Analysis and Synthesis of State-Feedback Controllers With Timing Jitter

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***Abstract*—We consider a continuous-time linear system with sampled constant linear state-feedback control and a convex quadratic performance measure. The sample times, however, are subject to variation within some known interval. We use linear matrix inequality (LMI) methods to derive a Lyapunov function that establishes an upper bound on performance degradation due to the timing jitter. The same Lyapunov function can be used in a heuristic for finding a bad timing jitter sequence, which gives a lower bound on the possible performance degradation. Numerical experiments show that these two bounds are often close, which means that our bound is tight. We show how LMI methods can be used to synthesize a constant state-feedback controller that minimizes the performance bound, for a given level of timing jitter.**

***Index Terms*—Linear matrix inequality (LMI), timing jitter.**

I. Introduction

We consider a continuous-time linear time-invariant control system, with plant given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state of the system, $u(t) \in \mathbf{R}^m$ is the input to the system, $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ and $x_0 \in \mathbf{R}^n$.

The plant is controlled by a sampled controller, with sample times $0 = t_0 < t_1 < t_2 < \ldots$, where we assume $t_i \to \infty$ as $i \to \infty$. The input is piecewise constant, given by

$$u(t) = u_i, \quad t_i \le t < t_{i+1}, \quad i = 0, 1, \ldots. \tag{2}$$

From (1) and (2) we can derive the equation

$$x(t) = A^{\mathrm{d}}(t - t_i)x(t_i) + B^{\mathrm{d}}(t - t_i)u(t_i), \quad t_i \le t \le t_{i+1} \tag{3}$$

where

$$A^{\mathrm{d}}(s) = e^{sA}, \quad B^{\mathrm{d}}(s) = Z(s)B \tag{4}$$

and

$$Z(s) = \int_0^s e^{\tau A} d\tau.$$

(The matrix Z can be computed either numerically or analytically, in terms of the matrix exponential; see Appendix A. For more on the matrix exponential, see [1].)

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In particular, (3) can be used to derive the equations for a discrete-time, linear time-varying system that describes the state at the sample times

$$x_{i+1} = A^{\mathrm{d}}(s_i)x_i + B^{\mathrm{d}}(s_i)u_i \tag{5}$$

where $s_i = t_{i+1} - t_i$ is the ith intersample time, $x_i = x(t_i)$, and $u_i = u(t_i)$.

Timing Model: In the case of a perfect clock with sampling period T, we have $t_i = iT$, and (5) reduces to the linear time-invariant system $x_{i+1} = A^{\mathrm{d}}_{\mathrm{nom}}x_i + B^{\mathrm{d}}_{\mathrm{nom}}u_i$, where $A^{\mathrm{d}}_{\mathrm{nom}} = A^{\mathrm{d}}(T)$ and $B^{\mathrm{d}}_{\mathrm{nom}} = B^{\mathrm{d}}(T)$. We will refer to this as the *nominal closed-loop system*.

We are interested, however, in the case where jitter and clock inaccuracies are present, where $t_{i+1} - t_i$ is near, but not exactly equal to, T. We will use the following model for sample times: they must satisfy

$$T - \Delta \le t_{i+1} - t_i \le T + \Delta, \quad i = 0, 1, \ldots \tag{6}$$

but are otherwise unknown. Here Δ, which is a parameter in our timing jitter model, gives the maximum possible jitter. (We assume $\Delta < T$.) This model includes changes in sampling rate: The sample time sequences $t_i = i(T - \Delta)$ and $t_i = i(T + \Delta)$, which correspond to uniform sampling with lower and higher periods, both satisfy (6). Another commonly used model for the sample times is the pure jitter model, described by

$$|t_i - iT| \le \Delta/2, \quad i = 0, 1, \ldots. \tag{7}$$

The pure jitter model is a special case of our timing model (6).

Performance Measure: We are interested in bounding the worst-case performance of the system under our timing model (6). We will use the traditional linear-quadratic regulator (LQR) continuous-time cost

$$J(x_0, u) = \int_0^\infty \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) d\tau. \tag{8}$$

Here $Q \in \mathbf{R}^{n \times n}$ and $R \in \mathbf{R}^{m \times m}$ are parameters in our cost function. We make the standard assumptions on Q and R: R is symmetric positive definite, Q is symmetric positive semidefinite and $(Q^{1/2}, A)$ is observable. The cost $J(x_0, u)$, which depends on the sample time sequence as well as the initial state x_0 and input sequence u, can be infinite.

Using (2) and (3), the cost can be expressed as

$$J(x_0, u) = \sum_{i=0}^\infty \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \Gamma(s_i) \begin{bmatrix} x_i \\ u_i \end{bmatrix} \tag{9}$$

where

$$\Gamma(s) = \begin{bmatrix} Q^{\mathrm{d}}(s) & S^{\mathrm{d}}(s) \\ S^{\mathrm{d}}(s)^T & R^{\mathrm{d}}(s) \end{bmatrix} = \int_0^s \Phi(\tau)^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \Phi(\tau) d\tau$$

and

$$\Phi(s) = \begin{bmatrix} e^{\tau A} & Z(\tau) \\ 0 & I \end{bmatrix}.$$

(Like $Z(s)$, the matrices $Q^{\mathrm{d}}(s)$, $S^{\mathrm{d}}(s)$, and $R^{\mathrm{d}}(s)$ can be computed either numerically or analytically, in terms of the matrix exponential;

see Appendix A.) It follows from our standard assumptions that the matrix $\Gamma(s)$ is positive definite.

For a given timing sequence, $J(x_0)$ is a convex functional in x_0 and the sequence $\{u_i\}$.

Linear State-Feedback Controller: We take the controller to be a state-feedback controller given by

$$u_i = Kx_i, \qquad i = 0, 1, \ldots \tag{10}$$

where $K \in \mathbf{R}^{m\times n}$ is the (constant) state feedback gain. The equations for the discrete-time, linear time-varying system that describes the state at the sample times becomes

$$x_{i+1} = \left(A^{\mathrm{d}}(s_i) + B^{\mathrm{d}}(s_i)K\right)x_i. \tag{11}$$

We will assume that this system is stable, i.e., the eigenvalues of $(A^{\mathrm{d}}_{\mathrm{nom}} + B^{\mathrm{d}}_{\mathrm{nom}}K)$ have magnitude smaller than one.

The cost (9), now only a function of the initial state and the sample time sequence, becomes

$$J(x_0) = \sum_{i=0}^{\infty} x_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(s_i) \begin{bmatrix} I \\ K \end{bmatrix} x_i. \tag{12}$$

For a fixed sample time sequence, the cost $J(x_0)$ is a convex quadratic function in the initial state x_0.

We define the *worst-case cost* as

$$J_{\mathrm{wc}}(x_0) = \sup\left\{J(x_0) | \{t_i\} \text{ satisfies (6)}\right\}. \tag{13}$$

Like $J(x_0)$, $J_{\mathrm{wc}}(x_0)$ can be infinite. Since J_{wc} is a supremum of a family of convex quadratic functions of x_0, it is convex, and also homogeneous of degree 2.

For the nominal system, i.e., when $t_i = iT$, the cost is

$$J_{\mathrm{nom}}(x_0) = x_0^T P_{\mathrm{nom}} x_0$$

where P_{nom} is the (unique) solution of the Lyapunov equation

$$\left(A^{\mathrm{d}}_{\mathrm{nom}} + B^{\mathrm{d}}_{\mathrm{nom}}K\right)^T P \left(A^{\mathrm{d}}_{\mathrm{nom}} + B^{\mathrm{d}}_{\mathrm{nom}}K\right) - P + Q^{\mathrm{d}}_{\mathrm{nom}} + S^{\mathrm{d}}_{\mathrm{nom}}K + K^T\left(S^{\mathrm{d}}_{\mathrm{nom}}\right)^T + K^T R^{\mathrm{d}}_{\mathrm{nom}}K = 0.$$

Here $Q^{\mathrm{d}}_{\mathrm{nom}} = Q^{\mathrm{d}}(T)$, $S^{\mathrm{d}}_{\mathrm{nom}} = S^{\mathrm{d}}(T)$, and $R_{\mathrm{nom}} = R^{\mathrm{d}}(T)$.

The relative performance degradation, compared to the nominal system, is

$$\frac{J_{\mathrm{wc}}(x_0) - J_{\mathrm{nom}}(x_0)}{J_{\mathrm{nom}}(x_0)}$$

(assuming $x_0 \neq 0$). This gives the relative increase in the cost due to jitter, for a specific initial state x_0. We define the *worst-case relative performance degradation* as

$$\eta = \sup_{x_0 \neq 0} \frac{J_{\mathrm{wc}}(x_0) - J_{\mathrm{nom}}(x_0)}{J_{\mathrm{nom}}(x_0)}. \tag{14}$$

This number is always nonnegative, since

$$J_{\mathrm{wc}}(x_0) \geq J_{\mathrm{nom}}(x_0)$$

for any x_0.

Upper and Lower Bounds: Our goal is to find a (computable) upper bound on η, given the problem data A, B, K, Q, R, T, and Δ. By computable we mean that the upper bound is obtained with modest computational complexity, e.g., solving a semidefinite program (SDP) or a standard problem involving LMIs.

We are also interested in obtaining lower bounds on η. A lower bound can be obtained by choosing a specific x_0 and a finite timing sequence $t_0, \ldots, t_M$ that satisfies our timing model (6). We then continue the chosen finite sequence infinitely, adding $t_i = t_M + (i - M)T$ for $i = M+1, M+2, \ldots$. This sample time sequence results in the cost

$$J_{\mathrm{lb}}(x_0) = \sum_{i=0}^{M-1} \left(x_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(s_i) \begin{bmatrix} I \\ K \end{bmatrix} x_i \right) + J_{\mathrm{nom}}(x(t_M)) \tag{15}$$

where x propagates according to (11). This cost is, of course, a lower bound on $J_{\mathrm{wc}}(x_0)$. We thus have

$$\eta \geq \frac{J_{\mathrm{lb}}(x_0) - J_{\mathrm{nom}}(x_0)}{J_{\mathrm{nom}}(x_0)}.$$

(If the timing sequence is chosen poorly, the right-hand side can be negative; in any case, it is a valid lower bound on η.) The challenge in getting a good lower bound is finding a 'bad' timing sequence $t_0, \ldots, t_M$, i.e., one that leads to large cost. We will address this question as well.

Previous and Related Work: For basic references on digital control of continuous time systems, see [2]–[4]; [5] describes the problem of timing jitter. There have been several approaches to analyzing the performance of systems in the presence of varying delays or jitter: early work on the subject of randomly sampled systems can be found in [6] and [7], information theoretic studies were presented in [8]; "Jitterbug," a computational toolbox for performance analysis, was described in [9]. A related topic is jitter compensation, which consists of adding a "compensator" to the existing control system to guarantee stability or a certain performance level in the presence of jitter; see, e.g., [10]–[12]. Other researchers have used LMIs, and robust control analysis and synthesis methods, to achieve stabilization of discrete uncertain systems [13], [14], or switched systems with unknown time-varying delays [15]. Recent work has also discussed the use of time-varying delay to model sample-and-hold circuits [16]. The issue of control in the presence of jitter has also been of interest in the field of scheduling: see [17] on the topic of jitter compensation in scheduling tasks and [18] on the design of real-time controllers under scheduling and timing constraints. The methodology we descibe in this technical note was subsequently used by Bhave and Krogh [19] in the context of state-feedback controllers with network delay.

II. Upper Bound

Semi-Infinite SDP Formulation: Our method is based on finding a quadratic (Lyapunov) function $V(z) = z^T P z$, which satisfies $V(z) \geq J(z)$ for all z, and all timing sequences that satisfy our timing model (6). It follows that $V(z) \geq J_{\mathrm{wc}}(z)$ for all z, and therefore

$$\frac{J_{\mathrm{wc}}(x_0) - J_{\mathrm{nom}}(x_0)}{J_{\mathrm{nom}}(x_0)} \leq \frac{V(x_0) - J_{\mathrm{nom}}(x_0)}{J_{\mathrm{nom}}(x_0)}. \tag{16}$$

The supremum of the right-hand side is $\lambda_{\max}(P, P_{\mathrm{nom}}) - 1$, where $\lambda_{\max}(X, Y)$ is the largest generalized eigenvalue of the pair (X, Y), for X symmetric and Y symmetric positive definite

$$\lambda_{\max}(X, Y) = \max\{t | \det(X - tY) = 0\} = \inf\{t | X \preceq tY\}.$$

(Here $\preceq$ denotes matrix inequality.) From (16), we see that

$$\eta \leq \lambda_{\max}(P, P_{\mathrm{nom}}) - 1 \tag{17}$$

provided $V(z) = z^T P z \geq J(z)$ holds for all z, and all possible timing sequences that satisfy our timing model (6).

A sufficient condition for the inequality $J_{\mathrm{wc}}(x_0) \leq V(x_0)$ to hold for all x_0 is $P \succeq 0$ and

$$\begin{bmatrix} I \\ K \end{bmatrix}^T (F(s) + \Gamma(s)) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0 \text{ for } T - \Delta \leq s \leq T + \Delta \tag{18}$$

where

$$F(s) = \begin{bmatrix} A^{\mathrm{d}}(s)^T P A^{\mathrm{d}}(s) - P & A^{\mathrm{d}}(s)^T P B^{\mathrm{d}}(s) \\ B^{\mathrm{d}}(s)^T P A^{\mathrm{d}}(s) & B^{\mathrm{d}}(s)^T P B^{\mathrm{d}}(s) \end{bmatrix}.$$

The inequality (18) is a semi-infinite LMI in the matrix P, i.e., a family of LMIs parametrized by the real number s, which ranges over an interval. For more on representing control system specifications via LMIs, see, e.g., [20]–[22].

To establish our claim, consider the Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$, defined as $V(z) = z^T P z$. Since P is positive semidefinite, $V(z) \geq 0$ for all $z \in \mathbf{R}^n$. For any $k > 0$

$$\begin{aligned} V(x_k) - V(x_0) &= \sum_{i=0}^{k-1} V(x_{i+1}) - V(x_i) \\ &= \sum_{i=0}^{k-1} x_{i+1}^T P x_{i+1} - x_i^T P x_i \\ &= \sum_{i=0}^{k-1} x_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T F(s_i) \begin{bmatrix} I \\ K \end{bmatrix} x_i. \end{aligned}$$

Now using (18)

$$V(x_k) - V(x_0) \leq -\sum_{i=0}^{M-1} x_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(s_i) \begin{bmatrix} I \\ K \end{bmatrix} x_i.$$

Reordering the terms in the inequality leads to

$$\begin{aligned} \sum_{i=0}^{k-1} x_i^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma(s_i) \begin{bmatrix} I \\ K \end{bmatrix} x_i &\leq V(x_0) - V(x_k) \\ &\leq V(x_0) \end{aligned}$$

where the last inequality follows because $V(x_k) \geq 0$. Letting k tend to infinity, we get

$$J(x_0) \leq V(x_0)$$

i.e., $J(x_0) \leq x_0^T P x_0$ for all $x_0 \in \mathbf{R}^n$ and for all timing sequences $\{t_i\}$. Therefore, it follows that $J_{\mathrm{wc}} \leq x_0^T P x_0$ for all x_0 and that (17) holds.

We can choose P to obtain the smallest possible upper bound on η, by solving the convex optimization problem

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(P, P_{\mathrm{nom}}) - 1 \\ \text{subject to} & \begin{bmatrix} I \\ K \end{bmatrix}^T (F(s) + \Gamma(s)) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0 \\ & \text{for } T - \Delta \leq s \leq T + \Delta \\ & P \succeq 0. \end{array} \tag{19}$$

The variable here is P. For more on convex problems, see [23].

The problem (19) can be expressed as a (semi-infinite) SDP by introducing a new scalar variable t:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & \begin{bmatrix} I \\ K \end{bmatrix}^T (F(s) + \Gamma(s)) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0 \\ & \text{for } T - \Delta \leq s \leq T + \Delta \\ & 0 \preceq P \preceq (1+t) P_{\mathrm{nom}}. \end{array} \tag{20}$$

Here the variables are P and t.

Discretization: We first describe a simple method for approximately solving (19) by discretizing (18). This approach is certainly adequate for any practical problem; in any case, we describe below a conservative discretization.

We define the discretized values of s to be

$$s_j = T + (2j - N)\Delta/N, \quad j = 0, 1, \ldots, N \tag{21}$$

where $N+1$ is the number of discretized values of s. Using this simple discretization, the problem (19) becomes

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(P, P_{\mathrm{nom}}) - 1 \\ \text{subject to} & \begin{bmatrix} I \\ K \end{bmatrix}^T (F_i + \Gamma_i) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0 \\ & j = 0, \ldots, N \\ & P \succeq 0 \end{array} \tag{22}$$

where $F_i = F(s_i)$ and $\Gamma_i = \Gamma(s_i)$. This is a tractable convex problem, readily transformed to an SDP and solved. Taking $N = 5$ is sufficient for any practical problem; in any case, the computational complexity grows linearly in N, so larger N (if it were needed) would not impose much computational burden.

Conservative Discretization: The simple discretized LMIs

$$\begin{bmatrix} I \\ K \end{bmatrix}^T (F_i + \Gamma_i) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0, \quad j = 0, 1, \ldots, N$$

do not imply the semi-infinite LMI (18). We can however modify the discretization slightly so the resulting convex inequalities do imply (18).

We define

$$H(s) = \begin{bmatrix} I \\ K \end{bmatrix}^T (F(s) + \Gamma(s)) \begin{bmatrix} I \\ K \end{bmatrix}$$

which is an analytic function of s over the interval $[T - \Delta, T + \Delta]$. Let $L \geq \|\dot{H}(s)\|$ for $s \in [T - \Delta, T + \Delta]$. Then we have $\|H(s) - H(t)\| \leq L|s - t|$, for all $s, t \in [T - \Delta, T + \Delta]$. It follows that with our uniform discretization (21), for any $s \in [T - \Delta, T + \Delta]$, there is an $i \in \{0, 1, \ldots, N\}$ for which $\|H(s) - H(s_i)\| \leq L\Delta/N$.

From this we conclude that if the $N+1$ LMIs

$$H(s_i) + (L\Delta/N) I \preceq 0, \quad i = 0, 1, \ldots, N$$

hold, then the semi-infinite LMI

$$H(s) \preceq 0 \quad \text{for } T - \Delta \leq s \leq T + \Delta$$

must hold.

In Appendix B we show that L can be chosen in the form $L = \epsilon + \tilde{\epsilon}\|P\|$, where ϵ and $\tilde{\epsilon}$ are constants that depend on the problem data. It follows that the $N+1$ convex inequalities

$$\begin{bmatrix} I \\ K \end{bmatrix}^T (F_i + \Gamma_i) \begin{bmatrix} I \\ K \end{bmatrix} + (\Delta/N) \left(\epsilon + \epsilon' \|P\|\right) I \preceq 0, \qquad j = 0, 1, \ldots, N$$

imply the semi-infinite LMI (18). Using these convex inequalities in the place of the LMIs appearing in (22) yields a tractable convex problem, readily transformed to an SDP, which gives an absolute guarantee.

III. Lower Bound

We describe a heuristic algorithm for generating a good lower bound on J_{wc}. This algorithm follows the guidelines described in Section I: it generates an initial state x_0 and a 'bad' timing sequence $\{t_i\}$ that result in a good lower bound of the form (15) on the worst-case cost $J_{\rm wc}(x_0)$. The basic idea behind the heuristic is to choose a sample time sequence that greedily maximizes the rate of increase of the Lyapunov function V at each time, i.e., that maximizes the quantity $(V(x(t_k)) - V(x(t_{k-1})))/(t_k - t_{k-1})$ at each time k.

The algorithm sets x_0 to be the eigenvector associated with the maximum generalized eigenvalue of the pair $(P, P_{\rm nom})$. It starts with $t_0 = 0$, $k = 0$, and $J = 0$. As long as $k < M$, it sets $t_{k+1} = t_k + s_j$, where

$$j = \underset{i=0,1,\ldots,N}{\arg\max} \left\{ x_k^T \begin{bmatrix} I \\ K \end{bmatrix}^T F_i \begin{bmatrix} I \\ K \end{bmatrix} x_k / s_i \right\}$$

and updates J as follows:

$$J := J + x_k^T \begin{bmatrix} I \\ K \end{bmatrix}^T \Gamma_j \begin{bmatrix} I \\ K \end{bmatrix} x_k.$$

It then propagates the state forward according to $x_{k+1} = (A_j^{\rm d} + B_j^{\rm d} K) x_k$, and increments k. When $k = M$, the algorithm adds a final cost term to J:

$$J := J + x_M^T P_{\rm nom} x_M$$

and exits. The final value of J is $J_{\rm lb}(x_0)$ and is a lower bound on $J_{\rm wc}(x_0)$. We therefore have

$$\eta \geq \frac{J_{\rm lb}(x_0) - J_{\rm nom}(x_0)}{J_{\rm nom}(x_0)}.$$

IV. State-Feedback Controller Synthesis

So far, the state feedback gain matrix K has been considered problem data, i.e., fixed and given. In this section we describe a method for *synthesizing* K that minimizes the performance degradation coefficient η, for some given $P_{\rm nom}$. The optimization problem that needs to be solved is (19) with variables P, K, and t. We will work with the discretized version (18), which is

$$\begin{array}{ll} \text{minimize} & \lambda_{\max}(P, P_{\rm nom}) - 1 \\ \text{subject to} & \begin{bmatrix} I \\ K \end{bmatrix}^T (F_i + \Gamma_i) \begin{bmatrix} I \\ K \end{bmatrix} \preceq 0 \\ & j = 0, 1, \ldots N \\ & P \succeq 0 \end{array} \qquad (23)$$

where the optimization variables are now both P and K.

The controller synthesis problem (23) is not convex as stated, but a change of variables yields an equivalent convex (and therefore tractable) problem. Let $Y = P^{-1}$ and $V = KP^{-1}$. Problem (23) is equivalent to the convex problem

$$\begin{array}{l} \text{maximize} \quad \lambda_{\min}\left(Y, P_{\rm nom}^{-1}\right) \\ \text{subject to} \\ \begin{bmatrix} Y & \left(A_i^{\rm d} Y + B_i^{\rm d} V\right)^T \\ A_i^{\rm d} Y + B_i^{\rm d} V & Y \end{bmatrix} \\ \quad \succeq \begin{bmatrix} W_i & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 0, 1, \ldots, N \\ \begin{bmatrix} W_i & Y & V^T \\ Y & & \\ V & & \Gamma_i^{-1} \end{bmatrix} \succeq 0, \quad i = 0, 1, \ldots, N. \end{array} \qquad (24)$$

with optimization variables Y, V, and W_i, for $i = 0, 1, \ldots, N$. Here $\lambda_{\min}(X, Y)$ is the smallest generalized eigenvalue of the pair (X, Y), for X symmetric and Y symmetric positive definite:

$$\begin{aligned} \lambda_{\min}(X, Y) &= \min\{t \,|\, \det(X - tY) = 0\} \\ &= \sup\{t \,|\, X \succeq tY\}. \end{aligned}$$

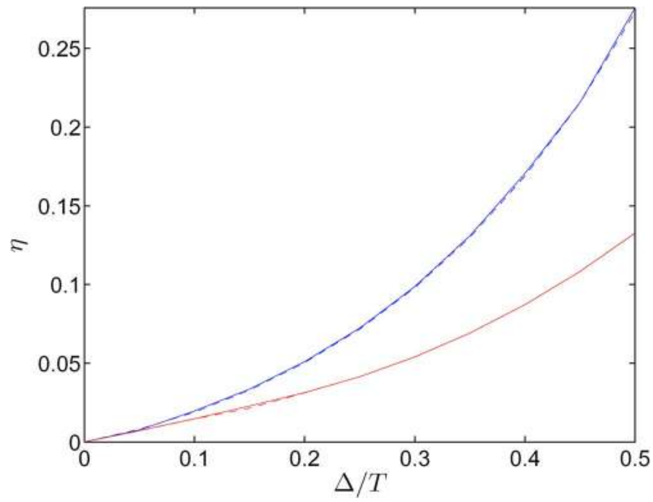

Fig. 1. *Top two curves*. Lower (dashed) and upper (solid) bounds on η with $K = K_{\rm lqr}$. *Bottom two curves*. Lower (dashed) and upper (solid) bounds on η, for $K = K^\star$, synthesized to minimize the bound on η (for each value of Δ).

Let $Y^\star$ and $Z^\star$ be solutions of (24). A solution of (23) can then be recovered as follows:

$$P^\star = (Y^\star)^{-1}, \qquad K^\star = V^\star (Y^\star)^{-1}.$$

The controller $K^\star$ minimizes the upper bound on η that we derived in Section II.

V. Example

In this section we present a simple example to illustrate the methods described above. Our example has dimensions $n = 10$ and $m = 2$. We generated the problem data A and B by choosing all entries independently from a standard normal distribution. We take LQR cost matrices $Q = I$, $R = I$, and sampling time $T = 0.1$. For discretizing the semi-infinite LMI, we use $N = 5$ (repeating the example with larger values of N had no effect on the results.) In our heuristic for finding a bad timing sequence, we use $M = 100$ steps before reverting to uniform nominal sampling.

In our first experiment, we choose the state-feedback controller to be the LQR-optimal feedback controller (for the nominal timing) $K_{\rm lqr}$, which minimizes $J_{\rm nom}(x_0)$ for any x_0. With this choice of controller, η represents the worst-case suboptimality of the LQR cost, over all possible timing sequences. For example, $\eta = 0.1$ means that for the given value Δ, the increase in LQR cost, due to the timing jitter, is at most 10%.

For each value of $\Delta \in [0, 0.5T]$, we compute an upper bound and a lower bound on η as described in Section II and Section III, respectively. The results are shown in the two upper plots in Fig. 1, with solid for the upper bound on η and dashed for the lower bound on η. These curves are almost on top of each other, which means that our method has (almost) exactly determined the value of η. It is a testament to the extraordinary robustness of LQR that the LQR cost rises only 27% over its optimal value, with extreme timing variations, for which s_i varies over a factor of 3 (from $0.5T$ to $1.5T$).

In our second experiment, we synthesize the feedback controller $K^\star$ as described in Section IV, for each value of $\Delta \in [0, 0.5T]$. This procedure gives the controller and an upper bound on η (referenced to the original nominal LQR controller); we also compute a lower bound using the method described in Section III. These results are shown as the two lower curves in Fig. 1, with the solid curve denoting the upper bound, and the lower curve denoting the lower bound. As in the previous experiment, the two curves are (almost) the same, meaning that we have determined η to within a very small interval. The plot shows that in this case, modifying the controller to take into account timing jitter leads to a reduction by a factor around two in the worst-case performance degradation due to jitter.

VI. Conclusion

We have described a computationally tractable method for bounding an LQR performance criterion in a state feedback controller with timing jitter. In many examples, the lower and upper bounds are very close, which means that we have actually computed (within a small approximation) the worst-case LQR cost under all possible timing sequences consistent with out timing model.

Some variations and extensions are simple; many others are not obvious, at least to us, at this time. The analysis can be extended to include constant linear dynamic controllers, and any set of performance measures that can be cast in the form of LMIs (e.g., a weighted L_2-norm gain). The synthesis problem with a general dynamic controller, though, would seem to be difficult.

In this technical note we have assumed that the timing jitter is the same for all actuators (or sensors; in this case we get the same model). If this were not the case, and individual actuators and sensors could have separate jitter values, the problem is much more difficult; in particular, the semi-infinite LMI we encounter is now parametrized by multiple parameters, instead of the one we have in this technical note. If the number of these parameters is small, we can still discretize (especially since 3 values of each jitter parameter is likely to be enough in practice); beyond that, the methods described here would have trouble.

Appendix A
Closed-Form Expressions

We make one simple assumption: the eigenvalues of A, denoted by $\lambda_i(A)$, $i = 1, 2, \ldots, n$, satisfy the property

$$\lambda_i(A) + \lambda_j(A) \neq 0, \quad i, j = 1, 2, \ldots, n. \tag{25}$$

This assumption implies that A has no zero eigenvalues and is therefore invertible.

We know that

$$Q(s) = \int_0^s A^{\mathrm{d}}(\tau)^T Q A^{\mathrm{d}}(\tau) d\tau$$
$$S(s) = \int_0^s A^{\mathrm{d}}(\tau)^T Q B^{\mathrm{d}}(\tau) d\tau$$
$$R(s) = sR + \int_0^s B^{\mathrm{d}}(\tau)^T Q B^{\mathrm{d}}(\tau) d\tau.$$

The closed-form expressions of $Q^{\mathrm{d}}(s)$, $S^{\mathrm{d}}(s)$, and $R^{\mathrm{d}}(s)$ are therefore

$$Q^{\mathrm{d}}(s) = \tilde{Z}(s) \tag{26}$$
$$S^{\mathrm{d}}(s) = \left(\tilde{Z}(s) - Z(s)^T\right) A^{-1} B \tag{27}$$
$$R^{\mathrm{d}}(s) = sR + B^T A^{-T} \Psi(s) A^{-1} B \tag{28}$$

where

$$\Psi(s) = \tilde{Z}(s) - Z(s)^T Q - Q Z(s) + sQ$$

and

$$\tilde{Z}(s) = \int_0^s e^{\tau A^T} Q e^{\tau A} d\tau$$
$$Z(s) = \int_0^s e^{\tau A} d\tau = (e^{sA} - I) A^{-1}.$$

Property (25) is equivalent to the Lyapunov operator $\mathcal{L} : X \to A^T X + XA$ being nonsingular. Therefore, $\tilde{Z}(s)$ is the (unique) solution of the Lyapunov equation

$$A^T Z(s) + Z(S) A = A^{\mathrm{d}}(s)^T Q A^{\mathrm{d}}(s) - Q.$$

If (25) doesn't hold, formulas (26), (27), and (28) are obviously not valid, but alternative formulas can be derived.

Appendix B
Constants in Conservative Discretization

To derive expressions for ϵ and $\tilde{\epsilon}$, we start by finding upper bounds on $\|\dot{F}(s)\|$ and $\|\dot{\Gamma}(s)\|$ over the interval $[T - \Delta, T + \Delta]$. We will use the following facts in our derivation:

$$\|Z(s)\| \leq \left(e^{|s|\|A\|} - 1\right) / \|A\|$$

and for any matrices X, Y, V, W of appropriate sizes,

$$\left\| \begin{bmatrix} X & Y \\ Z & W \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \|X\| & \|Y\| \\ \|Z\| & \|W\| \end{bmatrix} \right\| \leq \sqrt{\|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2}.$$

From

$$\dot{F}(s) = [A\ B]^T e^{sA^T} P \left[e^{sA}\ Z(s)\right] + \left[e^{sA}\ Z(s)\right]^T P e^{sA} [A\ B]$$

we have

$$\left\|\dot{F}(s)\right\| \leq \kappa \left(e^{|s|\|A\|} + \left(e^{|s|\|A\|} - 1\right) / \|A\|\right) e^{|s|\|A\|} \|P\| \tag{29}$$

where $\kappa = 2\|[A\ B]\|$. From

$$\dot{\Gamma}(s) = \Phi(s)^T \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \Phi(s)$$

we obtain

$$\left\|\dot{\Gamma}(s)\right\| \leq \left\| \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \right\| \times \left\| \begin{bmatrix} e^{|s|\|A\|} & \left(e^{|s|\|A\|} - 1\right) / \|A\| \\ 0 & 1 \end{bmatrix} \right\|^2 \tag{30}$$
$$\leq \mu \left(e^{|s|\|A\|} + 1 + \left(e^{|s|\|A\|} - 1\right) / \|A\|\right)^2 \tag{31}$$

where $\mu = \max\{\|Q\|, \|R\|\}$.

By computing the maximum of the right-hand sides of (29) and (31) over the interval $[T - \Delta, T + \Delta]$, we obtain the parameters

$$\epsilon = \mu \left(e^{(T+\Delta)\|A\|} + 1 + \left(e^{(T+\Delta)\|A\|} - 1\right) / \|A\|\right)^2$$
$$\tilde{\epsilon} = \kappa \left(e^{(T+\Delta)\|A\|} + \left(e^{(T+\Delta)\|A\|} - 1\right) / \|A\|\right) e^{(T+\Delta)\|A\|}.$$

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Inverse Agreement Protocols With Application to Distributed Multi-Agent Dispersion

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***Abstract*—We propose a distributed inverse agreement control law for multiple kinematic agents that forces the team members to disperse in the workspace. Both the cases of an unbounded and a circular, bounded workspace are considered. In the first case, we show that the closed-loop system reaches a configuration in which the minimum distance between any pair of agents is larger than a specific lower bound. It is proved that this lower bound coincides with the agents' sensing radius. In the case of a bounded circular workspace, the control law is redefined to force the agents to remain within the workspace boundary. Moreover the proposed control design guarantees collision avoidance between the team members in all cases. The results are supported through relevant computer simulations.**

***Index Terms*—Cooperative control, distributed multi-agents systems, swarm dispersion.**

I. INTRODUCTION

The emerging use of large-scale multi-robot/vehicle systems in various applications has raised recently the need for the design of control laws that force a team of multiple vehicles/robots (from now on called "agents") to achieve various goals. As the number of agents increases, centralized designs fail to guarantee robustness and are harder to implement than decentralized ones, which also provide a reduce in the computational complexity of the feedback scheme. Among the various objectives of the control design, convergence of the team to a common configuration, also known as the agreement problem, is a design specification that has been extensively pursued. Many distributed control schemes that achieve multi-agent agreement are found in literature; see [1], [2], [4], [8], [9], [14]–[16] for some recent results. In this technical note, we propose an algorithm for swarm dispersion which can be considered as an inverse agreement problem. Each agent follows a flow, whose inverse leads the multi-agent team to agreement. The design is distributed, since each agent only knows the relative positions of agents located within its sensing zone at each time. The sensing zone is a circular area around each agent whose radius is common for all agents. The application of this inverse agreement strategy is dispersion of the team members in the workspace, i.e., convergence to a configuration where the minimum distance between the swarm members is bounded from below by a controllable lower bound. It is shown that this lower bound coincides with the radius of the sensing zone of the agents in the case of an unbounded workspace. Furthermore, the results are extended in order to take into account the workspace boundary for the case of a circular bounded workspace.

Applications of the dispersion algorithm include coverage control [5], [11], [12], and optimal placement of a multi-robot team in small

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