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ANALYSIS OF A RECTANGULAR WAVEGUIDE USING FINITE ELEMENT METHOD

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Abstract—The characteristics impedance of the fundamental mode in a rectangular waveguide is computed using finite element method. The method is validated by comparison with the theoretical results. In addition to this, we have considered the problem of determining the modes of propagation of electromagnetic waves in a rectangular waveguide for the simple homogeneous dielectric case. The starting point is Maxwell's equations with an assumed exponential dependence of the fields on the Z-coordinates. From these equations we have arrived at the Helmholtz equation for the homogeneous case. Finite-element-method has been used to derive approximate values of the possible propagation constant for each frequency.

1. INTRODUCTION

The finite element method (FEM) has been widely used during the last two decades in the analysis of waveguide components. With this method propagation characteristics of arbitrarily shaped waveguides of different composition are easily attainable. The finite element method is based on a spatial discretization [1]. This approximation allows one to handle waveguide cross section geometries which are very similar to the real structures employed in practical devices. These complex structures do not lend themselves to analytical solutions. As a consequence, the FEM constitutes a promising tool to characterize such problems [2, 11]. Modern phased array radars imply the requirements for polarization agility of wideband array elements. One possible choice for a radiating element with this property is the rectangular waveguide. In this paper a formulation is proposed to solve waveguides problems. A numerically efficient finite element formulation is presented that shows propagation modes and which may be used to analyze problems involving linear systems of arbitrary complex tensor permittivity and permeability. The solution of these eigenvalue problems results in the approximate fields for all components of different eigenmodes in the waveguide which can further be used to obtain the corresponding eigenvalues [2]. A possible comparison of the proposed methodology with the available theoretical results has also been presented here in the paper to clear the accuracy and reliability of the solution method.

2. THE FINITE ELEMENT FORMULATION

In this paper, the rectangular cross section of the waveguide is divided into a number of finite elements. An element is considered to be first order triangular in shape [3]. An schematics of a triangular finite element in the rectangular waveguide is shown in Figure 1. Consider

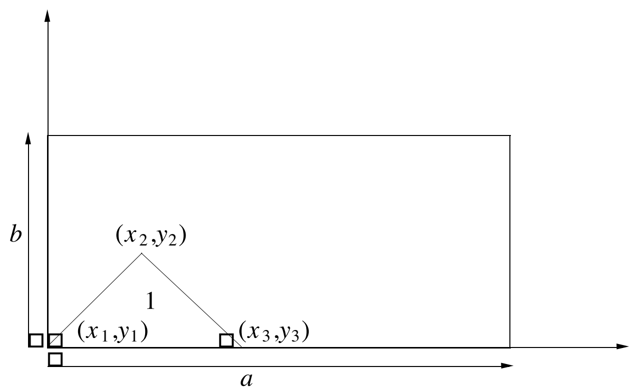

Figure 1. Schematic representation of a triangular finite element.

a triangle having vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Two vectors, $\overrightarrow{u}$ and $\overrightarrow{v}$ has been drawn by joining the vertices $[(x_1, y_1), (x_2, y_2)]$ and $[(x_1, y_1)$ and $(x_3, y_3)]$, respectively. Let

$$d_1 = |u| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

and

$$d_2 = |v| = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \tag{2}$$

The unit vector along the two directions u and v are

$$\hat{u} = \frac{u}{|u|} = \frac{(x_2 - x_1, y_2 - y_1)}{d_1} \tag{3}$$

and

$$\hat{v} = \frac{v}{|v|} = \frac{(x_3 - x_1, y_3 - y_1)}{d_2} \tag{4}$$

any point $(x,\, y)$ inside this triangle can be represented as

$$\begin{aligned}(x, y) &= (x_1, y_1) + u \cdot \hat{u} + v \cdot \hat{v} \\ &= (x_1, y_1) + \frac{u(x_2 - x_1, y_2 - y_1)}{d_1} + \frac{v(x_3 - x_1, y_3 - y_1)}{d_2}\end{aligned}$$

so

$$(x - x_1) = \frac{u(x_2 - x_1)}{d_1} + \frac{v(x_3 - x_1)}{d_2} \tag{5}$$

and

$$(y - y_1) = \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2} \tag{6}$$

These are two linear equations for the variable u and v and solving them gives us u, v as linear functions of $x,\, y$. The area measure is given by

$$ds(u, v) = |\vec{u} \times \vec{v}|\, du \cdot dv$$

where

$$|\vec{u} \times \vec{v}| = \sin(\alpha)$$

here angle α between the vectors u and v defined as

$$\begin{aligned}\cos(\alpha) &= \frac{u \cdot v}{d_1 \cdot d_2} \\ &= \frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1)}{d_1 \cdot d_2}\end{aligned} \tag{7}$$

The integral of a function can be evaluated as

$$\begin{aligned}I(\phi) &= \frac{1}{2} \int_0^{d_1} \int_0^{d_2} \phi \left[x_1 + \frac{u(x_2 - x_1)}{d_1} + \frac{v(x_3 - x_1)}{d_2},\right. \\ &\qquad \left. y_1 + \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2}\right] \sin\alpha \cdot dudv\end{aligned} \tag{8}$$

if $\phi = 1$ then we get

$$I(1) = \frac{d_1 \cdot d_2 \sin\alpha}{2} \tag{9}$$

which represents the area of the triangle. Suppose we write

$$V(x, y) = ax + by + c$$

for

$$x, y \in \Delta$$

with Δ as the area bounded by the triangle. a, b, c are chosen so that V at the vertices are given, i.e.,

$$\begin{aligned} V(x_1, y_1) &= V_1 \\ V(x_2, y_2) &= V_2 \\ V(x_3, y_3) &= V_3 \end{aligned}$$

Thus,

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

Thus we find that

$$a = \frac{V_1(y_2 - y_3) + V_2(y_3 - y_1) + V_3(y_1 - y_2)}{\Delta} \tag{10}$$

$$b = \frac{V_1(x_2 - x_3) + V_2(x_3 - x_1) + V_3(x_1 - x_2)}{\Delta} \tag{11}$$

$$c = \frac{V_1(x_2 \cdot y_3 - x_3 \cdot y_2) + V_2(x_3 \cdot y_1 - x_1 y_3) + V_3(x_1 \cdot y_2 - x_2 \cdot y_1)}{\Delta} \tag{12}$$

where

$$\Delta = x_2 \cdot y_3 - x_3 \cdot y_2 + x_3 \cdot y_1 - x_1 y_3 + x_1 \cdot y_2 - x_2 \cdot y_1 \tag{13}$$

thus for

$$x, y \in \Delta$$

we have

$$\begin{aligned} V(x, y) &= ax + by + c \\ &= V_1 \phi_1(x, y) + V_2 \phi_2(x, y) + V_3 \phi_3(x, y) \\ \phi_1(x, y) &= \frac{(y_2 - y_3)x + (x_2 - x_3)y + (x_2 \cdot y_3 - x_3 \cdot y_2)}{\Delta} && (14) \\ \phi_2(x, y) &= \frac{(y_3 - y_1)x + (x_3 - x_1)y + (x_3 \cdot y_1 - x_1 \cdot y_3)}{\Delta} && (15) \\ \phi_3(x, y)) &= \frac{(y_1 - y_2)x + (x_1 - x_2)y + (x_1 \cdot y_2 - x_2 \cdot y_1)}{\Delta} && (16) \end{aligned}$$

The following two integrals occur when one uses the finite element method

first

$$I_1 = \int_\Delta V^2_{(x,y)} dx \cdot dy$$

second

$$I_2 = \int_\Delta |\nabla V|^2 dx \cdot dy$$

Now

$$I_1 = \int_\Delta V^2_{(x,\,y)} dx \cdot dy = \int_\Delta (ax + by + c)^2 dx \cdot dy \tag{17}$$

By substituting the value of $(x,\, y)$ in terms of $(x_i,\, y_i)$ in Equation (17), we get

$$\begin{aligned} I(\phi) \;=\; & \frac{\sin\alpha}{2} \int_0^{d_1}\int_0^{d_2} \left[a\left(x_1 + \frac{u(x_2-x_1)}{d_1} + \frac{v(x_3-x_1)}{d_2}\right)\right. \\ & \left. +b\left(y_1 + \frac{u(y_2-y_1)}{d_1} + \frac{v(y_3-y_1)}{d_2}\right) + c\right]^2 dudv \end{aligned} \tag{18}$$

The use of method of variable separation for u and v results in the following.

$$\begin{aligned} I(\phi) \;=\; & \frac{\sin\alpha}{2} \int_0^{d_1}\int_0^{d_2} \left[u\left(\frac{a(x_2-x_1)+b(y_2-y_1)}{d_1}\right)\right. \\ & \left. +v\left(\frac{a(x_3-x_1)+b(y_3-y_1)}{d_2}\right) + c'\right]^2 \end{aligned} \tag{19}$$

where

$$c' = ax_1 + by_1 + c$$

Equation (19) can be written as

$$I(\phi) = \frac{\sin\alpha}{2}\int_0^{d_1}\int_0^{d_2}\Big[C_1u^2 + C_2uv + C_3v^2 + C_4u + C_5v + C_6\Big]dudv \tag{20}$$

Here

$$C_1 \;=\; \left[\frac{a(x_2-x_1)+b(y_2-y_1)}{d_1}\right]^2 \tag{21}$$

$$C_2 \;=\; 2\left[\frac{a(x_2-x_1)+b(y_2-y_1)}{d_1}\right]\left[\frac{a(x_3-x_1)+b(y_3-y_1)}{d_2}\right] \tag{22}$$

$$C_3 \;=\; \left[\frac{a(x_3-x_1)+b(y_3-y_1)}{d_2}\right]^2 \tag{23}$$

$$C_4 = \left[\frac{2C'(a(x_2 - x_1) + b(y_2 - y_1))}{d_1}\right] \quad (24)$$

$$C_5 = \left[\frac{2C'(a(x_3 - x_1) + b(y_3 - y_1))}{d_2}\right] \quad (25)$$

$$C_6 = C'^2 \quad (26)$$

Also note that

$$\int_0^{d_1}\int_0^{d_2} u^2 du dv = \frac{d_1^3 \cdot d_2}{3} \quad (27)$$

$$\int_0^{d_1}\int_0^{d_2} v^2 dv = \frac{d_1 \cdot d_2^3}{3} \quad (28)$$

$$\int_0^{d_1}\int_0^{d_2} u \cdot v du dv = \frac{d_1^2 \cdot d_2^2}{4} \quad (29)$$

and finally

$$\int_0^{d_1}\int_0^{d_2} du dv = d_1 \cdot d_2 \quad (30)$$

We also have

$$\nabla V(x,y) = (a,b) \quad (31)$$

and hence

$$I_2 = \int_\Delta |\nabla V|^2 dx dy = \frac{(a^2+b^2)\, d_1 \cdot d_2 \sin\alpha}{2} \quad (32)$$

Following the procedure stated above, these two integrals (I_1 and I_2) for each element are evaluated. We will find the summation of these integrals up to the number of elements in which we have divided the cross-section.

Characteristic power-voltage impedance Z can be defined as follows:

$$Z = \frac{U^2}{2P} \quad (33)$$

where P is the power carried along the waveguide and U is the voltage across the waveguide.

3. DETERMINATION OF EIGENVALUES OF THE MATRIX

The eigenvalues of the matrix are obtain as follows.

$$\left(V^T A V - k^2 V^T B V\right)$$

when minimized over V gives the quadratic form defined by

$$\int_{\Delta} |\nabla V|^2 dxdy - k^2 \int_{\Delta} V^2 dxdy \tag{34}$$

here

$$\delta \int_{\Delta} \left(\nabla \vec{V}, \nabla \vec{V}\right) dxdy = 2 \int_{\Delta} \left(\vec{\nabla}, \delta \vec{\nabla} V\right) dxdy = -2 \int \delta V \nabla^2 V dxdy \tag{35}$$

and

$$\delta \int_{\Delta} V^2 dxdy = \int 2V \delta V dxdy \tag{36}$$

From Equation (33)

$$-2 \int \delta V \nabla^2 V dxdy - 2k^2 \int \delta V \cdot V dxdy = 0 \tag{37}$$

or

$$\int \delta V \left(\nabla^2 + k^2\right) V dxdy = 0 \tag{38}$$

$$\left(\nabla^2 + k^2\right) V dxdy = 0 \tag{39}$$

An approximation of Equation (39) using the fem gives

$$AV - k^2 BV = 0 \tag{40}$$

$$\left(A - k^2 B\right) V = 0 \tag{41}$$

$$|A - k^2 B| = 0 \tag{42}$$

Here V is the vector of vertex nodal field values. Solution of this matrix will give the eigen values. These eigen values are the propagation modes of the waveguide. The above proposed method can be used to calculate the modes of a waveguide of any type of cross-section. This calculation procedure has been validated to present validity, accuracy and reliability of the solution, in the ensuing section.

4. SIMULATION RESULTS

In order to validate the procedure, the computed result is compared with those obtained from the theoretical analysis. Table 1 compares the eigenvalues of the rectangular waveguide for a range of rectangular cross-section. Also shown in the table are the value of characteristics impedance (z).

Table 1. Eigenvalue and characteristic impedance of the fundamental mode of the rectangular waveguide.

b/a	eigen value (Present)	eigen value (Theoretical)	characteristic impedance (z)
.1	1.266	1.1368	56.28
.2	1.6748	1.3475	97.42
.3	1.9717	1.8975	136.72
.4	2.2168	2.2834	173.78
.5	2.4288	2.4432	207.84
.6	2.7853	2.7582	278.44
.8	3.5839	2.9342	308.65
.9	3.8432	3.1502	342.43
1.0	3.6599	3.5422	401.87

5. CONCLUSION

In this paper an advantageous finite element method for the rectangular waveguide problem has been developed by which complex propagation characteristics may be obtained for arbitrarily shaped waveguide. The extension to higher order elements is straightforward, and by modifications of the method it is possible to treat other types of waveguides as well, e.g., dielectric waveguides with impedance walls and open unbounded dielectric waveguides properly treating the region of infinity.

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