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Analysis of a drinking epidemic model

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Abstract In this paper, we have developed a mathematical model of alcohol abuse which consists of four compartments corresponding to four population classes, namely, moderate and occasional drinkers, heavy drinkers, drinkers in treatment and temporarily recovered class. We have discussed about basic properties of the system. Sensitivity analysis of the system is also discussed. Next, Basic reproduction number (R_0) is calculated. The stability analysis of the model shows that the system is locally asymptotically stable at disease free equilibrium E_0 when $R_0 < 1$. When $R_0 > 1$, endemic equilibrium E^* exists and the system becomes locally asymptotically stable at E^* and E_0 becomes unstable. We have also discussed the global stability of the system at E_0. It is also found that a backward bifurcation may occur at $R_0 = 1$. Next we have discussed the drinking epidemic model with treatment control. An objective functional is considered which is based on a combination of minimizing the number of heavy drinkers and the cost of treatment. Then an optimal control is obtained which minimizes the objective functional. Our numerical findings are illustrated through computer simulations using MATLAB, which show the reliability of our model from the practical point of view. Epidemiological implications of our analytical findings are addressed critically.

Keywords Alcohol abuse · Basic reproduction number · Sensitivity · Local and global stability · Forward and backward bifurcations · Optimal control

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1 Introduction

Nowadays alcoholism has become an epidemic disease. Alcoholism is generally used to mean compulsive and uncontrolled consumption of alcoholic beverage which affects their work, health, education and social life. Alcohol abuse and alcoholism are two different forms of problems of drinking. Alcohol abuse is when alcohol drinking leads problems, but not physical addiction. On the other hand alcoholism is when one has signs of physical addiction to alcohol and continues to drink, despite problems with physical health, mental health, family or job responsibilities. In fact alcoholism is a long term effect of alcohol abuse. Alcohol abuse and alcoholism can affect all aspects of our life. Long-term alcohol use can cause serious health complications, damaging nearly every organ and system in the body including our brain. The World Health Organization (WHO) estimates that alcohol is supposed to cause about 60 types of diseases and injury like 20–30 % of esophageal cancer, liver cancer, cirrhosis of the liver, homicide, epileptic seizures and motor vehicle accidents worldwide. Drinking alcohol during pregnancy can cause an array of physical and mental birth defects and it is the leading preventable cause of mental retardation in children all over the world. The National Institute on Alcohol Abuse and Alcoholism estimates that 18 million Americans suffer from alcohol abuse, specially most beginning by their mid teens. Serious drinking often starts in adolescence; approximately 40 % of alcoholics develop their first symptoms between 15 and 19 years of age [1]. Over-consumption of alcohol is now the third leading cause of death all over the world. Alcohol related problems cost so much that it affects the economic structure of the countries [2,3].

Individuals who wish to overcome an alcohol abuse problem can enter into the treatment programmes. Completely stopping the use of alcohol is the ideal goal of treatment.

This is called abstinence. Completely stopping or avoiding alcohol is difficult for many people with alcoholism. Most of them seek outside help from treatment centers and therapy sessions. These programmes usually offer counselling and therapy to discuss alcoholism and its effects, mental health support, medical care etc. There are two major forms of intervention policy of alcohol abuse: (i) prevention initiation into alcohol abuse and (ii) rehabilitation of alcohol abusers. Among the many problems confronting these programmes, the most important is the very high rates of relapse after treatment. The National Institute on Alcohol Abuse and Alcoholism estimates that up to 70 % of treated alcohol abusers relapse after treatment which is indeed a big problem. Therefore prevention and control efforts that include treatment and education programmes should be improved so that the rate of relapse from treatment can be reduced.

It is obvious that alcohol abuse and alcoholism not only causes health problems but also has great social and economic impacts on the countries. Therefore, it is very important to understand the dynamics of alcoholism spread among the populations and identify the parameters of greater importance which will help the policy-makers in targeting prevention and treatment resources for maximum effectiveness. Although drinking is a problem of significant public health importance, not much has been analyzed in terms of using mathematical modelling to gain insight into its transmission dynamics at population level. Most of the existing works on alcohol abuse and alcoholism are of clinical aspects. Mathematical studies can be effective as it guides to the evaluation, testing and implementation of strategies over short or long time scales for chronic relapsing diseases such as alcohol addiction, drug abuse etc. There are many mathematical models for epidemic diseases like HIV [4–9], SARS [10], gonorrhea [11,12], dengue [13], cancer [14–16], Chlamydia [17], HFMD [18–21] etc. But the mathematical models on alcohol and drug abuse are very small in number though they have been referred to in terms of epidemics. There are some research works on drug abuse [22–26]. There is a very interesting model on alcohol abuse proposed by Sánchez et al. [27]. In this model the total population is divided into three classes, namely susceptible, alcoholics and alcoholics undergoing treatment. This model has generated useful insights about the role of basic reproduction number on the transmission dynamics of alcoholism. In this model author considered a three compartment drinking epidemic model. The population is divided into three classes, namely, moderate and occasional drinkers susceptible to drinking epidemic), heavy drinkers and recovered population. The model is a simple model which does not incorporate the relapse from recovered class to susceptible class, which is very unrealistic. The author did not introduce any treatment or intervention policy in this model. There must be an additional death rate for the heavy drinkers. But the authors did not introduce any additional drinking related death rate in this model, which is not realistic. A similar kind of work was done by Mulone and Straughan [23]. But in this paper also no treatment is introduced. As drinking is a chronic relapsing disease, the relapse of drinkers from treatment should be considered to make the work more realistic, which is missing in [23]. There are also some recent research works on this field [28–31].

To overcome those limitations we have developed an alcohol abuse model by introducing a treatment programme in the population and considered all possible relapses. We have divided the population into four classes, namely, moderate and occasional drinkers, heavy drinkers, drinkers in treatment and temporarily recovered class. Introduction of treatment in the system makes more realistic and significant biologically, which was missing in the previous works [23,27]. Next we have found the basic reproduction number R_0 [32]. We have also considered drinking related additional death rate. Sensitivity analysis of R_0 identifies the most useful parameters of the model. Then the stability analysis of the model is discussed using the basic reproduction number. It is observed that the model is locally asymptotically stable at the disease free equilibrium (DFE) E_0 when $R_0 < 1$. The system can undergo a backward bifurcation at $R_0 = 1$ considering R_0 as bifurcation parameter. When $R_0 > 1$, endemic equilibrium E^* exists and becomes stable and the DFE becomes unstable. We have also found the conditions of global asymptotic stability of E_0. It is obvious that alcohol abuse and alcoholism are not only public health problems but also has great social and economic impacts on the countries. So, it is a burning issue to control the spread of this disease by choosing optimal control strategy. From this aspect we have discussed optimal control problem relative to the alcohol abuse epidemic model to minimize the heavy drinkers as well as to minimize the cost of treatment. This kind of analysis was not discussed in any previous works [23,27]. We have considered treatment rate as a function of time and it is representing treatment control in the drinking model. This part of our research can help policy-makers and researchers to identify the parameters of greater interest and to make an optimal control strategy to control the epidemic. Next we have illustrated the key findings through numerical simulations using MATLAB followed by discussions and conclusions. Our comparative diagrams of controls help us to understand the effectiveness of controls to reduce the spread of alcohol abuse.

2 Mathematical model

In this section we have developed a mathematical model of alcohol abuse. In this model the adult human population is divided into four different classes, namely, moderate and occasional drinkers, heavy drinkers, drinkers in treatment and temporarily recovered class.

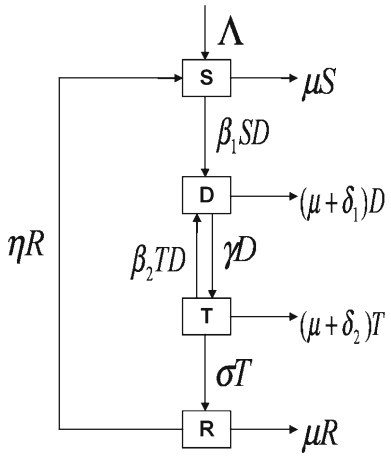


Fig. 1 Flow diagram of the alcohol abuse model

Figure 1 represents the flow of individuals from one class to the other.

The model can be presented by the following set of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta_1 S(t)D(t) - \mu S(t) + \eta R(t)\\
\frac{dD}{dt} &= \beta_1 S(t)D(t) + \beta_2 T(t)D(t) - (\mu + \delta_1 + \gamma)D(t)\\
\frac{dT}{dt} &= \gamma D(t) - \beta_2 T(t)D(t) - (\mu + \delta_2 + \sigma)T(t)\\
\frac{dR}{dt} &= \sigma T(t) - (\mu + \eta)R(t) \qquad (2.1)
\end{aligned}$$

with initial densities

$$S(0) > 0,\ D(0) \geq 0,\ T(0) \geq 0,\ R(0) \geq 0. \qquad (2.2)$$

Here $S(t)$, $D(t)$, $T(t)$ and $R(t)$ represent the numbers of moderate and occasional drinkers (susceptible to drinking epidemic), heavy drinkers, drinkers in treatment and temporarily recovered class respectively at time t. So, $N(t) = S(t) + D(t) + T(t) + R(t)$ denotes the total number of high-risk human population at time t.

The model parameters are described below:

- Λ: Recruitment rate of moderate and occasional drinkers,
- β_1: The transmission coefficient from moderate and occasional drinkers to heavy drinkers,
- β_2: The transmission coefficient from drinkers in treatment to heavy drinkers,
- μ: Natural death rate of each population class,
- δ_1: Drinking related death rate of heavy drinkers,
- δ_2: Drinking related death rate of drinkers in treatment,
- γ: The proportion of drinkers who enter in treatment,
- σ: Recovery rate of drinkers in treatment,
- η: The proportion of recovered class who re-enter into moderate and occasional drinkers class.

Clearly the model involves the following assumptions:

(i) All members of the population mix homogeneously. This means that each individual has an equal chance of becoming a heavy drinker.
(ii) The heavy drinking is passed to moderate and occasional drinkers by adequate contact with heavy drinkers not in treatment.
(iii) Heavy drinkers not in treatment are infectious to moderate and occasional drinkers and to drinkers in treatment.
(iv) Drinkers in treatment are not infectious to moderate and occasional drinkers.
(v) The drinkers in treatment most commonly relapse due to contact with heavy drinkers who are not in treatment.
(vi) Those who stop drinking alcohol enter into the temporarily recovered class and one part of which relapse to the moderate and occasional drinkers' class.

3 Basic properties

3.1 Non-negativity of solutions

Theorem 3.1 *Every solution of system* (2.1) *with initial conditions* (2.2) *exists in the interval* $[0, \infty)$ *and* $S(t) > 0$, $D(t) \geq 0$, $T(t) \geq 0$, $R(t) \geq 0$, *for all* $t \geq 0$.

Proof Since the right hand side of system (2.1) is completely continuous and locally Lipschitzian on C (space of continuous functions), the solution $(S(t), D(t), T(t), R(t))$ of (2.1) with initial conditions (2.2) exists and is unique on $[0, \xi)$, where $0 < \xi \leq +\infty$. From the second equation of (2.1), we have

$$D(t) = D(0)\exp\left[\int_0^t \{\beta_1 S(s) + \beta_2 T(s) - (\mu + \delta_1 + \gamma)\}ds\right] \geq 0$$

From the third equation of (2.1), we have

$$\begin{aligned}
\frac{dT}{dt} &\geq -[\beta_2 D(t) + (\mu + \delta_2 + \sigma)]\,T(t) \quad [\because D(t) \geq 0]\\
&\Rightarrow T(t) \geq T(0)\exp\left[-\left\{(\mu + \delta_2 + \sigma)t + \int_0^t \beta_2 D(s)ds\right\}\right] \geq 0.
\end{aligned}$$

Similarly, from the forth equation of (2.1), we have

$$\begin{aligned}
\frac{dR}{dt} &\geq -(\mu + \eta)R(t) \quad [\because T(t) \geq 0]\\
&\Rightarrow R(t) \geq R(0)\exp[-(\mu + \eta)t] \geq 0.
\end{aligned}$$

Finally, it follows from the first equation of the system (2.1) that,

$$\frac{dS}{dt} \geq \Lambda - [\beta_1 D(t) + \mu]\,S(t) \quad [\because R(t) \geq 0].$$

We thus have,

$$\frac{d}{dt}\left[S(t)\exp\left\{\mu t+\int_0^t\beta_1 D(s)ds\right\}\right] \geq \Lambda\exp\left\{\mu t+\int_0^t\beta_1 D(s)ds\right\}.$$

Hence

$$S(t)\exp\left\{\mu t+\int_0^t\beta_1 D(s)ds\right\}-S(0) \geq \int_0^t\Lambda\exp\left\{\mu t+\int_0^t\beta_1 D(s)ds\right\}dt,$$

so that

$$S(t)\geq S(0)\exp\left[-\left\{\mu t+\int_0^t\beta_1 D(s)ds\right\}\right] +\left[\exp\left\{-\left(\mu t+\int_0^t\beta_1 D(s)ds\right)\right\}\right] \times\left[\int_0^t\Lambda\exp\left\{\mu t+\int_0^t\beta_1 D(s)ds\right\}dt\right]>0.$$

$\therefore S(t)>0, D(t)\geq 0, T(t)\geq 0, R(t)\geq 0, \quad \forall t\geq 0.$
This completes the proof. □

3.2 Invariant region

Theorem 3.2 *The feasible region G defined by*

$$G=\left\{(S(t),D(t),T(t),R(t))\in\mathbb{R}_+^4: 0<N\leq\frac{\Lambda}{\mu}\right\}$$

with initial conditions $S(0)>0, D(0)\geq 0, T(0)\geq 0, R(0)\geq 0$, *is positively invariant.*

Proof Adding the equations of the system (2.1) we obtain

$$\frac{dN}{dt}=\Lambda-\mu N-\delta_1 D(t)-\delta_2 T(t) \leq \Lambda-\mu N \quad [\because D(t)\geq 0 \text{ and } T(t)\geq 0] \tag{3.1}$$

The solution $N(t)$ of the differential equation (3.1) has the following property,

$$0<N(t)\leq N(0)e^{-\mu t}+\frac{\Lambda}{\mu}(1-e^{-\mu t})$$

where $N(0)$ represents the sum of the initial values of the variables. As $t\to\infty, 0<N\leq\frac{\Lambda}{\mu}$. So, if $N(0)\leq\frac{\Lambda}{\mu}$, then $\lim_{t\to\infty}N(t)\leq\frac{\Lambda}{\mu}$. This means that $\frac{\Lambda}{\mu}$ is the upper bound of N. On the other hand, if $N(0)>\frac{\Lambda}{\mu}$, then $N(t)$ will decrease to $\frac{\Lambda}{\mu}$. This means that if $N(0)>\frac{\Lambda}{\mu}$, then the solution $(S(t), D(t), T(t), R(t))$ enters G or approach it asymptotically. Hence it is positively invariant under the flow induced by the system (2.1). Thus in G, the model (2.1) is well-posed epidemiologically and mathematically. Hence it is sufficient to study the dynamics of the model in G. □

4 The basic reproduction number R_0

Basic reproduction number R_0 [32] for drinking epidemic model is defined as the number of heavy drinkers produced when a single heavy drinker is introduced into moderate and occasional drinkers' population, i.e.,

R_0 = (effective contact rate) · (duration of heavy drinkers spend in the drinking class).

In the present model (2.1), $\frac{\Lambda\beta_1}{\mu}$ is the effective contact rate and $(\mu+\delta_1+\gamma)$ is the removal rate of the heavy drinkers from drinking class. By assumption all rates are constant. This means that the expected duration of heavy drinkers spend in the drinking class is simply the inverse of the removal rate, i.e., $\frac{1}{(\mu+\delta_1+\gamma)}$ Therefore, the basic reproduction number of system (2.1) is given by

$$R_0=\frac{\Lambda\beta_1}{\mu(\mu+\delta_1+\gamma)}. \tag{4.1}$$

5 Sensitivity analysis of R_0

The basic reproduction number (R_0) of system (2.1) depends on four parameters, namely, the transmission coefficient from moderate and occasional drinkers to heavy drinkers (β_1), drinking related death rate of heavy drinkers (δ_1), the proportion of drinkers who enter into treatment (γ) and the natural death rate of population (μ). Among those parameters, we can not control the natural death rate of population (μ). Therefore, to examine the sensitivity of R_0 to the parameters β_1, δ_1 and γ, normalized forward sensitivity index with respect to each of those parameters are computed as follows:

$$A_{\beta_1}=\frac{\frac{\partial R_0}{R_0}}{\frac{\partial\beta_1}{\beta_1}}=\frac{\beta_1}{R_0}\frac{\partial R_0}{\partial\beta_1} =\beta_1\left\{\frac{\mu(\mu+\delta_1+\gamma)}{\Lambda\beta_1}\right\}\left\{\frac{\Lambda}{\mu(\mu+\delta_1+\gamma)}\right\}=1,$$

$$A_{\delta_1}=\frac{\frac{\partial R_0}{R_0}}{\frac{\partial\delta_1}{\delta_1}}=\frac{\delta_1}{R_0}\frac{\partial R_0}{\partial\delta_1}=\frac{-\delta_1}{\mu+\delta_1+\gamma}\Rightarrow|A_{\delta_1}|<1,$$

$$A_{\gamma}=\frac{\frac{\partial R_0}{R_0}}{\frac{\partial\gamma}{\gamma}}=\frac{\gamma}{R_0}\frac{\partial R_0}{\partial\gamma}=\frac{-\gamma}{\mu+\delta_1+\gamma}\Rightarrow|A_{\gamma}|<1. \tag{5.1}$$

From the above discussion it is clear that the basic reproduction number (R_0) is most sensitive to changes in β_1, the transmission coefficient from moderate and occasional drinkers to heavy drinkers. If β_1 will increase R_0 will increase in same proportion and if β_1 will decrease R_0 will also decrease in same proportion. On the other hand, δ_1 and γ have an inversely proportional relationship with R_0, i.e., an increase in any of them will cause a decrease in R_0 and a decrease in any of them will cause an increase in R_0. But the

increase in δ_1, the drinking related death rate of the heavy drinkers not in treatment, is neither ethical nor practical. So, it is better to focus either on β_1,the transmission rate from moderate and occasional drinker to heavy drinker or γ, the proportion of drinkers who enter into treatment. As R_0 is more sensitive to changes in β_1 than γ, it seems sensible to focus on the reduction of β_1 to control the alcohol abuse. This sensitivity analysis tells us that efforts to increase prevention are more effective in controlling the spread of alcohol abuse in population than efforts to increase the numbers of heavy drinkers accessing treatment.

6 Equilibrium points: their existence and stability

In this section we will study the existence and stability behaviour of the system (2.1) at equilibrium points. The equilibrium points of the model system (2.1) are:

1. Drinking free equilibrium: $E_0(\frac{\Lambda}{\mu}, 0, 0, 0)$,
2. Endemic equilibrium: $E^*(S^*, D^*, T^*, R^*)$,

We use the term "drinking free equilibrium" to describe the state where a drinking culture does not exist as we have considered moderate and occasional drinkers as susceptible to drinking epidemic. On the other hand, "endemic equilibrium" stands for the state where a drinking culture exists.

6.1 Existence of endemic equilibrium $E^*(S^*, D^*, T^*, R^*)$

In this section, we analyze the existence of non-trivial endemic equilibrium of system (2.1). At an endemic equilibrium, disease is present and the followings hold:

$$S > 0, \quad D > 0, \quad T > 0, \quad R > 0,$$
$$\frac{dS}{dt} = \frac{dD}{dt} = \frac{dT}{dt} = \frac{dR}{dt} = 0.$$

Solving the equations of system (2.1) at equilibrium state we get,

$$S^* = \frac{(\mu+\delta_1)(\beta_2 D^* + \mu + \delta_2 + \sigma) + \gamma(\mu + \delta_2 + \sigma)}{\beta_1(\beta_2 D^* + \mu + \delta_2 + \sigma)},$$
$$T^* = \frac{\gamma D^*}{\beta_2 D^* + \mu + \delta_2 + \sigma},$$
$$R^* = \frac{\sigma\gamma D^*}{(\mu+\eta)(\beta_2 D^* + \mu + \delta_2 + \sigma)},$$

Now, putting the values of S^*, R^* into the second equation of (2.1) and simplifying we obtain,

$$a_1(D^*)^2 + a_2 D^* + a_3 = 0, \tag{6.1}$$

where

$$\begin{aligned}
a_1 &= -(\mu+\eta)(\mu+\delta_1)\beta_1\beta_2,\\
a_2 &= \Lambda\beta_1(\mu+\eta)\beta_2 - \{\beta_1(\mu+\delta_2+\sigma)\\
&\quad + \beta_2\mu\}(\mu+\eta)(\mu+\delta_1)\\
&\quad - \gamma(\mu+\eta)(\mu+\delta_2+\sigma)\beta_1 + \beta_1\eta\sigma\gamma,\\
a_3 &= (\mu+\eta)(\mu+\delta_2+\sigma)\{\Lambda\beta_1 - \mu(\mu+\delta_1+\gamma)\}\\
&= \mu(\mu+\eta)(\mu+\delta_2+\sigma)\\
&\quad \times(\mu+\delta_1+\gamma)\left[\frac{\Lambda\beta_1}{\mu(\mu+\delta_1+\gamma)} - 1\right]\\
&= \mu(\mu+\eta)(\mu+\delta_2+\sigma)(\mu+\delta_1+\gamma)[R_0 - 1]
\end{aligned}$$

Obviously a_1 is always negative and a_3 is positive if $R_0 > 1$ and negative if $R_0 < 1$. By applying the Descartes' rule of signs, we get the following observations:

(i) if $R_0 > 1$, i.e., $a_3 > 0$, one positive equilibria exists, whatever is the sign of a_2.
(ii) If $R_0 < 1$, i.e., $a_3 < 0$ with $a_2 < 0$, then the system has no positive equilibrium.
(iii) If $R_0 < 1$, i.e., $a_3 < 0$ with $a_2 > 0$, then we get two or no positive equilibrium, which may provide the backward bifurcation scenario, i.e., in the neighbourhood of 1, for $R_0 < 1$ a stable drinking free equilibrium coexists with two endemic equilibria: a smaller endemic equilibrium (with a smaller number of heavy drinkers) which is unstable and a large endemic equilibrium (with a large number of heavy drinkers) which is stable.

Summarizing the previous discussions we come to the following result:

Theorem 6.1 *The system (2.1) has a drinking free equilibrium* $E_0\left(\frac{\Lambda}{\mu}, 0, 0, 0\right)$, *which exists for all parameter values. If* $R_0 > 1$, *the system (2.1) also admits a unique endemic equilibrium* $E^*(S^*, D^*, T^*R^*)$. *If* $R_0 < 1$ *with* $a_2 < 0$ *in (6.1), then the system has no endemic equilibrium. If* $R_0 < 1$ *with* $a_2 > 0$ *in (6.1) then we get two or no positive equilibrium, which may provide the backward bifurcation scenario.*

Observation If we take $\beta_2 = 0$, then

$$\begin{aligned}
a_1 &= 0,\\
a_2 &= -\beta_1(\mu+\eta)(\mu+\delta_1)(\mu+\delta_2+\sigma)\\
&\quad - \gamma\beta_1\{\mu(\mu+\delta_2+\sigma) + \eta(\mu+\delta_2)\} < 0,\\
a_3 &= \mu(\mu+\eta)(\mu+\delta_2+\sigma)(\mu+\delta_1+\gamma)(R_0 - 1).
\end{aligned}$$

Then Eq. (6.1) becomes,

$$a_2 D^* + a_3 = 0 \tag{6.2}$$

Therefore, there exists only one root of the above equation, which is given by

$$D^* = -\frac{a_3}{a_2}.$$

Here, a_2 is always negative and $a_3 > 0$ if $R_0 > 1$ and $a_3 < 0$ if $R_0 < 1$. Therefore, if $R_0 > 1$ then the above equation has an unique positive root D^*. When $R_0 < 1$ there is no positive root of (6.2). This implies that there is a unique endemic equilibrium point of the system (2.1) when $\beta_2 = 0$ with $R_0 > 1$. Therefore in this case there is no existence of backward bifurcation as there is a unique endemic equilibrium of the system (2.1) when $\beta_2 = 0$ with $R_0 > 1$.

Therefore, we can conclude that the backward bifurcation occurs because of the insufficient capacity for treatment policies. As a result the drinkers in treatment come to the direct contact of heavy drinkers and they re-enter into the heavy drinkers class.

6.2 Local stability of drinking free equilibrium E_0

In this section we will study the local stability of the system (2.1) at drinking free equilibrium $E_0(\frac{\Lambda}{\mu}, 0, 0, 0)$.

The variational matrix of system (2.1) at $E_0(\frac{\Lambda}{\mu}, 0, 0, 0)$ is given by

$$V(E_0) = \begin{bmatrix} -\mu & -\beta_1\frac{\Lambda}{\mu} & 0 & \eta \\ 0 & \beta_1\frac{\Lambda}{\mu} - (\mu+\delta_1+\gamma) & 0 & 0 \\ 0 & \gamma & -(\mu+\delta_2+\sigma) & 0 \\ 0 & 0 & \sigma & -(\mu+\eta) \end{bmatrix}$$

Therefore, eigenvalues of the characteristic equation of $V(E_0)$ are

$$\lambda_1 = -\mu, \quad \lambda_2 = \beta_1\frac{\Lambda}{\mu} - (\mu+\delta_1+\gamma),$$
$$\lambda_3 = -(\mu+\sigma+\delta_2), \quad \lambda_4 = -(\mu+\eta).$$

Here, λ_1, λ_3 and λ_4 are clearly real and negative. Now, E_0 is stable if $\lambda_2 < 0$ i.e.,$\beta_1\frac{\Lambda}{\mu} - (\mu+\delta_1+\gamma) < 0$, i.e., $R_0 < 1$. Therefore the system (2.1) is local asymptotically stability at $E_0(\frac{\Lambda}{\mu}, 0, 0, 0)$ if $R_0 < 1$. So, we arrive to the following result:

Theorem 6.2 *The drinking free equilibrium E_0 of the model system (2.1) is locally asymptotically stable if $R_0 < 1$.*

6.3 Global stability analysis of drinking free equilibrium E_0

In this section we shall discuss about the global stability of the drinking free equilibrium E_0 when $R_0 < 1$. We consider the Lyapunov function as follows:

$$L_1(D, T, R) = \sigma D + \sigma T + (\mu+\delta_2+\sigma)R.$$

The derivative of $L_1(D, T, R)$ with respect to t gives

$$\begin{aligned}\frac{dL_1}{dt} &= \sigma[\beta_1 SD + \beta_2 TD - (\mu+\delta_1+\gamma)D] \\ &\quad + \sigma[\gamma D - \beta_2 TD - (\mu+\delta_2+\sigma)T] \\ &\quad + (\mu+\delta_2+\sigma)[\sigma T - (\mu+\eta)R] \\ &\le \sigma\beta_1 SD - \sigma(\mu+\delta_1)D \\ &\le \sigma\left[\beta_1\frac{\Lambda}{\mu} - (\mu+\delta_1)\right]D\end{aligned}$$

Now, $\frac{dL_1}{dt} < 0$ if $\beta_1\frac{\Lambda}{\mu} < \mu+\delta_1$.

Furthermore $\frac{dL_1}{dt} = 0$ if and only if $D = 0$. Therefore, the largest compact invariant set in $\{(S, D, T, R) \in \Gamma: \frac{dL_1}{dt} = 0\}$, when $\beta_1\frac{\Lambda}{\mu} < \mu+\delta_1$, is the singleton $\{E_0\}$. LaSalle's invariance principle [33] implies that E_0 is globally asymptotically stable in Γ when $\beta_1\frac{\Lambda}{\mu} < \mu+\delta_1$ (which implies $R_0 < 1$). So, we arrive to the following result:

Theorem 6.3 *If $\beta_1\frac{\Lambda}{\mu} < \mu+\delta_1$ then the drinking free equilibrium (DFE) E_0 of model system (2.1) is globally asymptotically stable.*

6.4 Bifurcation analysis at $R_0 = 1$

In this section, we discuss the stability of endemic equilibrium (E^*) and investigate the possibility of occurring backward bifurcation [13,32,34–37] due to existence of multiple equilibrium. There may exist two distinct bifurcations at $R_0 = 1$: (i) forward (supercritical) bifurcation and (ii) backward (subcritical) bifurcation [26,34,35,37,38]. A forward bifurcation happens when R_0 crosses unity from below. Then a small positive asymptotically stable endemic equilibrium (E^*) appears and the disease-free equilibrium (E_0) losses its stability. On the other hand, in backward bifurcation scenario, in the neighbourhood of 1 when R_0 is less than unity, a positive unstable endemic equilibrium (E_1) appears while a drinking-free equilibrium (E_0) and another positive endemic equilibrium (E^*) are locally asymptotically stable. These two endemic equilibria disappear by saddle-node bifurcation when the basic reproduction number R_0 is decreased below the critical value $R_0^c < 1$. It means that, though the necessary and sufficient condition for disease or drinking eradication is $R_0 < 1$, it is no longer sufficient when a backward bifurcation occurs.

To analyze it for the system (2.1), we use the center manifold theory [39]. To implement this method, we first change the variables of the model equations (2.1) so that $S = x_1, D = x_2, T = x_3, R = x_4$ with $\frac{dx_1}{dt} = f_1, \frac{dx_2}{dt} = f_2, \frac{dx_3}{dt} = f_3, \frac{dx_4}{dt} = f_4$.

Thus system (1) becomes:

$$\begin{aligned} f_1 &= \Lambda - \beta_1 x_1 x_2 - \mu x_1 + \eta x_4 \\ f_2 &= \beta_1 x_1 x_2 + \beta_2 x_3 x_2 - (\mu + \delta_1 + \gamma) x_2 \\ f_3 &= \gamma x_2 - \beta_2 x_3 x_2 - (\mu + \delta_2 + \sigma) x_3 \\ f_4 &= \sigma x_3 - (\mu + \eta) x_4 \end{aligned} \tag{6.3}$$

We choose $\beta_1^* = \beta_1$ as the bifurcation parameter, particularly as it has been shown in Eq. (5.1) that R_0 is more sensitive to change in β_1 than in its other parameters. If we consider $R_0 = 1$, then we obtain,

$$\beta_1^* = \frac{\mu(\mu + \delta_1 + \gamma)}{\Lambda}. \tag{6.4}$$

Now, the Jacobian of the linearized system (6.3) using identity (6.4) at drinking free equilibrium E_0 when $\beta_1^* = \beta_1$ is given by,

$$J\left(\beta_1^*\right) = \begin{bmatrix} -\mu & -\beta_1^* \frac{\Lambda}{\mu} & 0 & \eta \\ 0 & 0 & 0 & 0 \\ 0 & \eta & -(\mu + \delta_2 + \sigma) & 0 \\ 0 & 0 & \sigma & -(\mu + \eta) \end{bmatrix} \tag{6.5}$$

The matrix (6.5) has eigenvalues $(0, -\mu, -(\mu + \delta_2 + \sigma), -(\mu + \eta))^T$, which meets the requirement of simple zero eigenvalue and the others having negative real parts. We can thus use the center manifold theory [39] to analyze the dynamics of system (6.3). The right eigenvector associated with zero eigenvalue is given by, $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T$, where

$$\begin{aligned} \omega_1 &= \frac{1}{\mu}\left\{\eta - \frac{\Lambda \beta_1^* (\mu + \delta_2 + \sigma)(\mu + \eta)}{\mu \gamma \sigma}\right\} \omega_4 \\ \omega_2 &= \frac{(\mu + \delta_2 + \sigma)(\mu + \eta)}{\gamma \sigma} \omega_4 \\ \omega_3 &= \frac{(\mu + \eta)}{\sigma} \omega_4 \\ \omega_4 &= 1 \end{aligned}$$

with ω_4 free. Further, $J(\beta_1^*)$ has a corresponding left eigen vector $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$, where

$$\begin{aligned} \nu_1 &= 0 \\ \nu_2 &= \frac{\sigma \gamma}{(\mu + \delta_2 + \sigma)(\mu + \eta)} \\ \nu_3 &= 0 \\ \nu_4 &= 0. \end{aligned}$$

The value of ν_2 is chosen so that $\nu.\omega = 1$. In order to establish the local stability of E^*, we use the following theorem.

Theorem 6.4 [40] *Consider the following general system of ordinary differential equations with a parameter φ,*

$$\frac{dx}{dt} = f(x, \varphi), \quad f: R^n \times R \to R \text{ and } f \in C^2(R^n \times R), \tag{6.6}$$

where 0 is an equilibrium of the system that is $f(0, \varphi) = 0$, $\forall \varphi$ and assume:

A1. $A = D_x f(0, 0) = \left(\frac{\partial f_i}{\partial x_j}(0, 0)\right)$ *is linearization matrix of the system (6.6) around the equilibrium 0 with φ evaluated at 0. Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real parts;*

A2. Matrix A has a right eigenvector u and a left eigenvector ν corresponding to the zero eigenvalue.

Let f_k be the k-th component of f and

$$\begin{aligned} a &= \sum_{k,i,j=1}^{n} \nu_k u_i u_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0), \\ b &= \sum_{k,j=1}^{n} \nu_k u_i \frac{\partial^2 f_k}{\partial x_i \partial \varphi}(0, 0). \end{aligned} \tag{6.7}$$

The local dynamics of (6.6) around 0 are totally governed by a and b.

(i) $a, b > 0$. When $\varphi < 0$ with $|\varphi| \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when $0 < \varphi \ll 1$, 0 is unstable and there exists a negative and locally asymptotically stable equilibrium.

(ii) $a < 0, b < 0$. When $\varphi < 0$ with $|\varphi| \ll 1$, 0 is unstable; when $0 < \varphi \ll 1$, 0 is locally asymptotically stable, and there exists a positive unstable equilibrium.

(iii) $a > 0, b < 0$. When $\varphi < 0$ with $|\varphi| \ll 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \varphi \ll 1$, 0 is stable, and a positive unstable equilibrium appears.

(iv) $a < 0, b > 0$. When φ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

Now, we calculate the values of a and b to apply the Theorem 6.4.

In particular, since $\nu_1 = \nu_3 = \nu_4 = 0$,

$$a = \nu_2 \sum_{i,j=1}^{4} \omega_i \omega_j \frac{\partial^2 f_2}{\partial x_i \partial x_j}(0, 0)$$

$$\text{and} \quad b = \nu_2 \sum_{i=1}^{4} \omega_i \frac{\partial^2 f_2}{\partial x_i \partial \beta_1}(0, 0)$$

For the system (6.3), the associated non-zero partial derivatives at the drinking free equilibrium are given by:

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_1} = \frac{\mu(\mu+\delta_1+\gamma)}{\Lambda},$$
$$\frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3 \partial x_2} = \beta_2,$$
$$\frac{\partial^2 f_2}{\partial x_2 \partial \beta_1} = \frac{\Lambda}{\mu}.$$

It thus follows that,

$$a = \frac{2\nu_2\omega_2}{\Lambda\sigma\gamma}(X-\Gamma), \tag{6.8}$$

where

$$X = \gamma\{\eta\sigma(\mu+\delta_1+\gamma) + \Lambda\beta_2(\mu+\eta)\}, \tag{6.9}$$
$$\Gamma = (\mu+\eta)(\mu+\delta_1+\gamma)^2(\mu+\delta_2+\sigma), \tag{6.10}$$
$$\text{and } b = \nu_2\omega_2\frac{\Lambda}{\mu} > 0. \tag{6.11}$$

Hence the sign of a depends on the values of X and Γ, i.e., if $X > \Gamma$, then $a > 0$ and if $X < \Gamma$, then $a < 0$ while $b > 0$ always. Thus, we have the following result:

Theorem 6.5 *If $X > \Gamma$, then the system (2.1) has a backward bifurcation at $R_0 = 1$, otherwise if $X < \Gamma$, then it undergoes forward bifurcation and the endemic equilibrium is locally asymptotically stable for $R_0 > 1$, but close to one.*

7 Epidemic model with control

In the context of mathematical modelling in epidemiology, it is essential to frame an optimal control problem so that the total amount of drug is minimized. We have considered the epidemic model (2.1). We now let treatment rate be given as a function of time by $u(t)$. So, $u(t)$ is the rate of heavy drinkers entering in the treatment per unit time at t. Thus u is representing treatment control in the drinking model. Therefore our epidemic model with this treatment control becomes:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta_1 S(t)D(t) - \mu S(t) + \eta R(t)\\
\frac{dD}{dt} &= \beta_1 S(t)D(t) + \beta_2 T(t)D(t)\\
&\quad - (\mu+\delta_1)D(t) - u(t)D(t)\\
\frac{dT}{dt} &= u(t)D(t) - \beta_2 T(t)D(t) - (\mu+\delta_2+\sigma)T(t)\\
\frac{dR}{dt} &= \sigma T(t) - (\mu+\eta)R(t)
\end{aligned} \tag{7.1}$$

satisfying

$$S(0) = \bar{S}_0, \quad D(0) = \bar{D}_0, \quad T(0) = \bar{T}_0, R(0) = \bar{R}_0. \tag{7.2}$$

The objective functional [41–44] is defined as

$$J(u(t)) = \int_0^{t_f}\left[B_1 D + \frac{1}{2}B_2 u^2\right]dt \tag{7.3}$$

where B_1, B_2 are positive constants to keep a balance in the size of the terms. The square of the control variable reflects the severity of the side-effects of the treatment. Our aim is to minimize the objective functional or cost function $J(u(t))$ given in (7.3) so that the heavy drinkers as well as the cost of treatment can be minimized. So, we seek an optimal control u^* such that

$$J(u^*) = \min\{J(u): u \in U\}, \tag{7.4}$$

where $U = \{u: u \text{ is measurable}, 0 \le u \le 1, t \in [0, t_f],\}$ is the admissible control set.

7.1 Existence of an optimal control

Lemma 7.1 *Every solution of system (7.1) with initial conditions (7.2) exists in the interval $[0,\infty)$ and $S(t) > 0$, $D(t) \ge 0$, $T(t) \ge 0$, $R(t) \ge 0$, for all $t \ge 0$.*

Proof Since the right hand side of system (7.1) is completely continuous and locally Lipschitzian on C, the solution $(S(t), D(t), T(t), R(t))$ of (7.1) with initial conditions (7.2) exists and is unique on $[0, \xi)$, where $0 < \xi \le +\infty$. From the second equation of (7.1), we have

$$D(t) = D(0)\exp\left[\int_0^t \{\beta_1 S(s) + \beta_2 T(s) - (\mu+\delta_1+u(s))\}ds\right] \ge 0$$

From the third equation of (7.1), we have

$$\begin{aligned}
&\frac{dT}{dt} \ge -[\beta_2 D(t) + (\mu+\delta_2+\sigma)]\,T(t)\\
&\quad [\because D(t) \ge 0 \text{ and } 0 \le u \le 1]\\
&\Rightarrow T(t) \ge T(0)\exp\left[-\left\{(\mu+\delta_2+\sigma)t + \int_0^t \beta_2 D(s)ds\right\}\right] \ge 0.
\end{aligned}$$

Similarly, from the forth equation of (7.1), we have

$$\begin{aligned}
&\frac{dR}{dt} \ge -(\mu+\eta)R(t) \quad [\because T(t) \ge 0]\\
&\Rightarrow R(t) \ge R(0)\exp[-(\mu+\eta)t] \ge 0.
\end{aligned}$$

Finally, it follows from the first equation of the system (7.1) that,

$$\frac{dS}{dt} \ge \Lambda - [\beta_1 D(t) + \mu]\,S(t) \quad [\because R(t) \ge 0].$$

We thus have,

$$\begin{aligned}
S(t) \ge\ & S(0)\exp\left[-\left\{\mu t + \int_0^t \beta_1 D(s)ds\right\}\right]\\
&+\left[\exp\left\{-\left(\mu t + \int_0^t \beta_1 D(s)ds\right)\right\}\right]\\
&\times\left[\int_0^t \Lambda\exp\left\{\mu t + \int_0^t \beta_1 D(s)ds\right\}dt\right] > 0.
\end{aligned}$$

Similarly, therefore, we can see that $S(t) > 0, D(t) \geq 0, T(t) \geq 0, R(t) \geq 0, \quad \forall t \geq 0$. This completes the proof. □

Lemma 7.2 *The feasible region G defined by*

$$G = \left\{ (S(t), D(t), T(t), R(t)) \in \mathbb{R}^4_+ : 0 < N \leq \frac{\Lambda}{\mu} \right\}$$

with initial conditions $S(0) > 0, D(0) \geq 0, T(0) \geq 0, R(0) \geq 0$, *is positively invariant.*

Proof Adding the equations of the system (7.1) we obtain

$$\begin{aligned} \frac{dN}{dt} &= \Lambda - \mu N - \delta_1 D(t) - \delta_2 T(t) \\ &\leq \Lambda - \mu N \quad [\because D(t) \geq 0 \text{ and } T(t) \geq 0] \end{aligned} \tag{7.5}$$

The solution $N(t)$ of the differential equation (7.5) has the following property,

$$0 < N(t) \leq N(0)e^{-\mu t} + \frac{\Lambda}{\mu}(1 - e^{-\mu t})$$

where $N(0)$ represents the sum of the initial values of the variables. As $t \to \infty, 0 < N \leq \frac{\Lambda}{\mu}$. So, if $N(0) \leq \frac{\Lambda}{\mu}$, then $\lim_{t\to\infty} N(t) \leq \frac{\Lambda}{\mu}$. This means that $\frac{\Lambda}{\mu}$ is the upper bound of N. On the other hand, if $N(0) > \frac{\Lambda}{\mu}$, then $N(t)$ will decrease to $\frac{\Lambda}{\mu}$. This means that if $N(0) > \frac{\Lambda}{\mu}$, then the solution $(S(t), D(t), T(t), R(t))$ enters G_1 or approach it asymptotically, where

$$G_1 = \left\{ (S(t), D(t), T(t), R(t)) \in \mathbb{R}^4_+ : 0 < N \leq \frac{\Lambda}{\mu} \right\}.$$

Hence it is positively invariant under the flow induced by the system (7.1). Thus in G_1, the model (7.1) is bounded and well-posed epidemiologically and mathematically.

Theorem 7.1 *Given the objective functional*

$$J(u(t)) = \int_0^{t_f} \left[B_1 D + \frac{1}{2} B_2 u^2 \right] dt,$$

where $U = \{u : u \text{ is measurable}, 0 \leq u \leq 1, t \in [0, t_f], \}$ *subject to the system (7.1) with (7.2), then there exists an optimal control* u^* *such that* $J(u^*) = \min\{J(u) : u \in U\}$, *if the following conditions are satisfied:*

1. *The class of all initial conditions with a control* $u(t)$ *in the admissible control set along with each state equation being satisfied is not empty.*
2. *The admissible control set* U *is closed and convex.*
3. *Each right hand side of the state system (7.1) is continuous and is bounded above by a sum of the bounded control and the state and can be written as a linear function of* u *with coefficients depending on time and the state.*
4. *The integrand of* $J(u)$ *is convex on* U *and is bounded below by* $c_1 u^2 - c_2$ *with* $c_1, c_2 > 0$.

Proof In order to verify the first condition, we use a result by Lukes [45, Theorem 9.2.1] for the system (7.1) with bounded coefficients. The control set U is convex and closed by definition, which gives the condition 2.

Therefore, the right hand side of the state system (7.1) satisfies condition 3 as the state solutions are a priori bounded (see Lemmas 7.1 and 7.2).

For the fourth condition we need to show:

$$g((1-p)u + pv) \leq (1-p)g(u) + pg(v),$$

where $g(x) = B_1 D + \frac{1}{2} B_2 x^2$. Now,

$$\begin{aligned} &g((1-p)u + pv) - [(1-p)g(u) + pg(v)] \\ &= B_1 D(t) + \frac{B_2}{2} \left\{ (1-p)u + pv \right\}^2 - [(1-p)\{B_1 D(t) \\ &\quad + \frac{B_2}{2} u^2 \} + p \left\{ B_1 D(t) + \frac{B_2}{2} v^2 \right\}] \\ &= \frac{B_2}{2}(p^2 - p)(u - v)^2 \end{aligned}$$

Since $p \in (0, 1)$ implies $(p^2 - p) \leq 0$ and $(u - v)^2 > 0$, the expression $\frac{B_2}{2}(p^2 - p)(u - v)^2 \leq 0$, which implies that

$$g((1-p)u + pv) \leq (1-p)g(u) + pg(v).$$

Lastly,

$$B_1 D(t) + \frac{1}{2} B_2 u^2(t) \geq \frac{B_2}{2} u^2(t) \geq \frac{B_2}{2} u^2(t) - c_2 \geq c_1 u^2(t) - c_2,$$

which gives $c_1 u^2(t) - c_2$ as a lower bound of $g(u)$, for some $c_1 > 0, c_2 > 0$.

Therefore, we can conclude that there exists an optimal control u^* such that

$$J(u^*) = \min\{J(u) : u \in U\}.$$

□

7.2 Characterization of the optimal control pair

In order to derive the necessary conditions for the optimal control pair the Pontryagin's Maximum Principle [46] is used.

The Hamiltonian is defined as follows:

$$\begin{aligned} H = &\left(B_1 D + \frac{1}{2} B_2 u_1^2 \right) + \lambda_1 [\Lambda - \beta_1 SD - \mu S + \eta R] \\ &+ \lambda_2 [\beta_1 SD + \beta_2 TD - (\mu + \delta_1) D - uD] \\ &+ \lambda_3 [uD - \beta_2 TD - (\mu + \delta_2 + \sigma) T] \\ &+ \lambda_4 [\sigma T - (\mu + \eta) R], \end{aligned} \tag{7.6}$$

where $\lambda_i(t), i = 1, 2, 3, 4$ are the adjoint functions to be determined suitably.

The forms of the adjoint equations and transversality conditions are standard results from Pontryagin's Maximum

Principle [46]. The adjoint system can be obtained as follows:

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= -\left(\frac{\partial H}{\partial S}\right) = (\lambda_1-\lambda_2)\beta_1 D + \lambda_1\mu \\
\frac{d\lambda_2}{dt} &= -\left(\frac{\partial H}{\partial D}\right) = \lambda_1\beta_1 S - \lambda_2\{\beta_1 S + \beta_2 T - \mu - \delta_1 - u\} \\
&\quad -\lambda_3(u-\beta_2 T) - B_1 \\
\frac{d\lambda_3}{dt} &= -\left(\frac{\partial H}{\partial T}\right) = (\lambda_3-\lambda_2)\beta_2 D + \lambda_3(\mu+\delta_2+\sigma) - \lambda_4\sigma \\
\frac{d\lambda_4}{dt} &= -\left(\frac{\partial H}{\partial R}\right) = \lambda_4\mu + (\lambda_4-\lambda_1)\eta
\end{aligned} \tag{7.7}$$

The transversality conditions (or boundary conditions) are

$$\lambda_i(t_f) = 0, \quad \text{for } i = 1,2,3,4. \tag{7.8}$$

By the optimality condition, we have

$$\frac{\partial H}{\partial u} = B_2 u^* - (\lambda_2-\lambda_3)\bar{D}^* = 0 \quad at \ \ u = u^*(t) \tag{7.9}$$

By using the bounds for the control $u(t)$, we get

$$u^* = \begin{cases} \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2}, & \text{if } 0 \le \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2} \le 1 \\ 0, & \text{if } \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2} \le 0 \\ 1, & \text{if } \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2} \ge 1 \end{cases} \tag{7.10}$$

In compact notation:

$$u^* = \min\left\{\max\left\{0, \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2}\right\}, 1\right\} \tag{7.11}$$

Using (7.11), we obtain the following optimality system:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta_1 S D - \mu S + \eta R \\
\frac{dD}{dt} &= \beta_1 S D + \beta_2 T D - (\mu+\delta_1)D \\
&\quad - \min\left\{\max\left\{0, \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2}\right\}, 1\right\} D \\
\frac{dT}{dt} &= \min\left\{\max\left\{0, \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2}\right\}, 1\right\} \\
&\quad D - \beta_2 T D - (\mu+\delta_2+\sigma)T \\
\frac{dR}{dt} &= \sigma T - (\mu+\eta)R \\
\frac{d\lambda_1}{dt} &= (\lambda_1-\lambda_2)\beta_1 D + \lambda_1\mu \\
\frac{d\lambda_2}{dt} &= \lambda_1\beta_1 S - \lambda_2\{\beta_1 S + \beta_2 T - \mu - \delta_1\} + \lambda_3\beta_2 T \\
&\quad +(\lambda_2-\lambda_3)\min\left\{\max\left\{0, \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2}\right\}, 1\right\} - B_1 \\
\frac{d\lambda_3}{dt} &= (\lambda_3-\lambda_2)\beta_2 D + \lambda_3(\mu+\delta_2+\sigma) - \lambda_4\sigma \\
\frac{d\lambda_4}{dt} &= \lambda_4\mu + (\lambda_4-\lambda_1)\eta
\end{aligned} \tag{7.12}$$

subject to the following conditions:

$$S(0) = \bar{S}_0, \quad D(0) = \bar{D}_0, \quad T(0) = \bar{T}_0, \quad R(0) = \bar{R}_0$$

and

$$\lambda_i(t_f) = 0, \quad for \ \ i = 1,2,3,4.$$

The previous analysis can be summarized in the following theorem:

Theorem 7.2 *There exists an optimal control u^* and corresponding solutions $\bar{S}^*, \bar{D}^*, \bar{T}^*, \bar{R}^*$ of the system (7.1) with the initial conditions (7.2) that minimizes $J(u(t))$ over U. The explicit optimal controls are connected to the existence of continuous specific functions $\lambda_i(t), i = 1,2,3,4$, the solutions of the following adjoint system:*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1-\lambda_2)\beta_1 D + \lambda_1\mu \\
\frac{d\lambda_2}{dt} &= \lambda_1\beta_1 S - \lambda_2\{\beta_1 S + \beta_2 T - \mu - \delta_1\} \\
&\quad +\lambda_3\beta_2 T + (\lambda_2-\lambda_3)u - B_1 \\
\frac{d\lambda_3}{dt} &= (\lambda_3-\lambda_2)\beta_2 D + \lambda_3(\mu+\delta_2+\sigma) - \lambda_4\sigma \\
\frac{d\lambda_4}{dt} &= \lambda_4\mu + (\lambda_4-\lambda_1)\eta
\end{aligned}$$

subject to the transversality conditions:

$$\lambda_i(t_f) = 0, \quad for \ \ i = 1,2,3,4.$$

Furthermore, the following property holds:

$$u^* = \min\left\{\max\left\{0, \frac{(\lambda_2-\lambda_3)\bar{D}^*}{B_2}\right\}, 1\right\}$$

8 Numerical simulations

In this section, we first consider the case when $R_0 = 0.56 < 1$ using the parameter values given in Table 1. Using these parameter values, for different initial conditions, the dynamics of the model (2.1) is presented in Fig. 2a–d. These figures show that only moderate and occasional drinkers (S) persists and heavy drinkers (D), drinkers in treatment (T) and temporarily recovered class (R) decline to zero (extinct), i.e., the population approaches to the drinking free equilibrium or disease free equilibrium (DFE) $E_0(1.6, 0, 0, 0)$. This numerical verification supports the result stated in Theorem 6.2 on the stability of DFE (E_0).

Next, we consider the case when $R_0 = 2.24 > 1$ using the parameter values given in Table 2. Using these parameter values, for different initial conditions, the dynamics of the model is presented in Fig. 3a–d. These figures show that moderate and occasional drinkers (S), heavy drinkers (D), drinkers in treatment (T) and temporarily recovered population (R)

Table 1 Parameter values for Fig. 2(a)–(d)

Parameter	Values
Λ	0.4 population/year
β_1	0.35/population/year
β_2	0.3/population/year
μ	0.25/year
σ	0.1/year
η	0.1/year
γ	0.4/year
δ_1	0.35/year
δ_2	0.3/year

all persist, i.e., the population tends to endemic equilibrium $E^*(1.16469, 0.445752, 0.227504, 0.0650011)$ when $R_0 > 1$. These figures imply that the endemic equilibrium E^* is locally asymptotically stable and the DFE $E_0(2.8, 0, 0, 0)$ becomes unstable whenever $R_0 > 1$. In this case $X = 0.0334$ and $\Gamma = 0.2275$, i.e., $X < \Gamma$, so $a < 0$ (see Sect. 6.4). Therefore, it also shows the forward bifurcation of system (2.1) which is good agreement with Theorem 6.5.

After that, we consider the case when $R_0 = 0.833333 < 1$ using the parameter values given in Table 3. In this case for different initial conditions, the dynamics of the model is presented in Fig. 4a–d. These figures show that in this case, there exists three equilibria of the system (2.1), among them DFE $E_0(20, 0, 0, 0)$ and an endemic equilibrium $E^*(1.71634, 6.76286, 0.90035, 0.072028)$ are stable

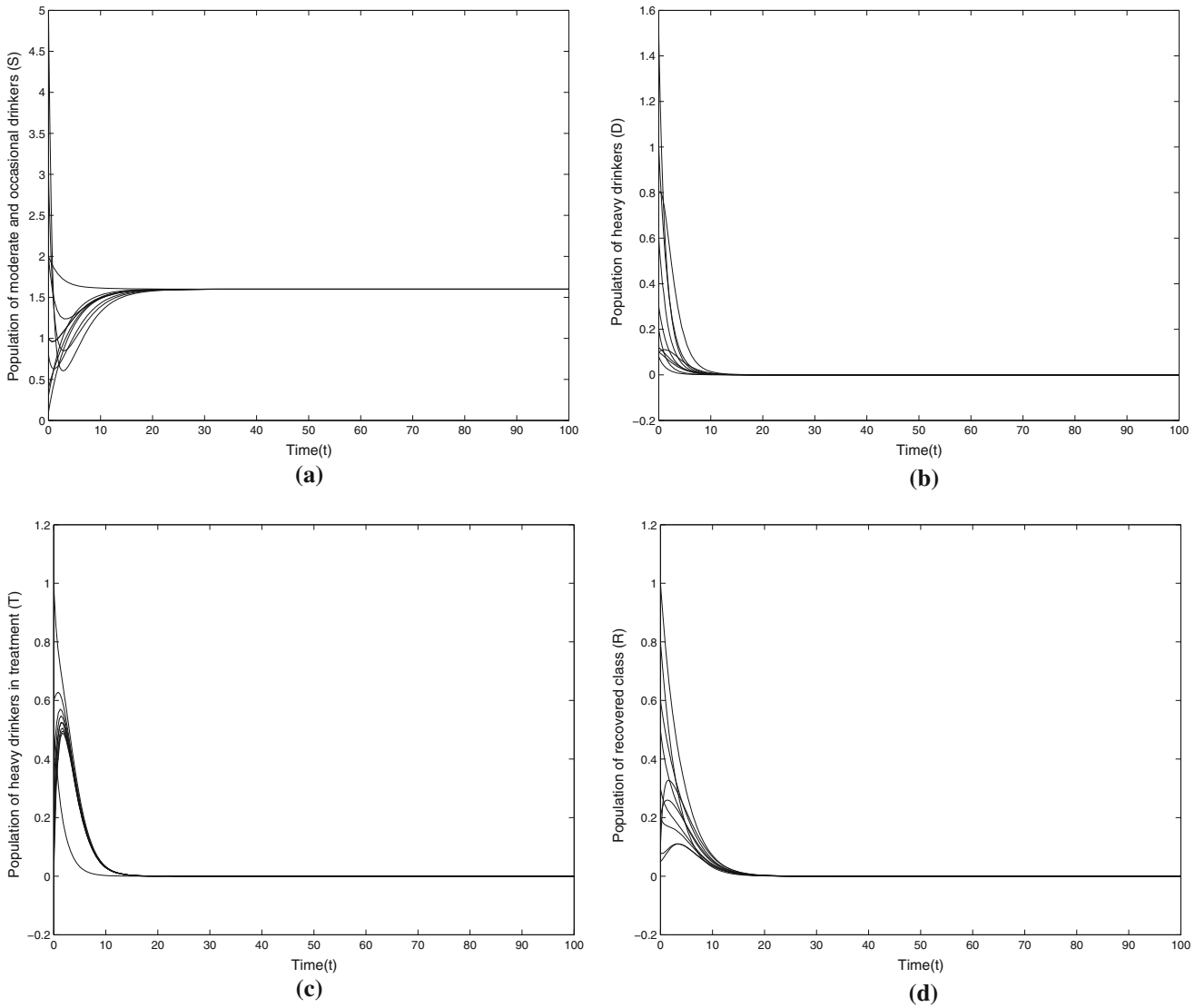


Fig. 2 Time series plot of **a** moderate and occasional drinkers (S), **b** heavy drinkers (D), **c** drinkers in treatment (T), **d** temporarily recovered class (R) respectively for $R_0 = 0.56 < 1$ with various initial conditions, parameter values are given in Table 1

Table 2 Parameter values for Fig. 3(a)–(d) and Fig. 5(b)

Parameter	Values
Λ	0.7 population/year
β_1	0.8/population/year
β_2	0.3/population/year
μ	0.25/year
σ	0.1/year
η	0.1/year
γ	0.4/year
δ_1	0.35/year
δ_2	0.3/year

and the other endemic equilibrium $E_1(19.5514, 0.0161807, 0.179743, 0.0143795)$ is unstable. Here $X = 0.0566$ and $\Gamma = 0.0075$, i.e., $X > \Gamma$, so $a > 0$ (see Sect. 6.4). Therefore, it also shows the backward bifurcation of system (2.1) which is also in good agreement with Theorem 6.2.

From the above analysis it is clear that when forward bifurcation occurs, for $R_0 > 1$, there are only two equilibria: the unstable disease free or drinking free equilibrium DFE (E_0) and a stable endemic equilibrium (E^*). It is also clear that in this case, the condition $R_0 < 1$ is a necessary and sufficient condition for disease eradication. But the condition $R_0 < 1$ is no longer sufficient when backward bifurcation occurs. In backward bifurcation scenario, when $R_0 < 1$, a positive unstable endemic equilibrium (E_1) appears while

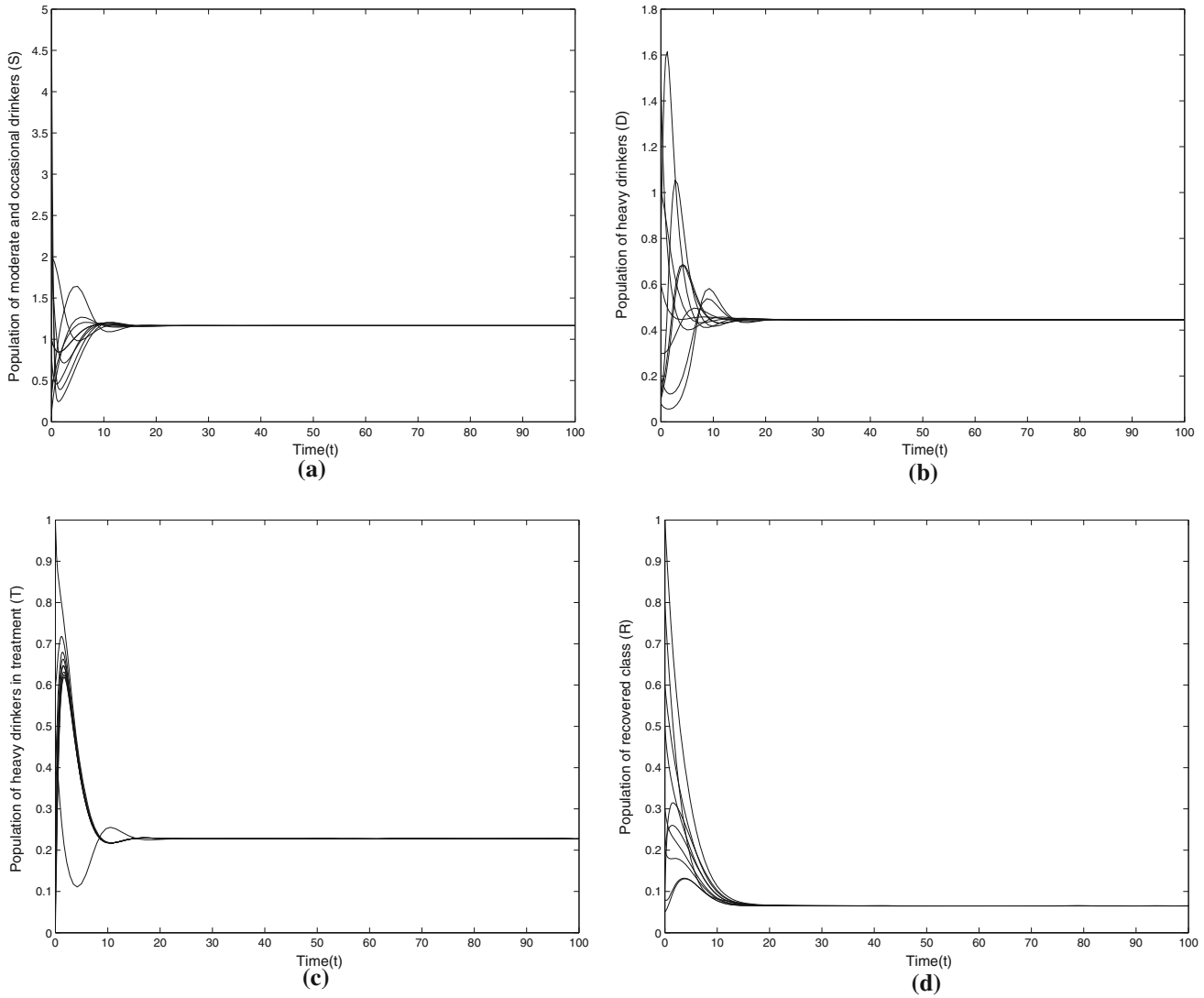


Fig. 3 Time series plot of **a** moderate and occasional drinkers (S), **b** heavy drinkers (D), **c** drinkers in treatment (T), **d** temporarily recovered class (R) respectively for $R_0 = 2.24 > 1$ with various initial conditions, parameter values are given in Table 2

Table 3 Parameter values for Fig. 4(a)–(d) and Fig. 5(a)

Parameter	Values
Λ	0.5 population/year
β_1	0.04/population/year
β_2	0.99/population/year
μ	0.025/year
σ	0.01/year
η	0.1/year
γ	0.9/year
δ_1	0.035/year
δ_2	0.03/year

a DFE (E_0) and another positive endemic equilibrium (E^*) are locally asymptotically stable. These two endemic equilibrium disappear when the basic reproduction number R_0 is decreased below the critical value $R_0^c < 1$. The qualitative bifurcation diagrams for backward bifurcation and forward bifurcation taking the bifurcation parameter R_0 are depicted in Fig. 5a, b respectively.

Next, the optimality system (7.1) has been solved numerically and the results have been presented graphically. This optimality system is a two-point boundary value problem with separated boundary conditions at times $t = 0$ and $t = t_f$. Here, we have solved this two-point boundary value optimality problem for $t_f = 10$. The value is chosen to represent the time in months at which treatment is stopped. To

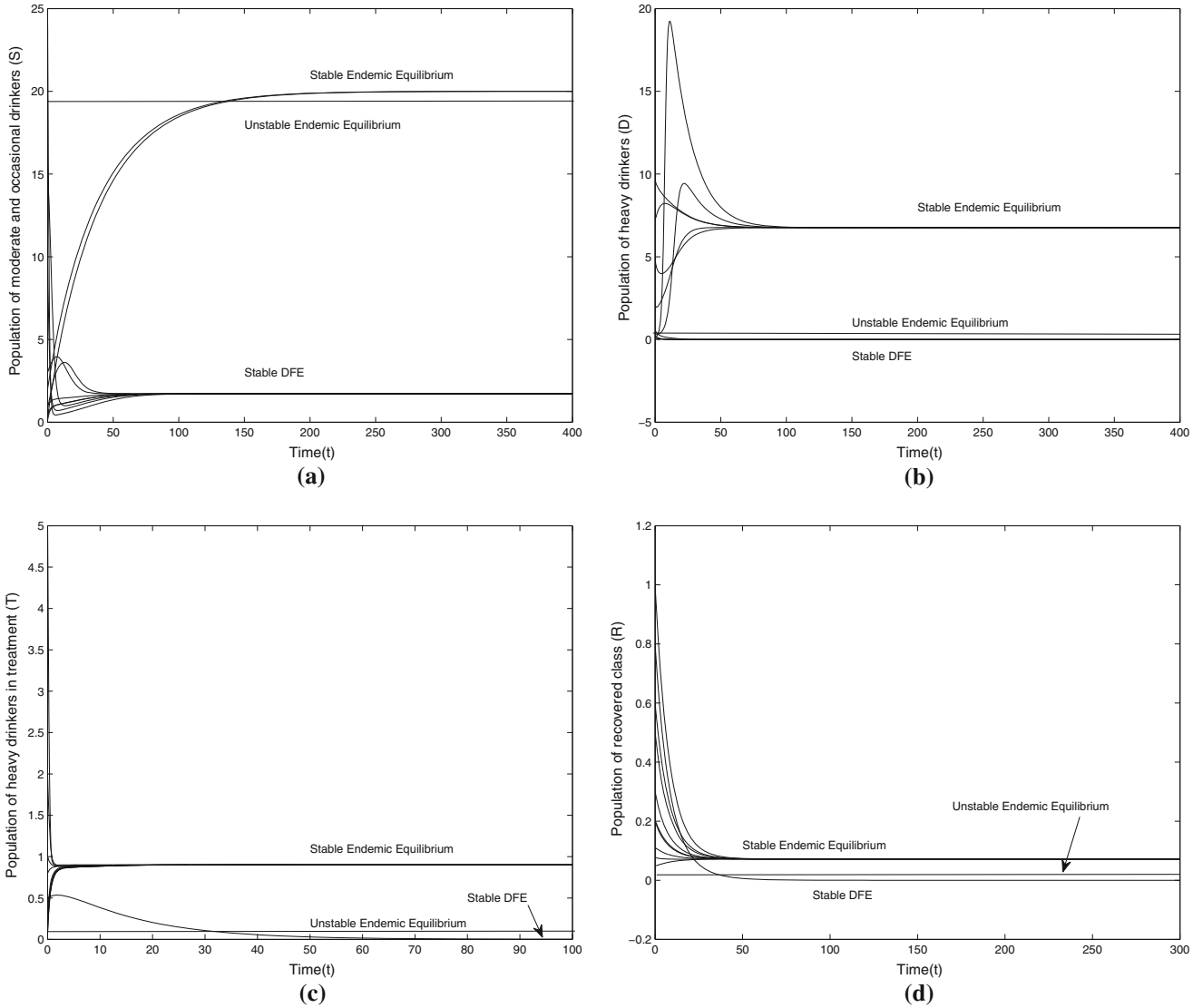
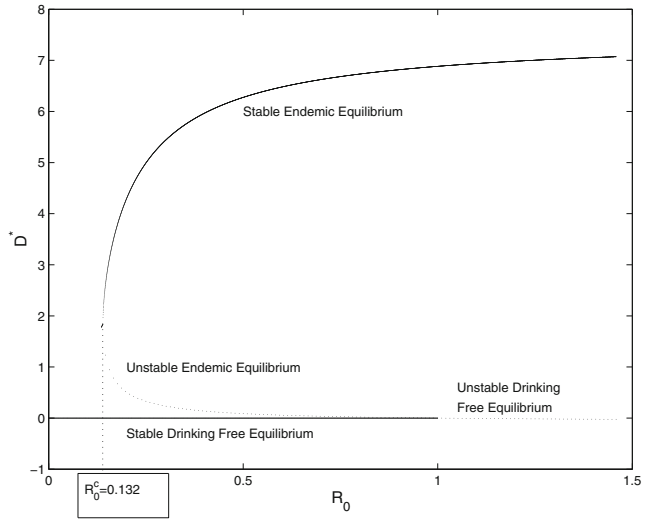


Fig. 4 Time series plot of **a** moderate and occasional drinkers (S), **b** heavy drinkers (D) , **c** drinkers in treatment (T), **d** temporarily recovered class (R) respectively for $R_0 = 0.833333 < 1$ with various initial conditions, parameter values are given in Table 3



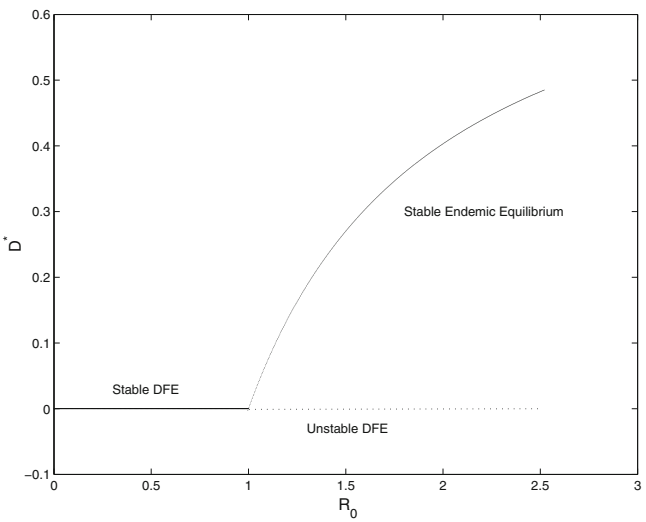


Fig. 5 Qualitative bifurcation diagrams for **a** backward bifurcation for $R_0 = 0.833333 < 1$ with parameter values given in Table 3 and **b** forward bifurcation for $R_0 = 2.24 > 1$ with parameter values given in Table 2 respectively. The bifurcation parameter is the basic reproductive number R_0. The *solid lines* denotes stability; the *dashed line* denotes instability

Table 4 Parameter values for Figs. 6–12

Parameter	Values
Λ	5.5 population/year
β_1	0.1/population/year
β_2	0.01/population/year
μ	0.3/year
σ	0.1/year
η	0.1/year
δ_1	0.35/year
δ_2	0.3/year

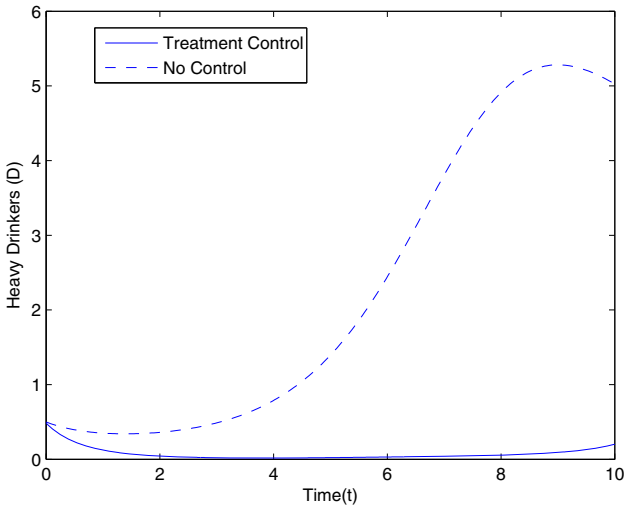


Fig. 6 The control diagrams for the heavy drinkers (D) using the parameter values given in Table 4 with $B_1 = 70$, $B_2 = 20$

solve our BVP, we have used collocation method with collocation code solver BVP4c in MATLAB. It is a powerful method to solve the two-point BVP resulting from the optimality conditions.

The different variables (population and control function) in the objective functional given in (7.3) have different scales. Hence they are balanced by choosing weight constants $B_1 = 70$, $B_2 = 20$ in the objective functional given in (7.3). The numerical results for the optimality problem are obtained by using the parameter values given in Table 4. At first we search for the optimal control function u, the treatment control. This optimal control function u is designed in such a way that it minimizes the objective functional (J) given by (7.3). In Fig. 6, we have presented the comparative time series diagram of heavy drinkers (D), with no control and treatment control. From this diagram, it is clearly seen that use of treatment control is more effective to control the disease than the case without any control. From these observations, we can conclude that this treatment control yields the best result to control the spread of the disease.

The optimal control graph for the treatment control u is presented in Fig. 7. In perspective, one could conclude from the optimal control diagram (Fig. 7) that we should give full effort in this treatment control at the beginning of the disease. This means that this control is very important at the beginning of disease outbreak than when the disease prevails. From Fig. 8, we observe that the treatment control function (u) minimizes the cost function J given in (7.3).

We have presented the time series plot of the adjoint variables (λ_1, λ_2, λ_3, λ_4) in Figs. 9, 10, 11 and 12. We know that, the time derivatives of the adjoint variables are negative of the corresponding partial derivatives of the Hamiltonian H with respect to the state variables. So, from those figures, we can

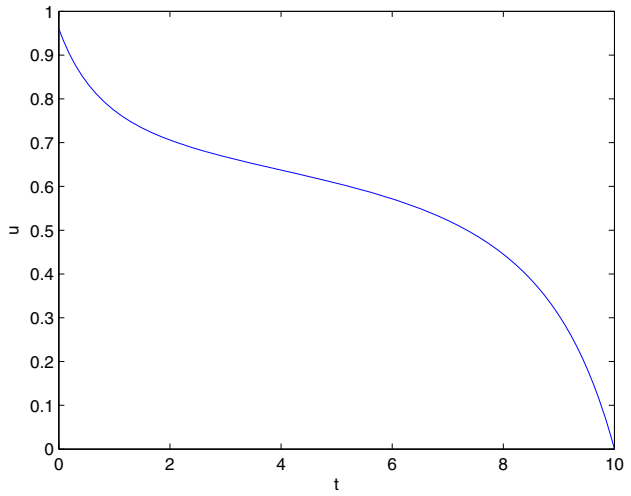


Fig. 7 The optimal control graph for the treatment control (u) using the parameter values given in Table 4 with $B_1 = 70$, $B_2 = 20$

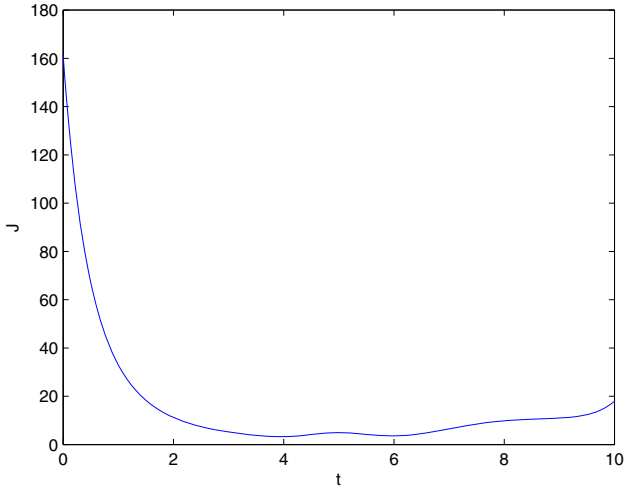


Fig. 8 The optimal control graph for the objective functional (J) using the parameter values given in Table 4 with $B_1 = 70$, $B_2 = 20$

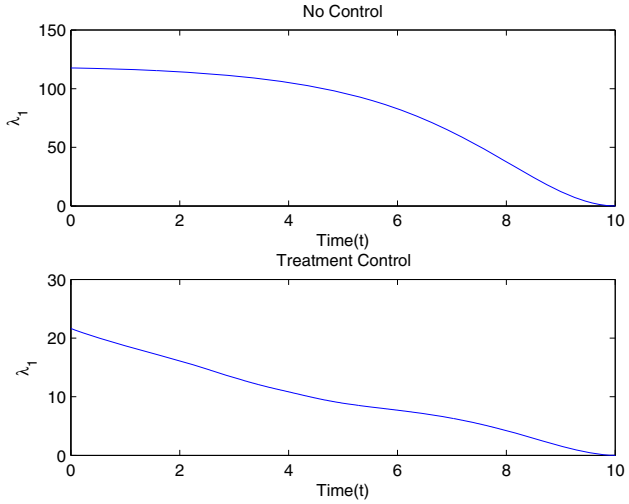


Fig. 9 The control diagrams for adjoint variable λ_1, using the parameter values given in Table 4 with $B_1 = 70$, $B_2 = 20$

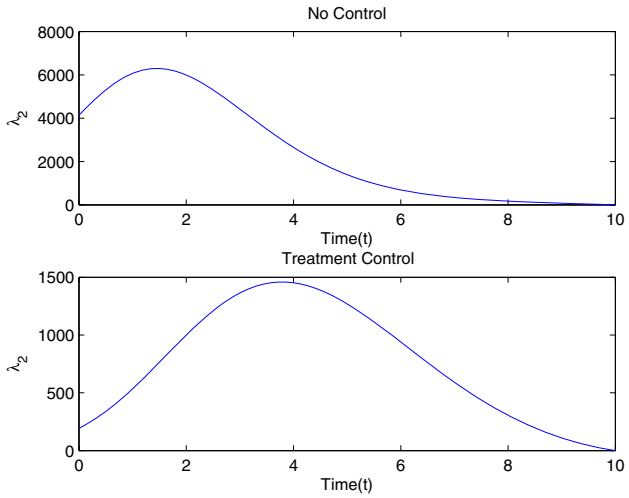


Fig. 10 The control diagrams for adjoint variable λ_2, using the parameter values given in Table 4 with $B_1 = 70$, $B_2 = 20$

clearly conclude that the adjoint variables are directly related to the change of the value of the Hamiltonian H. Also, in these figures it is observed that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are decreasing slowly to zero as time increases. These phenomena ensures that to get the minimum value of the objective functional J, the rate of change of Hamiltonian H must increase with the changes of the state variables (S, D, T, R).

9 Discussions and conclusions

In this paper we have considered an alcohol abuse consisting of four population classes, namely moderate and occasional drinkers (S), heavy drinkers (D), drinkers in treatment (T) and temporarily recovered class (R) and investigated the dynamical behaviour of this model. We have found

$$R_0 = \frac{\Lambda\beta_1}{\mu(\mu + \delta_1 + \gamma)}$$

as basic reproduction number of the system (2.1), which helps us to determine the dynamical behaviour of the system. Sensitivity analysis of R_0 identifies β_1, the transmission coefficient from moderate and occasional drinker to heavy drinker, as the most useful parameter to R_0. Then we have discussed the stability analysis of the model using the basic reproduction number. The system (2.1) is locally as well as globally asymptotically stable at disease free equilibrium (DFE) E_0 when $R_0 < 1$ under some conditions. When $R_0 > 1$, the endemic equilibrium E^* exists and the system becomes unstable at E_0 and locally asymptotically stable at E^*. The system can undergo a backward bifurcation at $R_0 = 1$. In this case, when $R_0 < 1$ a positive unstable endemic equilibrium (E_1) appears

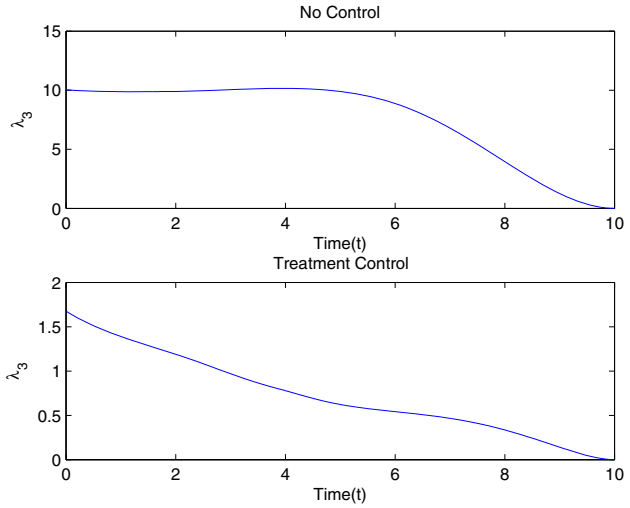


Fig. 11 The control diagrams for adjoint variable λ_3, using the parameter values given in Table 4 with $B_1 = 70$, $B_2 = 20$

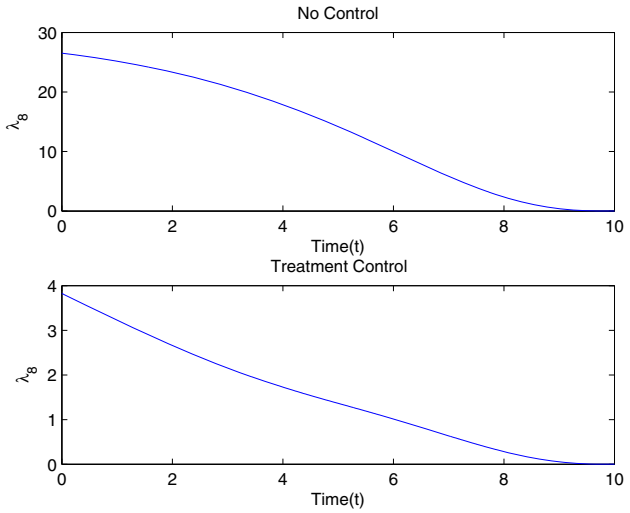


Fig. 12 The control diagrams for adjoint variable λ_4, using the parameter values given in Table 4 with $B_1 = 70$, $B_2 = 20$

while a drinking free or disease free equilibrium DFE (E_0) and another positive endemic equilibrium (E^*) are locally asymptotically stable. These two endemic equilibrium disappear when the basic reproduction number R_0 is decreased below the critical value $R_0^c < 1$.

The next focus of this paper is to set up an optimal control problem relative to the alcohol abuse epidemic model so as to minimize the heavy drinkers as well as to minimize the cost of treatment. We have considered treatment rate as a function of time by $u(t)$. So, $u(t)$ is the rate of heavy drinkers entering in the treatment per unit time at time t. Thus u is representing treatment control in the drinking model. The control function u is designed in such a way that it minimizes the objective functional (cost function) J as given in (7.3).

The important mathematical findings for the dynamical behaviour of the alcohol abuse model are also numerically verified using MATLAB. The dynamical behaviour of system (2.1) for $R_0 < 1$ and $R_0 > 1$ are clearly depicted through those numerical simulations. The situations in which backward and forward bifurcations may occur are also discussed very nicely through graphical representations. A comparison among the dynamical behaviour of heavy drinkers with no control and treatment control is presented graphically, which shows the effectiveness of the treatment control very clearly. It is observed that the optimal control is much more effective for reducing the number of heavy drinkers which implies that the treatment controls the spread of the drinking habit in population successfully. It can also be noticed that treatment programme is very important at the beginning of the outbreak to control the spread of the alcoholism.

Our model is not a case study. So, it is difficult to choose parameter values from quantitative estimation. We have used hypothetical sets of parameter values to verify our analytical results. Our analysis and simulations using these parameter values satisfy our analytical results and indicate that the optimal control is efficient to reduce the spread of the alcoholism.

The mathematical model presented in this paper shows backward bifurcation which occurs because of the insufficient capacity for treatment policies. As a result the drinkers in treatment come to the direct contact of heavy drinkers and they re-enter into the heavy drinkers' class. It shows that the prevention and control efforts that include treatment and education programmes should be improved so that the rate of relapse from treatment can be reduced. Alcoholism is one of the most common psychiatric disorders with a prevalence. The incidence of alcoholism is still more common in men. But it has been increasing in women. The female to male ratio for alcohol dependence has narrowed to one to two [3]. So an important effect that we want to include in our future work is the male/female distribution of alcohol abusers.

The mathematical model presented in this paper should be treated with circumspection like other papers due to the assumptions made and the difficulty in the estimation of the model parameters. Parameters are dependent on the environmental conditions, so they are rarely constant. But for the simplification of our model we have assumed that these parameters are constant. There may be a time lag as a moderate and occasional drinker may take some time to become heavy drinker and also a heavy drinker may take some time to become recovered through treatment. Therefore, as a part of the future work, the model considered here can be refined to incorporate time delays in the system to make it more realistic. Controlling the spread of alcoholism is now a challenging and important issue to study. So, to predict and identify the cost-effective strategies to control the spread of the habit of drinking and minimize the cost of the control programme are the primary goal of health administrators, policy-makers and

researchers. Our model study is a small step towards the goal by which we want to identify the parameters of interest for further study.

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