make effective use of the knowledge and skills discussed above (Schoenfeld 2004). Does the individual consciously select and exploit strategies and resources (Schoenfeld 1985b)? Can individuals monitor and regulate their attempts in ways indicative of both an understanding of and a warrant for the decisions they make (Schoenfeld 1992, 2004). That is, does the solver have sufficient meta-cognitive competence to control his or her decision making? Interestingly, while clearly agreeing with the need for self-regulatory competence, De Corte et al. argue for meta-knowledge, which embraces both meta-cognition and meta-volition, or understanding how one's motivation and mathematics-related affect influence problem solving engagement (De Corte 1995, 2004, De Corte et al 2000; Verschaffel, Luwel, Torbeyns & Van Dooren 2009). Finally, successful problem solvers see the problem as worth solving, are prepared to give it the time it requires and believe they have the competence to solve it (Schoenfeld 1992, 2004); they have an appropriately productive mathematical worldview (b 1985b), derived experientially, that not only shapes the mathematical knowledge and strategies they bring to bear on a problem but also the manner in which they exploit them (Schoenfeld 2010a). In short, problem solvers possess positive beliefs about themselves in relation to both mathematics itself and the context in which it is experienced (De Corte 1995, 2004, De Corte et al. 2000, 2008).

### **7.2. Creating a problem solving classroom**

In a problem solving classroom teachers encourage problematizing. That is, they encourage students to take on intellectually challenging problems (Schoenfeld 2007); mathematicians "are not passive recipients of knowledge", they "develop knowledge by asking why things work" and "whether particular examples can be extended or generalized (Schoenfeld 2012,

p.594). A problem solving classroom not only supports the “constructive, cumulative, goal-oriented acquisition processes in students” but encourages “active learning strategies in passive learners” (De Corte 1995, p. 41). Secondly, students are granted authority to work on such problems (Schoenfeld 2007), in classrooms that are “structured to help students develop agency and authority” (Schoenfeld 2012, p. 595). They are encouraged to reflect upon their solution strategies (De Corte et al. 2008) and acquire self-regulative skills (De Corte 1995; Depaepe et al. 2010). Thirdly, in contrast to the norm whereby students are accountable to their teachers (Schoenfeld 2012), a problem solving conducive classroom encourages students to expose their work to the scrutiny of others and, in particular, the disciplinary norms of mathematics (Schoenfeld, 2007) in the manner of a mathematician (Schoenfeld 2012). It encourages discussion about the nature of good problems and good solutions and allow the whole class to decide on what is optimal (De Corte et al. 2008). Fourthly, teachers need to ensure that students have the appropriate resources to achieve the above (Schoenfeld 2007, 2012). Such environments, creating different experiences from the traditional whereby “learning is an incidental by-product of ‘getting the work done’” (Schoenfeld 1988, p.151), places substantial demands on the teacher, not least because they have to provide flexible instructional support, manage the “balance between self-discovery and direct instruction, or between self-regulation and external regulation” (De Corte 1995, p. 41) and encourage the development of appropriate beliefs about the self and mathematical problem solving (Depaepe et al. 2010).

Putting the above together, and acknowledging the distinction between the roles of teacher and problem solver, we propose the framework for analysis shown in table 2. Data from each case were scrutinised repeatedly for evidence pertaining to the content of each cell and then gathered together to facilitate analysis. Inevitably, due to the nature of both the data and our aims, some evidence has been multiply assigned; that is, the same data was allocated to more than one cell. Once each cell had been completed for each case, the content of the cells in each row was integrated to give a sense of how the three constituent elements interact in respect of each of the eight characteristics. This process has produced an extensive but integrated summary of teachers’ problem solving-related beliefs, competence and practice, which forms the basis of what we report below.

		Perspectives on problem solving		
		Espoused beliefs (Interview)	Competence (Task)	Enacted beliefs (Observation)
Teachers as solvers of mathematical problems	Knowledge			
	Heuristics			
	Meta-knowledge			
	Beliefs			
Teacher as facilitators of problem solving	Problematising			
	Authority			
	Discipline			
	Resources			

Table 2: Summary of the analytical categories

## 8. Findings

In the following we present summaries for each teacher, framed by the rows of table 2. In so doing, the reader is alerted to the fact that throughout the analysis, *base* refers uniquely to the edge of the triangle through which the axis of symmetry passes, while *height* refers uniquely to the distance between the *base* and its opposite vertex, which we describe as the *symmetry vertex*. Also, as indicated above, and to avoid unnecessary repetition, a triangle of base  $a$  and height  $b$  is abbreviated to  $[a, b]$ . Finally, many of the examples we have selected to illustrate the various categories could have been used in more than one category; therefore we have tried to place them where we think their illustrative strength is greatest but acknowledge that readers may think differently.

## 8.1. Anna

### 8.1.1. Knowledge

Although it was not mentioned during her conversational interview, it became clear during her clinical interview that prior knowledge plays an important role in Anna's approach to solving a problem. She commented, as she read the problem, *first I shall write the formula that gives me the area of a triangle. The length of the base multiplied by the length of the height and then divided by two shall be nine*. At the start of her lesson, before introducing the problem to her class, Anna revisited the area of a triangle. This included her eliciting the formula for the area of a triangle and a public demonstration of the calculation for a  $[4, 5]$  triangle. Next, having drawn several isosceles triangles with bases in various orientations, selected students were invited to mark the equal sides.

In these two incidents, one from her clinical interview and one from her lesson, can be seen very different perspectives on knowledge as a prerequisite for problem solving. On the one hand, when placed in an authentic problem solving situation, Anna invoked, in entirely appropriate ways, the prior knowledge she thought would be useful for *her* to solve the problem. On the other hand, when facilitating her students' work on the same problem, she asserted, through her activation of their prior knowledge, the knowledge that she thought would be useful for *them* to solve the problem.

### 8.1.2. Heuristics

During her conversational interview Anna asserted that practice is her only approach to improving her problem solving skills, adding that when faced with difficult problems, which tend to make her anxious, she tries *to relax and concentrate on the given information* before trying to recall *previous knowledge regarding the solution of similar problems*. She began her clinical interview by commenting that *I have to start drawing triangles*. Later, following her rejection of  $[3, 6]$  triangles due to their symmetry vertex not lying on grid points but having found four of the twelve  $[6, 3]$  triangles, she turned her attention to those triangles falling within a 2 by 9 rectangle. She commented, after a few moments' reflection that, *I can't use nine as a base, for the same reason as before. I will use two as the base and nine as the height*. This led her to finding, analogously, four of the twelve  $[2, 9]$  triangles.

During her lesson, as part of her activation of prior knowledge, Anna asked, rhetorically, for the dimensions of a triangle with area  $8\text{cm}^2$ , before explaining that the procedure was to double the area and find factor pairs. Later, in relation to the problem itself, she drew two columns on the board, and asked for possible dimensions of a triangle of area  $9\text{cm}^2$ . A student volunteered the triangle  $[4, 4.5]$ , to which Anna responded, *No, we only want integers*. During the following minutes, drawing on contributions from her students, she completed the table for all possible integer pairs before asserting that *the number of the base must be even*. At this point, she deleted triangles with an odd base, and suggested students *try to find these triangles*.

Again, different perspectives on the nature and manifestation of heuristics can be inferred from these data. Firstly, Anna, knowing that previous problem solving experience can provide insights into the current, drew repeatedly on analogical reasoning. However, when faced with her class, she became a teacher who told, denying her students the possibility of seeing such isomorphic structures for themselves.

### 8.1.3. Meta-knowledge

Anna described herself as an averagely competent problem solver, explaining that she typically *tackled simple problems with simple solutions*. She tended to approach problems superficially, frequently revisiting them to understand what was required and leading her to *get bored of problems which require a more advanced level of thinking*. During her clinical interview several incidents occurred highlighting this lack of confidence and, as a consequence, a reluctance to assume control. For example, as indicated above, having found four of the twelve [6, 3] triangles, she analogously found the equivalent triangles of the [2, 9]. At this point she appeared nervous and paused, saying, *I can't think. I can't find any more triangles*. However, when asked how she found the previous two sets of triangles, she responded,

*I had to find isosceles triangles with area nine square units. In order to find the lengths of the height and the base, I had to multiply nine by two. Then I tried to find those numbers whose product is eighteen... Six times three, two times nine. Oh, there is also eighteen times one. I don't know how I forgot that.*

She found four such triangles, saying, *I have four more. That means I have twelve triangles in total*. She smiled, asking *are there any more?* She appeared to start thinking again, but soon seemed ready to abandon the problem. When prompted with a statement about all triangles satisfying all the criteria, she observed, *Hmm. One vertex should be the given point. That means the other vertices can be at the given point too*. Eventually, with some discomfort, she found all 36 solutions.

Shortly after posing the problem to her class, Anna announced that *I will help you. Start from the formula that gives us the area*. She then circulated the room repeating to each child what was to be done. One student had drawn a right-angle triangle with a base of 1 unit and a height of 18 units. On the grid paper the triangle looked isosceles and, in passing, Anna commented, *well done, you are right*.

The above highlight interesting perspectives on a problem solver's meta-knowledge. Firstly, Anna believes herself, implicitly at least, to have limited problem solving competence and, therefore, regulation. However, her use of isomorphic examples to warrant analogical reasoning seems to say something different. That is, she has greater regulatory competence than perhaps she thought. Secondly, despite this incongruity of espoused and enacted regulative competence, throughout the clinical interview Anna appeared hesitant and in need of external guidance, which, when given, enabled her to complete the task. Thus, it could be argued, she seems to enact her belief that problem solving skills improve with practice. Thirdly, her response to the incorrect student suggestion, admittedly given in haste, points to a classroom environment in which student regulation derives from the teacher and not the student.

### 8.1.4. Beliefs

Anna asserted, during her conversational interview, that a mathematical problem is *a situation, which requires you to discover which information is given, to order the given points, to find what the problem asks you to do and then solve it; problem solving is the achievement of a goal, either set by the problem solver or by others*. The information should

be clear and accurate so that children can understand what has to be done. However, she added that *students should not spend more than three to five minutes on a problem*, although sometimes she allows students *up to seven*. While there were implicit references to the nature of problems and problem solving during her clinical interview and lesson, nothing was explicit. For example, her actions during the clinical interview, despite ambiguities with regard to regulation, were indicative of a conflict of espoused and enacted beliefs about the nature of a mathematical problem, not least because she spent many minutes working on a task in contradiction to her belief that mathematical problems should not take more than a handful of minutes.

Thus, Anna believes that mathematical problems are well-structured and unambiguous tasks that should be amenable to completion in a few minutes. Drawing on all the above, her role as a teacher seems very much one of ensuring that the tasks she presents her students are unambiguous before providing the tools necessary for their completion.

#### 8.1.5. Problematising

During her conversational interview Anna commented that the role of students was to focus on obtaining correct answers because this is what they will be expected to do in life and in future mathematics examinations, because *if the steps they follow are correct but still the answer is wrong, then they will not attain a good grade*. During her lesson, as indicated in the ways in which she activated prior knowledge and generally managed her lesson, Anna removed all decision making from her students; her efforts seemed focused on ensuring that the task presented no problematic perspectives. In short, there was a consistency of belief and practice in that students will be successful if they follow the steps she provides for them; views that seem incompatible with the solving of any genuinely problematic task.

#### 8.1.6. Authority

During her conversational interview, Anna commented that teaching mathematics through problem solving is feasible only if children know appropriate mathematical procedures, for *if students do not know how to solve procedural exercises, they will be very disappointed and they will not have the motivation to try and solve more difficult problems*. Moreover, her students *seem nervous when they have to solve mathematical problems*, and that *only a few of them will reach a solution while the rest quickly abandon every effort* in the belief that *there is no point trying since the teacher will explain the solution on the board*. Therefore, in addition to verbal encouragement she gives *hints that will help them to proceed* and will often try to *simplify the situation* for them.

During her clinical interview Anna appeared to act as though she had little authority with respect to her engagement with the task. It could be argued, of course, that the particular problem was imposed, with limited opportunities for ownership; however, there were several occasions when Anna appeared hesitant and in need of prompts or interventions, implying a reluctance to accept the mantle of authority. When teaching her lesson, Anna seemed reluctant to grant authority to her students. For example, her activation of prior knowledge and the various ways in which structured her students' work on the problem were managed in ways that left students with little opportunity to assume any authority with respect to how they approached the problem.

In sum, Anna's beliefs and practices with respect to classroom authority were consistent. She did not espouse student autonomy, arguing that it made them uncomfortable. but a view that it was her role to structure student learning in ways that, we argue, run counter to an expectation that students be granted the authority to engage in bouts of authentic problem solving.

### 8.1.7. Discipline

Anna's conversational interview yielded the belief that *the school programme allows little time for problem solving*. She added that *there should be more teaching periods for mathematics*, so that she can *explain the solution of mathematical problems in more than one way*. In this way her *children would understand better and exhibit a more positive attitude towards mathematics*. During her clinical interview there were several occasions when Anna elected to defer to an external authority with regard to what was acceptable. For example, having found four triangles in each of the three families, she appeared nervous, paused, and said, *I can't think. I can't find any more triangles*. However, as discussed above, when prompted with a statement about all triangles satisfying all the criteria, she observed, *Hmm. One vertex should be the given point. That means the other vertices can be at the given point too*. Such incidents were not unique and indicated a dependency on others as to the accuracy and quality of her attempt. During her lesson, her actions presented her as source of mathematical authority in the room. For example, it was Anna who explained that the correct procedure for identifying a triangle's dimensions was to double the area of the triangle and then look for factor pairs; it was Anna who asserted that only integers would suffice and that the base must be even; it was Anna who, incorrectly, told a student, that [1, 18] was a valid triangle in the context of the problem.

Such beliefs and actions indicate that Anna was consistent in her being the mathematical authority in the room. She did not espouse a view that students' mathematical thinking should be held up for public scrutiny but that their performance would be enhanced if she had time to *explain the solution of mathematical problems in more than one way*. Her actions in the classroom confirmed this belief and, when positioned as a learner herself, she acted as though mathematical authority lay with an external authority and not something to be negotiated.

### 8.1.8. Resources

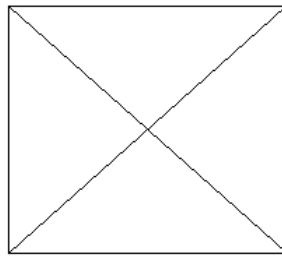
Anna's conversational interview indicated that she uses the national textbooks extensively, but, *because there are many problems within them*, is selective and supplements her choice with problems she devises herself; *it is easier and more practical to write problems similar to those in the national textbooks than to search for them on the internet or in books*. When teaching her lesson she provided resources in a variety of ways. On the one hand, she wrote the problem on the board, handed each student a sheet of grid paper and read aloud the problem. Thus, she provided the physical resources necessary for her students to complete the task. On the other hand, Anna provided all the cognitive resources she thought her students would need. For example, the activation of prior knowledge could be seen as her ensuring her children had the appropriate mathematical resources, as could her frequent telling her students what they should do next. In short, all decisions students could have made were preempted by her provision of resources.

## 8.2. Eleni

### 8.2.1. Knowledge

Eleni began her conversational interview by describing herself as an insecure problem solver, commenting, *solving problems is not something I enjoy doing and furthermore I do not have the skills for it*. At the start of her clinical interview she read the problem and murmured, *isosceles triangles are those which have two equal sides, aren't they? Hmm, and the three criteria must be satisfied at the same time. Hmm, isn't the formula for the area of a triangle base times the height divided by two? ....* She sketched the square below, saying, *I want to have a square with area 36 sq. units. Then I can have four isosceles triangles with area nine.*

However, despite its feasibility, she was unable to fit her drawing on the grid paper and quickly abandoned the idea.



In sum, Eleni's claim that she was an insecure problem solver found some evidence in her clinical interview. For example, her think aloud responses were hesitant and typically posed as questions rather than the assertions of someone confident in his or her subject knowledge. Moreover, the speed at which she abandoned her attempt to fit her sketched square - a square which could easily be fitted to the grid - also reflected, we argue, a lack of confidence in her ability; had she been more confident she would have been more likely to have persisted.

### 8.2.2. Heuristics

During her conversational interview Eleni spoke of how she helps students acquire *strategies that can be applied to the solution of the problem*. She does not explain what students should do, preferring students to *discover the way by themselves*, although she believes that *students can learn a lot through discussion and exchange of ideas while working with mathematical problems*. When working on the task itself during her clinical interview, Eleni's actions indicated an awareness of some generic problem solving strategies. For example, having decided that she was interested in finding factor pairs of 18, she quickly identified three possibilities before going on to find four members of the [2, 9] family. At this point, acknowledging structural similarities, she announced that she could find four triangles of the [6, 3] and four of the [18, 1] family. Moreover, she noted, in contradiction to her earlier claims about lack of confidence and appropriate mathematical knowledge, that she couldn't reverse the values of heights and bases because *then the vertices wouldn't be on grid points*. Shortly after this she realised that the given point could be any of the three vertices and drew all 12 [6, 3] triangles before commenting that there must be the same for [2, 9] and [18, 1] triangles. Finally, when asked if there were any more triangles, she commented, with a sense of pride in her voice, *yes, there are other triangles, but then I would have to use decimals. I can't do that, I did it before and the vertices won't be on grid points*.

In sum, despite her interview claims that she lacked the skills to solve problems, Eleni completed the task with minimal external prompts. In so doing she demonstrated a clear understanding of the structural similarities of the three families of triangles and exploited analogical reasoning with some authority. Moreover, this awareness of generic strategies accorded with her espoused beliefs concerning how she works with her students.

### 8.2.3. Meta-knowledge

During her conversational interview, Eleni mentioned that when students experience difficulties she helps *each student independently* by asking questions to help them think rationally. She added, as discussed above, that she does not explain solution approaches but encourages students to *discover the way by themselves*, particularly through *discussion and exchange of ideas*. Thus, it could be argued that Eleni espouses beliefs commensurate with their becoming independent and reflective solvers of mathematical problems.

When working on the task herself, there were occasions when her own regulatory skills seemed to fail her, as with her abandonment of her attempt to fit the square discussed above. However, there were also occasions, as with her awareness of structural analogies, when she was very much aware of her actions. Someway into her lesson Eleni stopped her class from working and, having asked for contributions, received several suggestions, which, through judicious questions, were shown to be irrelevant. Eventually, a girl demonstrated the 12 [6, 3] triangles, commenting that the base should always be an even number. On being asked to refine her argument, the girl added that if the base was odd then not all vertices would be on grid points. On another occasion, by means of Socratic questions, Eleni led her class to agree that the fixed point could be placed anywhere on their papers. Finally, throughout her lesson, Eleni told students to exchange their ideas with the other members of their group.

The above indicate several perspectives on the nature of control. As a problem solver, Eleni's inconsistent confidence resulted in inconsistent regulatory behaviour. At times she seemed to panic and abandon feasible avenues, while at others she acted with an intellectual surety. As a teacher, her espoused beliefs concerning a desire to help students develop problem solving strategies were reflected in her lesson, where her constant questioning enabled students not only to acquire those skills but to become aware of that process of acquisition and share in her sense of pride.

#### 8.2.4. Beliefs

During her interview, Eleni had much to say about problems and problem solving. She regarded a mathematical problem as being like any other, *a situation that requires a solution*. They take many forms from, *at one end of the scale ... very simple ones to, at the other, some which are much more complicated*. A mathematical problem comprises *a verbal and a numerical part*. *The ways the given information and the question of the problem are expressed comprise its verbal part. The numerical part is included in the verbal, in the form of numbers and symbols.*

Problem solving involves *understanding the problem, separating the given information from what one is asked to achieve and using all the elements of the problem in order to reach a solution*. She believed that *the journey towards the solution of a problem is much more important than the solution itself* and that this journey, related to critical thinking, *is not just a passive procedure*. The solver has to find a way *which will eventually lead to the solution*. *Only then has the problem solver really understood the problem and its solution and can explain the journey to others*. She viewed teaching through problem solving as not only a way of *connecting a mathematical concept with daily life situations* but also a way for *introducing new concepts*. However, she believed that *time parameters set by the school programme do not allow much time for problem solving activities, despite the fact that the national textbooks offer many problem solving activities*. Therefore, she does not spend time on problems *if they are not related to the particular mathematical concept she wants to teach* but may introduce new topics through problem solving activities.

In sum, despite her belief that she is not a good problem solver, a belief not entirely matched by her problem solving performance, Eleni has a clear view of what a problem comprises; it is located in text. It can be either difficult or hard and comprises both verbal and numerical elements. Such views are interesting, not least because the task she undertook during her clinical interview would have failed her definition but also because she seemed to think that being a problem is independent of the solver.

#### 8.2.5. Problematising



As indicated above, Eleni construed a mathematical problem as necessarily located in text and independent of the solver. Moreover, she indicated, during her conversational interview, that she sees herself as one who does *not actually know mathematics*, since she lacks an *advanced knowledge of the field*. Consequently, she is able to teach mathematics only if she is well prepared and would like to *feel more secure in teaching lessons based on non-routine problems*. During her lesson, Eleni began by giving each student a sheet of grid paper and a problem sheet, which she instructed them to read. After two minutes she asked if there was anything students did not understand, which led to her explaining the criterion concerning vertices on grid points. In such actions can be seen congruent espoused and enacted beliefs concerning problems as text-based and well-defined. Also in such actions can be seen limited opportunity for problematisation - Eleni had, in presenting the task in the way she did, pre-empted any opportunity for her students to problematise and thus avoid placing herself in the position of not being well-prepared. In other words, the lack of opportunity observed for students to problematise was entirely in accord with Eleni's low levels of mathematical confidence.

#### 8.2.6. Authority

As with problematisation, there was limited evidence of beliefs and actions associated with Eleni's shifting agency from herself to the student. During her conversational interview she commented that she would try to solve a difficult mathematical problem only if she had to *present it to students*, which, in essence is exactly what has been described above. She argued that mathematics and mathematical problems make her nervous, but *if I really had to solve it, I would try and use every possible means, because I am a really stubborn and proud person*. When faced with difficulties she either asks *for advice from someone more experienced* or leaves the problem until she can relax and revisit it. Also, most of her students *do not like solving problems due to the insecurity children feel, which derives from negative experiences of previous years' mathematics lessons*, and do not respond to problems with enthusiasm.

During her lesson, after ensuring the task as presented on paper was understood, students were asked *to write down any information that can be useful and think of a solution plan*. When a student asked if they should write the steps they would follow, she replied, *Yes, like we do for every problem*. After a few minutes, as indicated above with respect to control, Eleni asked for contributions and a full and well-warranted solution to the [6, 3] family was extensively discussed. In inviting the writing of a solution plan could be seen the germs of a shift in authority. However, such plans were subjected to public scrutiny and rejected if thought inappropriate. In other words, Eleni retained control over the directions students were permitted to take, with only those fitting her experience of the problem being encouraged.

When viewed alongside her perspectives on problematisation and both her and her students' problem solving-related anxieties, it is of little surprise that her data yielded little evidence of a shift in mathematical authority from Eleni to her students. That being said, her actions as a problem solver, and the manner in which she managed her lesson, indicated a deep understanding of the mathematics of the task, an understanding she may have denied if asked, and a desire to ensure that her students were inducted into a similar understanding. In other words, even though she were reluctant to hand over authority, she seemed keen not to deny her students the opportunity to engage meaningfully with the problem.

#### 8.2.7. Discipline

Eleni's conversational interview indicated a perception that *the time a student should spend on a problem depends on his abilities and on the degree of difficulty of the problem*. Some of her students can solve a difficult problem in less time than it would take her, *while others*

*need constant help and guidance.* Such views, we argue, indicate her acknowledging that some students - those who can solve a problem more rapidly than she - have acquired a disciplinary authority in which the norms of the subject rather than the teacher are privileged.

During her lesson, as we have already discussed, Eleni drew extensively on the work of one student who had a complete solution for the [6, 3] family of triangles. Moreover, having invited the student to demonstrate her solution, Eleni essentially suggested that the same strategies could be applied to other families. In actions such as these, particularly when set against repeated instructions for students to exchange ideas with other members of their groups, it is possible to see Eleni's desire for students to both share and draw on the nature of mathematics to both solve and understand the problem on which they work. That is, despite her extensive personal anxieties, Eleni seemed content, in a didactically controlled manner, to open up for scrutiny, both public and mathematical, the thinking of her students.

#### 8.2.8. Resources

At the start of her lesson, Eleni gave each student a sheet of grid paper and a problem sheet, which she instructed them to read. After two minutes she asked if there was anything students did not understand, which led to her explaining the criterion relating to vertices being on grid points. In short, Eleni ensured that her students were equipped with the physical resources necessary for the task ahead. More importantly, and acknowledging her personal concerns regard problem solving competence, the evidence presented above suggests an ambition that her students should acquire not only high level mathematical competence but personal autonomy also.

### 8.3. Stelios

#### 8.3.1. Knowledge

During his conversational interview Stelios commented that his mathematical knowledge and problem competence would allow him to solve the problems encountered in primary mathematics, but, acknowledging the relationship between the problem and the problem solver, was unsure if these would stretch to a problem requiring advanced mathematics. During his clinical interview he demonstrated a clear understanding of the mathematics necessary for solving the problem. For example, when presented with the problem he commented that *we want to find triangles with two equal sides and area of nine square units. First of all, let's start with the formula that gives us the area of a triangle.* Having written down the formula he drew two columns, which he labelled *base* and *height*, before commenting that the possible cases were base 2, height 9; base 3, height 6; base 9, height 2; base 6, height 3. At this point he did not mention the factor pair of 18 and 1 but explained that his goal was to create as many triangles as he could with those values, which he did, showing a clear understanding of the vertex constraints, the impossibility of [3, 6] and [9, 2] triangles and the role of rotations.

During his lesson he elicited, but did not tell, the mathematical insights necessary for solving the problem. For example, shortly after asking his students to read the problem, he asked,

*how can we use the first piece of information the problem gives us? Nine square units. Certainly, we are not going to draw triangles by chance.*

In response a student volunteered that *the length of the base times the length of the height divided by two.* This led to Stelios commenting,

*Good, we have a first clue here. The length of the base times the length of the height divided by two must be nine. Before doing something else, I want you to think. What are the possible values for the base and height so that we have an area of nine?*

In sum, despite his oversight with respect to the 18, 1 factor pair, the knowledge and mathematical understanding he brought to bear on the problem was not only appropriate but indicative of his being able to solve problems more sophisticated than those encountered in the typical primary classroom. In terms of his working with his students, his actions indicated a confidence that his students not only understood the mathematics necessary for working on the problem but he did not need to assert what he thought they needed to know.

### 8.3.2. Heuristics

During his clinical interview, having started with a [2, 9] triangle, he drew a [3, 6] but stopped after a few seconds and exclaimed, *this can't be one, otherwise the vertices won't be on grid points. Basically, we can't have an odd number for the length of the base because of the criterion stating that all vertices should be on grid points. So I rejected the case of base 3 and height 6 and, for the same reason, we can't have base 9 and height 2.* Having quickly found all twelve [2, 9] he found the twelve [6, 3] analogously. During his lesson there were several occasions indicative of his encouraging his students to consider and adopt problem solving strategies. At a very functional level he suggested to his students that it would be helpful if they discussed and shared their solutions with each other. At a more profound level he encouraged an awareness of the power of recognising in one context characteristics isomorphic to another. For example, once the class had found all members of the [6, 3] family, Stelios asked if this would help them in identifying the members of the [2, 9] family, reasoning that he encouraged them to adopt later with the [18, 1] family. Thus, while he had little to say with regard to heuristics during his conversational interview, both his clinical interview and his lesson provided evidence not only of his awareness of generic problem solving strategies but of an expectation that he expected his students to acquire them also.

### 8.3.3. Meta-knowledge

Stelios claimed, during his conversational interview, that difficult mathematical problems made him anxious because they *are not like the typical exercises used for practising.* Nevertheless, if he had to solve a difficult problem he would try to overcome his anxiety and do his best to revisit systematically the given information and try to connect it in ways that they will lead him to a solution. Such beliefs were matched in the manner in which he approached the problem, which he did systematically. Moreover, as indicated above, he was able to regulate his solution processes. For example, not only was he aware of the analogous relationships between the different families of solutions, as discussed above, but also, having exhausted the [2, 9] and [6, 3] families, reflected on what he had done and announced, *Oh, I forgot 18 times 1. Hmm, based on what I found for the other two cases, I am sure there are 12 more with base 18 and height 1. That means I can have 36 triangles in total.* In similar vein, also as indicated above, he quickly recognised the need for an even base, as *otherwise the vertices won't be on grid points.* The evidence of his lesson, however, was very much one of his scaffolding his students' meta-knowledge through elicitation and prompts. In other words, while he had demonstrated evidence of regulation in his approach to the problem, he appeared not to have reached a point where he expected the same of his students.

### 8.3.4. Beliefs

Stelios had very strong and clear views about the nature of mathematical problems, which he described as *situations* that lead the solver *to a mental adventure* involving a journey that is more important than the solution. Their value lies in the effort expended and not the outcomes, to the extent that procedural errors are unimportant as *the emphasis is on the mental processes in order to lead yourself to the solution.* In relation to students' learning of mathematics, he added that the time spent on a problem should depend on both problem

difficulty and student competence, although teachers should avoid creating the impression that *the fastest problem solver wins*. He added that a problem should be clearly posed so as not to confuse or lead to misunderstandings, although genuine problems have the potential for many routes to the solution and entail both the methodical examination and manipulation of given information. His behaviour during the clinical interview, as discussed above, confirmed these beliefs as evidence of, say, a *mental adventure* involving *mental processes*. With respect to the problem solving lesson, he announced at the beginning that one problem would occupy the whole lesson. He distributed a worksheet and asked them to write down any ideas they had for solving it. In these actions can be seen confirmation of several espoused beliefs; a problem lasting the whole lesson reflected a belief that a problem is a *situation* leading to a *journey*; and his worksheet reflected his belief in the importance of a clear problem presentation. Thus, Stelios indicated largely consistent espoused and enacted problem- and problem solving-related beliefs.

#### 8.3.5. *Problematizing*

During his conversational interview, Stelios spoke about problem solving being at the heart of mathematics teaching and learning, although teaching *via* problem solving is not feasible for all topics and *besides, I don't think that problem solving is one of the main goals of the National Textbooks*. In other words, while he made no explicit reference to the role of problematising in the development of a problem solving classroom, we argue that he was aware of such matters. Indeed, his actions during both the clinical interview and his lesson indicated a clear sense of the need to create a context in which problematisation was an everyday occurrence.

#### 8.3.6. *Authority*

Stelios believed that children learn mathematics in order to manage their daily lives and develop *a more organised way of thinking*. However, his view was that the teaching programme allowed few opportunities for problem solving, forcing teachers to *waive activities that require time, like problem solving*. In such statements, we believe, is evidence of a desire to pass authority from himself to the learner. Indeed, he commented during his conversational interview that when students encounter difficulties he gives hints that he believes will help them find their own pathways towards the solution. If such difficulties are common to several students, then he gives group feedback. For example, as shown earlier, having elicited the rule for the area of a triangle, Stelios asked his students what were the implications of this rule for their particular problem. Thus, the evidence of the lesson typically showed his attempting to prepare students for the taking of authority rather than their being granted it, but it could be argued that the task was so well defined as to make it difficult for students to grasp authority.

#### 8.3.7. *Discipline*

Although little was said in during his conversational interview concerning discipline, during his clinical interview there were several incidents indicative of his acknowledging the authority of mathematics as the warrant for his decisions. For example, as indicated above, he justified very quickly the fact that triangles' bases had to be even. Also, having found the 36 solutions and prompted as to the existence of other triangles, he commented, *look, we can have decimal numbers. For example, 4.5 times 4 divided by 2 gives us 9. But we can't use the decimals because of what I said before; the vertices won't be on grid points*. Interestingly, during his lesson a student asked if a triangle with base 4 and height 4.5 could be a solution. Stelios' response was to assert that only integers should be used, indicating that, for his students at least, he remained the source of mathematical authority in the classroom. Thus, his

espoused and enacted beliefs with respect to the discipline of mathematics seemed oddly at odds.

### 8.3.8. Resources

At the start of his lesson Stelios distributed a worksheet on which was written the task, in clear and unambiguous terms and squares on which students would work. In short, he provided the physical resources necessary for his students to solve the problem. In addition, he ensured, through elicitation, hints and occasional assertions, that his students were equipped with the mathematical insights necessary for solving the problem.

## 9. Discussion

In this paper we set out not only to examine the relationship between the problem solving beliefs, competence and practices of three Cypriot primary teachers but also to determine the efficacy of not only a framework derived from the different research traditions of Schoenfeld on the one hand and De Corte and Verschaffel on the other but also an inclusive definition of beliefs. Our view is that both have served those ends well, enabling the construction of detailed narratives for each of teacher in which could be seen the complex relationship between how teachers think and act (Cohen 1990; Skott 2009, 2013; Wilkins 2008). For example, and acknowledging that teachers are conditioned by the cultures in which they work (Andrews 2007; Cai & Nie 2007), it was reassuring to find substantial similarities in the problem-related beliefs, solution procedures and teaching practices of the three teachers that are, we suggest, reflective of a Cypriot perspective on mathematics education in general and problem solving in particular.

In this respect, all three teachers work within a centralised curriculum that not only privileges mathematical problem solving (Campbell & Kyriakides 2000) but, through its mandated textbooks, a particular perspective on the nature of problems and problem solving in which the word *problem* is synonymous with *arithmetical word problem* (Kyriakides, Charalambous, Philippou & Campbell 2006). Moreover, all problems, as construed by the Cypriot authorities, fall into one of four well-defined categories, each of which is amenable to a particular solution process that students are expected to master (Charalambous & Philippou 2010). Elements of these systemic perspectives were reflected in the three teachers beliefs about mathematical problems, which can be summarised as well-structured *situations* comprising verbal and numerical components that should be unambiguously expressed, views that accord with recent studies of Cypriot pre-service teachers' beliefs about mathematical problems and problem solving (Xenofontos & Andrews 2012, 2014). These views were also reflected in the manner in which, as discussed in relation to resources, all three teachers presented the problem to their students in clear and unambiguous written forms, whether on paper, the board or both. However, when discussing problem solving, none of the teachers mentioned problem categorisations or schematic solution strategies. In other words, despite working within a tightly regulated curriculum framework, their espoused and enacted beliefs about problem solving appeared independent of systemic perspectives on problem solving. In particular, the manner in which they attacked the problem reflected the more general heuristics discussed earlier, with all three exploiting, for example, analogical reasoning, the most widely used generic problem solving strategy (Antonietti, Ignazi & Perego 2000; Bernardo 2001; Metallidou 2009). Furthermore, similar substantive knowledge was exploited during teachers' cognitive interviews, with all three making explicit reference to the formula for the area of a triangle and how its use would lead to a search for particular factor pairs.

It was also reassuring to find differences. Anna, for example, was clear that problems should be time-limited and that students need to be encouraged to focus on getting the correct

answer. Such views reflected a strong belief that distinguished Anna from her colleagues who described problem solving as a journey where the process was more important than the outcome. Such matters are significant in interpreting the complexity of teachers' professional decision making. Other differences, despite superficial similarities, were found with respect to the personal knowledge of mathematics necessary for solving the problem. All three teachers were appropriately mathematically equipped, although the manner in which this was exploited seemed very much a function of individual confidence, with Anna, for example, somewhere on a continuum between the hesitancy of Eleni and the assurance of Stelios. Major differences were identified in the ways in which knowledge played out during their lessons, with Anna, by means of extensive and tightly managed activation of prior knowledge, appeared to believe that her students would be unable to address the problem unaided, while both Stelios and Eleni felt no such need, seemingly trusting that their students were appropriately prepared by their previous learning of mathematics. In other words, confirming earlier research, undertaking a problem prior to teaching it does not always translate into higher expectations for learners (Leikin & Kawass 2005).

Such matters are not unrelated to notions of meta-knowledge. Stelios exhibited a number of regulatory behaviours while solving the problem, as did Anna despite her arguing a limited problem solving competence and constant seeking of approval. However, Eleni's inconsistent confidence resulted in inconsistent regulatory behaviour. Interestingly, with respect to their classrooms, their behaviours offered additional insights into the relationship between beliefs, competence and practice. Both Eleni and Stelios, through various elicitations and prompts, acted in accordance with their espoused aims of helping students to develop problem solving competence. However, Anna remained in strict control, showing no evidence that her students were expected to consciously select and exploit strategies and resources (Schoenfeld 1985b). Similar comments could apply to the location of authority in their respective classrooms. Anna's beliefs and practices were consistent; she did not favour student autonomy, arguing that it made them uncomfortable, and acted accordingly. Eleni, despite her poor self-perception as a problem solver and subsequent reluctance to relinquish authority, seemed keen not to deny her students the opportunity to engage meaningfully with the problem. Similarly, Stelios, while not being observed to surrender authority to his students, acted in ways indicative of his preparing his students for it.

Of course, it could be said that the problem around which this study was based afforded limited opportunities for students to be granted authority, not least because it was, superficially, tightly defined. However, a loosening of some of the constraints would have created new and interesting challenges. For example, what if the area of nine was changed, or the isosceles became right-angled? In other words, the problem has the potential not only for problematisation (Schoenfeld 2007, 2012) but the development of mathematical habits of mind (Cuoco et al. 1996). However, in none of the three classrooms was this observed, although Stelios, during both clinical interview and lesson, indicated a desire to create a context in which it did. In related vein, the extent to which teachers encouraged alternatives to themselves as the disciplinary authority varied. Anna's beliefs and actions were consistent with her view of the teacher as the disciplinary authority, a perspective supported by her frequent deferral to others during her clinical interview. Alternatively, in line with the perspectives espoused by Cobb, Stephan, McClain and Gravemeijer (2011) and despite her personal anxieties, Eleni seemed content, in a didactically controlled manner, to open up for scrutiny, both public and mathematical, the thinking of her students, something that Stelios incongruously seemed not to encourage.

In conclusion, the framework developed for this study has facilitated a profound analysis of three Cypriot primary teachers' problem solving-related beliefs, competence and practice. It

has enabled us to identify regularities and irregularities, some of which are immediately explicable and others not. For example, all espoused beliefs were unambiguously informed by the curriculum within which our teachers work. However, going beyond such superficial similarities, the framework has allowed us to view each teacher in ways that permit, in somewhat speculative terms, personal summaries that point to future research. Anna's beliefs and practice appear coherently and consistently related to deep-seated beliefs not only about mathematical problems as routine and time-limited but also that the teacher is the sole arbiter of what is appropriate classroom mathematical behaviour. She works hard to ensure that her students are appropriately prepared for whatever task she gives them and will do all she can to support their attainment of the restricted goal she has set them. Such beliefs and practices are, in essence, independent of her competence. Eleni, despite a lack of personal problem solving confidence, appears to have a professional courage that enables her to take risks in her classroom; she will grant opportunities for her students to assume responsibility, even in situations that make her mathematically uncomfortable. Stelios, the most mathematically confident of the three, presents, initially at least, a coherent picture of belief, competence and practice, in which mathematics is placed at the centre of the learning experience. However, his failure to allow a shift of disciplinary authority alludes to a teacher who can *talk a good game* but fail to understand underlying principles in ways that will meaningfully change the learning experience (Collins 1996; Frykholm 1999). In short, our data have allowed us to construct, albeit tentatively, an account for each teacher that is not inconsistent with Skott's (2013) notion of pattern of participation. This has been facilitated not only by the analytical framework developed for the task but also the decision to adopt a broad and inclusive perspective on the nature of beliefs (Op 't Eynde et al. 2002; Richardson 1996). That being said, our failure to undertake delayed post observation interviews was a disappointing omission that we hope to rectify in future work.

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