Analysis of a class of nonlinear subdivision schemes and associated multiresolution transforms

S. Amat* K. Dadourian† J. Liandrat‡

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Abstract

This paper is devoted to the convergence and stability analysis of a class of nonlinear subdivision schemes and associated multiresolution transforms. As soon as a nonlinear scheme can be written as a specific perturbation of a linear and convergent subdivision scheme, we show that if some contractivity properties are satisfied, then stability and convergence can be achieved. This approach is applied to various schemes, which give different new results. More precisely, we study uncentered Lagrange interpolatory linear schemes, WENO scheme [24], PPH and Power-P schemes [2, 28] and a nonlinear scheme using local spherical coordinates [4]. Finally, a stability proof is given for the multiresolution transform associated to a nonlinear scheme of M. Marinov, N. Dyn and D. Levin [25].

Key Words. Nonlinear subdivision schemes, Nonlinear multiresolution, approximation, stability, convergence.

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*Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena (Spain).Research supported in part by 08662/PI/08 and MTM2007-62945. e-mail:`sergio.amat@upct.es`

†Ecole Centrale Marseille, laboratoire d'analyse topologie et probabilites (LATP). e-mail:`dadouria@cmi.univ-mrs.fr`

‡Ecole Centrale Marseille, laboratoire d'analyse topologie et probabilites (LATP), 38 rue Frédéric Joliot Curie, 13451 Marseille cedex 20, France. e-mail:`jacques.liandrat@centrale-marseille.fr`

1 Introduction

Multiresolution representations of discrete data are useful tools in several areas of application, such as image compression, computer-aided geometric design (CAGD) or adaptive methods for partial differential equations. In these applications, the ability of such representations to approximate the input data with high accuracy using a very small set of coefficients is a central property. Moreover, the stability of these representations in the presence of perturbations (generated by compression or due to approximations) is a key point.

In the last decade, several attempts to improve the properties of classical linear multiresolutions have led to nonlinear multiresolutions. In many cases, this nonlinear nature hinders the proofs of convergence and stability.

In [1], in the context of image compression, a new multiresolution transform has been presented. This bivariate multiresolution is based on a univariate nonlinear subdivision scheme called PPH introduced in [21] and also in [15], in the context of convexity-preserving schemes. The PPH scheme and its associated multiresolution transform have been analyzed in terms of convergence and stability in [1, 2]. Due to nonlinearity, the stability of the PPH multiresolution is not a consequence of the convergence of the associated subdivision scheme. It has been established in [2], presenting the PPH subdivision scheme as some perturbation of a linear scheme following [13], [27], [11] and [15].

The aim of the present paper is to generalize the results presented in [2] for a general family of nonlinear multiresolution schemes associated to a subdivision scheme $S_{NL} : l^\infty(\mathbb{Z}) \to l^\infty(\mathbb{Z})$ of the form:

$$\forall f \in l^\infty(\mathbb{R}), \quad \forall n \in \mathbb{Z} \qquad (S_{NL}f)_n = (Sf)_n + F(\rho f)_n, \tag{1}$$

where F is a (nonlinear) operator defined on $l^\infty(\mathbb{Z})$, ρ is a continuous linear operator on $l^\infty(\mathbb{Z})$ and S is a linear and convergent subdivision scheme.

This form is natural for two main reasons. On one hand, it can be easily shown that if S_1 and S_2 are two linear subdivision schemes reproducing constants (see Section 3) then for any f in $l^\infty(\mathbb{Z})$, $(S_1 - S_2)f$ is a function of df with $\forall n \in \mathbb{Z}, df_n = f_{n+1} - f_n$. On the other hand, many nonlinear subdivision schemes are derived by perturbing specifically linear schemes and are defined, from the beginning, in the form (1). To our knowledge, in all practical applications, ρf is an iterate of the first difference operator df. However, not every nonlinear subdivision scheme can be written in the form (1).

The paper is organized as follows. In Section 2, we briefly recall Harten's multiresolution framework, which is the natural setting for our applications. We specify the class of schemes under consideration and we establish the main results in Section 3. Theorems 1, 2 and 3 (see Section 3) establish that if F, ρ and S satisfy some natural properties, then the subdivision scheme is convergent, stable and the multiresolution is stable. We then establish different new results in Section 4: first we prove the convergence of non centered Lagrange interpolatory schemes if the interpolation polynomial degree is less or equal to 7. Secondly, we improve theoretical regularity results of the limit function for the WENO scheme [8]. Thirdly, we define and study the convergence of a new class of subdivision schemes that includes as a particular case the original PPH scheme. Next, we establish the convergence of a scheme available in the literature [4] but not yet proven to be convergent. Finally, we prove the stability of the multiresolution scheme associated to some geometrically controlled four-point interpolatory schemes [25].

2 Harten's framework and basic definitions

For sake of simplicity, we first describe Harten's interpolatory multiresolution framework. There, we consider a set of nested bi-infinite regular grids, $X^j = \{x^j_n\}_{n\in\mathbb{Z}}$ with $x^j_n = 2^{-j}n$. The parameter j is the scale parameter while n stands for the position parameter.

For any value of j, the space V_j stores the sampling at scale j of any continuous function f, i.e. $f^j = (f(x^j_n))_{n\in\mathbb{Z}}$. The sub-sampling operator SS is naturally defined from V_j to V_{j-1} for any element $f^j \in V_j$ by $f^{j-1}_n = SS(f^j)_n = f^j_{2n}$.

Any interpolatory subdivision scheme S_{NL} of our class can be considered as an operator from V_j to V_{j+1} with:

$$\forall f^j \in V_j, \forall n \in \mathbb{Z}, \qquad \left\{ \begin{array}{lcl} (S_{NL}f^j)_{2n+1} & = & (Sf^j)_{2n+1} + F(\rho f^j)_{2n+1}, \\ (S_{NL}f^j)_{2n} & = & f^j_n. \end{array} \right.$$

For any sequence $f^j \in V_j$, we can consider the detail $d^j \in V_{j+1}$ defined by $d^j_n = f^{j+1}_{2n+1} - S_{NL}(SS(f^j))_{2n+1}$. For any $J \in \mathbb{N}$, the multiresolution decomposition up to level $J_0 \leq J$ of any sequence $f^J \in V_J$ is the sequence $\{f^{J_0}, d^{J_0}, \ldots, d^{J-1}\}$ (see [3, 18, 19]).

In the general (non interpolatory) situation, for any j, the sampling operator is replaced by a linear discretization operator D_j from a suitable function space to the countable space V_j. A nesting condition is required and reads $\forall j, f, D_j f = 0 \Rightarrow D_{j-1} f = 0$. Interpolation is replaced by a reconstruction operator R_j acting from V_j to the suitable space of functions and satisfying $D_j R_j = Id_{V_j}$, where Id_{V_j} stands for the identity operator in V_j. Our class of nonlinear subdivision schemes reads $S_{NL} = D_j R_{j-1}$ with $R_j f^j = \tilde{R}_j f^j + \tilde{F}(\rho f^j)$, where $\tilde{R}_j$ is a linear reconstruction operator and $\tilde{F}(\rho f^j)$ stands for the perturbation.

As previously, the details appear at each scale from $D_j - D_j R_{j-1} D_{j-1}$ and a multiresolution can be defined.

We have the following definitions:

Definition 1 *Convergence of a subdivision scheme:*
We say that a subdivision scheme S_{NL} is uniformly convergent if:

$$\forall f \in l^\infty(\mathbb{Z}), \exists f^\infty \in C^0(\mathbb{R}) \quad \textit{such that} \quad \lim_{j \to +\infty} \sup_{n \in \mathbb{Z}} |(S^j_{NL} f)_n - f^\infty(2^{-j} n)| = 0.$$

We write $f^\infty = S^\infty_{NL} f$.

Definition 2 *$C^{\alpha-}$ convergence of a subdivision scheme:*
A convergent subdivision scheme S_{NL} is said to be $C^{\alpha-}$ convergent if for all $f \in l^\infty(\mathbb{Z}), S^\infty f \in C^{\alpha-}$ where
for $0 < \alpha \leq 1$,

$$\begin{aligned} C^{\alpha-} \;=\; & \{f \textit{ continuous, bounded and verifying } \forall \alpha_1 < \alpha, \exists C > 0, \forall x, y \in \mathbb{R}, \\ & |f(x) - f(y)| \leq C|x-y|^{\alpha_1}\}, \end{aligned}$$

and for $\alpha > 1$, writing $\alpha = p + r > 0$ with $p \in \mathbb{N}$ and $0 < r < 1$,

$$C^{\alpha-} = \{f \textit{ with } f^{(p)} \in C^{r-}\},$$

where $f^{(p)}$ stands for the derivative of order p.

Definition 3 *Reproduction of polynomials*
A subdivision scheme S_{NL} reproduces polynomials of degree P if, for any polynomial $\mathcal{P}$ of degree less than or equal to P, if $f_n = \mathcal{P}(x^j_n)$, there exists a polynomial $\mathcal{R}$ of same degree as $\mathcal{P}$ such that $(Sf)_n = \mathcal{R}(x^{j+1}_n)$.

Definition 4 *Stability of a subdivision scheme: We say that a convergent subdivision scheme S_{NL} is stable if:*

$$\exists C > 0 \textit{ such that } \forall f, g \in l^\infty(\mathbb{Z}) \quad ||S^\infty_{NL} f - S^\infty_{NL} g||_{L^\infty} \leq C||f-g||_\infty.$$

Definition 5 *Stability of a multiresolution transform: The multiresolution transform associated to the subdivision scheme S_{NL} is said to be stable if*

$$\text{there exists} \quad C \quad \text{such that} \quad \forall J \in \mathbb{N}, \forall j_0 \le J, \text{ and } \forall f^J, \tilde{f}^J, j_0 \le j \le J$$

$$\begin{aligned} ||f^j - \tilde{f}^j||_\infty &\le C \left(||f^{j_0} - \tilde{f}^{j_0}||_\infty + \sum_{k=j_0}^{j-1} ||d^k - \tilde{d}^k||_\infty \right), & (2) \\ ||f^{j_0} - \tilde{f}^{j_0}||_\infty &\le C||f^j - \tilde{f}^j||_\infty, & (3) \\ ||d^k - \tilde{d}^k||_\infty &\le C||f^j - \tilde{f}^j||_\infty, \quad \forall k,\, j_0 \le k \le j-1, & (4) \end{aligned}$$

where $\{\tilde{f}^{j_0}, \tilde{d}^{j_0}, \ldots, \tilde{d}^{J-1}\}$ is the multiresolution decomposition of $\tilde{f}^J$.

When the subdivision operator is linear, its convergence is equivalent to the existence of the so-called limit function $\phi = S_{NL}^\infty(\delta)$ where δ is an initial sequence of zeros except one coefficient equal to 1 for $n = 0, (\delta_0 = 1, \delta_n = 0$ if $n \neq 0)$ (see [14, 7]). Moreover, for a linear subdivision scheme, its convergence implies the stability of the associated multiresolution analysis. In the nonlinear case, things are much more complicated and there is no simple result either for the convergence of the subdivision or for the multiresolution analysis stability; the only general results we could find for nonlinear multiresolution analysis stability appeared in two very recent works of S. Harizanov and P. Oswald in [17] and P. Grohs in [16].

3 A Class of Nonlinear Subdivision Schemes

Introducing S a linear and convergent subdivision scheme reproducing polynomials up to degree P, we consider nonlinear subdivision schemes that read

$$(S_{NL}f)_n = (Sf)_n + F(\rho f)_n, \tag{5}$$

where F is a nonlinear operator defined on $l^\infty(\mathbb{Z})$ and ρ is a continuous linear operator on $l^\infty(\mathbb{Z})$.

3.1 Convergence analysis

We have the following theorem related to the convergence of the nonlinear subdivision scheme S_{NL}:

Theorem 1 *If F, S and ρ verify:*

$$\exists M > 0 \quad \text{such that} \quad \forall d \in l^\infty(\mathbb{Z}) \quad ||F(d)||_\infty \leq M||d||_\infty\,, \tag{6}$$
$$\exists L > 0,\ \exists c < 1 \text{ such that } \forall f \in l^\infty(\mathbb{Z}) \quad ||\rho S_{NL}^L(f)||_\infty \leq c||\rho f||_\infty \tag{7}$$

then the subdivision scheme S_{NL} is uniformly convergent.
Moreover, if S is $C^{\alpha-}$ convergent, then S_{NL} is $C^{\beta-}$ convergent with $\beta = \min\left(\alpha, -\frac{log_2(c)}{L}\right)$.

Proof

For all $f \in l^\infty(\mathbb{Z})$, using hypotheses (6) and (7) and the definition of S_{NL}, we get:

$$\begin{aligned} ||S_{NL}^{j+1}f - S(S_{NL}^j f)||_\infty &\leq M||\rho(S_{NL}^j f)||_\infty, \\ ||S_{NL}^{j+1}f - S(S_{NL}^j f)||_\infty &\leq M||\rho(S_{NL}^L(S_{NL}^{j-L}f))||_\infty, \\ ||S_{NL}^{j+1}f - S(S_{NL}^j f)||_\infty &\leq Mc||\rho(S_{NL}^{j-L}f)||_\infty, \end{aligned}$$

which can be rewritten, for $j \equiv j_0[L]$, as:

$$\begin{aligned} ||S_{NL}^{j+1}f - S(S_{NL}^j f)||_\infty &\leq M||\rho||_{l^\infty} \max\{||S_{NL}^l||_{l^\infty}, l = 0 \dots L-1\} c^{\frac{j-j_0}{L}} ||f||_\infty, \\ ||S_{NL}^{j+1}f - S(S_{NL}^j f)||_\infty &\leq M_1 2^{\frac{j}{L} log_2(c)} ||f||_\infty, \end{aligned}$$

with $M_1 = c^{-1}M||\rho||_{l^\infty} \max\{||S_{NL}^l||_{l^\infty}, l = 0 \dots L-1\}$.

The convergence of the subdivision scheme S_{NL} can then be derived by applying Theorem 3.3 of [11].

In our context, this theorem applies as follows:
If S is a linear $C^{\alpha-}$ convergent subdivision scheme reproducing polynomials up to degree P and if S_{NL} is a perturbation of S in the sense that, for all $f \in l^\infty(\mathbb{R})$,

$$||(S_{NL}^{j+1}f) - S((S_{NL}^j f))||_\infty = O(2^{-\nu j}),$$

then S_{NL} is $C^{\beta-}$convergent with $\beta \geq \min(P, s_L, \nu)$.

It follows that if S is $C^{\alpha-}$ convergent, then S_{NL} is at least $C^{\beta-}$ convergent with $\beta = \min\left(\alpha, -\frac{log_2(c)}{L}\right)$.
□

Remark 1 *When F is linear, Theorem 1 is a consequence of Theorem 6.2 in [14].*

Remark 2 *Another proof of Theorem 1 (see [9]) is based on the convergence of the function sequence*

$$f^j(x) = \sum_n f_n^j \phi(2^j x - n),$$

with $\phi = S^\infty(\delta)$ the limit function of the linear scheme.
The main advantage of this second proof is that it can be extended to multivariate schemes (see [10]); this is not the case with Theorem 3.3 of [11].

Remark 3 *We can also apply Theorem 1 to "coupled" schemes written as*

$$(S_{NL}(x,y))_n = \begin{pmatrix} S_{NL_1}(x,y)_n \\ S_{NL_2}(x,y)_n \end{pmatrix} = \begin{pmatrix} (Sx)_n + F_1(\rho x, \rho y)_n \\ (Sy)_n + F_2(\rho x, \rho y)_n \end{pmatrix}.$$

If for $i = 1, 2, \exists M_i > 0, L_i > 0$ and $c_i < 1$ such that

$$\begin{aligned} \forall d_1, d_2 \in l^\infty(\mathbb{Z}) \quad ||F_i(d_1, d_2)||_\infty &\leq M_i \max(||d_1||_\infty, ||d_2||_\infty), \\ \forall f, g \in l^\infty(\mathbb{Z}) \quad ||\rho(S_{NL_i}^{L_i}(f,g))||_\infty &\leq c_i \max(||\rho f||_\infty, ||\rho g||_\infty), \end{aligned}$$

then the scheme S_{NL} is uniformly convergent.

3.2 Stability analysis

We have the following theorem on the stability of the nonlinear operator S_{NL}^∞.

Theorem 2 *If the nonlinear scheme S_{NL} (5) reproduces constants and if F, ρ verify $\exists M > 0, L > 0$ and $c < 1$ such that*

$$\forall d_1, d_2 \in l^\infty(\mathbb{Z}) \quad ||F(d_1) - F(d_2)||_\infty \leq M||d_1 - d_2||_\infty\,, \tag{8}$$
$$\forall f, g \in l^\infty(\mathbb{Z})\ ||\rho(S_{NL}^L(f) - S_{NL}^L(g))||_\infty \leq c||\rho(f-g)||_\infty, \tag{9}$$

then the nonlinear scheme S_{NL} is stable.

Proof

We note that, under the assumption of reproducing constants on S_{NL}, the hypotheses of Theorem 2 imply the hypotheses of Theorem 1. Then, the

nonlinear scheme S_{NL} converges. We can study the stability of the nonlinear operator S_{NL}^{∞}.
We can also point out that with the scheme definition (5), we have:

$$\forall f,g\in l^{\infty}(\mathbb{Z}),\qquad ||S_{NL}f-S_{NL}g||_{\infty}\leq ||\rho||_{l^{\infty}}(||S||_{l^{\infty}}+M)||f-g||_{\infty}. \quad (10)$$

Let f and $g\in l^{\infty}(\mathbb{Z})$; we have $\forall j\in\mathbb{N}$,

$$||S_{NL}^{j}f-S_{NL}^{j}g||_{\infty} \;\leq\; ||S(S_{NL}^{j-1}f-S_{NL}^{j-1}g)||_{\infty}+||F(\rho S_{NL}^{j-1}f)-F(\rho S_{NL}^{j-1}g)||_{\infty}.$$

When we apply hypotheses (8) and (9),

$$||S_{NL}^{j}f-S_{NL}^{j}g||_{\infty} \;\leq\; ||S(S_{NL}^{j-1}f-S_{NL}^{j-1}g)||_{\infty}+Mc||\rho S_{NL}^{j-1-L}f-\rho S_{NL}^{j-1-L}g||_{\infty}.$$

By iterating, from hypothesis (9) and (10) we obtain

$$||S_{NL}^{j}f-S_{NL}^{j}g||_{\infty} \;\leq\; ||S(S_{NL}^{j-1}f-S_{NL}^{j-1}g)||_{\infty}+M_1c^{\frac{j-1}{L}}||f-g||_{\infty},$$

with $M_1=Mc^{-1}||\rho||_{l^{\infty}}\max\left\{(||\rho||_{l^{\infty}}(||S||_{l^{\infty}}+M))^{l}, l=0\ldots L-1\right\}$.

We write

$$\begin{aligned}||S_{NL}^{j}f-S_{NL}^{j}g||_{\infty} \;\leq\;& ||S^{2}(S_{NL}^{j-2}f-S_{NL}^{j-2}g)||_{\infty}+||S||_{\infty}||F(\rho S_{NL}^{j-2}f)-F(\rho S_{NL}^{j-2}g)||_{\infty}\\ &+M_1c^{\frac{j-1}{L}}||f-g||_{\infty}.\end{aligned}$$

Therefore, we get

$$||S_{NL}^{j}f-S_{NL}^{j}g||_{\infty} \;\leq\; ||S^{j}(f-g)||_{\infty}+M_1||f-g||_{\infty}\sum_{i=1}^{j-1}c^{\frac{i}{L}}||S^{j-1-i}||_{\infty}.$$

Since S is a linear convergent scheme, we have $||S^{j}||_{\infty}<M_S\ \forall j$ and the stability of the linear scheme. We get

$$lim_{j\to+\infty}||S^{j}(f-g)||_{\infty}=||S^{\infty}(f-g)||_{\infty}\leq M_{\infty}||f-g||_{\infty}.$$

This implies

$$||S_{NL}^{\infty}f-S_{NL}^{\infty}g||_{\infty}\leq\left(M_{\infty}+\frac{MM_Sc^{\frac{1}{L}}}{1-c^{\frac{1}{L}}}\right)||f-g||_{\infty},$$

and therefore the stability of the nonlinear scheme S_{NL}.

□

3.3 Stability analysis of the associated multiresolution transform

We now consider the multiresolution analysis associated to the subdivision scheme (5), recalling that f^j is the sequence which results from discretizing a function on the regular grid X^j (in the case of interpolatory schemes, $f^j = (f(k2^{-j}))_{k\in\mathbb{Z}})$ and that the details d^j are defined by $d^j_n = f^{j+1}_{2n+1} - S_{NL}(f^j)_{2n+1}$.

We have the following theorem:

Theorem 3 *If there exists $M > 0$ and $c < 1$ such that $\forall f, g, d_1, d_2 \in l^\infty(\mathbb{Z})$,*

$$||F(d_1) - F(d_2)||_\infty \leq M||d_1 - d_2||_\infty, \tag{11}$$

$$\|\rho(S_{NL}f - S_{NL}g)\|_\infty \leq c\|\rho(f - g)\|_\infty, \tag{12}$$

then the multiresolution transform associated to the nonlinear subdivision scheme S_{NL} is stable.

Proof

We first prove (2).
Since S is a convergent linear scheme, by using the stability of the linear scheme S, we get the existence of some $C' > 0$ such that

$$\begin{aligned} ||f^j - \tilde{f}^j||_\infty &\leq C'\left(||f^0 - \tilde{f}^0||_\infty + \sum_{k=1}^{j} ||f^k - S(f^{k-1}) - \tilde{f}^k + S(\tilde{f}^{k-1})||_\infty\right) \\ &\leq C'\left(||f^0 - \tilde{f}^0||_\infty + \sum_{k=1}^{j} ||d^{k-1} + F(\rho f^{k-1}) - \tilde{d}^{k-1} - F(\rho \tilde{f}^{k-1})||_\infty\right). \end{aligned}$$

From (11),

$$||f^j - \tilde{f}^j||_\infty \leq C'\left(||f^0 - \tilde{f}^0||_\infty + \sum_{k=0}^{j-1} ||d^k - \tilde{d}^k||_\infty + M\sum_{k=1}^{j} ||\rho(f^{k-1}) - \rho(\tilde{f}^{k-1})||_\infty\right).$$

Concentrating on the last right-hand-side term, we get

$$\sum_{k=1}^{j} ||\rho(f^{k-1}) - \rho(\tilde{f}^{k-1})||_\infty \leq \|\rho(f^0) - \rho(\tilde{f}^0)\|_\infty + \sum_{k=2}^{j} \left(\|\rho(S_{NL}f^{k-2}) - \rho(S_{NL}\tilde{f}^{k-2})\|_\infty + \|\rho d^{k-2} - \rho\tilde{d}^{k-2}\|_\infty\right).$$

From (12),

$$\begin{aligned}\sum_{k=1}^{j} ||\rho(f^{k-1}) - \rho(\tilde{f}^{k-1})||_\infty &\leq \|\rho(f^0) - \rho(\tilde{f}^0)\|_\infty \\ &\quad + \sum_{k=0}^{j-2} \left(c\|\rho(f^k) - \rho(\tilde{f}^k)\|_\infty + \|\rho d^k - \rho\tilde{d}^k\|_\infty\right) \\ &\leq \sum_{k=0}^{j-2} \left(c^k \|\rho f^0 - \rho\tilde{f}^0\|_\infty + \sum_{l=0}^{k} c^{k-l}\|\rho d^l - \rho\tilde{d}^l\|_\infty\right).\end{aligned}$$

Since $0 < c < 1$,

$$\begin{aligned}\|f^j - \tilde{f}^j\|_\infty &\leq C'||f^0 - \tilde{f}^0||_\infty + C'\sum_{k=0}^{j-1} ||d^k - \tilde{d}^k||_\infty \\ &\quad + MC'\frac{1}{1-c}\left(||\rho(f^0) - \rho(\tilde{f}^0)||_\infty + \sum_{k=0}^{j-2} \|\rho d^k - \rho\tilde{d}^k\|_\infty\right).\end{aligned}$$

Finally, we get (2) with the constant

$$C = C' + \frac{MC'\|\rho\|_\infty}{1-c}.$$

We now establish (3) and (4).
Eq. (3) is a direct consequence of the nestedness property of the discretization and is classical.
As regards (4), we have for $0 \leq k \leq j-1$

$$|d_n^k - \tilde{d}_n^k| \leq ||f^{k+1} - \tilde{f}^{k+1} - S(f^k) - S(\tilde{f}^k)||_\infty + ||F(\rho f^k) - F(\rho\tilde{f}^k)||_\infty.$$

Using the property (4) for the multiresolution associated to S, hypothesis (11) and the continuity of ρ,

$$|d_n^k - \tilde{d}_n^k| \leq C'||f^j - \tilde{f}^j||_\infty + M\|\rho\|_\infty \|f^k - \tilde{f}^k\|_\infty.$$

From (3) for the multiresolution associated to S_{NL}, we have

$$|d_n^k - \tilde{d}_n^k| \leq C'||f^j - \tilde{f}^j||_\infty + M\|\rho\|_\infty\|f^{j-1} - \tilde{f}^{j-1}\|_\infty.$$

Therefore, we get (4) with $C = C' + M||\rho||_\infty$.
□

Remark 4 *We cannot consider a weaker formulation for hypothesis (12) such as:*

$$\exists p \in \mathbb{N}, \quad \exists c < 1 \quad \textit{such that} \quad \|\rho(S^p_{NL}f - S^p_{NL}g)\|_\infty \leq c\|\rho(f-g)\|_\infty.$$

Under this hypothesis, we recall (Theorem 2) that the stability of the subdivision scheme can be established. However, the multiresolution stability is not ensured. To achieve this, a stronger hypothesis is required, such as

$$\exists p \in \mathbb{N}, M > 0 \quad \textit{and} \quad c < 1, \quad \textit{such that}$$
$$\|\rho(S^p_{NL}f - S^p_{NL}g)\|_\infty \leq c\|\rho(f-g)\|_\infty \quad + \quad M\sum_{k=0}^{p-1}||d^k(f) - d^k(g)||_\infty,$$

see [17]. This last hypothesis is satisfied for instance for the PPH scheme [2].

4 Applications

This section is devoted to applications of the previous results to several subdivision schemes (linear and nonlinear).
Throughout this section, given $f \in l^\infty(\mathbb{Z})$, we write:

$$\begin{aligned} df &= (df_n)_{n\in\mathbb{Z}} && \text{with} \quad df_n = f_{n+1} - f_n, \\ Df &= (Df_n)_{n\in\mathbb{Z}} && \text{with} \quad Df_n = f_{n+1} - 2f_n + f_{n-1}, \\ D^l f &= (D^l f_n)_{n\in\mathbb{Z}} && \text{with} \end{aligned}$$

$$D^l f_n = D(D^{l-1}f)_n = \sum_{i=0}^{2l}(-1)^i C^i_{2l} f_{n-l+i} \quad \text{and} \quad C^i_k = \frac{k!}{i!(k-i)!}.$$

4.1 Multiresolution analysis associated with a linear fully non-centered Lagrange interpolatory subdivision scheme

The convergence of centered linear Lagrange interpolatory schemes is well known since Deslauriers and Dubuc [12]. For non-centered schemes there is no general result of convergence. Moreover, the general tools proposed, for instance, in [14] are cumbersome to apply; in the framework of Lagrange interpolation, they must be applied case by case and cannot take into account the recursive structure of interpolation (from one degree to the next).

In this subsection, we focus on completely de-centered Lagrange interpolatory linear schemes using P points (or equivalently using polynomials of degree $P-1$). In order to apply our theoretical results, we consider S the two-point centered linear scheme, and we express any right-hand-side completely non-centered scheme S_P as a perturbation of S writing $(S_P f)_{2n+1} = (Sf)_{2n+1} + F_P(\rho f)_{2n+1}$. With $\rho f = Df$ we get

- if P is even

$$\begin{aligned} F_P(\rho f)_{2n+1} = \; & + \sum_{k=2,\ k\,even}^{P-2} D^{\frac{k}{2}} f_{n+\frac{k}{2}+1} \quad \tfrac{(2k-1)!}{2^{2k}(k-1)!(k+1)!} \\ & - \sum_{k=1,\ k\,odd}^{P-3} D^{\frac{k+1}{2}} f_{n+\frac{k+1}{2}} \quad \tfrac{(4k+5)(2k-1)!}{2^{2k+1}(k-1)!(k+2)!}, \end{aligned}$$

- if P is odd

$$\begin{aligned} F_P(\rho f^j)_{2n+1} = \; & + \sum_{k=2,\ k\,even}^{P-3} D^{\frac{k}{2}} f_{n+\frac{k}{2}+1} \quad \tfrac{(2k-1)!}{2^{2k}(k-1)!(k+1)!} \\ & - \sum_{k=1,\ k\,odd}^{P-4} D^{\frac{k+1}{2}} f_{n+\frac{k+1}{2}} \quad \tfrac{(4k+5)(2k-1)!}{2^{2k+1}(k-1)!(k+2)!} \\ & - D^{\frac{P-1}{2}} f_{n+\frac{N-1}{2}} \quad \tfrac{(2N-3)!}{2^{2(N-2)}(N-3)!(N-1)!}. \end{aligned}$$

We find that it is also possible to write S_P as a perturbation of S_{P-2} with $\rho = D^{\frac{P-2}{2}}$, for even values of P and as a perturbation of S_{P-1} with $\rho = D^{\frac{P-1}{2}}$ for odd values of P.
This gives

- if P is even

$$\begin{aligned} (S_P f)_{2n+1} \;=\; (S_{P-2} f)_{2n+1} \; & + \tfrac{(2P-3)!}{2^{2(P-2)}(P-3)!(P-1)!} \quad D^{\frac{P-2}{2}} f_{n+\frac{P}{2}} \\ & - \tfrac{(4P-8)(2P-7)!}{2^{2P-5}(P-4)!(P-1)!} \quad D^{\frac{P-2}{2}} f_{n+\frac{P}{2}-1}, \end{aligned}$$

P	perturbation term	contraction estimate	regularity estimate
4	$-\frac{3}{16}Df_{n+1}+\frac{1}{16}Df_{n+2}$	$\|\|D(S_4f)\|\|_\infty \leq \frac{1}{2}\|\|Df\|\|_\infty$	1
5	$-\frac{5}{128}D^2f_{n+2}$	$\|\|D^2(S_5f)\|\|_\infty \leq \frac{1}{2}\|\|D^2f\|\|_\infty$	1
6	$-\frac{17}{256}D^2f_{n+2}+\frac{7}{256}D^2f_{n+3}$	$\|\|D^2(S_6f)\|\|_\infty \leq \frac{87}{128}\|\|D^2f\|\|_\infty$	0.55
7	$-\frac{21}{1024}D^3f_{n+3}$	$\|\|D^3(S_7f)\|\|_\infty \leq \frac{367}{512}\|\|D^3f\|\|_\infty$	0.47
8	$-\frac{75}{2048}D^3f_{n+3}+\frac{33}{2048}D^3f_{n+4}$	$\|\|D^3(S_8f)\|\|_\infty \leq \frac{475}{512}\|\|D^3f\|\|_\infty$	0.1
9	$-\frac{429}{32768}D^4f_{n+4}$	$\|\|D^4(S_9f)\|\|_\infty \leq \frac{54734}{32768}\|\|D^4f\|\|_\infty$	

Table 1: Perturbation term, contraction, and regularity estimates for different values of P

- if P is odd

$$(S_Pf)_{2n+1} = (S_{P-1}f)_{2n+1} - \frac{(2P-3)!}{2^{2(P-2)}(P-3)!(P-1)!} D^{\frac{P-1}{2}} f_{n+\frac{P-1}{2}}.$$

Therefore, convergence and regularity can be analyzed using the above expressions and Theorem 1. Direct calculations provide the contraction estimate of Table 4.1 (center). Applying Theorem 1, convergence is then reached for $P \leq 8$ as well as regularity of the limit function (Table 4.1 (right)). Since these schemes are linear, the stability of the multiresolution is also ensured for the same range of P.

Remark 5 *It has been proved in [9] that convergence of completely decentered Lagrange interpolatory linear schemes using P points occurs if and only if $P \leq 9$.*

4.2 The six-point WENO subdivision scheme

For any given number of points P, the WENO subdivision schemes are interpolatory subdivision schemes constructed from a convex combination of all interpolatory Lagrange schemes using the same number of points (i.e. all the stencils of P points containing the position to be predicted; see [24, 20]). For $P = 4$, there are 3 possible stencils involving a total of 6 points. The WENO-6 scheme is defined as:

$$\begin{aligned}(S_{\text{WENO}}f)_{2n+1} &= \frac{\alpha_3}{16}f_{n-2} - \frac{5\alpha_3+\alpha_2}{16}f_{n-1} + (1+\frac{5\alpha_3+2\alpha_2}{8})f_n \\ &\quad +(1+\frac{5\alpha_1+2\alpha_2}{8})f_{n+1} - \frac{5\alpha_1+\alpha_2}{16}f_{n+2} + \frac{\alpha_1}{16}f_{n+3}.\end{aligned}$$

where the coefficients α_i satisfy $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $0 < \alpha_i < 1$.
This scheme can be rewritten as a perturbation of the two-point centered linear scheme $(Sf)_{2n+1} = \frac{f_n + f_{n+1}}{2}$ as

$$(S_{\text{WENO}}f)_{2n+1} = (Sf)_{2n+1} + F(df, Df)_{2n+1}, \tag{13}$$

with

$$F(df, Df)_{2n+1} = \frac{\alpha_0}{16} Df^j_{n+2} - \frac{3\alpha_1 + \alpha_0}{16} Df^j_{n+1} - \frac{\alpha_2 + 3\alpha_3}{16} Df^j_n + \frac{\alpha_3}{16} Df^j_{n-1}$$

and $\alpha_i = \alpha_{i,n}(df)$.
Indeed, these coefficients are defined as (see [24, 20])

$$\alpha_{i,n} = \frac{a_{i,n}}{a_{0,n} + a_{1,n} + a_{2,n}} \quad \text{with} \quad a_{i,n} = \frac{d_i}{(\epsilon + IS_{i,n})^2},$$

where b_i is a smoothness indicator defined as a function of the first difference df while d_i and ϵ are fixed positive constants.
Here, we will use a set of values suggested in [24, 20] and computed in [6]. These coefficients are defined for each level j by

$$\begin{aligned}
IS^j_{1,n} &= 2^{-(2j+1)} \left((Df^j_{n-1})^2 + (Df^j_n)^2 + (Df^j_{n-1} - Df^j_n)^2 \right), \\
IS^j_{2,n} &= 2^{-(2j+1)} \left((Df^j_n)^2 + (Df^j_{n+1})^2 + (Df^j_n - Df^j_{n+1})^2 \right), \\
IS^j_{3,n} &= 2^{-(2j+1)} \left((Df^j_{n+1})^2 + (Df^j_{n+2})^2 + (Df^j_{n+1} - Df^j_{n+2})^2 \right).
\end{aligned}$$

We can therefore simplify $F(df, Df)$ by $F(Df)$ in (13).
We then obtain the following proposition

Proposition 1 *The WENO-6 subdivision scheme (13) is convergent and its limit function is at least $C^{\beta-}$ with $\beta = 0.215$.*

Proof
First, according to the definition of F (13) and the properties of $(\alpha_{i,n})_{i=1\ldots 3}$, for all $f \in l^\infty(\mathbb{Z})$ and $n \in \mathbb{Z}$, we have

$$\begin{aligned}
|F(Df)_{2n+1}| &\leq \frac{4\alpha_{1,n} + 4\alpha_{2,n} + 4\alpha_{3,n}}{16} ||Df||_\infty \\
||F(Df)||_\infty &\leq \frac{1}{2} ||Df||_\infty.
\end{aligned}$$

Secondly, we are interested in establishing the contraction property relative to the operator D for S_{WENO} or S^L_{WENO} (7).
We consider two cases.
• For odd values, we have

$$\begin{aligned} (DS_{\text{WENO}}f)_{2n+1} &= f_{n+1} - 2(S_{\text{WENO}}f)_{2n+1} + f_n \\ &= -\frac{\alpha_{1,n}}{8}Df_{n+2} + \frac{3\alpha_{1,n}+\alpha_{2,n}}{8}Df_{n+1} \\ &\quad +\frac{\alpha_{2,n}+3\alpha_{3,n}}{8}Df_n - \frac{\alpha_{3,n}}{8}Df_{n-1}. \end{aligned} \tag{14}$$

From the coefficient properties,

$$\begin{aligned} |(DS_{\text{WENO}}f)_{2n+1}| &\leq \frac{4\alpha_{1,n}+2\alpha_{2,n}+4\alpha_{3,n}}{8}\|Df\|_\infty \\ &\leq \frac{4-2\alpha_{2,n}}{8}\|Df\|_\infty \\ &\leq \frac{1}{2}\|Df\|_\infty. \end{aligned} \tag{15}$$

• For even values, we have

$$\begin{aligned} (DS_{\text{WENO}}f)_{2n} &= (S_{\text{WENO}}f)_{2n+1} - 2f_n + (S_{\text{WENO}}f)_{2n-1} \\ &= \frac{f_{n+1}-2f_n+f_{n-1}}{2} + \frac{\alpha_{3,n}}{16}Df_{n+2} \\ &\quad -\frac{3\alpha_{3,n}+\alpha_{1,n}-\alpha_{3,n-1}}{16}Df_{n+1} \\ &\quad -\frac{\alpha_{2,n}+3\alpha_{1,n}+3\alpha_{2,n-1}+\alpha_{3,n-1}}{16}Df_n \\ &\quad -\frac{3\alpha_{1,n-1}+\alpha_{2,n-1}-\alpha_{1,n}}{16}Df_{n-1} + \frac{\alpha_{1,n-1}}{16}Df_{n-2}. \end{aligned}$$

$$\begin{aligned} (DS_{\text{WENO}}f)_{2n} &= \frac{\alpha_{3,n}}{16}Df_{n+2} \\ &\quad -\frac{3\alpha_{3,n}+\alpha_{1,n}-\alpha_{3,n-1}}{16}Df_{n+1} \\ &\quad +\frac{8-\alpha_{2,n}-3\alpha_{1,n}-3\alpha_{2,n-1}-\alpha_{3,n-1}}{16}Df_n \\ &\quad -\frac{3\alpha_{1,n-1}+\alpha_{2,n-1}-\alpha_{1,n}}{16}Df_{n-1} + \frac{\alpha_{1,n-1}}{16}Df_{n-2}. \end{aligned} \tag{16}$$

From the coefficient properties, $0 < 8 - \alpha_{2,n} - 3\alpha_{1,n} - 3\alpha_{2,n-1} - \alpha_{3,n-1}$

$$|(DS_{\text{WENO}}f)_{2n}| \leq \frac{4\alpha_{3,n} - 2\alpha_{1,n} + 4\alpha_{1,n-1} - 2\alpha_{2,n-1} + 8}{16}\|Df\|_\infty.$$

The same gives $0 < 8 - 2\alpha_{1,n} - 2\alpha_{2,n-1}$. Then

$$|(DS_{\text{WENO}}f)_{2n}| < \|Df\|_\infty. \tag{17}$$

This is not enough to establish a contraction of type (7) and we need to estimate $||D(S^2_{\text{WENO}})||_\infty$.

As in our case $\alpha_{i,n} = \alpha_{i,n}(Df)$, we can denote by $S_{D\text{WENO}}$ the scheme for the second differences defined by (14) and (16). We have

$$D(S^2_{\text{WENO}}f) = S^2_{D\text{WENO}}(Df) \quad \text{then} \quad ||D(S^2_{\text{WENO}})||_\infty = ||S^2_{D\text{WENO}}||_\infty.$$

From (14) and (16), we can write for $g \in l^\infty(\mathbb{Z})$

$$(S_{D\text{WENO}}g)_n = \sum_{i=-2}^{2} b_{i,n}(g) g_{[\frac{n}{2}]+i}.$$

Moreover, $S_{D\text{WENO}}$ satisfies the following properties from (15) and (17)

$$|(S_{D\text{WENO}}g)_{2n+1}| \leq \frac{1}{2}||g||_\infty, \tag{18}$$
$$|(S_{D\text{WENO}}g)_{2n}| < ||g||_\infty. \tag{19}$$

To estimate $||D(S^2_{\text{WENO}})||_\infty$ and thus $||S^2_{D\text{WENO}}||_\infty$, we need to study 4 cases.
• For $(S^2_{D\text{WENO}}g)_{4n}$,

$$\begin{aligned}(S^2_{D\text{WENO}}g)_{4n} &= \sum_{i=-2}^{2} b_{i,4n}(S_{D\text{WENO}}g)(S_{D\text{WENO}}g)_{2n+i} \\ &= \sum_{i=-2\ i\,even}^{2} b_{i,4n}(S_{D\text{WENO}}g)(S_{D\text{WENO}}g)_{2n+i} \\ &\quad + \sum_{i=-2\ i\,odd}^{2} b_{i,4n}(S_{D\text{WENO}}g)(S_{D\text{WENO}}g)_{2n+i}.\end{aligned}$$

From properties (18) and (19), we have

$$|(S^2_{D\text{WENO}}g)_{4n}| \leq \left(\sum_{i=-2\ i\,even}^{2} |b_{i,4n}(S_{D\text{WENO}}g)| + \frac{1}{2} \sum_{i=-2\ i\,odd}^{2} |b_{i,4n}(S_{D\text{WENO}}g)| \right) ||g||_\infty.$$

From (16), we can write

$$\begin{aligned}
|(S^2_{D\text{WENO}}g)_{4n}| &\leq (\frac{8+\alpha_{3,n}+\alpha_{1,n-1}}{16} \\
&\quad + \frac{1}{2}\frac{3\alpha_{3,n}+\alpha_{2,n}+\alpha_{1,n}+3\alpha_{1,n-1}+\alpha_{2,n-1}+\alpha_{3,n-1}}{16})||g||_\infty \\
&\leq \left(\frac{10}{16}+\frac{6}{32}\right)||g||_\infty \\
&\leq \frac{13}{16}||g||_\infty.
\end{aligned}$$

- For $(S^2_{D\text{WENO}}g)_{4n+1}$, using estimate (18) we get

$$\begin{aligned}
|(S^2_{D\text{WENO}}g)_{4n+1}| &\leq \frac{1}{2}||S_{D\text{WENO}}g||_\infty \\
&\leq \frac{1}{2}||g||_\infty.
\end{aligned}$$

- For $(S^2_{D\text{WENO}}g)_{4n+2}$,

$$\begin{aligned}
(S^2_{D\text{WENO}}g)_{4n+2} &= \sum_{i=-2}^{2} b_{i,4n}(S_{D\text{WENO}}g)(S_{D\text{WENO}}g)_{2n+1+i} \\
&= \sum_{i=-2\ i\,even}^{2} b_{i,4n}(S_{D\text{WENO}}g)(S_{D\text{WENO}}g)_{2n+1+i} \\
&\quad + \sum_{i=-2\ i\,odd}^{2} b_{i,4n}(S_{D\text{WENO}}g)(S_{D\text{WENO}}g)_{2n+1+i}.
\end{aligned}$$

From properties (18) and (19), we have

$$|(S^2_{D\text{WENO}}g)_{4n}| \leq \left(\frac{1}{2}\sum_{i=-2\ i\,even}^{2} |b_{i,4n}(S_{D\text{WENO}}g)| + \sum_{i=-2\ i\,odd}^{2} |b_{i,4n}(S_{D\text{WENO}}g)| \right) ||g||_\infty.$$

Following the discussion for $4n$, we get

$$\begin{aligned}
|(S^2_{D\text{WENO}}g)_{4n}| &\leq \frac{1}{2}(\frac{8+\alpha_{3,n}+\alpha_{1,n-1}}{16} \\
&\quad + \frac{3\alpha_{3,n}+\alpha_{2,n}+\alpha_{1,n}+3\alpha_{1,n-1}+\alpha_{2,n-1}+\alpha_{3,n-1}}{16})||g||_\infty \\
&\leq \left(\frac{10}{32}+\frac{6}{16}\right)||g||_\infty \\
&\leq \frac{11}{16}||g||_\infty .
\end{aligned}$$

• For $(S^2_{D\text{WENO}}g)_{4n+3}$, it is the same as the estimate $(S^2_{D\text{WENO}}g)_{4n+1}$.

To summarize,

$$||D(S^2_{\text{WENO}}f)||_\infty \leq \frac{13}{16}||Df||_\infty .$$

Using Theorem 1, the convergence is then verified.

Since the two-point linear scheme S is C^{1-} convergent, Theorem 1 provides for S_{NL} a regularity constant $\beta = -2^{-1}log_2(\frac{13}{16}) = 0.15$.
This constant can be improved by iterating the scheme. We can get the following estimate for $||S^5_{D\text{WENO}}||_\infty$

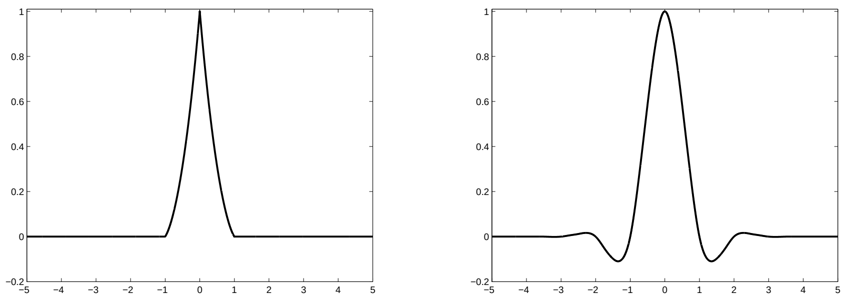
$$||S^5_{D\text{WENO}}||_\infty = \frac{3891}{8192} \quad \text{that leads to} \quad \beta = 0.215.$$

□

Remark 6 *Our estimate for the regularity of the limit function of the WENO-6 scheme is more accurate than the one provided in [8, 26]. However, numerical estimates obtained following a technique described in [22, 4] seem to indicate that our result is still not optimal (see Table 2 and Figure 1).*

iterations j	6	7	8	9	10	11	12
β	0.995	0.998	0.999	0.999	0.9997	0.9999	0.9999

Table 2: Numerical estimates of the WENO-6 regularity constant.



(a) WENO-6 scheme (b) six-point centered Lagrange scheme

Figure 1: Iterated function $S^8 f$ starting from f defined by $f = (\delta_{n,0})_n$ (with $\delta_{0,0} = 1$ and $\delta_{n,0} = 0$ if $n \neq 0$).

4.3 Powerp subdivision scheme: definition and convergence

In the same vein as the PPH scheme [1], the Powerp scheme is a four point scheme based on a piecewise degree-3 polynomial interpolation. If $S_{\mathcal{L}}$ is considered to be the centered four-point Lagrange interpolation prediction that reads

$$(S_{\mathcal{L}} f)_{2n+1} = \frac{f_n + f_{n+1}}{2} - \frac{1}{8}\frac{Df_{n+1} + Df_n}{2}, \tag{20}$$

the definition of the Powerp subdivision scheme is based on the substitution, in (20), of the arithmetic mean of second-order differences $\frac{Df_{n+1}+Df_n}{2}$ by a general mean $power_p(Df_n, Df_{n+1})$ defined in [28] for any integer $p \geq 1$, and any couple (x, y) as

$$power_p(x, y) \quad = \quad \frac{sign(x) + sign(y)}{2}\frac{x+y}{2}\left(1 - \left|\frac{x-y}{x+y}\right|^p\right). \tag{21}$$

Note that it coincides for $p = 2$ with the harmonic mean and therefore the Power2 scheme coincides with the PPH scheme.

The Powerp subdivision scheme then naturally appears as a perturbation of the linear two-point interpolation scheme, since it is defined by

$$(S_{power_p} f)_{2n+1} \quad = \quad \frac{f_n + f_{n+1}}{2} - \frac{1}{8} power_p(Df_n, Df_{n+1}). \tag{22}$$

Before establishing the convergence result, we first give the following lemma.

Lemma 1 *For any* $(x, y), (x', y') \in \mathbb{R}^2$*, the function* $power_p$ *satisfies the following properties :*

1. $power_p(x, y) = power_p(y, x)$,
2. $power_p(x, y) = 0 \quad if\ xy \leq 0$,
3. $power_p(-x, -y) = -power_p(x, y)$,
4. $|power_p(x, y)| \leq \max(|x|, |y|)$,
5. $\min(|x|, |y|) \leq |power_p(x, y)| \leq p \min(|x|, |y|)$,
6. *If* $x = O(1)$, $y = O(1)$ *and* $|y - x| = O(h)$, *then* $|\frac{x+y}{2} - power_p(x, y)| = O(h^p)$.

Proof

Claims of $1 - 4$ and 6 are obvious.
Inequality 5 comes from the equality (see [28])

$$power_p(x, y) \quad = \quad \frac{sign(x) + sign(y)}{2} \min(x, y) \left[1 + \left| \frac{x - y}{x + y} \right| + \cdots + \left| \frac{x - y}{x + y} \right|^{p-1} \right].$$

□

We then have the following proposition.

Proposition 2 *The Powerp subdivision scheme (22) is uniformly convergent and for any initial sequence the limit function belongs to* C^{1^-} *for all* p*.*

Proof
Here again the hypotheses of the general Theorem 1 must be checked.

We first check hypothesis (6). Using property 4 of Lemma 1, for $d \in l^\infty(\mathbb{Z})$ we get

$$\begin{aligned} |F(d)| &\leq \frac{1}{8} \max(|d_n|, |d_{n+1}|) \\ &\leq \frac{1}{8} ||d||_\infty. \end{aligned}$$

Then we consider hypothesis (7). As before, we study two different cases.

- For $k = 2n+1$,

$$\begin{aligned} D(S_{power_p} f)_{2n+1} &= f_n - 2(S_{powerp} f)_{2n+1} + f_{n+1} \\ &= f_{n+1} + f_n - 2\frac{f_n + f_{n+1}}{2} + 2\frac{1}{8} power_p(Df_n, Df_{n+1}) \\ &= \frac{1}{4} power_p(Df_n, Df_{n+1}). \end{aligned}$$

From property 4 of Lemma 1 we get:

$$|D(S_{power_p} f)_{2n+1}| \leq \frac{1}{4}\|Df\|_\infty. \tag{23}$$

- For $k = 2n$,

$$\begin{aligned} D(S_{power_p} f)_{2n} &= (S_{power_p} f)_{2n-1} - 2f_n + (S_{powerp} f)_{2n+1} \\ &= \frac{f_n + f_{n+1}}{2} - \frac{1}{8} power_p(Df_n, Df_{n+1}) - 2f_n \\ &\quad + \frac{f_{n-1} + f_n}{2} - \frac{1}{8} power_p(Df_{n-1}, Df_n) \\ &= \frac{Df_n}{2} - \frac{1}{8}\left(power_p(Df_n, Df_{n+1}) + power_p(Df_{n-1}, Df_n)\right). \end{aligned}$$

We note $D(S_{power_p} f)_{2n} = Z(Df_n, Df_{n+1}, Df_{n-1})$ with

$$Z(x, y, z) = \frac{x}{2} - \frac{1}{8}(power_p(x, y) + power_p(x, z)).$$

From (21) and properties 4 and 5 of Lemma 1, we have
if $x > 0$,

$$\begin{aligned} \frac{x}{2} - \frac{1}{8}(\max(x, y) + \max(x, z)) \leq \quad & Z(x, y, z) \quad \leq \frac{x}{2}, \\ \frac{x}{4} \leq \quad & Z(x, y, z) \quad \leq \frac{x}{2}, \\ 0 \leq \quad & |Z(x, y, z)| \quad \leq \frac{1}{2}|x|, \end{aligned}$$

if $x < 0$, the same result holds.

Finally,

$$|D(S_{power_p} f)_{2n}| \leq \frac{1}{2} \|Df\|_\infty. \tag{24}$$

From (23) and (24), we get

$$||DS_{power_p}(f)||_\infty \quad \leq \quad \frac{1}{2}||Df||_\infty \qquad \text{for all} \quad p.$$

Finally, Theorem 1 provides the convergence to a C^{1^-} function.
□

4.4 The convergence of a nonlinear scheme using spherical coordinates

The nonlinear subdivision scheme studied in this section is defined (but not proved to be convergent) in [4], where it is considered as a non-regular interpolatory subdivision scheme using local spherical coordinates. Here, we consider it to be a regular subdivision scheme applied to bivariate sequences $(x_n, y_n)^t_{n\in\mathbb{Z}}$. The resulting scheme reads (see [4]):

$$S_{spherical} \begin{pmatrix} x \\ y \end{pmatrix}_{2n+1} = \begin{pmatrix} S_1(x,y)_{2n+1} \\ S_2(x,y)_{2n+1} \end{pmatrix}$$

where

$$\begin{pmatrix} S_1(x,y)_{2n+1} \\ S_2(x,y)_{2n+1} \end{pmatrix} = \begin{pmatrix} \frac{x_n+x_{n+1}}{2} \\ \frac{y_n+y_{n+1}}{2} \end{pmatrix} + \frac{r_n}{4} \begin{pmatrix} \cos(\theta_n + h(\alpha_n)) - \cos(\theta_{n+1} + h(\beta_{n+1})) \\ \sin(\theta_n + h(\alpha_n)) - \sin(\theta_{n+1} + h(\beta_{n+1})) \end{pmatrix} \tag{25}$$

with:

$$\begin{aligned} r_n &= \sqrt{(x_{n+1}-x_n)^2 + (y_{n+1}-y_n)^2}, && (26) \\ \theta_n &= \arctan\left(\frac{y_{n+1}-y_{n-1}}{x_{n+1}-x_{n-1}}\right), && (27) \\ \gamma_n &= \arctan\left(\frac{y_{n+1}-y_n}{x_{n+1}-x_n}\right), && (28) \\ \alpha_n &= \gamma_n - \theta_n, && (29) \\ \beta_{n+1} &= \gamma_n - \theta_{n+1}, && (30) \end{aligned}$$

and, θ_n, $\gamma_n \in [-\frac{\pi}{2}; -\frac{\pi}{2}]$.
As explained in [4], the function h is *ad hoc*.

We can rewrite the above equations as

$$\begin{aligned}(S_1(x,y))_{2n+1} &= \frac{x_n + x_{n+1}}{2} + (F_1(dx,dy))_{2n+1},\\ (S_2(x,y))_{2n+1} &= \frac{y_n + y_{n+1}}{2} + (F_2(dx,dy))_{2n+1},\end{aligned}$$

with

$$\begin{aligned}(F_1(dx,dy))_{2n+1} &= \frac{r_n}{4}\left(\cos\left(\theta_n + h(\alpha_n)\right) - \cos\left(\theta_{n+1} + h(\beta_{n+1})\right)\right),\\ (F_2(dx,dy))_{2n+1} &= \frac{r_n}{4}\left(\sin\left(\theta_n + h(\alpha_n)\right) - \sin\left(\theta_{n+1} + h(\beta_{n+1})\right)\right).\end{aligned}$$

Due to (26), (27) and (28), r_n, θ_n, and γ_n can be written using the first divided difference (dx, dy) as

$$\begin{aligned}r_n &= \sqrt{(dx_n)^2 + (dy_n)^2},\\ \theta_n &= \arctan\left(\frac{dy_n + dy_{n-1}}{dx_n + dx_{n-1}}\right),\\ \gamma_n &= \arctan\left(\frac{dy_n}{dx_n}\right),\end{aligned}$$

as well as α_n and β_n due to (29) and (30).

Then, the scheme $S_{spherical}$ appears as a nonlinear perturbation of the natural bivariate version of the linear two-point interpolation scheme.

We then have the following proposition, where h' stands for the derivative of the function h:

Proposition 3 *If the function h used in the definition of $S_{spherical}$ satisfies $\max_{\theta\in[-\pi,\pi]} |1 - h'(\theta)| < \frac{\sqrt{2}}{\pi}$ the scheme $S_{spherical}$ in (25) is convergent.*

Proof

We again check the hypotheses of Theorem 1 generalized to $\mathbb{R}^2$ according to Remark 3. We have,

$$r_n \leq \sqrt{2}\max\left(|dx_n|, |dy_n|\right), \tag{31}$$

and therefore, for $i = 1, 2$, we have:

$$\begin{aligned} |(F_i(dx, dy))_{2n+1}| &\leq 2\frac{\sqrt{2}\max(|dx_n|, |dy_n|)}{4} \\ &\leq \frac{\sqrt{2}}{2}\max(||dx||_\infty, ||dy||_\infty), \end{aligned}$$

which shows that hypothesis (6) of Theorem 1 is satisfied.

We now check hypothesis (7), detailing the calculation for $d(S_1(x, y))_{2n}$.
For $(x, y) \in l^\infty(\mathbb{Z}^2)$ we have

$$\begin{aligned} d(S_1(x, y))_{2n} &= (S_1(x, y))_{2n+1} - (S_1(x, y))_{2n}, \\ &= \frac{x_n + x_{n+1}}{2} + \frac{r_n}{4}(\cos(\theta_n + h(\alpha_n)) - \cos(\theta_{n+1} + h(\beta_{n+1}))) - x_n \\ &= \frac{x_{n+1} - x_n}{2} + \frac{r_n}{4}(\cos(\theta_n + h(\alpha_n)) - \cos(\theta_{n+1} + h(\beta_{n+1}))), \end{aligned}$$

and therefore,

$$\begin{aligned} |d(S_1(x, y))_{2n}| &\leq \frac{||dx||_\infty}{2} \\ &+ \frac{\sqrt{2}\max(||dx||_\infty, ||df||_\infty)}{4}|\theta_n + h(\alpha_n) - \theta_{n+1} - h(\beta_{n+1})|. \end{aligned}$$

Using the definitions of α_n and β_n we get θ,

$$\begin{aligned} |d(S_1(x, y))_{2n}| &\leq \frac{||dx||_\infty}{2} \\ &+ \frac{\sqrt{2}\max(||dx||_\infty, ||dy||_\infty)}{4}|\theta_n + h(\gamma_n - \theta_n) - (\theta_{n+1} + h(\gamma_n - \theta_{n+1}))| \end{aligned}$$

and

$$\begin{aligned} |d(S_1(x, y))_{2n}| &\leq \frac{||dx||_\infty}{2} + \frac{\sqrt{2}\max(||dx||_\infty, ||dy||_\infty)}{4}\max_{\theta\in[-\pi,\pi]}\left|1 - h'(\theta)\right||\theta_n - \theta_{n+1}| \\ &\leq \left(\frac{1}{2} + \frac{\sqrt{2}\pi}{4}\max_{\theta\in[-\pi,\pi]}\left|1 - h'(\theta)\right|\right)\max(||dx||_\infty, ||dy||_\infty). \end{aligned}$$

Similar calculations provide the same estimate for $|d(S_1(x,y))_{2n}|$ and $||d(S_2(x,y))||$.

The contractivity (7) then follows as soon as the function h satisfies the hypothesis of the proposition.

$\square$

Remark 7 *According to [4], the criteria for the design of the function h are:*

- $|h'(\theta)| < 1$ *for small values of* θ*,*
- $h'(\theta) \to 1$ *when* $|\theta| \to \pi$*,*

and are compatible with the hypothesis of Proposition 3.

4.5 Stability of a nonlinear multiresolution transforms associated with the schemes of M. Marinov, N. Dyn and D. Levin [25]

Our last example is devoted to an application of Theorem 3. In [25], new schemes S_M, that reduce the artifacts generated by linear schemes in the drawing of curves in the presence of discontinuities, were derived. These four-point interpolatory schemes are defined using a nonlinear tension parameter adapted to control the difference with the two-point linear scheme. They read:

$$(S_M f)_{2n+1} = (f_n + f_{n+1})(w(f)_n + \frac{1}{2}) - w(f)_n(f_{n+1} + f_{n+2}) \tag{32}$$

with $w(f)_n = h(g(df)_n)$ and

$$g(df)_n = \begin{cases} \frac{D}{N}\frac{|df_n|}{|df_{n+1}-df_{n-1}|} & \text{if } |df_{n+1} - df_{n-1}| \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad , \qquad h(x) = N\min(x,1),$$

with the following constraints

$$0 < D < \frac{1}{2} \qquad \text{and} \qquad 0 < N < \frac{1}{8}. \tag{33}$$

Convergence and regularity of these schemes are achieved in [25] by using the analysis of Laurent polynomials for non-uniform schemes (see [23]). Moreover, when $\forall n,\ df_n \neq 0$, the limit function $S_M^\infty f$ is shown to be C^1.

The schemes S_M can be rewritten as a perturbation of the two-point centered linear scheme $(Sf)_{2n+1} = \frac{f_n+f_{n+1}}{2}$ as

$$(S_M f)_{2n+1} = (Sf)_{2n+1} + F(df)_{2n+1},$$

with the following nonlinear operator F:
if $df_n \neq 0$,

$$\begin{aligned} F(df)_{2n+1} &= w(f)_n(df_{n-1} - df_{n+1}) \\ &= \min\left(D\frac{|df_n|}{|df_{n-1} - df_{n+1}|}, N\right)(df_{n-1} - df_{n+1}) \\ &= sgn(df_{n-1} - df_{n+1}) \min\left(D|df_n|, N|df_{n-1} - df_{n+1}|\right). \end{aligned}$$

otherwise $F(df)_{2n+1} = 0$.

In other words, F is defined by:

$$F(df)_{2n+1} = \begin{cases} -D|df_n| & \text{if } df_{n-1} < df_{n+1} \text{ and } D|df_n| < N|df_{n-1} - df_{n+1}|, \\ D|df_n| & \text{if } df_{n-1} > df_{n+1} \text{ and } D|df_n| < N|df_{n-1} - df_{n+1}|, \\ w(df_{n-1} - df_{n+1}) & \text{if } df_{n-1} \neq df_{n+1} \text{ and } D|df_n| > w|df_{n-1} - df_{n+1}|, \\ 0 & \text{otherwise.} \end{cases}$$

The schemes S_M therefore satisfy the following properties (see [9]):

- Hypotheses (8) and (11) are satisfied with $M = \max(c, 2N)$.
- Hypotheses (9) and (12) are satisfied with $c = \frac{1}{2} + \max(D, 2N)$.

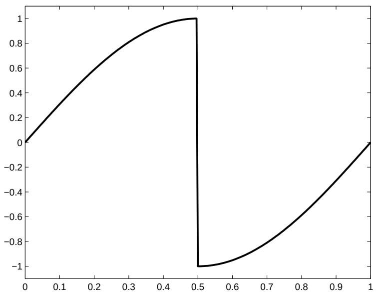
Due to (33) and using Theorems 2 and 3, we then have the following result:

Proposition 4 *The schemes S_M (32) and the associated multiresolution transforms are stable.*

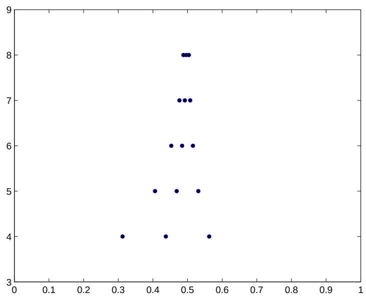
For the function

$$f(x) = \begin{cases} \sin(\pi x) & \text{if } x \leq 1/2, \\ -\sin(\pi x) & \text{if } x > 1/2. \end{cases}$$

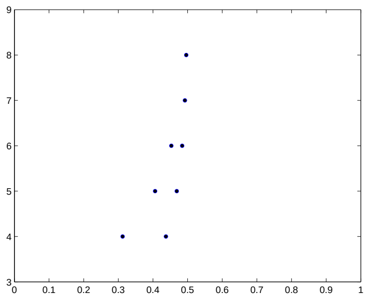
and the sequence $(f_n)_{n=0,\ldots,2^8} = (f(2^{-8}n))_n$, an example of multiresolution decomposition and thresholding is provided on Figure 2.



(a) Function f



(b) Lagrange four-point scheme. $nnz = 15$.



(c) nonlinear scheme S_M, $C = 0.2$, $w = \frac{1}{16}$. $nnz = 8$.

Figure 2: Nonzero detail coefficients (for each coefficient d_n^j a point is plotted at coordinates $(((2n+1)2^{-(j+1)}, j+1))$ after truncation of the coefficients less than $\epsilon = 10^{-3}$ in the multiresolution transform of f (Section 2 with $J = 8$ and $j_0 = 3$). The resulting number of nonzero coefficients among 2^J initial data is $2^{j_0} + nnz$

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