Usman Riaz* and Akbar Zada*

Analysis of (α, β)-order coupled implicit Caputo fractional differential equations using topological degree method

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Abstract: This article is devoted to establish the existence of solution of (α, β)-order coupled implicit fractional differential equation with initial conditions, using Laplace transform method. The topological degree theory is used to obtain sufficient conditions for uniqueness and at least one solution of the considered system. Beside this, Ulam's type stabilities are discussed for the proposed system. To support our main results, we present an example.

Keywords: Caputo fractional differential equation; Hyers–Ulam stability; Laplace transform method; topological degree method; Mittag–Leffler function.

Mathematics Subject Classifications: 2010; 26D10; 34A08; 35R11.

1 Introduction

Fractional order derivatives are the generalized forms of integer order derivatives. The idea about the fractional order derivative was introduced at the end of sixteenth century (1695) when Leibniz used the notation $\frac{d^n}{dx^n}$ for nth order derivative. By writing a letter to him, L'Hospital asked the question that, what would be the result if $n = \frac{1}{2}$? Leibniz answered in such words, "An apparent Paradox, from which one day useful consequences will be drawn" and this question becomes the foundation of fractional calculus, see [1]. In that time many mathematicians like Fourier and Laplace contributed in the development of fractional calculus. After that when Riemann and Liouville introduced Riemann-Liouville derivative which is a fundamental concept in fractional calculus, then fractional calculus became the most interested area for researchers. Fractional order derivative is a global operator, which is used as a tool for modeling different processes and physical phenomenon like mathematical biology [2], electro-chemistry [3], control theory [4], dynamical process [5], image and signal processing [6] etc. For more applications of fractional order differential equations, we refer the reader to [7, 8, 9, 10, 11, 12, 13, 14].

The most preferable research area in the field of fractional differential equations which received great attention from the researchers is the theory regarding the existence of solutions. Many researchers developed some interesting results about the existence of solutions of different boundary value problems, using different fixed point theorems. For details we refer the reader to [15, 16]. Most of the time, it is quite difficult to find the exact solutions of nonlinear differential equations, in such situation different approximation techniques were introduced. The difference between exact and approximate solutions is now a days dealing with the help of Hyers–Ulam(HU) type stabilities, which was first introduced in 1940 by Ulam [17] and then answered by Hyers in the following year, in the context of Banach spaces. Many researchers investigated HU type stabilities for different problems with different approaches, [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

***Corresponding authors: Usman Riaz**, Department of Mathematics, University of Peshawar, Peshawar, Khyber Pakhtunkhwa, Pakistan, E-mail: usmanriazstd15@uop.edu.pk. https://orcid.org/0000-0002-0231-8704; and **Akbar Zada**, Department of Mathematics, University of Peshawar, Peshawar, Khyber Pakhtunkhwa, Pakistan, E-mail: zadababo@yahoo.com. https://orcid.org/0000-0002-2556-2806

In the solution of fractional differential equations the Mittag–Leffler function naturally appears to play a role analogous to that of the exponential function as in the ordinary case. The Mittag–Leffler function was defined by Leffler [29]. In the last few decades, the Mittag–Leffler function has been the subject of generalization of many approaches, in finding the solutions of fractional differential equations [29, 30].

An effective and convenient method for solving fractional differential equations is needed. Methods in [31] for rational order fractional differential equations are not applicable to the case of arbitrary order. Some authors used the series method [32, 33, 34, 35], which allows solution of arbitrary order fractional differential equations, but it works only for relatively simple equations. Podlubny [36] introduced a method based on the Laplace transform technique, it is suitable for a large class of initial value problems for fractional differential equations. However, we found that the existence of Laplace transform is taken for granted in some papers to solve fractional differential equations see [37, 38]. In [39], Rezaei et al. presented HU stability of

$$u^n(\sigma) + \sum_{k=0}^{n} a_k u^k(\sigma) = \Phi(\sigma) \ \ \forall \ \ \sigma > 0\,,$$

by applying Laplace transform method, where n be positive integer and $a_0, a_1, \ldots, a_n \in \mathcal{R}^+$.

In [40], Kexue et al. studied the solution of the following fractional differential equations by Laplace transform

$$\begin{cases} {}^{c}_{0}\mathfrak{D}^{\alpha}_{\sigma} u(\sigma) = \mathrm{A}u(\sigma) + \Phi(\sigma) \quad 0 < \alpha < 1, \sigma > 0\,, \\ u(0) = \eta\,, \end{cases}$$

where ${}^{c}_{0}D^{\alpha}_{\sigma}$ is the Caputo fractional derivative, A is a constant square matrix, $\Phi(\sigma)$ is a continuous forcing term. In [41], Lin et al. obtained the solution by using Laplace transform of the fractional differential equation

$$\begin{cases} u''(\sigma) + au^{\alpha}(\sigma) + bu(\sigma) = 0 \quad 1 < \alpha < 2, \ \ \sigma > 0\,, \\ u(0) = \eta_0, \ \ u'(0) = \eta_1 \end{cases}$$

where α is the fractional order and $a, \ b \in \mathcal{R}$.

Recently in [42], using Laplace transformation Wang et al. studied the HU stability of following linear fractional differential equation

$$^{c}\mathfrak{D}^{\alpha} u(\sigma) - \xi \, {}^{c}\mathfrak{D}^{\beta} u(\sigma) = \Phi(\sigma)\,,$$

where $n - 1 < \alpha < n, m - 1 < \beta < m$ and $m < n, n, m \in \mathcal{R}$. Motivated from the above results, we study the following coupled fractional differential equations with the help of Laplace transform method

$$\begin{cases} {}^{c}\mathfrak{D}^{\alpha_1} u(\sigma) - \xi_1 \, {}^{c}\mathfrak{D}^{\beta_1} u(\sigma) = \Phi(\sigma, {}^{c}\mathfrak{D}^{\gamma_1} u(\sigma), {}^{c}\mathfrak{D}^{\gamma_2} v(\sigma)) \\ {}^{c}\mathfrak{D}^{\alpha_2} v(\sigma) - \xi_2 \, {}^{c}\mathfrak{D}^{\beta_2} v(\sigma) = \Psi(\sigma, {}^{c}\mathfrak{D}^{\gamma_3} u(\sigma), {}^{c}\mathfrak{D}^{\gamma_4} v(\sigma)) \\ {}^{c}\mathfrak{D}^{k} u(0) = u^{\star}_k, \, {}^{c}\mathfrak{D}^{k} v(0) = v^{\star}_k\,, \qquad \mathrm{k} = 0, 1, \ldots, \mathrm{p} - 1\,, \end{cases} \tag{1.1}$$

where $\xi_1, \xi_2 \in \mathcal{R}, \ p - 1 < \alpha_1, \alpha_2 \le p, \ q - 1 < \beta_1, \beta_2 \le q, \ p, q \in \mathcal{Z}^+, \ q \le p, \ 0 < \gamma_1, \gamma_2, \gamma_3, \gamma_4 \le 1$. The real functions $\Phi : \mathcal{J} \times \mathcal{R}^2 \to \mathcal{R}$ and $\Psi : \mathcal{J} \times \mathcal{R}^2 \to \mathcal{R} \ \forall \ t \in [0,1] = \mathcal{J}, \ u^{\star}_k, v^{\star}_k \in \mathcal{R}^+$ and $\alpha_1, \beta_1, \alpha_2, \beta_2$ are the order of Caputo fractional derivative. Using the method of Laplace transform [42] and the idea of Topological degree method [43], we obtain our results.

The paper is organized as follow: In Section 2, we give some definitions, theorems, topological degree theory and some necessary conditions. In Section 3 we focus on coupled fractional differential Equation (1.1), and investigate the existence of solution by using Laplace transform, while uniqueness and at least one solution with the help of topological degree method. In Section 4, we obtain the HU stabilities of system (1.1) by taking some assumptions. The last section contain an example which strengthen our results.

2 Preliminaries

In this Section, we present some helpful definitions and well known results of fractional calculus and theory of topological degree.

The Banach spaces of all continuous functions are denoted by $\mathcal{U} = \mathcal{C}(\mathcal{J}, \mathcal{R})$, $\mathcal{V} = \mathcal{C}(\mathcal{J}, \mathcal{R})$ and $\mathcal{W} = \mathcal{U} \times \mathcal{V}$ with topological norms $\|u\|_{\mathcal{U}} = \sup\{|u(\sigma)| : \sigma \in \mathcal{J}\}$, $\|v\|_{\mathcal{V}} = \sup\{|v(\sigma)| : \sigma \in \mathcal{J}\}$ and $\|(u, v)\|_{\mathcal{W}} = \|u\|_{\mathcal{U}} + \|v\|_{\mathcal{V}}$, respectively.

Definition 2.1. [1] For a function $u : \mathcal{J} \to \mathcal{R}$, the Caputo fractional integral of order $p - 1 < \alpha_1 \le p$ is defined as:

$$^{c}\mathcal{I}^{\alpha_1} u(\sigma) = \frac{1}{\Gamma(\alpha_1)} \int_0^{\sigma} (\sigma - s)^{\alpha_1 - 1} u(s) ds .$$

Definition 2.2. [1] For a function $u : \mathcal{J} \to \mathcal{R}$, the Caputo fractional derivative of order $p - 1 < \alpha_1 \le p$ is defined as:

$$^{c}\mathfrak{D}^{\alpha_1} u(\sigma) = \frac{1}{\Gamma(p - \alpha_1)} \int_0^{\sigma} (\sigma - s)^{p - \alpha_1 - 1} u^{p}(s) ds$$

where $p = [\alpha_1] + 1$ and $[\alpha_1]$ denote the integer part of the real number α_1.

A function $u \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ is said to be of exponential order if there are constants $a, b \in \mathcal{R}$ such that $|u(\sigma)| \le ae^{b\sigma}$ for all $\sigma > 0$. For each function u of exponential order, the Laplace transform of $u(\sigma)$ is defined by

$$\mathfrak{L}\{u(\sigma)\}(s) := \int_0^{\infty} e^{-s\sigma} u(\sigma) d\sigma, \quad s \in \mathbb{C}. \tag{2.1}$$

If the integral (2.1) is convergent at $s_0 \in \mathbb{C}$, then it converges absolutely for $s \in \mathbb{C}$ such that $\Re(s) > \Re(s_0)$. The convolution property of Laplace transform is given by

$$\mathfrak{L}\{u(\sigma) \star v(\sigma)\}(s) = \mathfrak{L}\{u(\sigma)\}\mathfrak{L}\{v(\sigma)\}(s),$$

where $u(\sigma) \star v(\sigma) = \int_0^{\sigma} u(\sigma - \zeta) v(\zeta) d\zeta$, see [44].

Results related to Laplace transform in the framework of Caputo fractional derivatives are as follows:

Lemma 2.3. [1] Let $\alpha_1 > 0$, $p - 1 < \alpha_1 \le p$, $p \in \mathcal{Z}^{+}$ be such that u and u^{p} are of exponential order functions, the Laplace transforms of $u(\sigma)$ and $^{c}D^{p} u(\sigma)$ exist, and $\lim_{t \to \infty} D^{k} u(\sigma) = 0$ for $k = 0, 1, \ldots, p - 1$. Then the following relation holds:

$$\mathfrak{L}\{^{c}D^{\alpha_1} u(\sigma)\}(s) = s^{\alpha_1} \mathfrak{L}\{u(\sigma)\}(s) - \sum_{k=0}^{p-1} s^{\alpha_1 - k - 1} \left(D^{k} u\right)(0).$$

If $p = 1$ then

$$\mathfrak{L}\{^{c}D^{\alpha_1} u(\sigma)\}(s) = s^{\alpha_1} \mathfrak{L}\{u(\sigma)\}(s) - s^{\alpha_1 - 1} u(0).$$

The Mittag–Leffler function E_{α_1, β_1} is defined as:

$$E_{\alpha_1, \beta_1}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha_1 k + \beta_1)}, \quad z, \beta_1 \in \mathbb{C}, \quad \Re(\alpha_1) > 0,$$

when $\alpha_1 = \beta_1 = 1$, we can see that $E_{1,1}(z) = e^{z}$. For more properties of the mentioned function see [1].

Lemma 2.4. [1] If $\Re(s) > 0$, $\xi_1 \in \mathbb{C}$, $\left|\xi_1 s^{-\alpha_1}\right| < 1$, then

$$\mathfrak{L}\left\{\sigma^{\beta_1-1}E_{\alpha_1,\beta_1}(\xi_1\sigma^{\alpha_1})\right\}(s)=\frac{s^{\alpha_1-\beta_1}}{s^{\alpha_1}-\xi_1},$$

where $E_{\alpha_1,\beta_1}(\xi_1\sigma^{\alpha_1})$ is the Mittag–Leffler function.

Let $\mathcal{K}$ be a family of all bounded sets of $\mathcal{B}(\mathcal{W})$, where $\mathcal{W}$ is a Banach space.

Definition 2.5. [43] The Kuratowski measure of non-compactness $\mu:\mathcal{K}\to\mathcal{R}$, is defined as

$$\mu(K)=\inf\{d>0:\text{admits a finite cover by sets of diameter }\le d\},$$

where $K\in\mathcal{K}$.

Proposition 2.6. [43] The Kuratowski measure μ satisfies the following properties:

(I) $\mu(K)=0$ iff K is relatively compact.

(II) μ is a semi norm, *i.e.*, $\mu(\eta K)=|\eta|\mu(K)$, $\eta\in\mathcal{R}$ and $\mu(K_1+K_2)\le\mu(K_1)+\mu(K_2)$.

(III) $K_1\subset K_2\Rightarrow\mu(K_1)\le\mu(K_2)$; $\mu(K_1\cup K_2)=\max\{\mu(K_1),\mu(K_2)\}$.

(IV) $\mu(conv\,K)=\mu(K)$.

(V) $\mu(\overline{K})=\mu(K)$.

Definition 2.7. [43] Consider $\mathbb{F}:\Delta\to\mathcal{W}$ is a continuous and bounded operator, where $\Delta\subset\mathcal{W}$. Then $\mathbb{F}$ be μ-Lipschitz if $\exists\,L\ge0$ such that

$$\mu(\mathbb{F}(K))\le L\mu(K),\ \text{for all}\ K\subset\Delta.$$

Along this, $\mathbb{F}$ will be strict μ-contraction if $L<1$.

Definition 2.8. [43] A function $\mathbb{F}$ is μ-condensing if

$$\mu(\mathbb{F}(K))<\mu(K),\ \text{for all}\ K\subset\Delta\ \text{bounded, with}\ \mu(K)>0.$$

another way, $\mu(\mathbb{F}(K))\ge\mu(K)\Rightarrow\mu(K)=0$.

The class of all strict μ-contractions and μ-condensing of $\mathbb{F}:\Delta\to\mathcal{W}$ is denoted by $\vartheta C_\mu(\Delta)$ and $C_\mu(\Delta)$, respectively.

Remark 2.9. [43] $\vartheta C_\mu(\Delta)\subset C_\mu(\Delta)$ and every $\mathbb{F}\in C_\mu(\Delta)$ is μ-Lipschitz with constant $L=1$.

Furthermore, we recalled that $\mathbb{F}:\Delta\to\mathcal{W}$ is Lipschitz if there exists $L>0$ such that

$$\|\mathbb{F}(u,v)-\mathbb{F}(\overline{u},\overline{v})\|\le L|(u,v)-(\overline{u},\overline{v})|,\ \text{for all}\ (u,v),(\overline{u},\overline{v})\in\Delta,$$

and that if $L<1$, then $\mathbb{F}$ is a strict contraction.

Proposition 2.10. [43] If $\mathbb{F},\mathbb{G}:\Delta\to\mathcal{W}$ are μ-Lipschitz with constants L and L', respectively, then $\mathbb{F}+\mathbb{G}:\Delta\to\mathcal{W}$ is μ-Lipschitz with constants $L+L'$.

Proposition 2.11. [43] If $\mathbb{F}:\Delta\to\mathcal{W}$ is compact, then $\mathbb{F}$ is μ-Lipschitz with constant $L=0$.

Proposition 2.12. [43] If $\mathbb{F}:\Delta\to\mathcal{W}$ is Lipschitz with constant $L\ldots$, then … is μ-Lipschitz with the same constant $L_{\mathbb{F}}$.

The following theorem, due to Isaia [45] plays an important role for our main result.

Theorem 2.13. [43] Let $\mathbb{F}:\mathcal{W}\to\mathcal{W}$ be μ-condensing and

$$\Upsilon = \{(u,v) \in \mathcal{W} : \exists \eta \in [0,1] \text{ such that } (u,v) = \eta \mathbb{F}(u,v)\}.$$

If Υ is a bounded set in $\mathcal{W}$, i.e., there exists $r > 0$ such that $\Upsilon \subset K_r(0)$, then the degree

$$D(1 - \eta\mathbb{F},\ K_r(0), 0) = 1, \quad \forall\ \eta \in [0,1].$$

Consequently, $\mathbb{F}$ has at least one fixed point and the set of the fixed points of $\mathbb{F}$ lies in $K_r(0)$.

In the sequel, we required the following assumptions:

(H_1). Let $\Phi, \Psi : \mathcal{J} \times \mathcal{R}^2 \to \mathcal{R}$ are continuous functions, then there exist constants $\mathfrak{L}_{\Phi 1}, \mathfrak{L}_{\Psi 1} > 0$ and $0 < \mathfrak{L}_{\Phi 2}, \mathfrak{L}_{\Psi 2} < 1$ such that $\sigma \in \mathcal{J}$ and $\forall\ u_1, u_2, v_1, v_2 \in \mathcal{R}$, the following holds

$$|\Phi(\sigma, u_1, u_2) - \Phi(\sigma, v_1, v_2)| \le \mathfrak{L}_{\Phi 1}|u_1 - v_1| + \mathfrak{L}_{\Phi 2}|u_2 - v_2|$$

and

$$|\Psi(\sigma, u_1, u_2) - \Psi(\sigma, v_1, v_2)| \le \mathfrak{L}_{\Psi 1}|u_1 - v_1| + \mathfrak{L}_{\Psi 2}|u_2 - v_2|.$$

(H_2). With the given continuous functions $\omega_{\Phi 1}, \omega_{\Phi 2}, \omega_{\Phi 3}, \omega_{\Psi 1}, \omega_{\Psi 2}, \omega_{\Psi 3} \in C(\mathcal{J}, \mathcal{R})$ for $u \in \mathcal{U}$, $v \in \mathcal{V}$, the nonlocal functions Φ, Ψ satisfy the following growth conditions:

$$|\Phi(\sigma, u_1(\sigma), u_2(\sigma))| \le \omega_{\Phi 1}(\sigma) + \omega_{\Phi 2}(\sigma)|u_1| + \omega_{\Phi 3}(\sigma)|u_2|$$

and

$$|\Psi(\sigma, v_1(\sigma), v_2(\sigma))| \le \omega_{\Psi 1}(\sigma) + \omega_{\Psi 2}(\sigma)|v_1| + \omega_{\Psi 3}(\sigma)|v_2|$$

with $\omega^*_{\Phi 1} = \sup_{\sigma \in \mathcal{J}} \omega_{\Phi 1}(\sigma)$, $\omega^*_{\Phi 2} = \sup_{\sigma \in \mathcal{J}} \omega_{\Phi 2}(\sigma)$, $\omega^*_{\Phi 3} = \sup_{\sigma \in \mathcal{J}} \omega_{\Phi 3}(\sigma)$, $\omega^*_{\Psi 1} = \sup_{\sigma \in \mathcal{J}} \omega_{\Psi 1}(\sigma)$, $\omega^*_{\Psi 2} = \sup_{\sigma \in \mathcal{J}} \omega_{\Psi 2}(\sigma)$, $\omega^*_{\Psi 3} = \sup_{\sigma \in \mathcal{J}} \omega_{\Psi 3}(\sigma)$ are positive constants.

3 Existence and uniqueness results

We prove the existence and uniqueness as follows.

Theorem 3.1. Let Φ be a continuous and linear function. The solution of

$$\begin{cases} {}^c\mathfrak{D}^{\alpha_1} u(\sigma) - \xi_1\, {}^c\mathfrak{D}^{\beta_1} u(\sigma) = \Phi(\sigma) \\ {}^c\mathfrak{D}^{k} u(0) = u^*_k, \qquad k = 0, 1, 2, \ldots, p-1, \end{cases} \tag{3.1}$$

is given as:

$$\begin{aligned} u(\sigma) = {} & \sum_{k=0}^{p-1} \sigma^k E_{\alpha_1-\beta_1, k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) u^*_k - \xi_1 \sum_{k=0}^{q-1} \sigma^{\alpha_1-\beta_1+k} E_{\alpha_1-\beta_1, \alpha_1-\beta_1+k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) u^*_k \\ & + \epsilon\sigma_0^\sigma (\sigma - s)^{\alpha_1-1} E_{\alpha_1-\beta_1, \alpha_1}\left(\xi_1 (\sigma - s)^{\alpha_1-\beta_1}\right)\Phi(s)ds, \end{aligned} \tag{3.2}$$

where $\xi_1, \xi_2 \in \mathcal{R}$, $p - 1 < \alpha_1 \le p$ and $q - 1 < \beta_1 \le q$ as $p, q \in \mathcal{Z}^+$ and $q \le p$.

Proof. Applying Laplace transform on fractional differential Equation (3.1) and using Lemma 2.3, we obtain

$$\begin{aligned}
&\mathfrak{L}\left\{{}^{c}D^{\alpha_1} u(\sigma) - \xi_1 \, {}^{c}D^{\beta_1} u(\sigma)\right\}(s) = \mathfrak{L}\{\Phi(\sigma)\}(s)\\
&\mathfrak{L}\{{}^{c}D^{\alpha_1} u(\sigma)\}(s) - \xi_1 \mathfrak{L}\left\{{}^{c}D^{\beta_1} u(\sigma)\right\}(s) = \mathfrak{L}\{\Phi(\sigma)\}(s)\\
&s^{\alpha_1} \mathfrak{L}\{u(\sigma)\}(s) - \sum_{k=0}^{p-1} s^{\alpha_1-k-1} u_k^\star - \xi_1 s^{\beta_1} \mathfrak{L}\{u(\sigma)\}(s) - \xi_1 \sum_{k=0}^{q-1} s^{\beta_1-k-1} u_k^\star = \mathfrak{L}\{\Phi(\sigma)\}(s)\\
&\left(s^{\alpha_1} - \xi_1 s^{\beta_1}\right)\mathfrak{L}\{u(\sigma)\}(s) = \sum_{k=0}^{p-1} s^{\alpha_1-k-1} u_k^\star - \xi_1 \sum_{k=0}^{q-1} s^{\beta_1-k-1} u_k^\star + \mathfrak{L}\{\Phi(\sigma)\}(s)\\
&\mathfrak{L}\{u(\sigma)\}(s) = \frac{\sum_{k=0}^{p-1} s^{\alpha_1-k-1} u_k^\star - \xi_1 \sum_{k=0}^{q-1} s^{\beta_1-k-1} u_k^\star + \mathfrak{L}\{\Phi(\sigma)\}(s)}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}.
\end{aligned} \tag{3.3}$$

Now by applying inverse Laplace transform to (3.3), we have

$$\begin{aligned}
\mathfrak{L}^{-1}\{\mathfrak{L}\{u(\sigma)\}(s)\} &= \mathfrak{L}^{-1}\left\{\frac{\sum_{k=0}^{p-1} s^{\alpha_1-k-1} u_k^\star}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\} - \mathfrak{L}^{-1}\left\{\frac{\xi_1 \sum_{k=0}^{q-1} s^{\beta_1-k-1} u_k^\star}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\} + \mathfrak{L}^{-1}\left\{\frac{\mathfrak{L}\{\Phi(\sigma)\}(s)}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\}\\
u(\sigma) &= \sum_{k=0}^{p-1} u_k^\star \mathfrak{L}^{-1}\left\{\frac{s^{\alpha_1-k-1}}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\} - \xi_1 \sum_{k=0}^{q-1} u_k^\star \mathfrak{L}^{-1}\left\{\frac{s^{\beta_1-k-1}}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\}\\
&\quad + \mathfrak{L}^{-1}\left\{\frac{1}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\} * \Phi(\sigma).
\end{aligned} \tag{3.4}$$

Lemma 2.4 help to find some inverse Laplace transforms that is

$$\mathfrak{L}^{-1}\left\{\frac{s^{\alpha_1-k-1}}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\} = \sigma^k E_{\alpha_1-\beta_1,k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right), \tag{3.5}$$

$$\mathfrak{L}^{-1}\left\{\frac{s^{\beta_1-k-1}}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\} = \sigma^{\alpha_1-\beta_1+k} E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) \text{ and} \tag{3.6}$$

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s^{\alpha_1} - \xi_1 s^{\beta_1})}\right\} = \sigma^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right). \tag{3.7}$$

Substituting (3.5), (3.6) and (3.7) in (3.4), we get

$$\begin{aligned}
u(\sigma) = &\sum_{k=0}^{p-1} \sigma^k E_{\alpha_1-\beta_1,k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) u_k^\star - \xi_1 \sum_{k=0}^{q-1} \sigma^{\alpha_1-\beta_1+k} E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) u_k^\star\\
&+ \sigma^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) * \Phi(\sigma).
\end{aligned}$$

From Theorem 3, we achieved that $(u, v) \in \mathcal{W}$ is the solution of system (1.1), where

$$\begin{cases}
u(\sigma) = \sum_{k=0}^{p-1} \sigma^k E_{\alpha_1-\beta_1,k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) u_k^* - \xi_1 \sum_{k=0}^{q-1} \sigma^{\alpha_1-\beta_1+k} E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) u_k^*\\
\qquad + \int_0^\sigma (\sigma - s)^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\left(\xi_1 (\sigma - s)^{\alpha_1-\beta_1}\right) \Phi(s, {}^{c}D^{\gamma_1} u(s), {}^{c}D^{\gamma_2} v(s))\, ds,\\
v(\sigma) = \sum_{k=0}^{p-1} \sigma^k E_{\alpha_2-\beta_2,k+1}\left(\xi_2 \sigma^{\alpha_2-\beta_2}\right) v_k^* - \xi_2 \sum_{k=0}^{q-1} \sigma^{\alpha_2-\beta_2+k} E_{\alpha_2-\beta_2,\alpha_2-\beta_2+k+1}\left(\xi_2 \sigma^{\alpha_2-\beta_2}\right) v_k^\star\\
\qquad + \int_0^\sigma (\sigma - s)^{\alpha_2-1} E_{\alpha_2-\beta_2,\alpha_2}\left(\xi_2 (\sigma - s)^{\alpha_2-\beta_2}\right) \Psi(s, {}^{c}D^{\gamma_3} u(s), {}^{c}D^{\gamma_4} v(s))\, ds.
\end{cases} \tag{3.8}$$

Define the operators $\mathbb{F}_1 : \mathcal{U} \to \mathcal{U}$, $\mathbb{F}_2 : \mathcal{V} \to \mathcal{V}$, as follows

$$\mathbb{F}_1(u)(\sigma) = \sum_{k=0}^{p-1} \sigma^k E_{\alpha_1-\beta_1,k+1}\big(\xi_1 \sigma^{\alpha_1-\beta_1}\big)u_k^* - \xi_1 \sum_{k=0}^{q-1} \sigma^{\alpha_1-\beta_1+k} E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\big(\xi_1 \sigma^{\alpha_1-\beta_1}\big)u_k^*$$
$$\mathbb{F}_2(v)(\sigma) = \sum_{k=0}^{p-1} \sigma^k E_{\alpha_2-\beta_2,k+1}\big(\xi_2 \sigma^{\alpha_2-\beta_2}\big)v_k^* - \xi_2 \sum_{k=0}^{q-1} \sigma^{\alpha_2-\beta_2+k} E_{\alpha_2-\beta_2,\alpha_2-\beta_2+k+1}\big(\xi_2 \sigma^{\alpha_2-\beta_2}\big)v_k^*,$$

and the operators $\mathbb{G}_1, \mathbb{G}_2 : \mathcal{W} \to \mathcal{W}$ as

$$\begin{aligned}\mathbb{G}_1(u,v)(\sigma) &= \int_0^\sigma (\sigma-s)^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Phi(s, {}^cD^{\gamma_1}u(s), {}^cD^{\gamma_2}v(s))\,ds \\ \mathbb{G}_2(u,v)(\sigma) &= \int_0^\sigma (\sigma-s)^{\alpha_2-1} E_{\alpha_2-\beta_2,\alpha_2}\big(\xi_2(\sigma-s)^{\alpha_2-\beta_2}\big)\Psi(s, {}^cD^{\gamma_3}u(s), {}^cD^{\gamma_4}v(s))\,ds.\end{aligned} \tag{3.9}$$

Moreover, we define $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2)$, $\mathbb{G} = (\mathbb{G}_1, \mathbb{G}_2)$ and $\mathbb{T} = \mathbb{F} + \mathbb{G}$. Then the system (3.8) can be written as an operator equation of the form

$$(u,v) = \mathbb{T}(u,v) = \mathbb{F}(u,v) + \mathbb{G}(u,v) \tag{3.10}$$

and solution of the system (3.8) is a fixed point of $\mathbb{T}$.

Lemma 3.2. The operator $\mathbb{F}$ satisfies the μ –Lipschitz condition and the following growth condition:

$$\|\mathbb{F}(u,v)\|_{\mathcal{W}} \le L_{\mathbb{F}} \|(u,v)\|_{\mathcal{W}} \text{ for every } (u,v) \in \mathcal{W}, \tag{3.11}$$

where

$$L_{\mathbb{F}} = \max\left\{\frac{p|\xi_1| + q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}, \frac{p|\xi_2| + q\xi_2^2}{\Gamma(\alpha_2-\beta_2)}\right\}.$$

Proof. For Lipschitz condition of $\mathbb{F}$, applying norm, we get

$$\begin{aligned}\|\mathbb{F}(u,v) - \mathbb{F}(\bar{u},\bar{v})\|_{\mathcal{W}} &\le \|\mathbb{F}_1(u) - \mathbb{F}_1(\bar{u})\|_{\mathcal{U}} + \|\mathbb{F}_2(v) - \mathbb{F}_2(\bar{v})\|_{\mathcal{V}} \\ &\le \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{p-1} \left|\sigma^k E_{\alpha_1-\beta_1,k+1}\big(\xi_1\sigma^{\alpha_1-\beta_1}\big)\right| \left|u_k^* - \bar{u}_k^*\right| \\ &\quad + \sup_{\sigma\in\mathcal{J}} |\xi_1| \sum_{k=0}^{q-1} \left|\sigma^{\alpha_1-\beta_1+k} E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\big(\xi_1\sigma^{\alpha_1-\beta_1}\big)\right\| u_k^* - \bar{u}_k^*\Big| \\ &\quad + \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{p-1} \left|\sigma^k E_{\alpha_2-\beta_2,k+1}\big(\xi_2\sigma^{\alpha_2-\beta_2}\big)\right\| v_k^* - \bar{v}_k^*\Big| \\ &\quad + \sup_{\sigma\in\mathcal{J}} |\xi_2| \sum_{k=0}^{q-1} \left|\sigma^{\alpha_2-\beta_2+k} E_{\alpha_2-\beta_2,\alpha_2-\beta_2+k+1}\big(\xi_2\sigma^{\alpha_2-\beta_2}\big)\right\| v_k^* - \bar{v}_k^*\Big| \\ &\le \frac{p|\xi_1| + q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} \|u - \bar{u}\|_{\mathcal{U}} + \frac{p|\xi_2| + q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} \|v - \bar{v}\|_{\mathcal{V}} \\ &\le L_{\mathbb{F}} \|(u,v) - (\bar{u},\bar{v})\|_{\mathcal{W}}.\end{aligned} \tag{3.12}$$

By Proposition 2.12, $\mathbb{F}$ is also μ-Lipschitz with constant $L_{\mathbb{F}}$. Now for the growth condition of $\mathbb{F}$, also applying norm, we obtain

$$\begin{aligned}
||\mathbb{F}(u,v)||_{\mathcal{W}} &\le ||\mathbb{F}_1(u)||_{\mathcal{U}} + \|\mathbb{F}_2(v)\|_{\mathcal{V}} \\
&\le \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{p-1} \left|\sigma^k E_{\alpha_1-\beta_1,k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) \| u_k^*\right| \\
&\quad + \sup_{\sigma\in\mathcal{J}} |\xi_1| \sum_{k=0}^{q-1} \left|\sigma^{\alpha_1-\beta_1+k} E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right) \| u_k^*\right| \\
&\quad + \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{p-1} \left|\sigma^k E_{\alpha_2-\beta_2,k+1}\left(\xi_2 \sigma^{\alpha_2-\beta_2}\right) \| v_k^*\right| \\
&\quad + \sup_{\sigma\in\mathcal{J}} |\xi_2| \sum_{k=0}^{q-1} \left|\sigma^{\alpha_2-\beta_2+k} E_{\alpha_2-\beta_2,\alpha_2-\beta_2+k+1}\left(\xi_2 \sigma^{\alpha_2-\beta_2}\right) \| v_k^*\right| \\
&\le \frac{p|\xi_1| + q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} u_{\mathcal{U}} + \frac{p|\xi_2| + q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} ||v||_{\mathcal{V}} \le L_{\mathbb{F}} ||(u,v)||_{\mathcal{W}}.
\end{aligned}$$

Lemma 3.3. Under the assumptions (H_1) and (H_2), the operator $\mathbb{G}$ is continuous and the following growth condition holds:

$$\|\mathbb{G}(u,v)\|_{\mathcal{W}} \le \Delta_{\mathbb{G}} \|(u,v)\|_{\mathcal{W}} + \Lambda_{\mathbb{G}} \quad \text{for every} \quad (u,v) \in \mathcal{W}, \tag{3.13}$$

where $\Delta_{\mathbb{G}} = \Delta_{\mathbb{G}}^* + \Delta_{\mathbb{G}}^{**}$, $\Lambda_{\mathbb{G}} = \omega_{\Phi 1}^* \mathcal{A}_1^* + \omega_{\Psi 1}^* \mathcal{A}_2^*$,

$$\Delta_{\mathbb{G}}^* = \max\left\{\frac{\omega_{\Phi 2}^* \mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}, \frac{\omega_{\Phi 3}^* \mathcal{A}_{\gamma_2}^*}{\Gamma(2-\gamma_2)}\right\}, \quad \Delta_{\mathbb{G}}^{**} = \max\left\{\frac{\omega_{\Psi 2}^* \mathcal{A}_{\gamma_3}^*}{\Gamma(2-\gamma_3)}, \frac{\omega_{\Psi 3}^* \mathcal{A}_{\gamma_4}^*}{\Gamma(2-\gamma_4)}\right\},$$

$$\mathcal{A}_1^* = \sup_{\sigma\in\mathcal{J}} \sigma^{\alpha_1} E_{\alpha_1-\beta_1,\alpha_1+1}\left(\xi_1 \sigma^{\alpha_1-\beta_1}\right), \quad \mathcal{A}_2^* = \sup_{\sigma\in\mathcal{J}} \sigma^{\alpha_2} E_{\alpha_2-\beta_2,\alpha_2+1}\left(\xi_2 \sigma^{\alpha_2-\beta_2}\right),$$

$$\mathcal{A}_{\gamma_1}^* = \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \frac{\xi_1^k \sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_1}}{\Gamma(k(\alpha_1-\beta_1)+\alpha_1)} B(2-\gamma_1, k(\alpha_1-\beta_1)+\alpha_1),$$

$$\mathcal{A}_{\gamma_2}^* = \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \frac{\xi_1^k \sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_2}}{\Gamma(k(\alpha_1-\beta_1)+\alpha_1)} B(2-\gamma_2, k(\alpha_1-\beta_1)+\alpha_1),$$

$$\mathcal{A}_{\gamma_3}^* = \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \frac{\xi_2^k \sigma^{k(\alpha_2-\beta_2)+\alpha_2-\gamma_3}}{\Gamma(k(\alpha_2-\beta_2)+\alpha_2)} B(2-\gamma_3, k(\alpha_2-\beta_2)+\alpha_2) \text{ and}$$

$$\mathcal{A}_{\gamma_4}^* = \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \frac{\xi_2^k \sigma^{k(\alpha_2-\beta_2)+\alpha_2-\gamma_4}}{\Gamma(k(\alpha_2-\beta_2)+\alpha_2)} B(2-\gamma_4, k(\alpha_2-\beta_2)+\alpha_2),$$

where $B(\cdot,\cdot)$ is beta function.

Proof. Consider sequence $\{(u_n,v_n)\}$ from a bounded set $\mathcal{X}_r = \{||(u,v)||_{\mathcal{W}} \le r : (u,v) \in \mathcal{W}\}$ such that $(u_n,v_n) \to (u,v) \in \mathcal{X}_r$ as $n \to \infty$. For proving the continuity of $\mathbb{G}$ use (H_1) and (3.9), it is enough to show that $\|\mathbb{G}(u_n,v_n) - \mathbb{G}(u,v)\| \to 0$ as $n \to \infty$. First we find the continuity of $\mathbb{G}_1$, i.e.,

$$\begin{aligned}
&\|\mathbb{G}_1(u_n,v_n)-\mathbb{G}_1(u,v)\|_{\mathcal{W}}\\
&\le \sup_{\sigma\in\mathcal{J}}\int_0^\sigma\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\,\|\,\Phi(s,{}^cD^{\gamma_1}u_n(s),{}^cD^{\gamma_2}v_n(s))-\Phi(s,{}^cD^{\gamma_1}u(s),{}^cD^{\gamma_2}v(s))\Big|\,ds\\
&\le \sup_{\sigma\in\mathcal{J}}\int_0^\sigma\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|[\mathfrak{L}_{\Phi1}|{}^cD^{\gamma_1}-{}^cD^{\gamma_2}u|+\mathfrak{L}_{\Phi2}|{}^cD^{\gamma_1}v_n-{}^cD^{\gamma_2}v|]\,ds\\
&\le \sup_{\sigma\in\mathcal{J}}\int_0^\sigma\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|\mathfrak{L}_{\Phi1}|{}^cD^{\gamma_1}u_n-{}^cD^{\gamma_1}u|\,ds\\
&\quad+\sup_{\sigma\in\mathcal{J}}\int_0^\sigma\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|\mathfrak{L}_{\Phi2}|{}^cD^{\gamma_2}v_n-{}^cD^{\gamma_2}v|\,ds\\
&\le \frac{\mathfrak{L}_{\Phi1}}{\Gamma(2-\gamma_1)}\sup_{\sigma\in\mathcal{J}}\int_0^\sigma s^{1-\gamma_1}\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|ds|u_n-u|\\
&\quad+\frac{\mathfrak{L}_{\Phi2}}{\Gamma(2-\gamma_2)}\sup_{\sigma\in\mathcal{J}}\int_0^\sigma s^{1-\gamma_2}\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|ds|v_n-v|\\
&\le \sup_{\sigma\in\mathcal{J}}\sum_{k=0}^{\infty}\frac{\mathfrak{L}_{\Phi1}\xi_1^k\sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_1}}{\Gamma(2-\gamma_1)\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}B(2-\gamma_1,\ k(\alpha_1-\beta_1)+\alpha_1)|u_n-u|\\
&\quad+\sup_{\sigma\in\mathcal{J}}\sum_{k=0}^{\infty}\frac{\mathfrak{L}_{\Phi2}\xi_1^k\sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_2}}{\Gamma(2-\gamma_2)\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}B(2-\gamma_2,\ k(\alpha_1-\beta_1)+\alpha_1)|v_n-v|,
\end{aligned}$$

it is clear that $u_n\to u$ and $v_n\to v$ as $n\to\infty \Rightarrow \|\mathbb{G}_1(u_n,v_n)-\mathbb{G}_1(u,v)\|_{\mathcal{W}}\to 0$ as $n\to\infty$. For $\mathbb{G}_2$ we use the same steps as above we can get $\|\mathbb{G}_2(u_n,v_n)-\mathbb{G}_2(u,v)\|_{\mathcal{W}}\to 0$ as $n\to\infty$. Hence $\mathbb{G}$ is continuous.

Next to show the growth bound condition of $\mathbb{G}$, we use (H_2) and (3.9), we achieve as

$$\begin{aligned}
&\|\mathbb{G}_1(u,v)\|_{\mathcal{W}}\\
&\le \sup_{\sigma\in\mathcal{J}}\int_0^\sigma(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)|\Phi(s,{}^cD^{\gamma_1}u(s),{}^cD^{\gamma_2}v(s))|\,ds\\
&\le \omega^*_{\Phi1}\sup_{\sigma\in\mathcal{J}}\int_0^\sigma(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\,ds\\
&\quad+\frac{\omega^*_{\Phi2}\|u\|_{\mathcal{U}}}{\Gamma(2-\gamma_1)}\sup_{\sigma\in\mathcal{J}}\int_0^\sigma s^{1-\gamma_1}(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\,ds\\
&\quad+\frac{\omega^*_{\Phi3}\|v\|_{\mathcal{V}}}{\Gamma(2-\gamma_2)}\sup_{\sigma\in\mathcal{J}}\int_0^\sigma s^{1-\gamma_2}(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\,ds\\
&\le \omega^*_{\Phi1}\sup_{\sigma\in\mathcal{J}}\sigma^{\alpha_1}E_{\alpha_1-\beta_1,\alpha_1+1}\big(\xi_1\sigma^{\alpha_1-\beta_1}\big)\\
&\quad+\frac{\omega^*_{\Phi2}\|u\|_{\mathcal{U}}}{\Gamma(2-\gamma_1)}\sup_{\sigma\in\mathcal{J}}\sum_{k=0}^{\infty}\frac{\xi_1^k\sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_1}}{\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}B(2-\gamma_1,\ k(\alpha_1-\beta_1)+\alpha_1)\\
&\quad+\frac{\omega^*_{\Phi3}\|v\|_{\mathcal{V}}}{\Gamma(2-\gamma_2)}\sup_{\sigma\in\mathcal{J}}\sum_{k=0}^{\infty}\frac{\xi_1^k\sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_2}}{\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}B(2-\gamma_2,k(\alpha_1-\beta_1)+\alpha_1)\\
&\le \omega^*_{\Phi1}\sup_{\sigma\in[0,1]}\mathcal{A}_1(\sigma)+\frac{\omega^*_{\Phi2}\|u\|_{\mathcal{U}}}{\Gamma(2-\gamma_1)}\sup_{\sigma\in[0,1]}\mathcal{A}_{\gamma_1}(\sigma)+\frac{\omega^*_{\Phi3}\|v\|_{\mathcal{V}}}{\Gamma(2-\gamma_2)}\sup_{\sigma\in[0,1]}\mathcal{A}_{\gamma_2}(\sigma)
\end{aligned}$$

$$
\begin{aligned}
&\leq \omega_{\Phi 1}^{*} \mathcal{A}_1^{*} + \frac{\omega_{\Phi 2}^{*} \mathcal{A}_{\gamma_1}^{*} \|u\|_{\mathcal{U}}}{\Gamma(2-\gamma_1)} + \frac{\omega_{\Phi 3}^{*} \mathcal{A}_{\gamma_2}^{*} \|v\|_{\mathcal{V}}}{\Gamma(2-\gamma_2)} \\
&\leq \Delta_{\mathbb{G}}^{*} \|(u,v)\|_{\mathcal{W}} + \omega_{\Phi 1}^{*} \mathcal{A}_1^{*}
\end{aligned}
$$

and

$$
\|\mathbb{G}_2(u,v)\|_{\mathcal{W}} \leq \Delta_{\mathbb{G}}^{**} \|(u,v)\|_{\mathcal{W}} + \omega_{\Psi 1}^{*} \mathcal{A}_2^{*},
$$

by the similar steps as for $\mathbb{G}_1$. Hence, it follows that

$$
\begin{aligned}
\|\mathbb{G}(u,v)\|_{\mathcal{W}} &\leq \|\mathbb{G}_1(u,v)\|_{\mathcal{W}} + \|\mathbb{G}_2(u,v)\|_{\mathcal{W}} \\
&\leq \Delta_{\mathbb{G}}^{*} \|(u,v)\|_{\mathcal{W}} + \Delta_{\mathbb{G}}^{**} \|(u,v)\|_{\mathcal{W}} + \omega_{\Phi 1}^{*} \mathcal{A}_1^{*} + \omega_{\Psi 1}^{*} \mathcal{A}_2^{*} \\
&\leq \Delta_{\mathbb{G}} \|(u,v)\|_{\mathcal{W}} + \Lambda_{\mathbb{G}}.
\end{aligned}
$$

Lemma 3.4. Under the bounds $s^{1-\gamma_1} \leq c_{\gamma_1}$, $s^{1-\gamma_2} \leq c_{\gamma_2}$, $s^{1-\gamma_3} \leq c_{\gamma_3}$ and $s^{1-\gamma_4} \leq c_{\gamma_4}$, the operator $\mathbb{G}: \mathcal{W} \to \mathcal{W}$ is compact. Consequently, $\mathbb{G}$ is μ-Lipschitz with zero constant.

Proof. Take a bounded $\Theta \subset \chi_r \subseteq \mathcal{W}$ and $\{(u_n, v_n)\} \in \Theta$ be a sequence, then (3.13) can be written as:

$$
\mathbb{G}(u,v)_{\mathcal{W}} \leq \Delta_{\mathbb{G}} r + \Lambda_{\mathbb{G}} \quad \text{for every} \quad (u,v) \in \mathcal{W}.
$$

So $\mathbb{G}(\Theta)$ is bounded. Now for equi-continuity and for given $\epsilon > 0$, take $\delta = \{\delta_1, \delta_2\}$, where

$$
\delta_1 = \left(\frac{\frac{\epsilon}{2} + \left(\omega_{\Phi 2}^{*} \frac{3A_{\gamma_1}^{*} + c_{\gamma_1} \mathcal{A}_1^{*}}{\Gamma(2-\gamma_1)} + \omega_{\Phi 3}^{*} \frac{3A_{\gamma_2}^{*} + c_{\gamma_2} \mathcal{A}_1^{*}}{\Gamma(2-\gamma_2)} \right) r}{4 \omega_{\Phi 1}^{*} \mathcal{A}_1^{*}} \right)^{\frac{1}{\alpha_1 - \gamma_1}}
$$

$$
\delta_2 = \left(\frac{\frac{\epsilon}{2} + \left(\omega_{\Psi 2}^{*} \frac{3A_{\gamma_3}^{*} + c_{\gamma_3} \mathcal{A}_2^{*}}{\Gamma(2-\gamma_3)} + \omega_{\Psi 3}^{*} \frac{3A_{\gamma_4}^{*} + c_{\gamma_4} \mathcal{A}_2^{*}}{\Gamma(2-\gamma_4)} \right) r}{4 \omega_{\Psi 1}^{*} \mathcal{A}_2^{*}} \right)^{\frac{1}{\alpha_2 - \gamma_3}}.
$$

For each $(u_n, v_n) \in \Theta$, we claim that if $\sigma, \tau \in \mathcal{J}$ and $0 < \tau - \sigma < \delta_1$, then

$$
\sup_{\sigma \in \mathcal{J}} |\mathbb{G}_1(u_n, v_n)(\sigma) - \mathbb{G}_1(u_n, v_n)(\tau)| < \frac{\epsilon}{2}.
$$

Now consider

$$
\begin{aligned}
&\sup_{\sigma \in \mathcal{J}} |\mathbb{G}_1(u_n, v_n)(\sigma) - \mathbb{G}_1(u_n, v_n)(\tau)| \\
&= \sup_{\sigma \in \mathcal{J}} \Bigg| \int_0^{\sigma} (\sigma - s)^{\alpha_1 - 1} E_{\alpha_1 - \beta_1, \alpha_1}\left(\xi_1 (\sigma - s)^{\alpha_1 - \beta_1}\right) \Phi(s, {}^{c}D^{\gamma_1} u(s), {}^{c}D^{\gamma_2} v(s))\, ds \\
&\quad - \int_0^{\tau} (\tau - s)^{\alpha_1 - 1} E_{\alpha_1 - \beta_1, \alpha_1}\left(\xi_1 (\tau - s)^{\alpha_1 - \beta_1}\right) \Phi(s, {}^{c}D^{\gamma_1} u(s), {}^{c}D^{\gamma_2} v(s))\, ds \Bigg| \\
&\leq \sup_{\sigma \in \mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_0^{\sigma} \frac{\left[(\sigma - s)^{k(\alpha_1 - \beta_1) + \alpha_1 - 1} - (\tau - s)^{k(\alpha_1 - \beta_1) + \alpha_1 - 1} \right]}{\Gamma(k(\alpha_1 - \beta_1) + \alpha_1)} |\Phi(s, {}^{c}D^{\gamma_1} u(s), {}^{c}D^{\gamma_2} v(s))|\, ds \\
&\quad + \sup_{\sigma \in \mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_{\sigma}^{\tau} \frac{(\tau - s)^{k(\alpha_1 - \beta_1) + \alpha_1 - 1}}{\Gamma(k(\alpha_1 - \beta_1) + \alpha_1)} |\Phi(s, {}^{c}D^{\gamma_1} u(s), {}^{c}D^{\gamma_2} v(s))|\, ds \\
&\leq \omega_{\Phi 1}^{*} \sup_{\sigma \in \mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_0^{\sigma} \frac{\left[(\sigma - s)^{k(\alpha_1 - \beta_1) + \alpha_1 - 1} - (\tau - s)^{k(\alpha_1 - \beta_1) + \alpha_1 - 1} \right]}{\Gamma(k(\alpha_1 - \beta_1) + \alpha_1)}\, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\omega^*_{\Phi 2}\, u_{\mathcal{U}} \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_0^{\sigma} \frac{\left[(\sigma-s)^{k(\alpha_1-\beta_1)+\alpha_1-1}-(\tau-s)^{k(\alpha_1-\beta_1)+\alpha_1-1}\right]s^{1-\gamma_1}}{\Gamma(2-\gamma_1)\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}\,ds \\
&+\omega^*_{\Phi 3}\|v\|_{\mathcal{V}} \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_0^{\sigma} \frac{\left[(\sigma-s)^{k(\alpha_1-\beta_1)+\alpha_1-1}-(\tau-s)^{k(\alpha_1-\beta_1)+\alpha_1-1}\right]s^{1-\gamma_2}}{\Gamma(2-\gamma_2)\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}\,ds \\
&+\omega^*_{\Phi 1} \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_{\sigma}^{\tau} \frac{(\tau-s)^{k(\alpha_1-\beta_1)+\alpha_1-1}}{\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}\,ds + \omega^*_{\Phi 2}\|u\|_{\mathcal{U}} \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_{\sigma}^{\tau} \frac{(\tau-s)^{k(\alpha_1-\beta_1)+\alpha_1-1}s^{1-\gamma_1}}{\Gamma(2-\gamma_1)\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}\,ds \\
&+\omega^*_{\Phi 3}\|v\|_{\mathcal{V}} \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \xi_1^k \int_{\sigma}^{\tau} \frac{(\tau-s)^{k(\alpha_1-\beta_1)+\alpha_1-1}s^{1-\gamma_2}}{\Gamma(2-\gamma_2)\Gamma(k(\alpha_1-\beta_1)+\alpha_1)}\,ds \\
&\le \omega^*_{\Phi 1} \sup_{\sigma\in\mathcal{J}} \left[\sigma^{\alpha_1} E_{\alpha_1-\beta_1,\alpha_1+1}\left(\xi_1\sigma^{\alpha_1-\beta_1}\right) + \tau^{\alpha_1} E_{\alpha_1-\beta_1,\alpha_1+1}\left(\xi_1\tau^{\alpha_1-\beta_1}\right) + (\tau-\sigma)^{\alpha_1} E_{\alpha_1-\beta_1,\alpha_1+1}\left(\xi_1(\tau-\sigma)^{\alpha_1-\beta_1}\right)\right] \\
&+\omega^*_{\Phi 2}\|u\|_{\mathcal{U}} \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \frac{\xi_1^k\left[\sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_1}+\tau^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_1}-(\tau-\sigma)^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_1}\right]}{\Gamma(2-\gamma_1)\Gamma(k(\alpha_1-\beta_1)+\alpha_1+1)} \\
&\qquad\qquad \times B(2-\gamma_1,\; k(\alpha_1-\beta_1)+\alpha_1) \\
&+\omega^*_{\Phi 3}\, v_{\mathcal{V}} \sup_{\sigma\in\mathcal{J}} \sum_{k=0}^{\infty} \frac{\xi_1^k\left[\sigma^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_2}+\tau^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_2}-(\tau-\sigma)^{k(\alpha_1-\beta_1)+\alpha_1-\gamma_2}\right]}{\Gamma(2-\gamma_2)\Gamma(k(\alpha_1-\beta_1)+\alpha_1+1)} \\
&\qquad\qquad \times B(2-\gamma_2,\; k(\alpha_1-\beta_1)+\alpha_1) \\
&+\left(\omega^*_{\Phi 1}+\frac{\omega^*_{\Phi 2}c_{\gamma_1}\|u\|_{\mathcal{U}}}{\Gamma(2-\gamma_1)}+\frac{\omega^*_{\Phi 3}c_{\gamma_2}\|v\|_{\mathcal{V}}}{\Gamma(2-\gamma_2)}\right)\sup_{\sigma\in\mathcal{J}}(\tau-\sigma)^{\alpha_1} E_{\alpha_1-\beta_1,\alpha_1+1}\left(\xi_1(\tau-\sigma)^{\alpha_1-\beta_1}\right) \\
&\le 4\omega^*_{\Phi 1}\mathcal{A}_1^*\delta_1^{\alpha_1-\gamma_1}+\omega^*_{\Phi 2}\frac{3A^*_{\gamma_1}+c_{\gamma_1}\mathcal{A}_1^*}{\Gamma(2-\gamma_1)}\|u\|_{\mathcal{U}}+\omega^*_{\Phi 3}\frac{3A^*_{\gamma_2}+c_{\gamma_2}A_1^*}{\Gamma(2-\gamma_2)}\|v\|_{\mathcal{V}} \\
&\le 4\omega^*_{\Phi 1}\mathcal{A}_1^*\delta_1^{\alpha_1-\gamma_1}+\left(\omega^*_{\Phi 2}\frac{3A^*_{\gamma_1}+c_{\gamma_1}\mathcal{A}_1^*}{\Gamma(2-\gamma_1)}+\omega^*_{\Phi 3}\frac{3A^*_{\gamma_2}+c_{\gamma_2}A_1^*}{\Gamma(2-\gamma_2)}\right)r=\frac{\varepsilon}{2}.
\end{aligned}
$$

Similarly for $\mathbb{G}_2$, we assert that if $t,\tau\in\mathcal{J}$ and $0<\tau-\sigma<\delta_2$, then

$$\|\mathbb{G}_2(u_n,v_n)(\sigma)-\mathbb{G}_2(u_n,v_n)(\tau)\|_{\mathcal{W}}\le 4\omega^*_{\Psi 1}\mathcal{A}_2^*\delta_2^{\alpha_2-\gamma_3}+\left(\omega^*_{\Psi 2}\frac{3A^*_{\gamma_3}+c_{\gamma_3}\mathcal{A}_2^*}{\Gamma(2-\gamma_3)}+\omega^*_{\Psi 3}\frac{3A^*_{\gamma_4}+c_{\gamma_4}A_2^*}{\Gamma(2-\gamma_4)}\right)r=\frac{\varepsilon}{2}.$$

Hence, we have

$$\|\mathbb{G}(u_n,v_n)(\sigma)-\mathbb{G}(u_n,v_n)(\tau)\|_{\mathcal{W}}\le\varepsilon.$$

Thus $\mathbb{G}(\Theta)$ is equi-continuous. In view of the Arzelà-Ascoli Theorem $\mathbb{G}(\Theta)$ is compact. In addition, by Proposition 2.11 the operator $\mathbb{G}$ is μ–Lipschitz with constant zero.

Theorem 3.5. Under the assumptions (H_1)–(H_2), the system (1.1) has at least one solution $(u,v)\in\mathcal{W}$ provided $L_{\mathbb{F}}+\Delta_{\mathbb{G}}<1$. Moreover, the set of solutions of (1.1) is bounded in $\mathcal{W}$.

Proof. By Lemma 3.2 and 3.3, $\mathbb{F}$ is μ–Lipschitz with constant $L\in[0,1)$ and $\mathbb{G}$ is μ–Lipschitz with constant zero, respectively. Proposition 2.10 implies that $\mathbb{T}$ is strict μ–contraction with constant L. Define

$$\mathbf{W}=\{(u,v)\in\mathcal{W}:\ \text{there exists}\ \lambda\in[0,1]\ \text{such that}\ (u,v)=\lambda\mathbb{T}(u,v)\}.$$

We have to show the boundedness of $\mathbf{W}$ in $\mathcal{W}$. For this, choose $(u,v)\in\mathbf{W}$, then from the growth conditions on $\mathbb{F}$ and $\mathbb{G}$ in Lemma 3.2 and 3.3, we get

$$
\begin{aligned}
\|(u,v)\|_{\mathcal{W}}=\|\lambda\mathbb{T}(u,v)\|_{\mathcal{W}}=\lambda\|\mathbb{T}(u,v)\|_{\mathcal{W}}&\le\lambda\|\mathbb{F}(u,v)\|_{\mathcal{W}}+\lambda\|\mathbb{G}(u,v)\|_{\mathcal{W}} \\
&\le\lambda L_{\mathbb{F}}\|(u,v)\|_{\mathcal{W}}+\lambda\Delta_{\mathbb{G}}\|(u,v)\|_{\mathcal{W}}+\lambda\Lambda_{\mathbb{G}} \\
&\le\lambda(L_{\mathbb{F}}+\Delta_{\mathbb{G}})\|(u,v)\|_{\mathcal{W}}+\lambda\Lambda_{\mathbb{G}},
\end{aligned}
$$

implies that **W** is bounded in $\mathcal{W}$. Thus, by Theorem 2.13 we get that $\mathbb{T}$ has at least one fixed point and the set of fixed points is bounded in $\mathcal{W}$.

Theorem 3.6. Assume that (H_1) hold and

$$L_{\mathbb{F}} + \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{\mathfrak{L}_{\Phi 2}\mathcal{A}^*_{\gamma_2}}{\Gamma(2-\gamma_2)} + \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)} + \frac{\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_4}}{\Gamma(2-\gamma_4)} < 1, \tag{3.14}$$

then system (1.1) has a unique solution.

Proof. For (u,v), $(\overline{u},\overline{v}) \in \mathcal{R}\times\mathcal{R}$, we use Banach contraction theorem and (3.12), we get

$$\|\mathbb{F}(u,v) - \mathbb{F}(\overline{u},\overline{v})\|_{\mathcal{W}} \le L_{\mathbb{F}}\|(u,v) - (\overline{u},\overline{v})\|_{\mathcal{W}}. \tag{3.15}$$

Using (H_1) and (3.9), we have

$$\begin{aligned}
&\|\mathbb{G}_1(u,v) - \mathbb{G}_1(\overline{u},\overline{v})\|_{\mathcal{W}} \\
&\le \sup_{\sigma\in\mathcal{J}} \int_0^\sigma \Big|(\sigma-s)^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\,\|\,\Phi(s,{}^cD^{\gamma_1}u(s),{}^cD^{\gamma_2}v(s)) - \Phi(s,{}^cD^{\gamma_1}\overline{u}(s),{}^cD^{\gamma_2}\overline{v}(s))\Big|\,ds \\
&\le \sup_{\sigma\in\mathcal{J}} \int_0^\sigma \Big|(\sigma-s)^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|\,[\mathfrak{L}_{\Phi 1}|{}^cD^{\gamma_1}u - {}^cD^{\gamma_1}\overline{u}| + \mathfrak{L}_{\Phi 2}|{}^cD^{\gamma_2}v - {}^cD^{\gamma_2}\overline{v}|]\,ds \\
&\le \sup_{\sigma\in\mathcal{J}} \int_0^\sigma \Big|(\sigma-s)^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|\mathfrak{L}_{\Phi 1}|{}^cD^{\gamma_1}u - {}^cD^{\gamma_1}\overline{u}|\,ds \\
&\quad + \sup_{\sigma\in\mathcal{J}} \int_0^\sigma \Big|(\sigma-s)^{\alpha_1-1} E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|\mathfrak{L}_{\Phi 2}|{}^cD^{\gamma_2}v - {}^cD^{\gamma_2}\overline{v}|\,ds \\
&\le \frac{\mathfrak{L}_{\Phi 1}}{\Gamma(2-\gamma_1)} \sup_{\sigma\in\mathcal{J}} \int_0^\sigma s^{1-\gamma_1}\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|\,ds|u-\overline{u}| \\
&\quad + \frac{\mathfrak{L}_{\Phi 2}}{\Gamma(2-\gamma_2)} \sup_{\sigma\in\mathcal{J}} \int_0^\sigma s^{1-\gamma_2}\Big|(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)\Big|ds|v-\overline{v}| \\
&\le \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}\|u-\overline{u}\|_{\mathcal{U}} + \frac{\mathfrak{L}_{\Phi 2}\mathcal{A}^*_{\gamma_2}}{\Gamma(2-\gamma_2)}\|v-\overline{v}\|_{\mathcal{V}} \\
&\le \left(\frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{\mathfrak{L}_{\Phi 2}\mathcal{A}^*_{\gamma_2}}{\Gamma(2-\gamma_2)}\right)\|(u,v) - (\overline{u},\overline{v})\|_{\mathcal{W}}.
\end{aligned} \tag{3.16}$$

On the same way, we can achieve

$$\|\mathbb{G}_2(u,v) - \mathbb{G}_2(\overline{u},\overline{v})\|_{\mathcal{W}} \le \left(\frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)} + \frac{\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_4}}{\Gamma(2-\gamma_4)}\right)\|(u,v) - (\overline{u},\overline{v})\|_{\mathcal{W}}. \tag{3.17}$$

From (3.16) and (3.17), it gain that

$$\begin{aligned}
\|\mathbb{G}(u,v) - \mathbb{G}(\overline{u},\overline{v})\|_{\mathcal{W}} &\le \|\mathbb{G}_1(u,v) - \mathbb{G}_1(\overline{u},\overline{v})\|_{\mathcal{W}} + \|\mathbb{G}_2(u,v) - \mathbb{G}_2(\overline{u},\overline{v})\|_{\mathcal{W}} \\
&\le \left(\frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{\mathfrak{L}_{\Phi 2}\mathcal{A}^*_{\gamma_2}}{\Gamma(2-\gamma_2)} + \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)} + \frac{\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_4}}{\Gamma(2-\gamma_4)}\right)\|(u,v) - (\overline{u},\overline{v})\|_{\mathcal{W}}.
\end{aligned} \tag{3.18}$$

Hence, by combining (3.15) and (3.18), we attain

$$\|\mathbb{T}(u,v)-\mathbb{T}(\overline{u},\overline{v})\|_{\mathcal{W}} \le \|\mathbb{F}(u,v)-\mathbb{F}(\overline{u},\overline{v})\|_{\mathcal{W}} + \|\mathbb{G}(u,v)-\mathbb{G}(\overline{u},\overline{v})\|_{\mathcal{W}}$$
$$\le\left(L_{\mathbb{F}}+\frac{\mathfrak{L}_{\Phi1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}+\frac{\mathfrak{L}_{\Phi2}\mathcal{A}^*_{\gamma_2}}{\Gamma(2-\gamma_2)}+\frac{\mathfrak{L}_{\Psi1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)}+\frac{\mathfrak{L}_{\Psi2}\mathcal{A}^*_{\gamma_4}}{\Gamma(2-\gamma_4)}\right)\|(u,v)-(\overline{u},\overline{v})\|_{\mathcal{W}},$$

which implies that $\mathbb{T}$ is a contractive operator. By the Banach contraction principle, the system (1.1) has a unique solution.

4 Hyers–Ulam stability

In this fragment, we state and prove four types of HU stabilities.

Definition 4.1. [46] The coupled fractional differential Equation (1.1) is said to be HU stable if there exists $K_{u,v}=\max\{K_u,K_v\}>0$ such that, for $\varrho=\max\{\varrho_u,\varrho_v\}>0$ and for every solution $(\tilde{u},\tilde{v})\in\mathcal{W}$ of the inequality

$$\begin{cases} \left|{}^cD^{\alpha_1}\tilde{u}(\sigma)-\xi_1\,{}^cD^{\beta_1}\tilde{u}(\sigma)-\Phi(\sigma,{}^cD^{\gamma_1}\tilde{u}(\sigma),{}^cD^{\gamma_2}\tilde{v}(\sigma))\right|\le\varrho_u, & \sigma\in\mathcal{J},\\ \left|{}^cD^{\alpha_2}\tilde{v}(\sigma)-\xi_2\,{}^cD^{\beta_2}\tilde{v}(\sigma)-\Psi(\sigma,{}^cD^{\gamma_3}\tilde{u}(\sigma),{}^cD^{\gamma_4}\tilde{v}(\sigma))\right|\le\varrho_v, & \sigma\in\mathcal{J},\end{cases} \tag{4.1}$$

there exists a unique solution $(u,v)\in\mathcal{W}$ with

$$\|(u,v)-(\tilde{u},\tilde{v})\|_{\mathcal{W}}\le K_{u,v}\varrho,\quad \sigma\in\mathcal{J}.$$

Definition 4.2. [46] The coupled fractional differential Equation (1.1) is said to be generalized HU stable if there exists $\phi\in\mathcal{C}(\mathcal{R}^+,\mathcal{R}^+)$ with $\phi(0)=0$ such that, for any approximate solution $(\tilde{u},\tilde{v})\in\mathcal{W}$ of inequality (4.1), there exists a unique solution $(u,v)\in\mathcal{W}$ of (1.1) satisfying

$$\|(u,v)-(\tilde{u},\tilde{v})\|_{\mathcal{W}}\le\phi(\varrho),\quad \sigma\in\mathcal{J}.$$

Denote $\varphi_{u,v}=\max\{\varphi_u,\varphi_v\}\in\mathcal{C}(\mathcal{J},\mathcal{R})$ and $K_{\varphi_u,\varphi_v}=\max\{K_{\varphi_u},K_{\varphi_v}\}>0$.

Definition 4.3. [46] The coupled fractional differential Equation (1.1) is said to be HU–Rassias stable with respect to $\varphi_{u,v}$ if there exists a constant K_{φ_u,φ_v} such that, for some $\varrho=\max\{\varrho_u,\varrho_v\}>0$ and for any approximate solution $(\tilde{u},\tilde{v})\in\mathcal{W}$ of the inequality

$$\begin{cases} \left|{}^cD^{\alpha_1}\tilde{u}(\sigma)-\xi_1\,{}^cD^{\beta_1}\tilde{u}(\sigma)-\Phi(\sigma,{}^cD^{\gamma_1}\tilde{u}(\sigma),{}^cD^{\gamma_2}\tilde{v}(\sigma))\right|\le\varphi_u(\sigma)\varrho_u, & \sigma\in\mathcal{J},\\ \left|{}^cD^{\alpha_2}\tilde{v}(\sigma)-\xi_2\,{}^cD^{\beta_2}\tilde{v}(\sigma)-\Psi(\sigma,{}^cD^{\gamma_3}\tilde{u}(\sigma),{}^cD^{\gamma_4}\tilde{v}(\sigma))\right|\le\varphi_v(\sigma)\varrho_v, & \sigma\in\mathcal{J},\end{cases} \tag{4.2}$$

there exists a unique solution $(u,v)\in\mathcal{W}$ with

$$\|(u,v)-(\tilde{u},\tilde{v})\|_{\mathcal{W}}\le K_{\varphi_u,\varphi_v}\varphi_{u,v}(\sigma)\varrho,\quad \sigma\in\mathcal{J}.$$

Definition 4.4. [46] The coupled fractional differential Equation (1.1) is said to be generalized HU–Rassias stable with respect to $\varphi_{u,v}$ if there exists a constant K_{φ_u,φ_v} such that, for any approximate solution $(\tilde{u},\tilde{v})\in\mathcal{W}$ of inequality (4.2), there exists a unique solution $(u,v)\in\mathcal{W}$ of (1.1) satisfying

$$\|(u,v)-(\tilde{u},\tilde{v})\|_{\mathcal{W}}\le K_{\varphi_u,\varphi_v}\varphi_{u,v}(\sigma),\quad \sigma\in\mathcal{J}.$$

Remark 4.5. We say that $(\tilde{u},\tilde{v})\in\mathcal{W}$ is a solution of the system of inequalities (4.1) if there exist functions Θ_Φ, $\Theta_\Psi\in\mathcal{C}(\mathcal{J},\mathcal{R})$ depending upon u,v, respectively, such that

(I) $|\Theta_\Phi(\sigma)|\le\varrho_u$, $|\Theta_\Psi(\sigma)|\le\varrho_v$, $\sigma\in\mathcal{J}$;

$$\text{(II)} \begin{cases} {}^cD^{\alpha_1}\tilde{u}(\sigma) - \xi_1 {}^cD^{\beta_1}\tilde{u}(\sigma) = \Phi(\sigma, {}^cD^{\gamma_1}\tilde{u}(\sigma), {}^cD^{\gamma_2}\tilde{v}(\sigma)) + \Theta_\Phi(\sigma), \\ {}^cD^{\alpha_2}\tilde{v}(\sigma) - \xi_2 {}^cD^{\beta_2}\tilde{v}(\sigma) = \Psi(\sigma, {}^cD^{\gamma_3}\tilde{u}(\sigma), {}^cD^{\gamma_4}\tilde{v}(\sigma)) + \Theta_\Psi(\sigma). \end{cases}$$

Theorem 4.6. If assumption (H_1) and inequality (3.14) are satisfied and

$$\mathrm{F} = 1 - \frac{\mathfrak{L}_{\Phi 2}\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_2}\mathcal{A}^*_{\gamma_4}}{\left(1 - \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}(p\xi_1 - q\xi_1^2)}{\Gamma(\alpha_1-\beta_1)\Gamma(2-\gamma_1)}\right)\left(1 - \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}(p\xi_2 - q\xi_2^2)}{\Gamma(\alpha_2-\beta_2)\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_2)\Gamma(2-\gamma_4)} > 0,$$

then the unique solution of the coupled fractional differential Equation (1.1) is HU stable and consequently generalized HU stable.

Proof. Let $(\tilde{u}, \tilde{v}) \in \mathcal{W}$ be an approximate solution of inequality (4.1) and let $(u, v) \in \mathcal{W}$ be the unique solution of the coupled system (1.1). By Remark 4.5, we have

$$\begin{cases} {}^cD^{\alpha_1}\tilde{u}(\sigma) - \xi_1 {}^cD^{\beta_2}\tilde{u}(\sigma) = \Phi(\sigma, {}^cD^{\gamma_1}\tilde{u}(\sigma), {}^cD^{\gamma_2}\tilde{v}(\sigma)) + \Theta_\Phi(\sigma) {}^cD^{\alpha_2}\tilde{v}(\sigma) - \xi_2 {}^cD^{\beta_2}\tilde{v}(\sigma) \\ \qquad = \Psi(\sigma, {}^cD^{\gamma_3}\tilde{u}(\sigma), {}^cD^{\gamma_4}\tilde{v}(\sigma)) + \Theta_\Psi(\sigma). \end{cases} \tag{4.3}$$

By Equation (3.8), the solution of problem (4.3) is

$$\begin{cases} \tilde{u}(\sigma) = \sum\limits_{k=0}^{p-1}\sigma^k E_{\alpha_1-\beta_1,k+1}\left(\xi_1\sigma^{\alpha_1-\beta_1}\right)\tilde{u}_k^* - \xi_1\sum\limits_{k=0}^{q-1}\sigma^{\alpha_1-\beta_1+k}E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\left(\xi_1\sigma^{\alpha_1-\beta_1}\right)\tilde{u}_k^* \\ + \int\limits_0^\sigma (\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\left(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\right)\Phi(s, {}^cD^{\gamma_1}\tilde{u}(s), {}^cD^{\gamma_2}\tilde{v}(s))\,ds \\ + \int\limits_0^\sigma (\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\left(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\right)\Theta_\Phi(s)\,ds \\ \tilde{v}(\sigma) = \sum\limits_{k=0}^{p-1}\sigma^k E_{\alpha_2-\beta_2,k+1}\left(\xi_2\sigma^{\alpha_2-\beta_2}\right)\tilde{v}_k^* - \xi_2\sum\limits_{k=0}^{q-1}\sigma^{\alpha_2-\beta_2+k}E_{\alpha_2-\beta_2,\alpha_2-\beta_2+k+1}\left(\xi_2\sigma^{\alpha_2-\beta_2}\right)\tilde{v}_k^* \\ + \int\limits_0^\sigma (\sigma-s)^{\alpha_2-1}E_{\alpha_2-\beta_2,\alpha_2}\left(\xi_2(\sigma-s)^{\alpha_2-\beta_2}\right)\Psi(s, {}^cD^{\gamma_3}\tilde{u}(s), {}^cD^{\gamma_4}\tilde{v}(s))\,ds \\ + \int\limits_0^\sigma (\sigma-s)^{\alpha_2-1}E_{\alpha_2-\beta_2,\alpha_2}\left(\xi_2(\sigma-s)^{\alpha_2-\beta_2}\right)\Theta_\Psi(s)\,ds. \end{cases} \tag{4.4}$$

With the help of Theorem 3.5, we consider

$$\begin{aligned}
&\|u-\tilde{u}\|_{\mathcal{W}}\\
&\le \sup_{\sigma\in\mathcal{J}}\sum_{k=0}^{p-1}\sigma^{k}E_{\alpha_1-\beta_1,k+1}\big(\xi_1\sigma^{\alpha_1-\beta_1}\big)\big|u_k-\tilde{u}_k^*\big| + \sup_{\sigma\in\mathcal{J}}|\xi_1|\sum_{k=0}^{q-1}\sigma^{\alpha_1-\beta_1+k}E_{\alpha_1-\beta_1,\alpha_1-\beta_1+k+1}\big(\xi_1\sigma^{\alpha_1-\beta_1}\big)\big|u_k-\tilde{u}_k^*\big|\\
&\quad+\sup_{\sigma\in\mathcal{J}}\int_0^{\sigma}(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)|\Phi(s,{}^cD^{\gamma_1}u(s),{}^cD^{\gamma_2}v(s))-\Phi(s,{}^cD^{\gamma_1}\tilde{u}(s),{}^cD^{\gamma_2}\tilde{v}(s))|\,ds\\
&\quad+\sup_{\sigma\in\mathcal{J}}\int_0^{\sigma}(\sigma-s)^{\alpha_1-1}E_{\alpha_1-\beta_1,\alpha_1}\big(\xi_1(\sigma-s)^{\alpha_1-\beta_1}\big)|\Theta_{\Phi}(s)|\,ds\\
&\le \frac{p\xi_1-q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}\|u-\tilde{u}\|_{\mathcal{U}}+\frac{\mathfrak{L}_{\Phi1}\mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}u-\tilde{u}_{\mathcal{U}}+\frac{\mathfrak{L}_{\Phi2}\mathcal{A}_{\gamma_2}^*}{\Gamma(2-\gamma_2)}\|v-\tilde{v}\|_{\mathcal{V}}+\mathcal{A}_1^*\rho_u\\
&\le\left(\frac{p\xi_1-q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}+\frac{\mathfrak{L}_{\Phi1}\mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}\right)\|u-\tilde{u}\|_{\mathcal{U}}+\frac{\mathfrak{L}_{\Phi2}\mathcal{A}_{\gamma_2}^*}{\Gamma(2-\gamma_2)}\|v-\tilde{v}\|_{\mathcal{V}}+\mathcal{A}_1^*\rho_u
\end{aligned}\tag{4.5}$$

and

$$\|v-\tilde{v}\|_{\mathcal{V}}\le\left(\frac{p\xi_2-q\xi_2^2}{\Gamma(\alpha_2-\beta_2)}+\frac{\mathfrak{L}_{\Psi1}\mathcal{A}_{\gamma_3}^*}{\Gamma(2-\gamma_3)}\right)\|v-\tilde{v}\|_{\mathcal{V}}+\frac{\mathfrak{L}_{\Psi2}\mathcal{A}_{\gamma_4}^*}{\Gamma(2-\gamma_4)}\|u-\tilde{u}\|_{\mathcal{U}}+\mathcal{A}_2^*\rho_v.\tag{4.6}$$

From (4.5) and (4.6), we have

$$\|u-\tilde{u}\|_{\mathcal{U}}-\frac{\mathfrak{L}_{\Phi2}\mathcal{A}_{\gamma_2}^*}{\left(1-\frac{p\xi_1-q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}-\frac{\mathfrak{L}_{\Phi1}\mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}\right)\Gamma(2-\gamma_2)}\|v-\tilde{v}\|_{\mathcal{V}}\le\frac{\mathcal{A}_1^*}{1-\frac{p\xi_1-q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}-\frac{\mathfrak{L}_{\Phi1}\mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}}\rho_u$$

and

$$\|v-\tilde{v}\|_{\mathcal{V}}-\frac{\mathfrak{L}_{\Psi2}\mathcal{A}_{\gamma_4}^*}{\left(1-\frac{p\xi_2-q\xi_2^2}{\Gamma(\alpha_2-\beta_2)}-\frac{\mathfrak{L}_{\Psi1}\mathcal{A}_{\gamma_3}^*}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_4)}\|u-\tilde{u}\|_{\mathcal{U}}\le\frac{\mathcal{A}_2^*}{1-\frac{p\xi_2-q\xi_2^2}{\Gamma(\alpha_2-\beta_2)}-\frac{\mathfrak{L}_{\Psi1}\mathcal{A}_{\gamma_3}^*}{\Gamma(2-\gamma_3)}}\rho_v,$$

respectively. Let $\mathcal{G}_u=\frac{\mathcal{A}_1^*}{1-\frac{p\xi_1-q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}-\frac{\mathfrak{L}_{\Phi1}\mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}}$ and $\mathcal{G}_v=\frac{\mathcal{A}_2^*}{1-\frac{p\xi_2-q\xi_2^2}{\Gamma(\alpha_2-\beta_2)}-\frac{\mathfrak{L}_{\Psi1}\mathcal{A}_{\gamma_3}^*}{\Gamma(2-\gamma_3)}}$. Then the last two inequalities can be written in matrix form as

$$\begin{bmatrix}1 & -\frac{\mathfrak{L}_{\Phi2}\mathcal{A}_{\gamma_2}^*}{\left(1-\frac{p\xi_1-q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}-\frac{\mathfrak{L}_{\Phi1}\mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}\right)\Gamma(2-\gamma_2)}\\ -\frac{\mathfrak{L}_{\Psi2}\mathcal{A}_{\gamma_4}^*}{\left(1-\frac{p\xi_2-q\xi_2^2}{\Gamma(\alpha_2-\beta_2)}-\frac{\mathfrak{L}_{\Psi1}\mathcal{A}_{\gamma_3}^*}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_4)} & 1\end{bmatrix}\begin{bmatrix}\|u-\tilde{u}\|_{\mathcal{U}}\\ \|v-\tilde{v}\|_{\mathcal{V}}\end{bmatrix}\le\begin{bmatrix}\mathcal{G}_u\rho_u\\ \mathcal{G}_v\rho_v\end{bmatrix},$$

$$\begin{aligned}
&\begin{bmatrix}\|u-\tilde{u}\|_{\mathcal{U}}\\ \|v-\tilde{v}\|_{\mathcal{V}}\end{bmatrix}\\
&\le\begin{bmatrix}\frac{1}{\mathrm{F}} & \frac{\mathfrak{L}_{\Phi2}\mathcal{A}_{\gamma_2}^*}{\mathrm{F}\left(1-\frac{p\xi_1-q\xi_1^2}{\Gamma(\alpha_1-\beta_1)}-\frac{\mathfrak{L}_{\Phi1}\mathcal{A}_{\gamma_1}^*}{\Gamma(2-\gamma_1)}\right)\Gamma(2-\gamma_2)}\\ \frac{\mathfrak{L}_{\Psi2}\mathcal{A}_{\gamma_4}^*}{\mathrm{F}\left(1-\frac{p\xi_2-q\xi_2^2}{\Gamma(\alpha_2-\beta_2)}-\frac{\mathfrak{L}_{\Psi1}\mathcal{A}_{\gamma_3}^*}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_4)} & \frac{1}{\mathrm{F}}\end{bmatrix}\begin{bmatrix}\mathcal{G}_u\rho_u\\ \mathcal{G}_v\rho_v\end{bmatrix},
\end{aligned}\tag{4.7}$$

where

$$\mathrm{F} = 1 - \frac{\mathfrak{L}_{\Phi 2}\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_2}\mathcal{A}^*_{\gamma_4}}{\left(1 - \frac{p\xi_1 - q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} - \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}\right)\left(1 - \frac{p\xi_2 - q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} - \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_2)\Gamma(2-\gamma_4)}.$$

From system (4.7), we have

$$\|u - \tilde{u}\|_{\mathcal{U}} \le \frac{\mathcal{G}_u\rho_u}{\mathrm{F}} + \frac{\mathfrak{L}_{\Phi 2}\mathcal{A}^*_{\gamma_2}\mathcal{G}_v\rho_v}{\mathrm{F}\left(1 - \frac{p\xi_1 - q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} - \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}\right)\Gamma(2-\gamma_2)},$$

$$\|v - \tilde{v}\|_{\mathcal{V}} \le \frac{\mathcal{G}_v\rho_v}{\mathrm{F}} + \frac{\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_4}\mathcal{G}_u\rho_u}{\mathrm{F}\left(1 - \frac{p\xi_2 - q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} - \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_4)},$$

which implies that

$$\|u - \tilde{u}\|_{\mathcal{U}} + \|v - \tilde{v}\|_{\mathcal{V}} \le \frac{\mathcal{G}_u\rho_u}{\mathrm{F}} + \frac{\mathcal{G}_v\rho_v}{\mathrm{F}} + \frac{\mathfrak{L}_{\Phi 2}\mathcal{A}^*_{\gamma_2}\mathcal{G}_v\rho_v}{\mathrm{F}\left(1 - \frac{p\xi_1 - q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} - \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}\right)\Gamma(2-\gamma_2)} + \frac{\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_4}\mathcal{G}_u\rho_u}{\mathrm{F}\left(1 - \frac{p\xi_2 - q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} - \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_4)}.$$

If $\max\{\rho_u, \rho_v\} = \rho$ and $\frac{\mathcal{G}_u}{\mathrm{F}} + \frac{\mathcal{G}_v}{\mathrm{F}} + \frac{\mathfrak{L}_{\Phi 2}\mathcal{A}^*_{\gamma_2}\mathcal{G}_v}{\mathrm{F}\left(1 - \frac{p\xi_1 - q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} - \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}\right)\Gamma(2-\gamma_2)} + \frac{\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_4}\mathcal{G}_u}{\mathrm{F}\left(1 - \frac{p\xi_2 - q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} - \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_4)} = \mathcal{G}_{u,v}$, then

$$\|(u, v) - (\tilde{u}, \tilde{v})\|_{\mathcal{W}} \le \mathcal{G}_{u,v}\rho.$$

This shows that system (1.1) is HU stable. Also, if

$$\|(u, v) - (\tilde{u}, \tilde{v})\|_{\mathcal{W}} \le \mathcal{G}_{u,v}\phi(\rho),$$

with $\phi(0) = 0$, then the solution of (1.1) is generalized HU stable.

Theorem 4.7. If assumption (H_1) and inequality (3.14) are satisfied and

$$\mathrm{F} = 1 - \frac{\mathfrak{L}_{\Phi 2}\mathfrak{L}_{\Psi 2}\mathcal{A}^*_{\gamma_2}\mathcal{A}^*_{\gamma_4}}{\left(1 - \frac{p\xi_1 - q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} - \frac{\mathfrak{L}_{\Phi 1}\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}\right)\left(1 - \frac{p\xi_2 - q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} - \frac{\mathfrak{L}_{\Psi 1}\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_2)\Gamma(2-\gamma_4)} > 0,$$

then the unique solution of (1.1) is HU–Rassias stable and consequently generalized HU–Rassias stable.

Proof. Using Definition 4.3 and 4.4, we can get our result by performing the same steps as in Theorem 4.6.

5 Example

Here, we present an example to verify our results.

Example 5.1. Consider

$$\begin{cases} {}^{c}D^{\frac{7}{2}}u(\sigma) - \xi_1\, {}^{c}D^{\frac{5}{2}}u(\sigma) = \dfrac{3 + {}^{c}D^{\frac{1}{2}}u(\sigma) + {}^{c}D^{\frac{1}{3}}v(\sigma)}{75e^{t+10}\left(1 + {}^{c}D^{\frac{1}{2}}u(\sigma) + {}^{c}D^{\frac{1}{3}}\right.}\, v(\sigma)\Big) \\ {}^{c}D^{\frac{10}{3}}v(\sigma) - \xi_2\, {}^{c}D^{\frac{8}{3}}v(\sigma) = \dfrac{t\cos\left({}^{c}D^{\frac{2}{3}}u(\sigma)\right) + {}^{c}D^{\frac{2}{5}}v(\sigma)\sin(\sigma)}{40} + \dfrac{{}^{c}D^{\frac{2}{3}}u(\sigma)}{35 + {}^{c}D^{\frac{2}{3}}u(\sigma)} \\ u(0) = u'(0) = u''(0) = u'''(0) = v(0) = v'(0) = v''(0) = v'''(0) = 1, \end{cases} \tag{5.1}$$

where $\sigma \in [0,1]$ and $\xi_1 = \xi_2 = \frac{4}{9}$. From system (5.1), we can see that $p = 4$, $q = 3$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 > 0$. Also we can easily find $\mathfrak{L}_{\Phi1} = \mathfrak{L}_{\Phi1} = \frac{1}{75e^{10}}$, $\mathfrak{L}_{\Psi1} = \mathfrak{L}_{\Psi1} = \frac{1}{35}$. Here $L_{\mathbb{F}} \approx 4.59$, which is calculated from

$$\begin{cases} u(\sigma) = \sum\limits_{k=0}^{3} \sigma^k E_{\frac{7}{2}-\frac{5}{2},k+1}\left(\frac{4}{9}\sigma^{\frac{7}{2}-\frac{5}{2}}\right)u^k - \frac{4}{9}\sum\limits_{k=0}^{2} \sigma^{\frac{7}{2}-\frac{5}{2}+k} E_{\frac{7}{2}-\frac{5}{2},\frac{7}{2}-\frac{5}{2}+k+1}\left(\frac{4}{9}\sigma^{\frac{7}{2}-\frac{5}{2}}\right)u^k \\ \quad + \int\limits_0^{\sigma} (\sigma - s)^{\frac{7}{2}-1} E_{\frac{7}{2}-\frac{5}{2},\frac{7}{2}}\left(\frac{4}{9}(\sigma - s)^{\frac{7}{2}-\frac{5}{2}}\right) \dfrac{3 + {}^{c}D^{\frac{1}{2}}u(s) + {}^{c}D^{\frac{1}{3}}v(s)}{75e^{s+10}\left(1 + {}^{c}D^{\frac{1}{2}}u(s) + {}^{c}D^{\frac{1}{3}}\right.}\, v(s)\Big)\, ds, \\ v(\sigma) = \sum\limits_{k=0}^{3} \sigma^k E_{\frac{10}{3}-\frac{8}{3},k+1}\left(\frac{4}{9}\sigma^{\frac{10}{3}-\frac{8}{3}}\right)v^k - \frac{4}{9}\sum\limits_{k=0}^{2} \sigma^{\frac{10}{3}-\frac{8}{3}+k} E_{\frac{10}{3}-\frac{8}{3},\frac{10}{3}-\frac{8}{3}+k+1}\left(\frac{4}{9}\sigma^{\frac{10}{3}-\frac{8}{3}}\right)v^k \\ \quad + \int\limits_0^{\sigma} (\sigma - s)^{\frac{10}{3}-1} E_{\frac{10}{3}-\frac{8}{3},\frac{10}{3}}\left(\frac{4}{9}(\sigma - s)^{\frac{10}{3}-\frac{8}{3}}\right)\left(\dfrac{s\cos\left({}^{c}D^{\frac{2}{3}}u(s)\right) + {}^{c}D^{\frac{2}{5}}v(s)\sin(s)}{40} + \dfrac{{}^{c}D^{\frac{2}{3}}u(s)}{35 + {}^{c}D^{\frac{2}{3}}u(s)}\right) ds. \end{cases} \tag{5.2}$$

With the help of Theorem 3.5, we have

$$L_{\mathbb{F}} + \frac{\mathfrak{L}_{\Phi1}\,\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)} + \frac{\mathfrak{L}_{\Phi2}\,\mathcal{A}^*_{\gamma_2}}{\Gamma(2-\gamma_2)} + \frac{\mathfrak{L}_{\Psi1}\,\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)} + \frac{\mathfrak{L}_{\Psi2}\,\mathcal{A}^*_{\gamma_4}}{\Gamma(2-\gamma_4)} \approx 0.08456 < 1.$$

Hence (5.1) has a unique solution. Also,

$$\mathrm{F} = 1 - \frac{\mathfrak{L}_{\Phi2}\,\mathfrak{L}_{\Psi2}\,\mathcal{A}^*_{\gamma_2}\,\mathcal{A}^*_{\gamma_4}}{\left(1 - \frac{p\xi_1 - q\xi_1^2}{\Gamma(\alpha_1-\beta_1)} - \frac{\mathfrak{L}_{\Phi1}\,\mathcal{A}^*_{\gamma_1}}{\Gamma(2-\gamma_1)}\right)\left(1 - \frac{p\xi_2 - q\xi_2^2}{\Gamma(\alpha_2-\beta_2)} - \frac{\mathfrak{L}_{\Psi1}\,\mathcal{A}^*_{\gamma_3}}{\Gamma(2-\gamma_3)}\right)\Gamma(2-\gamma_2)\Gamma(2-\gamma_4)} \approx 0.967672 > 0,$$

Hence by Theorem 4.6 the system (5.1) is HU stable and thus generalized HU stable. Similarly, we can verify the conditions of Theorem 3.6 and 4.7.

6. Conclusion

In this manuscript, we studied an implicit coupled (α, β)–order fractional differential equation with initial conditions. We obtained the existence of solution by Laplace transform method. For uniqueness of solution we used contraction principle. We also presented at least one solution of (1.1) with the help of topological method. We set few conditions to obtained the HU stable, generalized HU stable, HU–Rassias stable and generalized HU–Rassias stable of (1.1). With the help of an example, we illustrated existence, uniqueness and stabilities of system (1.1).

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