Analysis of a Class of Superconvergence Patch Recovery Techniques for Linear and Bilinear Finite Elements

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Abstract

Mathematical proofs are presented for the derivative superconvergence obtained by a class of patch recovery techniques for both linear and bilinear finite elements in the approximation of second order elliptic problems.

Keywords: finite element, superconvergence patch recovery, least-squares, local projection, error estimate.

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1 Introduction

Recently, Zienkiewicz and Zhu have developed a superconvergence patch recovery (SPR) technique for finite element approximations of second order

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elliptic problems [34, 35, 36]. It is a discrete version of a traditional post-processing technique by local L^2 projection. Both techniques can be described briefly as follows. Let u_h be a standard Galerkin finite element approximation of the exact solution u of a second order elliptic problem. Let (α_0, β_0) be an interior nodal point surrounded by elements $K_1, \ldots, K_m$ of the underlying finite element mesh τ_h of size h. Set $\omega_0 = \cup_{i=1}^m K_i$. Suppose on each element $K \in \tau_h$, u_h belongs to the polynomial space $P(K)$ which is the restriction on K of a fixed polynomial space P. Now two polynomials $p_1, p_2 \in P(\omega_0)$ are determined to approximate respectively $\partial_1 u$ and $\partial_2 u$ on ω_0, where $\partial_1 = \frac{\partial}{\partial x}$ and $\partial_2 = \frac{\partial}{\partial y}$. In the Zienkiewicz-Zhu SPR technique, p_1 and p_2 are obtained by solving the locally discrete least-squares problems

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{s}\left[p_l(\xi_{ij},\eta_{ij}) - \partial_l u_h(\xi_{ij},\eta_{ij})\right]^2 \\
&\qquad = \min_{q\in P(\omega_0)} \sum_{i=1}^{m}\sum_{j=1}^{s}\left[q(\xi_{ij},\eta_{ij}) - \partial_l u_h(\xi_{ij},\eta_{ij})\right]^2, \qquad l = 1, 2, \qquad (1)
\end{aligned}$$

where $\{(\xi_{ij}, \eta_{ij}) : j = 1, \ldots, s\}$ is a set of points in K_i for each $i = 1, \ldots, m$. In the traditional post-processing technique, p_1 and p_2 are the local L^2 projections onto $P(\omega_0)$ of $\partial_1 u_h$ and $\partial_2 u_h$, respectively,

$$\begin{aligned}
&\int_{\omega_0} |p_l(x,y) - \partial_l u_h(x,y)|^2 \, dxdy \\
&\qquad = \min_{q\in P(\omega_0)} \int_{\omega_0} |q(x,y) - \partial_l u_h(x,y)|^2 \, dxdy, \qquad l = 1, 2. \qquad (2)
\end{aligned}$$

Such p_1 and p_2 are often found to be superconvergent to $\partial_1 u$ and $\partial_2 u$, respectively, on a set of points, S_i, in each element $K_i, i = 1, \ldots, m$. Solve the equation (1) or (2) for all interior nodes of the mesh. If there are enough recovered superconvergence points on each element, we can then obtain by interpolation two globally continuous functions G_1 and G_2 whose restrictions on each element $K \in \tau_h$ are in $P(K)$ such that G_1 and G_2 globally superconverge to $\partial_1 u$ and $\partial_2 u$, respectively, except possibly on a boundary layer of size h.

We call (cf. [34, 35]) (α_0, β_0) a patch assembly point, ω_0 the element patch surrounding (α_0, β_0), $(\xi_{ij}, \eta_{ij}), j = 1, \ldots, s$, the least-squares sampling points of the element $K_i, i = 1, \ldots, m$, and the points in $S_i, i = 1, \ldots, m$, the recovered derivative superconvergent points. We also call $p_1, p_2 \in P(\omega_0)$

the recovered derivatives by the locally discrete least-squares if problem (1) is solved or the recovered derivatives by the local L^2 projection if problem (2) is solved.

Numerical experiments by Zienkiewicz and Zhu [34] have shown the derivative superconvergence for various types of finite elements in the case of locally discrete least-squares recovery but only for lower order finite elements in the case of local L^2 projection recovery. The ultraconvergence (superconvergence of order two) by the locally discrete least-squares recovery for quadratic or biquadratic elements discovered in these experiments is especially of mathematical interest. In practice, both of the recovery techniques are cost effective because of the locality of their treatment. Such techniques, especially the Zienkiewice-Zhu discrete least-squares based SPR, are also applicable to the design of a robust error estimator for the adaptive finite element method because of the global superconvergence of the recovered derivatives [33, 35, 36].

We notice that a different kind of superconvergence recovery technique for finite element approximations based on correction by interpolation has been mathematically developed by Lin, Yan, Yang, and Zhou [15, 16, 17, 18, 19, 29, 30]. Both this interpolation based local correction technique and the least-squares based patch recovery technique are economic and practical, and both of them recover, in most practical cases, the global superconvergence. A common feature of these two classes of techniques is that the recovered derivatives of a finite element solution lie in a space of piecewise polynomials same as or even larger than that the solution itself lies in. We refer to the recent work [1, 2, 3, 4] for a series of studies in a computer based approach on the superconvergence for finite element approximations.

Mathematical analysis for the superconvergence patch recovery techniques first appear in [25, 27] for the recovered derivatives by the locally discrete least-squares for one-dimensional problems. Generalization to two-dimensional tensor product finite elements has been made in [26, 28].

In this paper, we consider both the locally discrete least-squares recovery and the local L^2 projection recovery for both triangular linear and rectangular bilinear finite element approximations of general second order elliptic problems on two-dimensional convex polygonal domains. We prove that the derivative superconvergence is achieved by both of the methods for the triangular linear element on a strongly regular family of meshes, by the locally discrete least-squares recovery for the rectangular bilinear element on a quasi-uniform family of meshes, and by the local L^2 projection recovery for the rectangular bilinear element on a unidirectionally uniform family of meshes. We also give an example which strongly suggests that, for the local L^2 projection recovery by the rectangular bilinear element, the recovered derivatives will not be superconvergent if the underlying family of meshes are only quasi-uniform, a phenomenon that has been in fact numerically observed in [34].

The key argument in our proofs is based on an observation on the two recovery techniques as well as an exploitation of the earlier work of the mathematical analysis on the so-called natural superconvergence for the linear and bilinear finite elements, see, e.g., [5, 6, 10, 11, 12, 13, 14, 23, 24, 31, 32]. For a uniformly regular family of meshes, the idea of such an argument is clear and the proof is almost trivial. However, difficulties arise when a more general family of meshes are considered.

We remark that we consider affine rectangular bilinear elements rather than general isoparametric quadrilateral bilinear elements. This is simply for the convenience of the exposition of the main idea. The generalization can be easily made within our framework via the work by Lin and Whiteman [13], see also [8, 26].

In Section 2, we state our main results which include local superconvergence estimates that only involve the local smoothness of the exact solution and the local regularity of the underlying family of meshes. In Section 3, we give proofs of our main results. In Section 4, we present an example in a one-dimensional setting concerning the local L^2 projection recovery.

2 Main Results

For the simplicity of exposition, let Ω be a convex polygonal domain in the xy-plane. We consider the following boundary value problem:

$$\left\{ \begin{array}{rcll} -\sum_{i,j=1}^{2} \partial_j \left(a_{ij}\partial_i u\right) + \sum_{i=1}^{2} b_i \partial_i u + cu & = & f, & \text{in } \Omega, \\ u & = & 0, & \text{on } \partial\Omega, \end{array} \right. \tag{3}$$

where a_{ij}, b_i, c, and f are sufficiently smooth functions defined on $\bar{\Omega}$, and

$$\sum_{i,j=1}^{2} a_{ij}(x,y) d_i d_j \geq a_0 \left(d_1^2 + d_2^2\right), \qquad \forall (d_1, d_2) \in \mathbf{R}^2, \quad \forall (x,y) \in \Omega,$$

for some constant $a_0 > 0$.

As usual, for an integer $k \geq 0$ and an extended real number p with $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are the Sobolev spaces over the domain Ω, and $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$ the corresponding norms and seminorms. If $p = 2$ we write $H^k(\Omega)$ and $H_0^k(\Omega)$ instead of $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$, respectively. We define the bilinear form $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbf{R}$ associated with the elliptic problem (3) by

$$A(v,w) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}\partial_i v \partial_j w + \sum_{i=1}^{2} b_i \partial_i v w + cvw \right) dxdy, \qquad \forall v,w \in H_0^1(\Omega),$$

and assume that A is $H_0^1(\Omega)$-elliptic, i.e., there exists a constant $\sigma > 0$ such that

$$A(v,v) \geq \sigma \|v\|_{1,2,\Omega}^2, \qquad \forall v \in H_0^1(\Omega).$$

We let $u \in H_0^1(\Omega)$ be the unique generalized solution of (3) which is defined by

$$A(u,v) = (f,v), \qquad \forall v \in H_0^1(\Omega),$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

Now let $\{\tau_h : 0 < h \leq h_0\}$, where h_0 is a positive constant, be a quasi-uniform family of finite element meshes parameterized by the mesh size h of τ_h, covering the domain Ω, i.e., $\bar{\Omega} = \cup_{K \in \tau_h} K$ for each $h \in (0, h_0]$ [9]. We consider two cases. In the first case, we assume that all elements of $\tau_h\,(0 < h \leq h_0)$ are triangles and we consider the affine linear finite element approximation. Thus, we define for each $h \in (0, h_0]$ the finite element space to be

$$V_h = \left\{ v \in C(\bar{\Omega}) : v|_K \in P_1(K), \forall K \in \tau_h; v = 0 \text{ on } \partial\Omega \right\},$$

where $P_1(K)$ denotes the restriction on K of P_1, the space of all linear polynomials. In the second case, we assume that all elements of $\tau_h\,(0 < h \leq h_0)$ are rectangles with sides parallel to the coordinate axes, respectively, and we consider the affine bilinear finite element approximation. (We have implicitly assumed in this case that the boundary of Ω is composed of line segments parallel to the coordinate axes. This in turn implies that Ω is just a rectangular domain by its convexity.) Thus, we define for each $h \in (0, h_0]$ the finite element space to be

$$V_h = \left\{ v \in C(\bar{\Omega}) : v|_K \in Q_1(K), \forall K \in \tau_h; v = 0 \text{ on } \partial\Omega \right\},$$

where $Q_1(K)$ denotes the restriction on K of Q_1, the space of all bilinear polynomials. In both cases, we have $V_h \subset H_0^1(\Omega), 0 < h \leq h_0$. We also let $u_h \in V_h$ for each $h \in (0, h_0]$ be the unique finite element solution defined by

$$A(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h.$$

Let us fix $h \in (0, h_0]$ throughout the paper. The regularity of the mesh τ_h will always be referred to that of the whole family of meshes containing τ_h. As in Section 1, for an interior nodal point (α_0, β_0) of τ_h which is surrounded by elements $K_1, \ldots, K_m$, we set the element patch $\omega_0 = \cup_{i=1}^m K_i$. According to Zienkiewicz and Zhu [34], we choose the gravity center of element K_i, denoted by (ξ_i, η_i), as the only sampling point of the element $K_i, i = 1, \ldots, m$, for both the linear and bilinear finite element approximations. For convenience, we define in the sequel the polynomial space $P = P_1$ when considering the triangular linear element and $P = Q_1$ when considering the rectangular bilinear element. For the element patch ω_0, we define the recovered derivatives to be the polynomials $p_1, p_2 \in P(\omega_0)$ that satisfy the equation (1) which becomes in the present setting

$$\begin{aligned} &\sum_{i=1}^m \left[p_l(\xi_i, \eta_i) - \partial_l u_h(\xi_i, \eta_i)\right]^2 \\ &\qquad = \min_{q \in P(\omega_0)} \sum_{i=1}^m \left[q(\xi_i, \eta_i) - \partial_l u_h(\xi_i, \eta_i)\right]^2, \qquad l = 1, 2, \end{aligned} \tag{4}$$

for the locally discrete least-squares recovery and that satisfy the equation (2) for the local L^2 projection recovery.

In what follows, the symbol C will be used as a generic constant varying with the context and will be always assumed to be independent of the solution u, the mesh size h, and the element patch ω_0, except when the dependence is otherwise indicated.

Our first result is the existence and uniqueness of the recovered derivatives.

Lemma 1 *For any element patch $\omega_0 = \cup_{i=1}^m K_i$ surrounding a patch assembly point (α_0, β_0), both the minimization problems* (4) *and* (2) *admit unique minimizers $p_1, p_2 \in P(\omega_0)$. Moreover, p_1 and p_2 are characterized by*

$$\sum_{i=1}^m \left[p_l(\xi_i, \eta_i) - \partial_l u_h(\xi_i, \eta_i)\right] q(\xi_i, \eta_i) = 0, \qquad \forall q \in P(\omega_0), \quad l = 1, 2, \tag{5}$$

for the locally discrete least-squares recovery and by

$$\int_{\omega_0} [p_l(x,y) - \partial_l u_h(x,y)]\, q(x,y)\, dxdy = 0, \qquad \forall q \in P(\omega_0), \quad l = 1,2, \quad (6)$$

for the local L^2 projection recovery.

We recall that a quasi-uniform triangular mesh is strongly regular if any two adjacent elements in the mesh form an approximate parallelogram in which the difference of the two vectors corresponding to any two opposite sides of the parallelogram is bounded above by Ch^2, cf. [5, 6, 14, 32]. Obviously, a uniform triangular mesh is always strongly regular but the reverse is not true in general. Practically, strongly regular triangular meshes can cover domains such as convex quadrilaterals that can not be covered by uniform triangular meshes.

The following theorem validates the recovered superconvergence by the triangular linear element.

Theorem 1 *Suppose that the solution $u \in H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$. Suppose also that the triangular mesh τ_h is strongly regular. Then, for any element patch $\omega_0 = \cup_{i=1}^m K_i$ surrounding a patch assembly point (α_0, β_0), we have the following superconvergence estimate for the recovered derivatives $p_1, p_2 \in P(\omega_0) = P_1(\omega_0)$ defined by* (4) *or* (2)

$$|p_l(\alpha_0, \beta_0) - \partial_l u(\alpha_0, \beta_0)| \le Ch^2 \, |\ln h| \, \|u\|_{3,\infty,\Omega}, \qquad l = 1,2. \quad (7)$$

We recall that a quasi-uniform rectangular mesh covering a rectangular domain is called unidirectionally uniform if all parallel element edges in the mesh have the same length [7]. Clearly, such a mesh is in fact a Cartesian product of two one-dimensional, uniform meshes along the x and y axes, respectively.

The following theorem validates the recovered superconvergence by the rectangular bilinear element. Notice that, for the local L^2 projection recovery, we need the assumption that the underlying rectangular mesh is unidirectionally uniform. An example will be given in Section 4 to strongly suggest that such a stronger regularity assumption on the mesh can not be replaced by quasi-uniformity.

Theorem 2 *Suppose that the solution $u \in H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$. Suppose also that the rectangular mesh τ_h is quasi-uniform when considering the locally discrete least-squares recovery and is unidirectionally uniform when considering the local L^2 projection recovery. Then, for any element patch $\omega_0 = \cup_{i=1}^m K_i$ surrounding a patch assembly point (α_0, β_0), we have the following superconvergence estimate for the recovered derivatives $p_1, p_2 \in P(\omega_0) = Q_1(\omega_0)$ defined by (4) or (2)*

$$|p_l(\alpha_0, \beta_0) - \partial_l u(\alpha_0, \beta_0)| \leq Ch^2 \left|\ln h\right| \|u\|_{3,\infty,\Omega}, \qquad l = 1, 2. \tag{8}$$

Now let G_1 and G_2 be the two continuous functions that are piecewise linear or bilinear defined on the union of all interior elements of τ_h by interpolating the two recovered derivatives at all assembly points, respectively. Directly from the above two theorems, we have the following result on the globally uniform superconvergence estimate on interior elements of τ_h.

Corollary 1 *With the same hypothesis as in Theorem 1 and Theorem 2 on the smoothness of the solution u and the regularity of the mesh τ_h, we have*

$$\max_{(x,y)\in K} |G_l(x, y) - \partial_l u(x, y)| \leq Ch^2 \left|\ln h\right| \|u\|_{3,\infty,\Omega}, \qquad l = 1, 2,$$

for any $K \in \tau_h$ such that $\partial K \cap \partial\Omega = \emptyset$.

We now state the result of local estimate with regard to the local smoothness of the solution u and the local regularity of the mesh τ_h.

Theorem 3 *Let Ω_0 and Ω_1 be two Lipschitz domains such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. Suppose that the solution $u \in H_0^1(\Omega) \cap H^2(\Omega) \cap W^{3,\infty}(\Omega_1)$. Suppose also that the triangular mesh τ_h is quasi-uniform in Ω and is strongly regular in Ω_1. Then, for any element patch $\omega_0 = \cup_{i=1}^m K_i$ surrounding a patch assembly point (α_0, β_0) such that $\omega_0 \subset \bar{\Omega}_0$, we have the following local superconvergence estimate for the recovered derivatives by the triangular linear element $p_1, p_2 \in P(\omega_0) = P_1(\omega_0)$ defined by (4) or (2)*

$$|p_l(\alpha_0, \beta_0) - \partial_l u(\alpha_0, \beta_0)| \leq Ch^2 \left|\ln h\right| \left(\|u\|_{3,\infty,\Omega_1} + \|u\|_{2,2,\Omega}\right), \quad l = 1, 2,$$

where the constant C may depend on Ω_0 and Ω_1.

Theorem 4 *Let Ω_0 and Ω_1 be two Lipschitz domains such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$. Suppose that the solution $u \in H_0^1(\Omega) \cap H^2(\Omega) \cap W^{3,\infty}(\Omega_1)$. Suppose also that the rectangular mesh τ_h is quasi-uniform in Ω, and is unidirectionally uniform in Ω_1 when considering the local L^2 projection recovery. Then, for any element patch $\omega_0 = \cup_{i=1}^m K_i$ surrounding a patch assembly point (α_0, β_0) such that $\omega_0 \subset \bar{\Omega}_0$, we have the following local superconvergence estimate for the recovered derivatives by the rectangular bilinear element $p_1, p_2 \in P(\omega_0) = Q_1(\omega_0)$ defined by (4) or (2)*

$$|p_l(\alpha_0, \beta_0) - \partial_l u(\alpha_0, \beta_0)| \leq Ch^2 |\ln h| \left(\|u\|_{3,\infty,\Omega_1} + \|u\|_{2,2,\Omega} \right), \quad l = 1, 2,$$

where the constant C may depend on Ω_0 and Ω_1.

A direct consequence of the above two theorems is the following result on the locally uniform superconvergence.

Corollary 2 *With the same hypothesis as in Theorem 3 and Theorem 4 on the domains Ω_0 and Ω_1, the smoothness of the solution u, and the regularity of the mesh τ_h, we have*

$$\max_{(x,y)\in K} |G_l(x, y) - \partial_l u(x, y)| \leq Ch^2 |\ln h| \left(\|u\|_{3,\infty,\Omega_1} + \|u\|_{2,2,\Omega} \right), \quad l = 1, 2,$$

for any $K \in \tau_h$ such that $K \subset \bar{\Omega}_0$, where the constant C may depend on Ω_0 and Ω_1.

3 Proofs

Proof of Lemma 1 We consider two cases.

Case 1. The locally discrete least-squares recovery. Recall that $(\xi_i, \eta_i), i = 1, \ldots, m$, are the gravity centers of elements K_i in the element patch $\omega_0 = \cup_{j=1}^m K_j$ surrounding the patch assembly point (α_0, β_0). Notice that if $q \in P(\omega_0)$ satisfies

$$q(\xi_i, \eta_i) = 0, \qquad i = 1, \ldots, m,$$

then $q = 0$ identically on ω_0 for both the triangular linear element and the rectangular bilinear element. Thus, the mapping

$$q \to \|q\| \equiv \left\{ \sum_{i=1}^m [q(\xi_i, \eta_i)]^2 \right\}^{\frac{1}{2}} \tag{9}$$

defines a norm on the space $P(\omega_0)$. With this norm the nonnegative functional $F_l : P(\omega_0) \to \mathbf{R}$ defined for $l = 1$ or 2 by (cf. (4))

$$F_l(q) = \sum_{i=1}^{m} |q(\xi_i, \eta_i) - \partial_l u_h(\xi_i, \eta_i)|^2, \qquad q \in P(\omega_0), \tag{10}$$

is obviously continuous. Moreover, $F_l(q) \to \infty$ as $\|q\| \to \infty$. Therefore, the local compactness of the finite-dimensional space $P(\omega_0)$ implies the existence of minimizers of F_l on $P(\omega_0)$.

It is easy to verify that the second variation of F_l satisfies

$$\delta^2 F_l(q)(p,p) = 2\|p\|^2 > 0, \qquad \forall q, p \in P(\omega_0), p \neq 0.$$

Consequently, the functional $F_l : P(\omega_0) \to \mathbf{R}$ is strictly convex. This implies the uniqueness of the minimizer of F_l on $P(\omega_0)$.

Now the unique minimizer $p_l \in P(\omega_0)$ of F_l on $P(\omega_0)$ is characterized by

$$\begin{aligned} &F_l(p_l + tq) - F_l(p_l) \\ &\quad = \|q\|^2 t^2 + 2 \left\{ \sum_{i=1}^{m} q(\xi_i, \eta_i) \left[p_l(\xi_i, \eta_i) - \partial_l u_h(\xi_i, \eta_i) \right] \right\} t \geq 0 \end{aligned}$$

for any $q \in P(\omega_0)$ and any $t \in \mathbf{R}$. This is equivalent to (5).

Case 2. The local L^2 projection recovery. We can proceed the proof for this case similarly by using the $L^2(\omega_0)$ norm instead of the norm defined by (9) and by using the functional $F_l : P(\omega_0) \to \mathbf{R}$ defined by (cf. (2))

$$F_l(q) = \int_{\omega_0} |q(x,y) - \partial_l u_h(x,y)|^2 \, dxdy, \qquad q \in P(\omega_0), \tag{11}$$

instead of that defined by (10). The proof is complete.

Remark By the same argument as in the above proof, we can easily generalize Lemma 1 to obtain the existence and uniqueness for the minimization problems (1) and (2) for any type of finite element approximation. However, for the problem (1) we need to choose all the sampling points $(\xi_{ij}, \eta_{ij}) \in K_i$ $(1 \leq i \leq m, 1 \leq j \leq s)$ in such a way that, for any $q \in P(\omega_0)$, we have $q = 0$ identically on ω_0 whenever $q = 0$ at all these points $(\xi_{ij}, \eta_{ij}), 1 \leq i \leq m, 1 \leq j \leq s$.

To prove Theorem 1 we need the following result on the stability of recovered derivatives for the triangular finite element.

Lemma 2 *Suppose that the solution $u \in H_0^1(\Omega) \cap W^{3,\infty}(\Omega)$. Suppose also that the triangular mesh τ_h is quasi-uniform. For any element patch $\omega_0 = \cup_{i=1}^m K_i$ surrounding an assembly point (α_0, β_0), let $p_1, p_2 \in P(\omega_0) = P_1(\omega_0)$ be the recovered derivatives defined by (4) or (2). Then, we have the stability estimate*

$$|\nabla p_l(\alpha_0, \beta_0)| \le C\|u\|_{3,\infty,\Omega}, \qquad l = 1, 2. \tag{12}$$

Proof We consider two cases separately.

Case 1. The locally discrete least-squares recovery. The mapping $p \to |\nabla p(\alpha_0, \beta_0)|$ defines a seminorm on $P_1(\omega_0)$. Since $P_1(\omega_0)$ is finite-dimensional and the mesh τ_h is quasi-uniform, by an affine transformation from the patch ω_0 to a reference patch $\hat{\omega}_0$ with diameter of size $O(1)$, we have the inverse estimate

$$|\nabla q(\alpha_0, \beta_0)| \le Ch^{-1}\|q\|, \qquad \forall q \in P_1(\omega_0), \tag{13}$$

where $\|\cdot\|$ is the norm on $P_1(\omega_0)$ defined by (9).

Let $L_l \in P_1(\omega_0)$ be the linear part of $\partial_l u$ around the assembly point (α_0, β_0):

$$L_l(x, y) = \partial_l u(\alpha_0, \beta_0) + \nabla\partial_l u(\alpha_0, \beta_0) \cdot (x - \alpha_0, y - \beta_0), \qquad (x, y) \in \omega_0.$$

Obviously, $\nabla L_l(\alpha_0, \beta_0) = \nabla\partial_l u(\alpha_0, \beta_0)$, and

$$|L_l(x, y) - \partial_l u(x, y)| \le Ch^2\|u\|_{3,\infty,\omega_0}, \qquad \forall (x, y) \in \omega_0. \tag{14}$$

We thus have by (13) that

$$\begin{aligned} |\nabla p_l(\alpha_0, \beta_0)| &\le |\nabla(p_l - L_l)(\alpha_0, \beta_0)| + |\nabla L_l(\alpha_0, \beta_0)| \\ &\le Ch^{-1}\|p_l - L_l\| + |\nabla\partial_l u(\alpha_0, \beta_0)|. \end{aligned} \tag{15}$$

Since $p_l \in P_1(\omega_0)$ is the minimizer of the functional $F_l : P_1(\omega_0) \to \mathbf{R}$ defined by (10) on $P_1(\omega_0)$, we have

$$\begin{aligned} &\|p_l - L_l\| \\ &\quad \le \left[\sum_{i=1}^m |p_l(\xi_i, \eta_i) - \partial_l u_h(\xi_i, \eta_i)|^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^m |\partial_l u_h(\xi_i, \eta_i) - L_l(\xi_i, \eta_i)|^2\right]^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le 2\left[\sum_{i=1}^{m} |L_l(\xi_i,\eta_i) - \partial_l u_h(\xi_i,\eta_i)|^2\right]^{\frac{1}{2}} \\
&\le 2\left[\sum_{i=1}^{m} |L_l(\xi_i,\eta_i) - \partial_l u(\xi_i,\eta_i)|^2\right]^{\frac{1}{2}} + 2\left[\sum_{i=1}^{m} |\partial_l u(\xi_i,\eta_i) - \partial_l u_h(\xi_i,\eta_i)|^2\right]^{\frac{1}{2}} \\
&\le Ch\|u\|_{3,\infty,\Omega}, \qquad (16)
\end{aligned}
$$

where in the last step we used (14) as well as the $W^{1,\infty}$ error estimate for the linear finite element approximation [6, 9, 20, 21, 22, 32]. The stability estimate (12) now follows from (15) and (16) in this case.

Case 2. The local L^2 projection recovery. The proof of (12) in this case is similar to that in Case 1 except we replace the norm (9) by the $L^2(\omega_0)$ norm and the functional $F_l : P_1(\omega_0) \to \mathbf{R}$ defined by (10) by that defined by (11). The proof is complete.

Now we are ready to prove Theorem 1.

Proof of Theorem 1 Let u, (α_0, β_0), K_i, and $\omega_0 = \cup_{i=1}^m K_i$ be all the same as in the theorem. We may assume without loss of generality that $m = 6$, i.e., the patch ω_0 consists exactly of six elements $K_i, i = 1, \dots, 6$, since this is so with h sufficiently small by the strongly regular property of the triangular finite element mesh τ_h.

For $i = 1, \dots, 6$, we assume that the element K_i is adjacent to the element K_{i+1} and denote by (γ_i, δ_i) the midpoint of $\partial K_i \cap \partial K_{i+1}$, the common edge of the two elements K_i and K_{i+1}, where we adopt the convention that two indices i_1 and i_2 are the same if and only if $i_1 \equiv i_2 \pmod 6$. Fix $l = 1$ or 2. We have by the previous works [5, 6, 10, 12, 14, 24, 32] that

$$
\begin{aligned}
&\left|\partial_l u(\gamma_i,\delta_i) - \tfrac{1}{2}\left[\partial_l u_h|_{K_i}(\gamma_i,\delta_i) + \partial_l u_h|_{K_{i+1}}(\gamma_i,\delta_i)\right]\right| \\
&\qquad \le Ch^2\,|\ln h|\,\|u\|_{3,\infty,\Omega}, \qquad i = 1, \dots, 6. \qquad (17)
\end{aligned}
$$

Since u_h is piecewise linear on $\omega_0 = \cup_{i=1}^6 K_i$, we have

$$
\sum_{i=1}^{6} \frac{1}{2}\left[\partial_l u_h|_{K_i}(\gamma_i,\delta_i) + \partial_l u_h|_{K_{i+1}}(\gamma_i,\delta_i)\right] = \sum_{i=1}^{6} \partial_l u_h(\xi_i,\eta_i),
$$

which together with (17) leads to

$$\left|\sum_{i=1}^{6}\left[\partial_l u(\gamma_i,\delta_i)-\partial_l u_h(\xi_i,\eta_i)\right]\right| \le Ch^2\left|\ln h\right| \|u\|_{3,\infty,\Omega}. \tag{18}$$

By the Taylor expansion we have

$$\begin{gathered}\partial_l u(\gamma_i,\delta_i)=\partial_l u(\alpha_0,\beta_0)+\nabla\partial_l u(\alpha_0,\beta_0)\cdot[(\gamma_i,\delta_i)-(\alpha_0,\beta_0)]+R_i,\\ |R_i|\le Ch^2\|u\|_{3,\infty,\omega_0},\qquad i=1,\ldots,6.\end{gathered}$$

Since τ_h is strongly regular, we also have

$$\left|\sum_{i=1}^{6}\left[(\gamma_i,\delta_i)-(\alpha_0,\beta_0)\right]\right|\le Ch^2.$$

Consequently, we have by (18) that

$$\begin{aligned}&\left|\partial_l u(\alpha_0,\beta_0)-\frac{1}{6}\sum_{i=1}^{6}\partial_l u_h(\xi_i,\eta_i)\right|\\ &\qquad\le\frac{1}{6}\left|\sum_{i=1}^{6}\left[\partial_l u(\gamma_i,\delta_i)-\partial_l u_h(\xi_i,\eta_i)\right]\right|\\ &\qquad\qquad+\frac{1}{6}\left|\nabla\partial_l u(\alpha_0,\beta_0)\right|\left|\sum_{i=1}^{6}\left[(\gamma_i,\delta_i)-(\alpha_0,\beta_0)\right]\right|+\frac{1}{6}\sum_{i=1}^{6}|R_i|\\ &\qquad\le Ch^2\left|\ln h\right|\|u\|_{3,\infty,\Omega}.\end{aligned} \tag{19}$$

Let $F_l : P(\omega_0)\to\mathbf{R}$ be the functional as defined by (10) for the locally discrete least-squares recovery and as defined by (11) for the local L^2 projection recovery. Since $p_l\in P_1(\omega_0)$ is the unique minimizer of F_l by the assumption of the theorem, it satisfies the corresponding equation (5) or (6).

We now consider two cases.

Case 1. The locally discrete least-squares recovery. Taking $q=1$ identically on ω_0 in (5) with $m=6$ and $P=P_1$, we obtain in this case that

$$\sum_{i=1}^{6}p_l(\xi_i,\eta_i)=\sum_{i=1}^{6}\partial_l u_h(\xi_i,\eta_i). \tag{20}$$

By the strongly regular property of the triangular mesh τ_h, we have

$$\left|\sum_{i=1}^{6}\left[(\xi_i,\eta_i)-(\alpha_0,\beta_0)\right]\right|\le Ch^2. \tag{21}$$

Since p_l is linear on ω_0 in this case, we thus conclude that

$$\begin{aligned}
&\left| p_l(\alpha_0,\beta_0) - \frac{1}{6}\sum_{i=6}^{6} p_l(\xi_i,\eta_i) \right| \\
&\quad = \frac{1}{6}\left| \sum_{i=1}^{6} \left[p_l(\alpha_0,\beta_0) - p_l(\xi,\eta_i) \right] \right| \\
&\quad = \frac{1}{6}\left| \nabla p_l(\alpha_0,\beta_0) \cdot \sum_{i=1}^{6} \left[(\xi_i,\eta_i) - (\alpha_0,\beta_0) \right] \right| \\
&\quad \le Ch^2 \left| \nabla p_l(\alpha_0,\beta_0) \right|,
\end{aligned}$$

which together with (12), (20) and (19) implies (7) in this case.

Case 2. The local L^2 projection recovery. Since p_l is linear on ω_0, we have by taking $q = 1$ identically on ω_0 in (6) with $P = P_1$ that

$$\sum_{i=1}^{6} |K_i|\, p_l(\xi_i,\eta_i) = \sum_{i=1}^{6} |K_i|\, \partial_l u_h(\xi_i,\eta_i), \tag{22}$$

where we denote by $|\omega|$ the measure of a measurable set $\omega \subset \bar{\Omega}$.

By the strongly regular property of the mesh τ_h, we deduce easily that

$$\left| |K_i| - \frac{1}{6}|\omega_0| \right| \le Ch^3, \qquad i = 1,\ldots,6. \tag{23}$$

By the Taylor expansion we have

$$\begin{gathered}
\partial_l u(\xi_i,\eta_i) = \partial_l u(\alpha_0,\beta_0) + \nabla \partial_l u(\alpha_0,\beta_0)\cdot(\xi_i-\alpha_0,\eta_i-\beta_0) + S_i, \\
|S_i| \le Ch^2 \|u\|_{3,\infty,\omega_0}, \qquad i = 1,\ldots,6.
\end{gathered}$$

Consequently, by the fact that $\sum_{i=1}^{6} |K_i| = |\omega_0|$, we get

$$\begin{aligned}
&\left| \sum_{i=1}^{6} \left(|K_i| - \frac{1}{6}|\omega_0| \right) \partial_l u(\xi_i,\eta_i) \right| \\
&\quad = \left| \sum_{i=1}^{6} \left(|K_i| - \frac{1}{6}|\omega_0| \right) \left[\partial_l u(\alpha_0,\beta_0) + \nabla \partial_l u(\alpha_0,\beta_0)\cdot(\xi_i-\alpha_0,\eta_i-\beta_0) + S_i \right] \right| \\
&\quad = \left| \sum_{i=1}^{6} \left(|K_i| - \frac{1}{6}|\omega_0| \right) \left[\nabla \partial_l u(\alpha_0,\beta_0)\cdot(\xi_i-\alpha_0,\eta_i-\beta_0) + S_i \right] \right| \\
&\quad \le Ch^4 \|u\|_{3,\infty,\omega_0}.
\end{aligned}$$

We thus have by (23) and the $W^{1,\infty}$ estimate for the linear finite element approximation [6, 9, 20, 21, 22, 32] that

$$
\begin{aligned}
&\left|\sum_{i=1}^{6}\left(|K_i| - \frac{1}{6}|\omega_0|\right)\partial_l u_h(\xi_i,\eta_i)\right| \\
&\quad \le \left|\sum_{i=1}^{6}\left(|K_i| - \frac{1}{6}|\omega_0|\right)\left[\partial_l u_h(\xi_i,\eta_i) - \partial_l u(\xi_i,\eta_i)\right]\right| \\
&\qquad + \left|\sum_{i=1}^{6}\left(|K_i| - \frac{1}{6}|\omega_0|\right)\partial_l u(\xi_i,\eta_i)\right| \\
&\quad \le Ch^4\|u\|_{3,\infty,\Omega}.
\end{aligned}
$$

This together with (19) leads to

$$
\begin{aligned}
&\left||\omega_0|\,\partial_l u(\alpha_0,\beta_0) - \sum_{i=1}^{6}|K_i|\,\partial_l u_h(\xi_i,\eta_i)\right| \\
&\quad \le \left||\omega_0|\,\partial_l u(\alpha_0,\beta_0) - \frac{1}{6}|\omega_0|\sum_{i=1}^{6}\partial_l u_h(\xi_i,\eta_i)\right| \\
&\qquad + \left|\sum_{i=1}^{6}\left(|K_i| - \frac{1}{6}|\omega_0|\right)\partial_l u_h(\xi_i,\eta_i)\right| \\
&\quad \le Ch^4\,|\ln h|\,\|u\|_{3,\infty,\Omega}.
\end{aligned}
\tag{24}
$$

Finally, we have by (23), (21) and the stability estimate (12) that

$$
\begin{aligned}
&\left||\omega_0|\,p_l(\alpha_0,\beta_0) - \sum_{i=1}^{6}|K_i|\,p_l(\xi_i,\eta_i)\right| \\
&\quad = \left||\omega_0|\,p_l(\alpha_0,\beta_0) - \sum_{i=1}^{6}|K_i|\left[p_l(\alpha_0,\beta_0) + \nabla p_l(\alpha_0,\beta_0)\cdot(\xi_i-\alpha_0,\eta_i-\beta_0)\right]\right| \\
&\quad = \left|\nabla p_l(\alpha_0,\beta_0)\cdot\sum_{i=1}^{6}|K_i|\,(\xi_i-\alpha_0,\eta_i-\beta_0)\right| \\
&\quad \le |\nabla p_l(\alpha_0,\beta_0)|\left|\sum_{i=1}^{6}\left(|K_i| - \frac{1}{6}|\omega_0|\right)(\xi_i-\alpha_0,\eta_i-\beta_0)\right| \\
&\qquad + \frac{1}{6}|\nabla p_l(\alpha_0,\beta_0)|\,|\omega_0|\left|\sum_{i=1}^{6}(\xi_i-\alpha_0,\eta_i-\beta_0)\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le Ch^4 \left|\nabla p_l(\alpha_0, \beta_0)\right| \\
&\le Ch^4 \|u\|_{3,\infty,\Omega},
\end{aligned}
$$

which together with (22), (24), and the quasi-uniformity of the mesh implies (7) in this case. The proof is complete.

Proof of Theorem 2 Let u, (α_0, β_0), K_i, and $\omega_0 = \cup_{i=1}^m K_i$ be all the same as in the theorem. Since the mesh τ_h is a rectangular mesh, we have $m = 4$. Denoting again by (ξ_i, η_i) the center of the element K_i, $i = 1, \ldots, 4$, we recall the superconvergence estimate in the present setting from the previous work [6, 10, 11, 13, 23, 32, 37]

$$
|\partial_l u(\xi_i, \eta_i) - \partial_l u_h(\xi_i, \eta_i)| \le Ch^2 \left|\ln h\right| \|u\|_{3,\infty,\Omega}, \qquad i = 1, \ldots, 4. \tag{25}
$$

Fix $l = 1$ or 2. Let $F_l : P(\omega_0) \to \mathbf{R}$ be the functional as defined by (10) for the locally discrete least-squares recovery and as defined by (11) for the local L^2 projection recovery. Since $p_l \in P_1(\omega_0)$ is the unique minimizer of F_l by the assumption of the theorem, it satisfies the corresponding equation (5) or (6).

We now consider two cases.

Case 1. The locally discrete least-squares recovery on a quasi-uniform rectangular mesh. For convenience, we number all the four elements in the element patch ω_0 counterclockwise starting from the lower left one. Define $q_i \in Q_1(\omega_0)$ for $i = 1, \ldots, 4$ by

$$
q_i(x, y) = (-1)^{i+1} \frac{4}{|\omega_0|} \left(x - \xi_{i+2}\right) \left(y - \eta_{i+2}\right), \qquad (x, y) \in \omega_0,
$$

where we adopt the convention that two indices i_1 and i_2 are the same if $i_1 \equiv i_2 \pmod 4$. Since the four centers $(\xi_i, \eta_i) \in K_i, i = 1, \ldots, 4$, are vertices of a rectangle with its sides parallel to the coordinate axes, respectively, it is easy to verify that $q_i(\xi_j, \eta_j) = \delta_{ij}, i, j = 1, \ldots, 4$. So, taking $q = q_j$ in (5) with $m = 4$ and $P = Q_1$, we get

$$
p_l(\xi_j, \eta_j) = \partial_l u_h(\xi_j, \eta_j), \qquad j = 1, \ldots, 4. \tag{26}
$$

Let h_{xi} and h_{yi} be the length of sides of K_i along Ox and Oy coordinate directions, respectively, i.e.,

$$
h_{xi} = 2\left|\alpha_0 - \xi_i\right|, \quad h_{yi} = 2\left|\beta_0 - \eta_i\right|, \qquad i = 1, \ldots, 4.
$$

We have by an easy calculation that

$$q_i(\alpha_0, \beta_0) = \frac{h_{x\,i+2} h_{y\,i+2}}{|\omega_0|}, \qquad i = 1, \ldots, 4.$$

Therefore,

$$\begin{aligned} q(\alpha_0, \beta_0) &= \sum_{i=1}^{4} q(\xi_i, \eta_i) q_i(\alpha_0, \beta_0) \\ &= \frac{1}{|\omega_0|} \sum_{i=1}^{4} h_{x\,i+2} h_{y\,i+2}\, q(\xi_i, \eta_i), \qquad \forall q \in Q_1(\omega_0). \end{aligned} \tag{27}$$

Now let $B_l \in Q_1(\omega_0)$ be the bilinear part of $\partial_l u$ on ω_0 around the assembly point (α_0, β_0):

$$\begin{aligned} B_l(x, y) = \partial_l u(\alpha_0, \beta_0) + \nabla \partial_l u(\alpha_0, \beta_0) \cdot (x - \alpha_0, y - \beta_0) \\ + \partial_1 \partial_2 \partial_l u(\alpha_0, \beta_0)(x - \alpha_0)(y - \beta_0), \qquad (x, y) \in \omega_0. \end{aligned}$$

Obviously, $B_l(\alpha_0, \beta_0) = \partial_l u(\alpha_0, \beta_0)$, and

$$|B_l(x, y) - \partial_l u(x, y)| \le Ch^2 \|u\|_{3,\infty,\omega_0}, \qquad \forall (x, y) \in \omega_0. \tag{28}$$

Therefore, using the expression (27) for p_l and B_l, respectively, we have by (26), (25), and (28) that

$$\begin{aligned} &|p_l(\alpha_0, \beta_0) - \partial_l u(\alpha_0, \beta_0)| \\ &\quad = |p_l(\alpha_0, \beta_0) - B_l u(\alpha_0, \beta_0)| \\ &\quad = \frac{1}{|\omega_0|} \left| \sum_{i=1}^{4} h_{x\,i+2} h_{y\,i+2} \left[p_l(\xi_i, \eta_i) - B_l(\xi_i, \eta_i) \right] \right| \\ &\quad \le \sum_{i=1}^{4} |\partial_l u_h(\xi_i, \eta_i) - \partial_l u(\xi_i, \eta_i)| + \sum_{i=1}^{4} |\partial_l u(\xi_i, \eta_i) - B_l(\xi_i, \eta_i)| \\ &\quad \le Ch^2 \, |\ln h| \, \|u\|_{3,\infty,\Omega}, \end{aligned}$$

proving (8) in this case.

Case 2. The local L^2 projection recovery on a unidirectionally uniform rectangular mesh. We have in this special case that $|K_i| = \frac{1}{4}|\omega_0|$ for $i =$

$1,\ldots,4$. Setting $q=1$ identically on ω_0 in (6) with $P=Q_1$, by the fact that p_l is bilinear and u_h piecewise bilinear, we obtain that

$$\sum_{i=1}^{4} p_l(\xi_i,\eta_i) = \sum_{i=1}^{4} \partial_l u_h(\xi_i,\eta_i). \tag{29}$$

By a simple calculation we also have that

$$q(\alpha_0,\beta_0) = \frac{1}{4}\sum_{i=1}^{4} q(\xi_i,\eta_i), \qquad \forall q\in Q_1(\omega_0).$$

It therefore follows from (29), (28) and (25) that

$$\begin{aligned}
&|p_l(\alpha_0,\beta_0) - \partial_l u(\alpha_0,\beta_0)| \\
&\quad = |p_l(\alpha_0,\beta_0) - B_l(\alpha_0,\beta_0)| \\
&\quad = \frac{1}{4}\left|\sum_{i=1}^{4} p_l(\xi_i,\eta_i) - \sum_{i=1}^{4} B_l(\xi_i,\eta_i)\right| \\
&\quad \le \frac{1}{4}\left|\sum_{i=1}^{4} \partial_l u_h(\xi_i,\eta_i) - \sum_{i=1}^{4} B_l(\xi_i,\eta_i)\right| \\
&\quad \le \frac{1}{4}\sum_{i=1}^{4} |\partial_l u_h(\xi_i,\eta_i) - \partial_l u(\xi_i,\eta_i)| + \frac{1}{4}\sum_{i=1}^{4} |\partial_l u(\xi_i,\eta_i) - B_l(\xi_i,\eta_i)| \\
&\quad \le Ch^2\,|\ln h|\,\|u\|_{3,\infty,\Omega},
\end{aligned}$$

which implies (8) in this case. The proof is finished.

Proof of Theorem 3 and Theorem 4 It follows from the definition of $W^{-1,2}(\Omega)$-norm and the standard L^2 error estimate that

$$\|u-u_h\|_{-1,2,\Omega} \le \|u-u_h\|_{0,2,\Omega} \le Ch^2\|u\|_{2,2,\Omega}.$$

Therefore, under the assumption of the theorems, we have the local $W^{1,\infty}$ estimate [6, 22, 23, 32]

$$\|u-u_h\|_{1,\infty,\Omega_0} \le Ch\left(\|u\|_{2,\infty,\Omega_1} + \|u\|_{2,2,\Omega}\right). \tag{30}$$

Since the element patch $\omega_0 = \cup_{i=1}^{m} K_i \subset \bar{\Omega}_0$, the stability estimate (12) becomes now (cf. (14) and (16))

$$|\nabla p_l(\alpha_0,\beta_0)| \le C\left(\|u\|_{3,\infty,\Omega_1} + \|u\|_{2,2,\Omega}\right), \qquad l=1,2. \tag{31}$$

Further, the superconvergence estimates (17) and (25) become respectively [6, 22, 23, 24, 31, 32]

$$\begin{aligned} &\left|\partial_l u(\gamma_i,\delta_i) - \tfrac{1}{2}\left[\partial_l u_h|_{K_i}(\gamma_i,\delta_i) + \partial_l u_h|_{K_{i+1}}(\gamma_i,\delta_i)\right]\right| \\ &\quad \le Ch^2\left|\ln h\right|\left(\|u\|_{3,\infty,\Omega_1} + \|u\|_{2,2,\Omega}\right), \qquad i = 1,\ldots,6, \end{aligned} \tag{32}$$

and

$$\begin{aligned} &\left|\partial_l u(\xi_i,\eta_i) - \partial_l u_h(\xi_i,\eta_i)\right| \\ &\quad \le Ch^2\left|\ln h\right|\left(\|u\|_{3,\infty,\Omega_1} + \|u\|_{2,2,\Omega}\right), \qquad i = 1,\ldots,4. \end{aligned} \tag{33}$$

Using (30) – (33), we can obtain the desired local superconvergence estimates by repeating the corresponding proofs of Theorem 1 and Theorem 2.

4 An Example

In this section, we give a simple example to show that in the one-dimensional case the quasi-uniformity of a mesh is in general not sufficient to result in the derivative superconvergence by the local L^2 projection recovery. It also strongly suggests that in the two-dimensional case the derivative superconvergence will not be recovered in general by the rectangular bilinear element using the local L^2 projection if the underlying rectangular mesh is only quasi-uniform, cf., the numerical experiments reported in [34].

We consider the two-point boundary value problem

$$\begin{cases} -u'' = f, & \text{in}\,(0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{34}$$

where $f \in L^2(0,1)$. Let u be the exact solution of (34) and assume that u is smooth enough on $[0,1]$.

Let $\tau_h : 0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary mesh of the interval $[0,1]$, where $h = \max\{h_i : 1 \le i \le N\}$ and $h_i = x_i - x_{i-1}, i = 1,\ldots,N$. Denote by V_h the corresponding linear finite element space:

$$V_h = \left\{v_h \in H_0^1(0,1) : v_h|_{(x_{i-1},x_i)} \in P_1(x_{i-1},x_i), i = 1,\ldots,N\right\},$$

where P_1 denotes the space of all one-variable, linear polynomials. Let $u_h \in V_h$ be the finite element approximation of the exact solution u defined by

$$\int_0^1 (u' - u_h')\, v_h' dx = 0, \qquad \forall v_h \in V_h. \tag{35}$$

Let $I_h u \in V_h$ be the Lagrange interpolant of u defined by

$$I_h u(x_i) = u(x_i), \qquad i = 0, 1, \ldots, N.$$

For any $v_h \in V_h$, we have by (35) that

$$\begin{aligned}\int_0^1 (u_h - I_h u)' \, v_h' dx &= \int_0^1 (u - I_h u)' \, v_h' dx \\ &= \sum_{i=1}^N \int_{x_{i-1}}^{x_i} (u - I_h u)' \, v_h' dx = \sum_{i=1}^N (u - I_h u) \, v_h' |_{x_{i-1}}^{x_i} = 0.\end{aligned}$$

Setting $v_h = u_h - I_h u$, we obtain by the constraint of zero boundary value that in this case $u_h = I_h u$ identically on $[0, 1]$, a well-known result.

Now we consider an element patch $\omega_i = (x_{i-1}, x_{i+1})$ surrounding the assembly point $x_i (1 \le i \le N-1)$. Let $p \in P_1(x_{i-1}, x_{i+1})$ be the recovered derivative by the local L^2 projection which is determined by (cf. Lemma 1)

$$\int_{x_{i-1}}^{x_{i+1}} (p - u_h') \, q \, dx = 0, \qquad \forall q \in P_1(x_{i-1}, x_{i+1}).$$

Assuming that $p(x) = a_0 + a_1(x - x_i)$ for $x \in (x_{i-1}, x_{i+1})$, where a_0 and a_1 are constants, we obtain since $u_h = I_h u$ that

$$\int_{x_{i-1}}^{x_{i+1}} \left[a_0 + a_1(x - x_i) - (I_h u)'\right] q \, dx = 0, \qquad \forall q \in P_1(x_{i-1}, x_{i+1}).$$

Setting $q(x) = 1$ and $q(x) = x - x_i$, respectively, we have by a series of calculations that

$$\begin{aligned}(h_i + h_{i+1}) \, a_0 + \frac{1}{2}\left(h_{i+1}^2 - h_i^2\right) a_1 &= u_{i+1} - u_{i-1}, \\ \frac{1}{2}\left(h_{i+1}^2 - h_i^2\right) a_0 + \frac{1}{3}\left(h_i^3 + h_{i+1}^3\right) a_1 &= \frac{1}{2} h_{i+1} \left(u_{i+1} - u_i\right) - \frac{1}{2} h_i \left(u_i - u_{i-1}\right),\end{aligned}$$

where we recall $h_j = x_j - x_{j-1}, j = 1, \ldots, N$, and we denote $u_j = u(x_j)$ for $j = 0, 1, \ldots, N$.

Notice that $p(x_i) = a_0$. So, we need only to find a_0. Solving the above two equations, we obtain that

$$\begin{aligned}a_0 = \frac{1}{(h_i + h_{i+1})^3} &\left[(u_{i+1} - u_i)\left(4h_i^2 + h_{i+1}^2 - h_i h_{i+1}\right)\right. \\ &\left. + (u_i - u_{i-1})\left(4h_{i+1}^2 + h_i^2 - h_i h_{i+1}\right)\right].\end{aligned}$$

By the Taylor expansion we have

$$u_{i+1} = u_i + h_{i+1}u'(x_i) + \frac{1}{2}h_{i+1}^2 u''(x_i) + \frac{1}{6}h_{i+1}^3 u'''(\bar{x}_{i+1})$$

for some $\bar{x}_{i+1} \in [x_i, x_{i+1}]$, and

$$u_{i-1} = u_i - h_i u'(x_i) + \frac{1}{2}h_i^2 u''(x_i) - \frac{1}{6}h_i^3 u'''(\bar{x}_i)$$

for some $\bar{x}_i \in [x_{i-1}, x_i]$. Consequently, we have by a series of calculations that

$$\begin{aligned} p(x_i) - u'(x_i) &= a_0 - u'(x_i) \\ &= \frac{h_i^3 + h_{i+1}^3}{2\left(h_i + h_{i+1}\right)^3}\left(h_{i+1} - h_i\right) u''(x_i) + r_i \end{aligned}$$

with

$$|r_i| \le \frac{1}{6}\left(h_i + h_{i+1}\right)^2 \max_{x_{i-1} \le x \le x_{i+1}} \left|u'''(x)\right|.$$

It is easy to verify that

$$\frac{1}{8} \le \left|\frac{h_i^3 + h_{i+1}^3}{2\left(h_i + h_{i+1}\right)^3}\right| \le \frac{1}{2}.$$

Therefore, to recover the derivative superconvergence at all assembly points x_i by the local L^2 projection, we need to have the condition that $|h_{i+1} - h_i| = O(h^2)$ for all $i = 1, \ldots, N-1$. This is, however, certainly much stronger than the quasi-uniformity condition which only guarantees in general that $|h_{i+1} - h_i| = O(h)$ for all $i = 1, \ldots, N-1$.

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