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ANALYSIS AND GEOMETRY ON MANIFOLDS WITH INTEGRAL RICCI CURVATURE BOUNDS. II

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Abstract. We extend several geometrical results for manifolds with lower Ricci curvature bounds to situations where one has integral lower bounds. In particular we generalize Colding's volume convergence results and extend the Cheeger-Colding splitting theorem.

1. Introduction

We shall in this paper establish several geometrical results for manifolds with integral bounds for their Ricci curvature. Our notation for the integral curvature bounds on a Riemannian manifold (M,g) is as follows. For each $x \in M$ let $r(x)$ denote the smallest eigenvalue for the Ricci tensor $\mathrm{Ric} : T_xM \to T_xM$, and define $\rho(x) = |\min\{0, r(x)\}|$,

$$\begin{aligned} k(p,R) &= \sup_{x\in M}\left(\int_{B(x,R)}\rho^p\right)^{\frac{1}{p}}, \\ \bar{k}(p,R) &= \sup_{x\in M}\left(\frac{1}{\mathrm{vol}B(x,R)}\cdot\int_{B(x,R)}\rho^p\right)^{\frac{1}{p}} = \sup_{x\in M}\|\rho\|_{p,B(x,R)}. \end{aligned}$$

Here and in the rest of the paper, the L^p norm on a domain is always normalized. Note that in earlier works the curvature quantities were defined as $(k(p,R))^p$ and $\left(\bar{k}(p,R)\right)^p$.

These curvature quantities evidently measure how much Ricci curvature lies below zero in the (normalized) integral sense. In our earlier work [15] we worked with the more general curvature quantity $\bar{k}(\lambda,p,R)$,

$$\bar{k}(\lambda,p,R) = \sup_{x\in M}\left(\frac{1}{\mathrm{vol}B(x,R)}\cdot\int_{B(x,R)}|\min\{0, r-\lambda\}|^p\right)^{\frac{1}{p}},$$

which measures how much curvature lies below a fixed number λ. It is a very simple matter to extend the appropriate results to this more general context. Therefore, we have for notational convenience only handled the case of $\lambda = 0$.

The main tools we use to get our results are D. Yang's estimates on Sobolev constants from [16] and the relative volume comparison for integral Ricci curvature

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we established earlier in [15]. Using these bounds for Sobolev constants, we can derive a maximum principle and gradient estimate in the integral setting. The maximum principle we need is the following:

Theorem 1.1. *Let M be an n-dimensional Riemannian manifold, let $B(x,1) \subset M$ have $\mathrm{vol}B(x,1) \geq v$, and let $p > n/2$. Then there exist $\varepsilon(n,p,v) > 0$ and $K(n,p,q,v) > 1$ such that, for $R \leq 1$, if $R^2 \cdot \bar{k}(p,R) \leq \varepsilon$, then any function $u : \Omega \subset B(x,R) \to \mathbb{R}$ with $\Delta u \geq -f$ satisfies*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + R^2 \cdot K \cdot \|f\|_q ,$$

for any $q > p$.

Note that $R^2 \cdot \bar{k}(p,R)$ is scale invariant, and its smallness condition, ε, doesn't depend on R.

The gradient estimate can be paraphrased as

Theorem 1.2. *Let M be an n-dimensional Riemannian manifold, let $B(x,1) \subset M$ have $\mathrm{vol}B(x,1) \geq v$, and let $p > n/2$. Then there exist $K(n,p,v) > 1$ and $\varepsilon(n,p,v) > 0$ such that, for $R \leq 1$, if $R^2 \cdot \bar{k}(p,R) \leq \varepsilon$, then any harmonic function $u : \Omega \subset B(x,R) \to \mathbb{R}$ satisfies*

$$\sup_{B\left(x,\frac{1}{2}R\right)} |\nabla u| \leq R^{-1} \cdot K \cdot \sup_{B(x,R)} |u|$$

These can be thought as the classical maximal principle and the Cheng-Yau gradient estimate [3] in the integral curvature setting.

The proofs are fairly standard iteration arguments and can found in [11]. Quick proofs of these estimates will be given in Section 3. These results are very useful in applications, even in a non-integral curvature setting. See [13] for more application. A global gradient estimate for sections of vector bundles was also obtained in [9].

The maximum principle will lead, in Section 4, to an extension of the Abresch-Gromoll excess estimate from [1] to a situation where one has integral curvature bounds.

Combining these three tools (maximum principle, gradient estimate and excess estimate) with the mean curvature comparison estimate in [15], we can generalize some of Colding's and Cheeger-Colding's work from [8], [6], [7], [5] and [4].

In Section 5 we prove the necessary changes which make it possible to establish Colding's volume convergence results from [8], [6].

Theorem 1.3. *Suppose a sequence of complete Riemannian n-manifolds (M_i, g_i) converges to a Riemannian n-manifold (N,g) in the pointed Gromov-Hausdorff topology. Then we can find $\varepsilon(n,p,\lambda,R) > 0$ such that if for all the manifolds we have $\bar{k}(p,\lambda,R) \leq \varepsilon$ and the points $x_i \in M_i$ converge to $x \in N$, then*

$$\mathrm{vol}B(x_i,r) \to \mathrm{vol}B(x,r)$$

for all $r < \frac{R}{8}$.

This together with the results from [5] immediately yields

Corollary 1.4. *Suppose a sequence of closed Riemannian n-manifolds (M_i, g_i) converges to a Riemannian n-manifold (N,g) in the Gromov-Hausdorff topology. For each $R > 0$ there is an $\varepsilon(n,p,\lambda,R) > 0$ such that if all the manifolds satisfy $\bar{k}(p,\lambda,R) \leq \varepsilon$, then they must be diffeomorphic to N for large i.*

As a converse to the volume convergence result we also have the following extension of results by Colding and Cheeger-Colding from [8], [7] and [5]. Let $v(n,\lambda,R)$ be the volume of an R-ball in S^n_λ, the simply connected constant curvature λ space form of dimension n.

Theorem 1.5. *Given $\epsilon > 0$, $R > 0$ and $\lambda \in \mathbb{R}$, then we can find $\varepsilon(n,p,R,\lambda) > 0$ and $\delta(n,p,R,\lambda) > 0$ such that if (M,g) is a Riemannian n-manifold with $\bar{k}(p,\lambda,R) \leq \varepsilon$ and $\operatorname{vol}B(x,R) \geq (1-\delta)\cdot v(n,\lambda,R)$ for some $x \in M$, then $B(x,r)$, $r < \frac{R}{8}$, is ϵ Gromov-Hausdorff close to an r-ball in the constant curvature λ simply connected space form of dimension n.*

Combining this with the above corollary, we get the following volume/curvature pinching result, which generalizes earlier work by Perel′man from [12].

Corollary 1.6. *Given $\lambda > 0$ and $0 < R < \pi/\sqrt{\lambda}$, we can find $\varepsilon(n,p,R,\lambda) > 0$ and $\delta(n,p,R,\lambda) > 0$ such that if (M,g) is a complete Riemannian n-manifold with $\bar{k}(p,\lambda,R) \leq \varepsilon$ and $\operatorname{vol}B(x,R) \geq (1-\delta)\cdot v(n,\lambda,R)$ for all $x \in M$, then M is diffeomorphic to a manifold with constant curvature $\lambda > 0$.*

This corollary evidently also holds in case $\lambda \leq 0$, but in this case we need to assume that the diameter of M is bounded, and consequently ε and δ will also depend on this diameter bound. The reason why we can get rid of this diameter bound in the positive case is that Petersen and Sprouse in [14] showed that the diameter bound is automatic in this setting.

In Section 6 we establish some of the estimates from [4] which lead to an extension of the splitting theorem. With these results one can in particular recapture the results from [5, Section 5 and 6] without further ado for classes of manifolds which do not collapse. As explained in [4], it is necessary to use some of the techniques from [8] that were devised for the noncollapsed situation.

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2. Preliminary material

2.1. **A Word on Scaling.** It is convenient to work with the scale invariant curvature quantity $R^2 \cdot \bar{k}(p,R)$. This makes many things easier, as one can then simply scale the metric and assume that one works under the assumption that $\bar{k}(p,1)$ is small. Note also that if one has a lower bound for the Ricci curvature $\operatorname{Ric} \geq (n-1)\lambda$, then the quantity $R^2 \cdot \bar{k}(p,R)$ will be small for sufficiently small R. Similarily one can also show that if $\bar{k}(p,\lambda,R)$ is small, then for small enough r we have that $r^2\bar{k}(p,r)$ is small.

2.2. **Relative Volume Comparison.** First we need a scale invariant modification (or local version) of the relative volume comparison theorem established in [15, Corollary 2.4]. This result also holds, as almost everything in this paper, when one merely has smallness on $R^2 \cdot \bar{k}(\lambda,p,R)$, but in this case the constants ε and α will necessarily also depend on R.

Theorem 2.1. *Given $\alpha < 1$ and $p > n/2$, there is an $\varepsilon = \varepsilon(n, p, \alpha) > 0$ such that if $R^2 \cdot \bar{k}(p, R) < \varepsilon$, then for all $x \in M^n$ and $r_1 < r_2 \leq R$ the following relative volume comparison holds:*

$$\alpha \cdot \frac{r_1^n}{r_2^n} \leq \frac{\mathrm{vol}B(x, r_1)}{\mathrm{vol}B(x, r_2)}.$$

Proof. From [15, Lemma 2.1] we know that

$$\frac{d}{dr}\frac{\mathrm{vol}B(x, r)}{r^n} \leq C_1(n, p)\left(\frac{\mathrm{vol}B(x, r)}{r^n}\right)^{1-\frac{1}{2p}}(k(p, R))^{\frac{1}{2}} r^{-\frac{n}{2p}}.$$

Thus

$$\left(\frac{\mathrm{vol}B(x, r_2)}{r_2^n}\right)^{\frac{1}{2p}} - \left(\frac{\mathrm{vol}B(x, r_1)}{r_1^n}\right)^{\frac{1}{2p}} \leq C_2(n, p)(k(p, R))^{\frac{1}{2}} R^{1-\frac{n}{2p}}. \tag{2.1}$$

Cross multiplying yields

$$\begin{aligned}
\left(\frac{r_1^n}{r_2^n}\right)^{\frac{1}{2p}} - \left(\frac{\mathrm{vol}B(x, r_1)}{\mathrm{vol}B(x, r_2)}\right)^{\frac{1}{2p}} &\leq \left(\frac{r_1^n}{\mathrm{vol}B(x, r_2)}\right)^{\frac{1}{2p}} C_2(n, p)(k(p, R))^{\frac{1}{2}} R^{1-\frac{n}{2p}} \\
&\leq c\left(\frac{r_1^n}{r_2^n}\right)^{\frac{1}{2p}},
\end{aligned}$$

$$c = \left(\frac{r_2^n}{\mathrm{vol}B(x, r_2)}\right)^{\frac{1}{2p}} C_2(n, p)(k(p, R))^{\frac{1}{2}} R^{1-\frac{n}{2p}},$$

which implies

$$(1-c)\left(\frac{r_1^n}{r_2^n}\right)^{\frac{1}{2p}} \leq \left(\frac{\mathrm{vol}B(x, r_1)}{\mathrm{vol}B(x, r_2)}\right)^{\frac{1}{2p}}.$$

Now if we consider c as a function of r_2, then we almost have that c is increasing. In fact using (2.1) again but replacing $r_1,\ r_2$ with $r_2,\ R$, we have

$$\begin{aligned}
\left(\frac{r_2^n}{\mathrm{vol}B(x, r_2)}\right)^{\frac{1}{2p}} &\leq \left(\left(\frac{\mathrm{vol}B(x, R)}{R^n}\right)^{\frac{1}{2p}} - C_2(n, p)(k(p, R))^{\frac{1}{2}} R^{1-\frac{n}{2p}}\right)^{-1} \\
&= \left(\frac{R^n}{\mathrm{vol}B(x, R)}\right)^{\frac{1}{2p}}\left(1 - C_2(n, p)\left(\bar{k}(p, R)\right)^{\frac{1}{2}} R\right)^{-1} \\
&\leq \left(\frac{R^n}{\mathrm{vol}B(x, R)}\right)^{\frac{1}{2p}}\left(1 - C_2(n, p)\,\varepsilon^{\frac{1}{2}}\right)^{-1} \\
&\leq 2\left(\frac{R^n}{\mathrm{vol}B(x, R)}\right)^{\frac{1}{2p}},
\end{aligned}$$

as long as we have assumed that ε is small enough. Using this estimate for c in the above situation now yields

$$\begin{aligned}
c &\leq 2\left(\frac{R^n}{\mathrm{vol}B(x, R)}\right)^{\frac{1}{2p}} C_2(n, p)(k(p, R))^{\frac{1}{2}} R^{1-\frac{n}{2p}} \\
&\leq 2C_2(n, p)\left(\bar{k}(p, R)\right)^{\frac{1}{2}} R \\
&\leq 2C_2(n, p)\,\varepsilon^{\frac{1}{2}}.
\end{aligned}$$

This can be made as small as we please, and so the theorem follows immediately. $\square$

If we merely assume that $\bar{k}(p,\lambda,R)$ is small and that $R<\pi/2\sqrt{\lambda}$ when $\lambda>0$, then we get a relative volume comparison of the type

$$\alpha\cdot\frac{v(n,\lambda,r_1)}{v(n,\lambda,r_2)}\leq\frac{\mathrm{vol}B(x,r_1)}{\mathrm{vol}B(x,r_2)},$$

except that the smallness of curvature now also depends on R.

2.3. **Curvature Inequalities.** It is perhaps worthwhile pointing out how the relative volume comparison can be used to compare the quantities $\bar{k}(p,r_1)$ and $\bar{k}(p,r_2)$ when $r_1<r_2$. Our claim is that

$$\bar{k}(p,r_1)\leq C_1(n,p)\cdot\left(\frac{r_2}{r_1}\right)^{\frac{n}{p}}\cdot\bar{k}(p,r_2),$$

or

$$r_1^2\bar{k}(p,r_1)\leq C_1(n,p)\cdot\left(\frac{r_2}{r_1}\right)^{\frac{n}{p}-2}\cdot r_2^2\bar{k}(p,r_2),$$

provided $r_2^2\cdot\bar{k}(p,r_2)\leq\varepsilon(n,p,1/2)$ for the ε in Theorem 2.1.

To prove this inequality we use the trivial inequality

$$\bar{k}(p,r_1)\leq\sup_{x\in M}\left(\frac{\mathrm{vol}B(x,r_2)}{\mathrm{vol}B(x,r_1)}\right)^{\frac{1}{p}}\cdot\bar{k}(p,r_2)$$

and then use Theorem 2.1 on the ratio $\frac{\mathrm{vol}B(x,r_2)}{\mathrm{vol}B(x,r_1)}$ with $\alpha=1/2$.

The importance of this curvature inequality lies in the fact when working with closed manifolds with bounded diameter there is no restriction in working with $\bar{k}(p,1)$ rather than the more global constant $\bar{k}(p,D_M)$.

In the other direction one can also bound $\bar{k}(p,r_2)$ in terms of $\bar{k}(p,r_1)$. To do so requires a packing argument (see also [2, Lemma 1.4]). Assuming that $\bar{k}(p,r_1)$ is small, we get a relative volume comparison on balls of radius $<r_1$. Now select a maximal family of disjoint balls $B(x_i,r_1/2)$ with centers in $B(x,r_2)$; then the balls $B(x_i,r_1)$ cover $B(x,r_2)$. Thus

$$\begin{aligned}
\left(\bar{k}(p,r_2)\right)^p &= \frac{1}{\mathrm{vol}B(x,r_2)}\int_{B(x,r_2)}\rho^p\\
&\leq \sum_i\frac{1}{\mathrm{vol}B(x,r_2)}\int_{B(x_i,r_1)}\rho^p\\
&\leq \sum_i\frac{\mathrm{vol}B(x_i,r_1)}{\mathrm{vol}B(x,r_2)}\left(\bar{k}(p,r_1)\right)^p\\
&\leq \max\frac{\mathrm{vol}B(x_i,r_1)}{\mathrm{vol}B(x_i,r_1/2)}\sum_i\frac{\mathrm{vol}B(x_i,r_1/2)}{\mathrm{vol}B(x,r_2)}\left(\bar{k}(p,r_1)\right)^p\\
&\leq \max\frac{\mathrm{vol}B(x_i,r_1)}{\mathrm{vol}B(x_i,r_1/2)}\left(\bar{k}(p,r_1)\right)^p.
\end{aligned}$$

Now the quantity

$$\max\frac{\mathrm{vol}B(x_i,r_1)}{\mathrm{vol}B(x_i,r_1/2)}$$

is bounded just in terms of the smallness of $r_1^2\bar{k}(p,r_1)$.

2.4. **Sobolev Constants and Eigenvalues.** We shall also need to use some of the estimates for Sobolev constants obtained in [16, Theorem 7.4]. Recall that the L^p norm on a domain Ω is normalized, i.e.

$$\|u\|_{p,\Omega} = \left(\frac{1}{\mathrm{vol}\Omega} \int_\Omega |u|^p \right)^{\frac{1}{p}}.$$

The precise statement of what we need is

Theorem 2.2. *Suppose $R = 1$. Given $p > n/2$ and $v > 0$, there is an $\varepsilon(n,p,v) > 0$ such that if $B(x,1) \subset M^n$ has $\mathrm{vol}\, B(x,1) \geq v$ and $\bar{k}(p,1) \leq \varepsilon$, then the Sobolev constant $\lambda(2p,2)$ which occurs in*

$$\lambda(2p,2) \cdot \|u\|_{\frac{2p}{p-1},B(x,1)} \leq \|\nabla u\|_{2,B(x,1)}$$

can be bounded from below:

$$\lambda(2p,2) \geq C(n,p,v).$$

To get a bound for the Sobolev constant for general R we need only rescale the metric g by R^{-2}; then the scale invariant curvature quantity $\bar{k}_g(p,R) \cdot R^2$ becomes $\bar{k}_{R^{-2}g}(p,1)$. Thus we need to assume that $\bar{k}_g(p,R) \cdot R^2 \leq \varepsilon$ and $\mathrm{vol}B(x,R) \geq vR^n$. Using Theorem 2.1, for $R \leq 1$, the local volume growth automatically follows from smallness of curvature and $B(x,1) \geq v$. Now let $\lambda(2p,2,R)$ be the Sobolev constant in the inequality

$$\lambda(2p,2,R) \cdot \|u\|_{\frac{2p}{p-1},B(x,R)} \leq \|\nabla u\|_{2,B(x,R)}.$$

Here the only term that makes a difference when scaling is $|\nabla u|$, and this term scales like R^{-1}. Thus we must have

$$R \cdot \lambda_g(2p,2,R) = \lambda_{R^{-2}g}(2p,2).$$

Hence we obtain the estimate

$$\lambda_g(2p,2,R) \geq R^{-1} C(n,p,v).$$

Similarly to the analysis above, we also get a bound for the first eigenvalue $\lambda_1(R)$. Namely for $\lambda_1(R)$, defined by

$$\lambda_1(R) \cdot \int_{B(x,R)} |u|^2 d\mathrm{vol}_g \leq \int_{B(x,R)} |\nabla u|^2 d\mathrm{vol}_g, \text{ for all } x \in M,$$

we have

$$\lambda_1(R) \geq R^{-2} C(n,p,v)$$

provided $\bar{k}_g(p,R) \cdot R^2 \leq \varepsilon$ is small enough and $\mathrm{vol}B(x,1) \geq v$.

Finally we should point out that in case $\lambda > 0$ we can use the diameter bound from [14] together with the global Sobolev and eigenvalue constant bounds from [10] to get bounds for these quantities only in terms of λ, n, p. Thus in this case they do not depend on a lower volume bound. For our applicatioons, however, this makes no significant difference.

3. Maximum principle and gradient estimate

In this section we shall use the above Sobolev constant estimate (Theorem 2.2) to establish a maximum principle and a gradient estimate in the integral setting. The proofs of these are the standard iteration and are probably known to experts. But the exact setup is not in the literature. For completeness, we present a proof.

3.1. **The Maximum Principle.**

Theorem 3.1. *Let M be an n-dimensional manifold and $\Omega \subset B(x,R) \subset M$. Then for any function u on Ω with $u_{|\partial\Omega} = 0$ we have*

$$\|u\|_\infty \le \lambda^{-2}(2s,2,R) \cdot K(s,q) \cdot \|\Delta u\|_{q,B(x,R)},$$

where $q > s \ge \frac{n}{2}$, and $K(s,q)$ is a constant depending on s and q.

Proof. Since the domain we use is $B(x,R)$ throughout the proof, we omit it from the norm. To conform with our notations, our L^2-inner product is also normalized. The theorem follows directly from iteration and the Sobolev inequality

$$\lambda(2s,2,R) \cdot \|v\|_{\frac{2s}{s-1}} \le \|\nabla v\|_2, \tag{3.1}$$

for $s \ge \frac{n}{2}$ and v any function which vanishes on $\partial\Omega$. In the sequel we shall abbreviate $\lambda = \lambda(2s,2,R)$.

Using (3.1), we have on one hand that for any $l > 0$

$$\begin{aligned} \left\|u^l \Delta u\right\|_1 \ge \left|\left(u^l, \Delta u\right)\right| &= l \cdot \left|\left(u^{l-1}\nabla u, \nabla u\right)\right| \\ &= \frac{4l}{(l+1)^2} \cdot \left\|\nabla\left(u^{\frac{l+1}{2}}\right)\right\|_2^2 \\ &\ge \frac{4l}{(l+1)^2} \cdot \lambda^2 \cdot \left\|u^{\frac{l+1}{2}}\right\|_{\frac{2s}{s-1}}^2 . \end{aligned}$$

On the other hand,

$$\left\|u^l \Delta u\right\|_1 \le \left\|u^l\right\|_{\frac{q}{q-1}} \cdot \|\Delta u\|_q .$$

Thus

$$\frac{4l}{(l+1)^2} \cdot \lambda^2 \cdot \left\|u^{\frac{l+1}{2}}\right\|_{\frac{2s}{s-1}}^2 \le \left\|u^l\right\|_{\frac{q}{q-1}} \cdot \|\Delta u\|_q .$$

Suppose $q > s \ge \frac{n}{2}$ are fixed; then

$$\begin{aligned} \frac{(l+1)\cdot s}{s-1} &= \frac{l \cdot s}{s-1} + \frac{s}{s-1} \\ &= \nu \cdot \frac{l \cdot q}{q-1} + \frac{s}{s-1}, \end{aligned}$$

where

$$\nu = \frac{s}{s-1} \cdot \frac{q-1}{q} > 1.$$

Now for any $p_0 > 0$, define

$$\begin{aligned} p_k &= \nu \cdot p_{k-1} + \frac{s}{s-1} \\ &= \nu^k \cdot p_0 + \left(\nu^{k-1} + \cdots + 1\right) \cdot \frac{s}{s-1} \\ &= \nu^k \cdot p_0 + \frac{\nu^k - 1}{\nu - 1} \cdot \frac{s}{s-1}. \end{aligned}$$

The above formula can then be written as

$$\|u\|_{p_k} \le \left(\|u\|_{p_{k-1}}\right)^{\nu \frac{p_{k-1}}{p_k}} \cdot \left(\lambda^{-2} \cdot \|\Delta u\|_q\right)^{\frac{s}{(s-1)p_k}} \cdot \left(\frac{p_k^2}{p_{k-1}}\right)^{\frac{s}{(s-1)p_k}} .$$

Iteration of this inequality from 0 to k yields

$$\|u\|_{p_k} \leq \left(\|u\|_{p_0}\right)^{\frac{\nu^k}{p_k}p_0} \cdot \left(\lambda^{-2} \cdot \|\Delta u\|_q\right)^{\frac{s}{(s-1)}\frac{1}{p_k}\sum_{i=0}^{k-1}\nu^i} \cdot \left(\prod_{i=1}^{k}\left(\frac{p_i^2}{p_{i-1}}\right)^{\frac{\nu^{k-i}}{p_k}}\right)^{\frac{s}{(s-1)}}.$$

Now let $k \to \infty$; then

$$\|u\|_\infty \leq \left(\|u\|_{p_0}\right)^{\frac{(\nu-1)(s-1)p_0}{(\nu-1)(s-1)p_0+s}} \cdot \left(\lambda^{-2} \cdot \|\Delta u\|_q \cdot \frac{1}{p_0}\prod_{i=1}^{\infty}(p_i)^{\frac{1}{\nu^i}}\right)^{\frac{s}{(\nu-1)(s-1)p_0+s}}.$$

Here we should observe that $\prod_{i=1}^{\infty}(p_i)^{\frac{1}{\nu^i}}$ converges, as we can compare the series $\sum \frac{\log p_i}{\nu^i}$ with the convergent series $\sum \frac{i}{\nu^i}$. Now

$$\|u\|_{p_0} \leq \|u\|_\infty .$$

Using this in the above, we have

$$\|u\|_\infty \leq \|\Delta u\|_q \cdot \lambda^{-2}(2s, 2, R) \cdot K(p_0, s, q).$$

Letting $p_0 = 1$ finishes our proof. □

Corollary 3.2. *Let M be an n-dimensional manifold and $\Omega \subset B(x,R) \subset M$ a domain. Then for any function u on Ω with $\Delta u \geq -f$, where f is non-negative on Ω, we have*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u + \lambda^{-2}(2s,2,R) \cdot K(s,q) \cdot \|f\|_q$$

for any $q > s \geq \frac{n}{2}$.

Proof. We can without loss of generality assume that $\sup\{u(x) : x \in \partial\Omega\} = 0$. Now solve the Dirichlet problem

$$\begin{aligned} \triangle v &= -f, \\ v &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then we have that $u - v$ is subharmonic, i.e., $\triangle(u-v) \geq 0$, and $u - v \leq 0$ on $\partial\Omega$. The maximum principle then implies that $\sup u \leq \|v\|_\infty$. We can then use the above theorem on v to obtain the desired inequality. □

Proof of Theorem 1.1. By Theorem 2.2, for $p > \frac{n}{2}$, the smallness of $R^2 \cdot \bar{k}(p,R)$ implies that $\lambda^{-2}(2p,2,R) \leq R^2 (C(n,p,v))^{-2}$. This gives Theorem 1.1. □

3.2. **The Gradient Estimate.** We shall suppose that M is a complete manifold. The Sobolev constant $\lambda(2p,2,2)$ for balls of radius 2 is simply denoted by λ. Let $u : B(x,2) \to \mathbb{R}$ be a harmonic function. Fix a bump function $\varphi : B(x,2) \to \mathbb{R}$ such that $\varphi = 1$ on $B(x,1)$ and $\mathrm{supp}\varphi \subset B(x,2)$. Thus all derivatives of φ vanish near the boundary of $B(x,2)$. Such a bump function can be chosen in such a way that $|\nabla\varphi|$ is bounded independently of the manifold.

Theorem 3.3. *Suppose $m > p \geq \frac{n}{2}$ are fixed. There is a constant $C(n,m,p)$ such that any harmonic function $u : B(x,2) \to \mathbb{R}$ satisfies*

$$\begin{aligned} \sup_{B(x,1)} |\nabla u| &\leq C \cdot \|u\|_{2,B(x,2)} \cdot \left(1 + \bar{k}(m,2)\right)^{\frac{1}{2}\frac{mp}{m-p}} \cdot \lambda^{-\frac{mp}{m-p}} \\ &\leq C \cdot \sup_{B(x,2)} |u| \cdot \left(1 + \bar{k}(m,2)\right)^{\frac{1}{2}\frac{mp}{m-p}} \cdot \lambda^{-\frac{mp}{m-p}}. \end{aligned}$$

As a corollary we get a similar estimate for the gradient on balls of arbitrary size. If we wish to consider a ball of size $2R$, we can simply change the metric from g to $R^{-2}g$, use the above estimate, and then scale back. We could also change the size to R instead of $2R$; doing so will give us the estimate

$$\sup_{B\left(x,\frac{1}{2}R\right)} |\nabla u| \\ \leq R^{-1} \cdot C \cdot \sup_{B(x,R)} |u| \cdot \left(1 + R^2 \cdot \bar{k}(m,R)\right)^{\frac{1}{2}\frac{mp}{m-p}} \cdot (R \cdot \lambda(2p,2,R))^{-\frac{mp}{m-p}}.$$

We know that smallness of $R^2 \cdot \bar{k}(p,R)$ yields a bound on the Sobolev constant $\lambda(2p,2,R) \geq R^{-1}C(n,p,v)$ for any $p > \frac{n}{2}$. Therefore, we can, with some change in notation, conclude that if $R^2 \cdot \bar{k}(p,R) \leq \varepsilon(n,p,q)$ is sufficiently small to give a bound for the Sobolev constant $\lambda(2q,2,R)$ with $p > q > \frac{n}{2}$, then there is an estimate of the form

$$\sup_{B\left(x,\frac{1}{2}R\right)} |\nabla u| \leq R^{-1} \cdot C \cdot \sup_{B(x,R)} |u| \cdot (1+\varepsilon)^{\frac{1}{2}\frac{pq}{p-q}}.$$

Here we have some freedom in choosing q, but this will not be important. The only thing to notice is that the exponent must increase with the dimension. Thus we can't just have it be a fixed number. However, as the exponent is an increasing function of q, it must always be larger than $\frac{pn}{4p-2n}$.

3.3. **Proof of the Gradient Estimate.** Applying the Bochner formula to ∇u, we get

$$|\nabla u|(\Delta|\nabla u|) + |\nabla|\nabla u||^2 = \Delta\frac{1}{2}|\nabla u|^2 = |\mathrm{Hess}u|^2 + \mathrm{Ric}(\nabla u, \nabla u).$$

Using the notation that $\rho = |\min\{0, r(x)\}|$, where $r(x)$ is the smallest eigenvalue of Ric, we have, from Kato's inequality,

$$|\nabla u|(\Delta|\nabla u|) + |\nabla|\nabla u||^2 = \Delta\frac{1}{2}|\nabla u|^2 \geq |\nabla|\nabla u||^2 - \rho|\nabla u|^2,$$

which immediately implies that

$$\Delta|\nabla u| \geq -\rho|\nabla u|.$$

Using the Sobolev inequality and the iteration as in [17, Appendix B] (note that D. Yang uses a stronger Sobolev constant, but this doesn't alter the proof significantly), we have

$$\|\varphi\nabla u\|_\infty \leq \|\varphi^\alpha \nabla u\|_2 \cdot C(p,q,m) \cdot \left(\|1+\rho\|_m \cdot \lambda^{-2}\right)^{\frac{1}{2}\frac{mp}{m-p}}, \quad \alpha = \exp(-\frac{mp}{m-p}).$$

Since $\Delta u = 0$, we have

$$\begin{aligned}\int_{B(x,2)} \varphi^2|\nabla u|^2 &\leq 2\int_{B(x,2)} (|\nabla\varphi|u)(\varphi|\nabla u|) \\ &\leq 2\left(\int_{B(x,2)} |\nabla\varphi|^2 u^2\right)^{\frac{1}{2}} \left(\int_{B(x,2)} \varphi^2|\nabla u|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\|\varphi \cdot \nabla u\|_2 \leq 2\|\nabla\varphi \cdot u\|_2.$$

Therefore

$$\|\varphi\nabla u\|_\infty \leq C(p,m)\cdot \|u\cdot|\nabla\varphi^\alpha|\|_2\cdot\left(\|1+\rho\|_m\cdot\lambda^{-2}\right)^{\frac{1}{2}\frac{mp}{m-p}},$$

$$\begin{aligned}\sup_{B(x,1)}|\nabla u| &\leq C\cdot\|u\|_2\cdot\left(\|1+\rho\|_m\cdot\lambda^{-2}\right)^{\frac{1}{2}\frac{mp}{m-p}}\\ &\leq C\cdot\sup_{B(x,2)}|u|\cdot\left(1+\bar{k}(m,2)\right)^{\frac{1}{2}\frac{mp}{m-p}}\cdot\lambda^{-\frac{mp}{m-p}}.\end{aligned}$$

4. The excess estimate

4.1. **The Result.** We shall in this section extend the Abresch-Gromoll excess estimate. The treatment is self-contained, but runs parallel to their proof with the modification that we must use Theorem 1.1 in place of the standard maximum principle.

We are given a complete Riemannian manifold M^n and two points $q_-, q_+ \in M$. The excess function for this pair is

$$e_{q_-,q_+}(x) = d(q_-,x) + d(x,q_+) - d(q_-,q_+).$$

This is a non-negative function which measures the defect in the triangle inequality. The object is to find an estimate for $e_{q_-,q_+}(x)$ in terms of the height function $h(x)$, which measures the shortest distance from x to a segment from q_- to q_+. In [1] the authors obtain

Theorem 4.1. *Suppose that* $\mathrm{Ric}M \geq (n-1)\lambda$. *Then there is a continuous function* $E(r)$ *with* $E(0)=0$, *which only depends on* $\lambda\cdot R^2$ *and* n, *such that*

$$e_{q_-,q_+}(x) \leq h(x)\cdot E\left(\frac{h(x)}{s(x)}\right),$$

where $s(x) = \min\{d(q_-,x), d(x,q_+)\}$ *and* $d(x,q_-), d(x,q_+) \leq R/2$.

This inequality was generalized in [4, Section 6] to a slightly more general situation. Since we are primarily interested in extending the results from [4], we shall present this generalized form here.

Theorem 4.2. *Given* $\tau > 0$, $r > 0, v > 0$, *there exist* $L = L(n,p,\tau,v) > 1$ *and* $\varepsilon = \varepsilon(n,p,\tau,v) \in \left(0,(L-1)^{-1}\right)$ *such that for any* $y_0, q_-, q_+ \in M$ *with the properties*

$$\max\{d(y_0,q_+), d(y_0,q_-)\} \leq R/3,$$

$$\min\{d(y_0,q_+), d(y_0,q_-)\} \geq Lr,$$

$$e_{q_-,q_+}(y_0) \leq \varepsilon r,$$

$$R^2\bar{k}(p,R) \leq \varepsilon,$$

$$\mathrm{vol}\,B(y_0,R) > vR^n,$$

we have

$$\sup_{x\in B(y_0,r/2)} e_{q_-,q_+}(x) \leq \tau r.$$

4.2. **Some Applications.** It is worthwhile clarifying the importance of this excess estimate in a few different contexts.

Suppose that we have a sequence of pointed complete Riemannian manifolds (M_i, g_i, x_i) which converges to a limit space (X, d, x) in the pointed Gromov-Hausdorff topology. First observe that a unit speed curve $\ell : \mathbb{R} \to X$ is a line $\ell(t)$ iff there are points $y_i, p_+^i, p_-^i \in M_i$, $\varepsilon_i \to 0$, and $L_i \to \infty$ such that $y_i \to \ell(0)$, segments from y_i to p_+^i and p_-^i respectively converge to the rays $\ell^+(t) = \ell(t), t \geq 0$, and $\ell^-(t) = \ell(-t), t \geq 0$, respectively, and

$$e_{p_+^i, p_-^i}(y_i) \leq \varepsilon_i,$$

$$\min\left\{d\left(y_i, p_+^i\right), d\left(y_i, p_-^i\right)\right\} \geq L_i.$$

If we know also that $R_i^2 \bar{k}_{g_i}(p, R_i) \to 0$, with $R_i \geq 3\max\left\{d\left(y_i, p_+^i\right), d\left(y_i, p_-^i\right)\right\}$, and the sequence doesn't collapse, the excess estimates on M_i with $r = 1$ carry over to the limit space X. More precisely, the sum of the Busemann functions $b^+ + b^-$ in X is simply the limit of the excess functions $e_{p_+^i, p_-^i}$. The excess estimates for these functions carry over to the limit space. However, we know that for each point y_0 on the line the sum $(b^+ + b^-)(y_0)$ is zero. As both $L_i \to \infty$ and $R_i^2 \bar{k}_{g_i}(p, R_i) \to 0$, it must therefore follow that $b^+ + b^-$ is zero in a tubular neighborhood of size $1/2$ around ℓ. This in turn makes it extremely plausible that X must split along the line ℓ. However, some more work is needed in order to show this. Having shown the splitting on the size $1/2$ neighborhood of the line, we then obtain a global splitting by continuing the argument for nearby lines using the same argument just described. In order to pass to the limit it is of course necessary to have that $R_i^2 \bar{k}_{g_i}(p, R_i)$ is somewhat smaller than L_i^{-1}. This can be achieved artificially by simply decreasing each of the L_i while still making sure that they go to infinity.

The most important case where this is used is in the situation where we have a sequence of metrics (M_i, g_i) where $\bar{k}_{g_i}(p, 1) \leq \varepsilon$ is small. If we rescale these metrics by letting $\tilde{g}_i = r_i^{-2} g_i$, where $r_i \to \infty$, then we obtain a sequence of metrics with $r_i^2 \bar{k}_{\tilde{g}_i}(p, r_i) \leq \varepsilon$. Now let $R_i < r_i$; then our curvature inequality from Section 2.3 tells us that

$$R_i^2 \bar{k}_{\tilde{g}_i}(p, R_i) \leq C(n, p)\left(\frac{R_i}{r_i}\right)^{2p-n} \varepsilon.$$

Therefore, if, e.g., $R_i = \sqrt{r_i}$, then $R_i^2 \bar{k}_{g_i}(p, R_i) \to 0$ and $R_i \to \infty$. We are therefore in a situation where the above discussion applies.

In fact it follows from the proof of the excess estimate that it suffices to assume that $r_i^2 \bar{k}_{\tilde{g}_i}(p, r_i) \leq \varepsilon$ as long as $r_i \to \infty$. Thus it is not necessary to worry about choosing R_i. It turns out that the condition $r_i \to \infty$ together with $r_i^2 \bar{k}_{\tilde{g}_i}(p, r_i) \leq \varepsilon$ still gives us the desired excess estimate.

This is particularly useful in case we have pointed Gromov-Hausdorff convergence $(M_i, g_i, x_i) \to (X, d, x)$ with the additional information that $\bar{k}_{g_i}(p, 1)$ is small. The local structure of X is governed by its infinitesimal structure, which in turn can be studied using rescaled sequences $\left(M_i, r_i^{-2} g_i, p_i\right)$. For such sequences we evidently have that $R_i^2 \bar{k}_{r_i^{-2} g_i}(p, R_i)$ goes to zero for appropriately chosen R_i. Thus, whatever techniques are developed for such sequences can be applied to study the infinitesimal structure of the limit space.

4.3. **Proof of the Excess Estimate.** We now give the proof of the above stated excess estimate.

We can assume without loss of generality that $r = 1$. Define

$$u(x) = e_{q_-,q_+}(x).$$

Then $u(x)$ satisfies:

1) $u \geq 0$,
2) $u(y_0) \leq \varepsilon$,
3) u is a Lipschitz function with Lipschitz constant 2, and
4) $\Delta u \leq 2(n-1)/(L-1) + \psi_+ + \psi_-$.

Here $\psi_\pm$ is defined as the positive difference between the Laplacian of the distance function $d(\cdot, q_\pm)$ and the comparison Laplacian in Euclidean space $(n-1)/d(\cdot, q_\pm)$. In other words, for the distance function $d(\cdot, q_\pm)$ we have

$$\Delta d(\cdot, q_\pm) \leq (n-1)/d(\cdot, q_\pm) + \psi_\pm.$$

Since u is the sum of two distance functions which are both larger than $L-1$ on the ball $B(y_0, 1/2)$, we obtain the desired Laplacian estimate. Note that, in addition, if $d(y, y_0) \leq 1/2$, then each $\psi_\pm$ can be estimated by

$$\begin{aligned}
&\int_{B(y,1)} \psi_\pm^{2p} d\mathrm{vol} \\
&\leq \int_{B(q_\pm, R/3+1/2+1)} \psi_\pm^{2p} d\mathrm{vol} \\
&\leq C_1(n,p) \int_{B(q_\pm, R/3+1/2+1)} \rho^p d\mathrm{vol} \\
&\leq C_1(n,p) \cdot \mathrm{vol}B(q_\pm, R/3+1/2+1) \cdot \left(\bar{k}(p, R/3+1/2+1)\right)^p \\
&\leq C_2(n,p) \cdot \mathrm{vol}B(q_\pm, R/3+1/2+1) \cdot \left(\frac{R}{R/3+1/2+1}\right)^n \cdot \left(\bar{k}(p,R)\right)^p \\
&\leq C_3(n,p) \cdot \mathrm{vol}B(q_\pm, R/3+1/2+1) \cdot R^{-2p} \cdot \varepsilon^p;
\end{aligned}$$

here we have used the estimate of Lemma 2.2 in [15] in the second inequality and the curvature inequality of Section 2.3 in the fourth inequality. Thus we get

$$\begin{aligned}
\|\psi_\pm\|_{2p,B(y,1)} &\leq C_4(n,p) \cdot \left(\frac{\mathrm{vol}B(q_\pm, R/3+1/2+1)}{\mathrm{vol}B(y,1)}\right)^{\frac{1}{2p}} \cdot R^{-1} \cdot \varepsilon^{\frac{1}{2}} \\
&\leq C_4(n,p) \cdot \left(\frac{\mathrm{vol}B(y,R)}{\mathrm{vol}B(y,1)}\right)^{\frac{1}{2p}} \cdot R^{-1} \cdot \varepsilon^{\frac{1}{2}} \\
&\leq C_5(n,p) \cdot R^{\frac{n}{2p}} \cdot R^{-1} \cdot \varepsilon^{\frac{1}{2}} \\
&= C_5(n,p) \cdot R^{\frac{n}{2p}-1} \cdot \varepsilon^{\frac{1}{2}}.
\end{aligned}$$

If $G : \mathbb{R} \to \mathbb{R}$, then

$$\triangle\, G \circ d = G'' \circ d + \triangle\, d \cdot G' \circ d.$$

In the case where $G' \leq 0$ we therefore have

$$\begin{aligned}
\triangle\, G \circ d &= G'' \circ d + \triangle\, d \cdot G' \circ d \\
&\geq G'' \circ d + \frac{n-1}{d} \cdot G' \circ d + \psi \cdot G' \circ d,
\end{aligned}$$

where $\psi = (\triangle\, d - \frac{n-1}{d})_+$. We can define the function G on $[0,1]$ as

$$\begin{aligned} G(r) &= \frac{2(n-1)}{L-1}\int_r^1\int_r^t\left(\frac{t}{s}\right)^{n-1} dsdt \\ &= \frac{n-1}{L-1}\frac{2r^{2-n}-n+(n-2)r^2}{n(n-2)}. \end{aligned}$$

This function clearly has the properties

$$\begin{aligned} G &> 0 \quad \text{on } (0,1), \\ G' &< 0 \quad \text{on } (0,1), \\ G(1) &= 0, \\ G'' + \frac{n-1}{r}\cdot G' &= \frac{2(n-1)}{L-1}. \end{aligned}$$

Consequently we obtain

$$\triangle\, G\circ d \geq \frac{2(n-1)}{L-1} + \psi\cdot G'\circ d.$$

Thus the function $v = G\circ d - u$ satisfies

$$\triangle\, v \geq \psi\cdot G'\circ d - \psi_+ - \psi_-. \tag{4.1}$$

Now fix $y \in B(y_0, 1/2)$, consider a domain of the type $\Omega = B(y,1) - B(y,c)$, and let d be the distance function to y. If $y_0 \in \Omega$, then we have that $d(y_0) < 1/2$, so, as long as $n > 2$, we have

$$\begin{aligned} v(y_0) &= G\circ d(y_0) - u(y_0) \\ &> G(1/2) - \varepsilon \\ &= \frac{n-1}{L-1}\frac{2^{n-1}-n+(n-2)1/4}{n(n-2)} - \varepsilon \\ &> 0, \end{aligned}$$

provided ε is chosen sufficiently small. This is the place where ε gets to depend on L. Applying Theorem 1.1 (maximum principle) to (4.1), we can then conclude that

$$\begin{aligned} 0 &< \sup_\Omega v \leq \sup_{\partial\Omega} v + K\cdot\|\psi\cdot G'\circ d - \psi_+ - \psi_-\|_{2p,B(y,1)} \\ &\leq \sup_{\partial\Omega} v + K\cdot\left(\|\psi\|_{2p,B(y,1)}\cdot\sup_{[c,1]}|G'| + \|\psi_+ + \psi_-\|_{2p,B(y,1)}\right) \\ &\leq \sup_{\partial\Omega} v + K_2\cdot R^{\frac{n}{2p}-1}\cdot\varepsilon^{\frac12}\cdot\left(\sup_{[c,1]}|G'| + 2\right). \end{aligned}$$

Here we used estimate (4.1) for $\psi_\pm$, Lemma 2.2 in [15] for ψ, and K_2 depends only on n, p and the smallness of ε. All L^{2p} norms are taken on the domain Ω. In other words,

$$-\sup_{\partial\Omega} v \leq K_2\cdot R^{\frac{n}{2p}-1}\cdot\varepsilon^{\frac12}\cdot\left(\sup_{[c,1]}|G'| + 2\right).$$

Now we have

$$\sup_{\partial\Omega} v \leq G(c) - \inf_{\partial B(y,c)} u$$

since $G-u$ is non-positive on $\partial B(y,1)$. In addition we always have

$$u(y) \leq 2c + \inf_{\partial B(y,c)} u,$$

since dil$u \leq 2$. Therefore, we obtain

$$\begin{aligned} u(y) &\leq 2c + \inf_{\partial B(y,c)} u \\ &\leq 2c + G(c) - \sup_{\partial\Omega} v \\ &\leq 2c + G(c) + \varepsilon^{\frac{1}{2}} \cdot R^{\frac{n}{2p}-1} \cdot K_2 \cdot \left(\sup_{[c,1]} |G'| + 2\right). \end{aligned}$$

If on the other hand $y_0 \in B(y,c)$, then we certainly have

$$u(y) \leq 2c,$$

since we assumed that u has dil$u \leq 2$.

It then remains to compute the two quantities $G(c)$, $\sup_{[c,1]} |G'|$ and select c appropriately in such a way that the desired conclusion will hold. First we have

$$\begin{aligned} G(c) &= \frac{2(n-1)}{L-1} \int_c^1 \int_c^t \left(\frac{t}{s}\right)^{n-1} dsdt \\ &\leq \frac{2(n-1)}{L-1} \left(\frac{1}{c}\right)^{n-1}. \end{aligned}$$

and

$$\begin{aligned} \sup_{r\in[c,1]} |G'(r)| &= \frac{2(n-1)}{L-1} \sup_{r\in[c,1]} \left|\int_r^1 \left(\frac{r}{s}\right)^{n-1} ds\right| \\ &\leq \frac{2(n-1)}{L-1} \left(\frac{1}{c}\right)^{n-1}. \end{aligned}$$

Now let $c = L^{-\frac{1}{n}}$; then we get

$$u(y) \leq 2L^{-\frac{1}{n}} + \frac{2(n-1)}{L-1} L^{\frac{n-1}{n}} + \varepsilon^{\frac{1}{2}} \cdot R^{\frac{n}{2p}-1} \cdot K_2 \cdot \left(\frac{2(n-1)}{L-1} L^{\frac{n-1}{n}} + 2\right).$$

Note that this expression goes to zero as $L \to \infty$ and $\varepsilon \to 0$. Therefore, it is possible to make $u(y)$ smaller than any prescribed τ, given that ε is sufficiently small and L sufficiently large. This finishes the proof. □

Note that we could also make u small while keeping ε fixed if instead we let $R \to \infty$. Letting $R \to \infty$ and keeping ε fixed of course still forces the amount of Ricci curvature which lies below zero to converge to zero in the integral sense, so it is not as if we are getting a free lunch out of this. However, as pointed out above, it is convenient when rescaling not to have to worry about getting new curvature estimate on smaller scales.

5. Volume convergence

We shall now use the gradient estimate to generalize some of Colding's work in [8], which is in the noncollapsing situation. Our proofs are along the same lines. Namely, as soon as one has relative volume comparison and a gradient estimate, then most of his results follow almost immediately.

5.1. **Hessian Estimates.** We present here some of the key inequalities. As usual we assume that $R^2\bar{k}(p,R) \leq \varepsilon$ is small. The results can easily be done in the context where one has smallness for the amount of curvature that lies below a fixed number, but for simplicity we only cover the case where $R^2\bar{k}(p,R)$ is small.

Lemma 5.1. *Let $\Omega \subset M$ be a domain with boundary $\partial\Omega$ in a complete Riemannian manifold. If $d(x,y) = d(x)$ denotes the distance to $y \notin \Omega$ and $f:(0,\infty) \to (0,\infty)$ is a nondecreasing function, then*

$$\begin{aligned}\|\Delta(f\circ d)\|_{1,\Omega} \leq 2\max_{x\in\Omega}&\left(f''(d(x)) + \frac{n-1}{d(x)}\cdot f'(d(x))\right)_+\\ &+\frac{\mathrm{vol}\partial\Omega}{\mathrm{vol}\Omega}\cdot\max_{x\in\partial\Omega} f'(d(x))\\ &+2\cdot\left\|\left(m-\frac{n-1}{d(x)}\right)_+\right\|_{2p,\Omega}\cdot\max_{x\in\Omega} f'(d(x)),\end{aligned}$$

where $m = \Delta d$, and $(\ )_+$ means taking the positive part.

Proof. For a function $\bar{f}$ on Ω we write

$$\begin{aligned}\left|\bar{f}\right| &= \bar{f}_+ + \bar{f}_-,\\ \bar{f}_+ &= \max\{f,0\},\\ \bar{f}_- &= \max\{-f,0\}.\end{aligned}$$

With this notation we have

$$\int_\Omega\left|\bar{f}\right| \leq \left|\int_\Omega \bar{f}\right| + 2\int_\Omega \bar{f}_+.$$

Using this on $\bar{f} = \Delta(f\circ d)$, we get, after applying Stokes' theorem,

$$\begin{aligned}\int_\Omega|\Delta(f\circ d)| &\leq \left|\int_\Omega\Delta(f\circ d)\right| + 2\int_\Omega(\Delta(f\circ d))_+\\ &\leq \mathrm{vol}\partial\Omega\cdot\max_{x\in\partial\Omega} f'(d(x)) + 2\int_\Omega(\Delta(f\circ d))_+.\end{aligned}$$

Next we observe that

$$\begin{aligned}\int_\Omega(\Delta(f\circ d))_+ &\leq \int_\Omega\left(f''\circ d + \frac{n-1}{d}\cdot f'\circ d\right)_+\\ &\quad+\int_\Omega\left(m\circ d - \frac{n-1}{d}\right)_+\cdot f'\circ d\\ &\leq \mathrm{vol}\Omega\cdot\max_\Omega\left(f''\circ d + \frac{n-1}{d}\cdot f'\circ d\right)_+\\ &\quad+\max_\Omega f'\circ d\cdot\int_\Omega\left(m-\frac{n-1}{d}\right)_+\\ &\leq \mathrm{vol}\Omega\cdot\max_\Omega\left(f''\circ d + \frac{n-1}{d}\cdot f'\circ d\right)_+\\ &\quad+\max_\Omega f'\circ d\cdot\mathrm{vol}\Omega\cdot\left\|\left(m-\frac{n-1}{d}\right)_+\right\|_{2p,\Omega}\end{aligned}$$

This gives the desired estimate. $\square$

The crucial Hessian estimates are contained in the next two lemmas.

Lemma 5.2. *For $x, y \in M$ with $d(x,y) > 2R$, let $b^+ = d(y,\cdot) - d(x,y)$. Define b to be the harmonic function such that $b = b^+$ on $\partial B(x,R)$. Then we have the estimate*

$$\begin{aligned} &\int_{B(x,R/4)} |\mathrm{Hess}\,(b)|^2 \\ &\leq \sup_{B(x,R/2)} |\nabla b|^2 \cdot \mathrm{vol} B(x,R) \cdot \left\{ \max_{0 \leq t \leq R} \left(-f''(t) - f'(t) \cdot \frac{n-1}{t} \right)_+ \right. \\ &\qquad \left. + \frac{1}{R} \left\| \left(m - \frac{n-1}{d} \right)_+ \right\|_{2p, B(x,R)} + \|\rho\|_{1,B(x,R)} \right\} \end{aligned}$$

Here $f : (0,\infty) \to (0,\infty)$ is a bump function with $f = 1$ on $(0, R/4)$, $f = 0$ on (R,∞), $f' \in [-2R^{-1}, 0]$, and $|f''| \leq 16R^{-2}$ everywhere. Recall that $\rho = |\min\{0, r(x)\}|$, where $r(x)$ is the smallest eigenvalue of the Ricci tensor.

Proof. Bochner's formula for the harmonic function b gives

$$|\mathrm{Hess} b|^2 = \frac{1}{2}\Delta |\nabla b|^2 - \mathrm{Ric}(\nabla b, \nabla b)$$

If we denote $d = d(\cdot, y)$, then we have

$$\begin{aligned} \int_{B(x,R/4)} |\mathrm{Hess} b|^2 &\leq \int_{B(x,R/2)} |\mathrm{Hess} b|^2 \cdot f \circ d \\ &= \int_{B(x,R/2)} \left(\frac{1}{2}\Delta |\nabla b|^2 - \mathrm{Ric}(\nabla b, \nabla b) \right) \cdot f \circ d \\ &\leq \int_{B(x,R/2)} \frac{1}{2} |\nabla b|^2 \cdot |\Delta (f \circ d)| + \int_{B(x,R/2)} \rho \cdot |\nabla b|^2 \cdot f \circ d \\ &\leq \sup_{B(x,R/2)} |\nabla b|^2 \cdot \left(\int_{B(x,R)} \frac{1}{2} \cdot |\Delta (f \circ d)| + \int_{B(x,R)} \rho \right). \end{aligned}$$

We can now apply the first lemma to the expression $\int_{B(x,R)} \frac{1}{2} \cdot |\Delta (f \circ d)|$ to get the desired inequality. □

Lemma 5.3. *Let b^+ and b be as above. There is a constant $C(n,p)$ such that*

$$\int_{B(x,R/4)} |\mathrm{Hess} b|^2 \leq C \cdot \mathrm{vol} B(x, R/4) \cdot R^{-2}$$

Proof. From the above lemma we first observe that our gradient estimate gives us

$$\begin{aligned} \sup_{B(x,R/2)} |\nabla b|^2 &\leq R^{-2} \cdot C_1 \cdot \sup_{B(x,R)} |b^+|^2 \\ &\leq C_2 \end{aligned}$$

The volume term can simply be estimated using relative volume comparison:

$$\mathrm{vol} B(x,R) \leq C_3 \cdot \mathrm{vol} B(x, R/4)$$

We also clearly have

$$\max_{0 \leq t \leq R} \left(-f''(t) - f'(t) \cdot \frac{n-1}{t} \right)_+ \leq 16R^{-2}.$$

Estimating as in (4.1) gives

$$\left\| \left(m - \frac{n-1}{d(\cdot, y)} \right)_+ \right\|_{2p, B(x,R)} \leq C_4 R^{-1} \varepsilon^{\frac{1}{2}}.$$

Finally, Hölder's inequality yields

$$\|\rho\|_{1,B(x,R)} \leq \|\rho\|_{p,B(x,R)} \leq R^{-2}\varepsilon.$$

Applying all of these inequalities together to Lemma 5.2 yields

$$\begin{aligned}
&\int_{B(x,R/4)} |\mathrm{Hess}\,(b)|^2 \\
&\leq \sup_{B(x,R/2)} |\nabla b|^2 \cdot \mathrm{vol} B\,(x,R) \cdot \left\{ \max_{0\leq t\leq R} \left(-f''(t) - f'(t) \cdot \frac{n-1}{t} \right)_+ \right. \\
&\qquad\qquad \left. + \frac{1}{R} \left\| \left(m - \frac{n-1}{d} \right)_+ \right\|_{2p,B(x,R)} + \|\rho\|_{1,B(x,R)} \right\} \\
&\leq C_2 \cdot C_3 \cdot \mathrm{vol} B\,(x, R/4) \cdot \left\{ 16R^{-2} + C_4 R^{-2} \varepsilon^{\frac{1}{2}} + R^{-2}\varepsilon \right\} \\
&\leq C_5 \cdot R^{-2} \cdot \mathrm{vol} B\,(x, R/4)
\end{aligned}$$

□

Another estimate which is needed to make Colding's methods go through is simply that one needs to have a bound for the first eigenvalue as described in the section on Sobolev constants.

5.2. **Volume Convergence.** Given the above Hessian estimate and the relative volume comparison estimate, the volume convergence and curvature/volume pinching results now follow immediately without any further work. The diffeomorphism results from [5, Appendix A] which are used to get the diffeomorphism stability results also carry over without trouble, as they only depend on the volume convergence results of Colding.

6. The splitting theorem

In order to generalize the work from [5, Section 5 and 6], we need to establish the refined Hessian estimate [4, Prop. 6.60]. This can be proved along the same lines, using the tools we worked out in the previous sections, namely, the maximum principle, the excess estimate and the gradient estimate, and the mean curvature comparison estimate in our earlier work [15].

We will work under the setup of the excess estimate (in Section 4) with $r = 1$. Namely, for any $y_0, q_-, q_+ \in M^n$, $p > n/2, v > 0$ and $R \geq 1$, we will work on a ball $B(y_0, R)$ and assume

$$\max\{d(y_0, q_+), d(y_0, q_-)\} \leq R/3,$$

$$\min\{d(y_0, q_+), d(y_0, q_-)\} \geq L,$$

$$e_{q_-,q_+}(y_0) \leq \varepsilon,$$

$$R^2 \bar{k}(p, R) \leq \varepsilon,$$

$$\operatorname{vol} B(y_0, R) \geq vR^n.$$

Let

$$b_\pm(x) = d(x, q_\pm) - d(y_0, q_\pm)$$

and b the harmonic function on $B(y_0, 1)$ such that $b|\partial B\,(y_0, 1) = b_+|\partial B\,(y_0, 1)$.

Denote by $\Phi(u_1, \cdots, u_k|\cdot, \cdots)$ a nonnegative function depending on the numbers $u_1, \cdots, u_k$, and some additional parameters, such that when these additional parameters are fixed, we have

$$\lim_{u_1, \cdots, u_k \to 0} \Phi(u_1, \cdots, u_k|\cdot, \cdots) = 0.$$

We will first establish

Lemma 6.1.

$$|b - b_+| \leq \Phi(L^{-1}|n, p) + C(n, p, v) R^{\frac{n}{2p}-1} \varepsilon^{\frac{1}{2}}.$$

Proof. Let $\psi_\pm$ be the positive part of $\left(\triangle b_\pm - \frac{n-1}{d(x,q_\pm)}\right)$. Then $\triangle b_\pm \leq \Phi(L^{-1}|n) + \psi_\pm$, and we have

$$\triangle(b - b_+) \geq -\Phi - \psi_+.$$

By the maximum principle (Theorem 1.1),

$$\begin{aligned} b - b_+ &\leq \max_{\partial B(y_0,1)} (b - b_+) + K(n, p, v) \|\Phi + \psi_+\|_{2p, B(y_0,1)} \\ &\leq \Phi(L^{-1}|n) + K(n, p, v) \|\psi_+\|_{2p, B(y_0,1)}. \end{aligned}$$

Note that, by (4.1),

$$\|\psi_+\|_{2p, B(y_0,1)} \leq C(n, p, v) R^{\frac{n}{2p}-1} \varepsilon^{\frac{1}{2}}.$$

This proves one side of our inequality. To prove the other side of the inequality, consider the harmonic function β on $B(y_0, 1)$ such that $\beta|\partial B\,(y_0, 1) = b_-|\partial B\,(y_0, 1)$. Similarly we can show that

$$\beta - b_- \leq \Phi(L^{-1}|n, p) + C(n, p, v) R^{\frac{n}{2p}-1} \varepsilon^{\frac{1}{2}}. \tag{6.1}$$

Now $b - b_+ = b + \beta - (\beta - b_-) - (b_- + b_+)$, and

$$b_+ + b_- = e(x) - e(y_0).$$

Using the excess estimate for $e(x)$ in Theorem 4.2 and the maximum principle gives the desired bounds for $|b + \beta|$ and $|b_- + b_+|$. These combined with (6.1) gives the other side of our inequality. ☐

Next we give an L_1 estimate for the Laplacian of b_+.

Lemma 6.2.

$$\|\triangle b_+\|_{1, B(y_0,1)} \leq \Phi(L^{-1}|n) + c(n) + C(n, p, v) R^{\frac{n}{2p}-1} \varepsilon^{\frac{1}{2}}.$$

Proof. By applying Lemma 5.1 in the previous section to the ball $B(y_0, 1)$ with the functions $f(x) = x$ and $d(x) = d(x, q_+)$, we have

$$\|\triangle b_+\|_{1, B(y_0,1)} \leq \Phi(L^{-1}|n) + \frac{\operatorname{vol}(\partial B(y_0, 1))}{\operatorname{vol}(B(y_0, 1))} + 2\, \|\psi_+\|_{2p, B(y_0,1)}\,.$$

From the proof of Lemma 2.1 in [15], one can easily derive that

$$\frac{\mathrm{vol}(\partial B(y_0,1))}{\mathrm{vol}(B(y_0,1))} \le c(n) + n\,\|\psi\|_{2p,B(y_0,1)}\,,$$

where $\psi = \left(\triangle d(y_0,x) - \frac{n-1}{d(y_0,x)}\right)_+$. Now the lemma follows by using the estimates for ψ_+ and ψ. □

Lemmas 6.1 and 6.2, combined with Stokes' theorem give

Lemma 6.3.

$$\frac{1}{\mathrm{vol}B\,(y_0,1)}\int_{B(y_0,1)} |\nabla(b-b_+)|^2 \le \Phi(L^{-1}|n,p) + c(n,p,v)R^{\frac{n}{2p}-1}\varepsilon^{\frac12}.$$

To show the Hessian estimate, we need to construct a cut-off function with uniform bounds on its gradient and Laplacian.

Theorem 6.4. *Given any $r>0, p>n/2, v>0$, there exist $c(n,p,r,v)$, $\varepsilon(n,p,v)$ such that if $(2r)^{2p}\bar{k}(p,2r)\le\varepsilon$, $y\in M^n$ and $\mathrm{vol}\,B(y,2r)\ge v(2r^n)$, then there exists $\phi:\ M^n\to[0,1]$ such that $\phi|_{B(y,\frac12 r)}\equiv 1$, $\mathrm{supp}\,\phi\subset B(y,r)$, and*

$$|\nabla\phi| \le c(n,p,r,v),$$

$$|\triangle\phi| \le c(n,p,r,v).$$

Proof. Let $\delta=\delta(n,r)>0$ be such that there exists $G:(0,\infty)\to(0,\infty)$ (singular at $t=0$) such that

$$\begin{aligned} G' &< 0 \quad \text{on } (0,r)\,,\\ G(\tfrac{1}{2}r) &= 1,\\ G\,(r) &= 0,\\ G'' + \frac{n-1}{t}\cdot G' &= \delta. \end{aligned}$$

Essentially, G is the same function that appeared in the proof of Theorem 4.2.

If we set $t(x)=d(x,y)$, then

$$\triangle G(t) \ge \delta + \psi_y G'(t),$$

where $\psi_y = \left(\triangle t(x) - \frac{n-1}{d(x,y)}\right)_+$ and

$$\|\psi_y\|_{2p,B(y,r)} \le c(n,p)(\bar{k}(p,r))^{\frac12} \le c(n,p)r^{-1}\varepsilon^{\frac12}.$$

Let the function, $k:\ \overline{B(y,r)}\setminus B(y,\frac12 R)\to\mathbb{R}$, satisfy

$$\begin{aligned} k &= 1 \quad \text{on } \partial B\left(y,\tfrac12 r\right),\\ k &= 0 \quad \text{on } \partial B\,(y,r)\,,\\ \Delta k &= \delta. \end{aligned}$$

By applying the maximum principle to $G-k$, we get

$$k \ge G(t) - r^2K(n,p,v)\|\psi_y G'(t)\|_{2p,B(y,r)}$$

on $\overline{B(y,r)}\setminus B(y,\frac12 r)$.

Let $K:\ (0,\infty)\to(0,\infty)$ satisfy

$$\begin{aligned} K' &> 0,\\ K(0) &= 0,\\ K''+\frac{n-1}{t}\cdot K' &= 1. \end{aligned}$$

Put $s(z)=d(x,z)$, where $z\in B(y,R)$. Then

$$\triangle K(s)\le 1+\psi_x K'(s),$$

and so

$$\triangle(k-\delta K(s))\ge -\delta\psi_x K'(s),$$

where $\psi_x=\left(\triangle s(z)-\frac{n-1}{d(x,z)}\right)_+$ and

$$\|\psi_x\|_{2p,B(y,r)}\le c(n,p)r^{-1}\varepsilon^{\frac{1}{2}}.$$

By applying the maximum principle to $k-\delta K(s)$ on $\overline{B(y,r)}\setminus B(y,\frac{1}{2}r)$, we get

$$k(x)\le 1-\delta K\left(t(x)-\frac{1}{2}r\right)+r^2\delta K(n,p,v)\|\psi_x K'(s)\|_{2p,B(y,r)}.$$

Choose $\eta=\eta(n,r)$ such that

$$G\left(\frac{1+\eta}{2}R\right)>1-\delta K\left(\left(\frac{1}{2}-\frac{1}{4}\eta\right)R\right).$$

Let $\varphi:\ [0,1]\to[0,1]$ satisfy

$$\begin{aligned} \varphi &= 1 \quad \text{on } [G(\tfrac{1+\eta}{2}r),1],\\ \varphi &= 0 \quad \text{on } [0,1-\delta K\left(\left(\tfrac{1}{2}-\tfrac{1}{4}\eta\right)r\right)]. \end{aligned}$$

Then $\phi=\varphi\circ K$ satisfies

$$\begin{aligned} \phi &= 1 \text{ on } \overline{B(y,(\tfrac{1}{2}+\tfrac{\eta}{4})r)}-B(y,\tfrac{1}{2}r),\\ \phi &= 0 \text{ on } \overline{B(y,r)}-B(y,(1-\tfrac{\eta}{2})r), \end{aligned}$$

if

$$\begin{aligned} r^2K(n,p)\|\psi_y G'(t)\|_{2p,B(y,r)} &\le G((\tfrac{1}{2}+\tfrac{\eta}{4})r)-G((\tfrac{1}{2}+\tfrac{\eta}{2})r),\\ r^2K(n,p)\|\psi_x K'(s)\|_{2p,B(y,r)} &\le K((\tfrac{1}{2}-\tfrac{\eta}{4})r)-K((\tfrac{1}{2}-\tfrac{\eta}{2})r). \end{aligned}$$

By the estimates for $\psi_y,\ \psi_x$ and the construction of the functions G,K these are satisfied when ε is sufficiently small.

Note that

$$\begin{aligned} \nabla\phi &= \varphi'\nabla K,\\ \triangle\phi &= \varphi''|\nabla k|^2+\varphi'\delta. \end{aligned}$$

Thus $\operatorname{supp}\nabla\phi$, $\operatorname{supp}\triangle\phi\subset\overline{B(y,(1-\frac{\eta}{2})r)}\setminus B(y,(\frac{1}{2}+\frac{\eta}{4})r)$. By our gradient estimate (adjusted to the annulus), the theorem follows. □

Now we are ready to prove the crucial Hessian estimate.

Proposition 6.5.

$$\frac{1}{\operatorname{vol}(B(y,1/4))}\int_{B(y,1/4)}|\mathrm{Hess}_b|^2\le\Phi(L^{-1}|n,p)+c(n,p,v)R^{\frac{n}{2p}-1}\varepsilon^{\frac{1}{2}}.$$

Proof. Since b is harmonic, Bochner's formula gives

$$\begin{aligned}\frac{1}{2}\triangle|\nabla b|^2 &= |\mathrm{Hess}_b|^2 + \mathrm{Ric}(\nabla b, \nabla b)\\ &\geq |\mathrm{Hess}_b|^2 - \rho|\nabla b|^2.\end{aligned}$$

Let $\phi : B(y, 1/2) \to [0, 1]$ be the function constructed above in Theorem 6.4. Multiply the left side of the inequality by ϕ and integrate. We get

$$\int_{B(y,1/2)} \phi\frac{1}{2}\triangle|\nabla b|^2 = \int_{B(y,1/2)} \frac{1}{2}\triangle\phi(|\nabla b|^2 - 1).$$

Now

$$\int_{B(y,1/2)} (|\nabla b|^2 - 1) \leq \int_{B(y,1/2)} |\nabla(b - b_+)|^2|\nabla(b + b_+)|^2.$$

Using Lemma 6.3 and Theorem 6.4, we then have

$$\int_{B(y,1/2)} \phi\frac{1}{2}\triangle|\nabla b|^2 \leq \Phi(L^{-1}|n, p) + c(n, p, v)R^{\frac{n}{2p}-1}\varepsilon^{\frac{1}{2}}.$$

By our gradient estimate

$$\sup_{B(y,1/2)} |\nabla b| \leq c(n, p, v).$$

Combining the above gives the Hessian estimate. □

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