ANALYSIS OF BINARY NUMBER ASSOCIATION SCHEME PARTIALLY BALANCED DESIGNS[1]

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1. SUMMARY AND INTRODUCTION

This paper is concerned with the application of the Properties A and B associated with the incidence matrix to the analysis of Partially Balanced Designs having Binary Number Association Scheme (BNAS) or Balanced Factorial Experiments (BFE).

We present a practical method of intra-and inter-block analysis of Partially Balanced Block Designs (PBBD) having BNAS. All group divisible association scheme designs, rectangular association scheme designs, hierarchical group divisible block designs, and direct product designs are BNAS PBB designs. Thus, the method presented in this paper will unify and simplify the calculations for the above various designs when compared with presently available procedures. Also, we extend the method to the PAB type rectangular designs. The various steps in the computation in the analysis are presented with numerical examples.

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2. PRELIMINARIES

The following notation will be used:

$\underset{\sim}{1}_{m_i}$: $m_i \times 1$ column vector having all elements unity,

$J_{m_i} = \underset{\sim}{1}_{m_i}\underset{\sim}{1}'_{m_i}$: $m_i \times m_i$ matrix with all elements unity,

I_{m_i} : $m_i \times m_i$ identity matrix,

$$D_i^{\delta_i} = \begin{cases} I_{m_i} & \text{if } \delta_i = 0 \\ J_{m_i} & \text{if } \delta_i = 1 . \end{cases}$$

The direct product or Kronecker product of $D_i^{\delta_i}$ and $D_j^{\delta_j}$ will be written as $D_i^{\delta_i} \otimes D_j^{\delta_j}$ and in general, the joint direct product of n $D_i^{\delta_i}(i=1,2,\cdots,n)$ will be written as $\prod_{i=1}^{n} \otimes D_i^{\delta_i}$.

Let there be v treatments, each replicated r times in b blocks of k plots each. Let $N = \|n_{ij}\|$, $i=1,2,\cdots,v$; $j=1,2,\cdots,b$, be the incidence matrix of the design, where n_{ij} is equal to the number of times the i^{th} treatment occurs in the j^{th} block. The set up assumed is

$$y_{ij} = \mu + t_i + \beta_j + \epsilon_{ij} , \tag{2.1}$$

where y_{ij} is the yield of the plot in the j^{th} block to which the i^{th} treatment is applied, μ is the overall effect, t_i is the effect of the i^{th} treatment, β_j is the effect of the j^{th} block, and ϵ_{ij} is the experimental error. The effects μ, t_i, and β_j are assumed to be fixed constants, while the errors $\{\epsilon_{ij}\}$ are assumed to be independent normal variates with mean zero and variance σ^2 (in section 4.2, the block effects $\{\beta_j\}$ will be assumed to be independent random variates with mean zero and variance σ_β^2).

Let T_i be the total yield of all plots having the i^{th} treatment, B_j be the total yield of all plots of the j^{th} block, and $\hat{t}_i$ be a solution for t_i in the normal equations. Further, we denote the column vectors with elements

$(T_1, T_2, \cdots, T_v)$, $(B_1, B_2, \cdots, B_b)$, $(t_1, t_2, \cdots, t_v)$, and $(\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v)$ by $\underline{T}$, $\underline{B}$, $\underline{t}$, and $\hat{\underline{t}}$ respectively. It is well known that the reduced normal equations for intra-block estimates of the treatment effects are

$$C\hat{\underline{t}} = \underline{Q} , \tag{2.2}$$

where

$$C = rI_v - \frac{1}{k} NN' , \tag{2.3}$$

and

$$\underline{Q} = \underline{T} - \frac{1}{k} N\underline{B} . \tag{2.4}$$

The matrix C defined in (2.3) will be called the C matrix of the design. The solution of (2.2) is

$$\hat{\underline{t}} = C^{+}\underline{Q} , \tag{2.5}$$

where C^{+} is a generalized inverse of C .

Consider a factorial experiment with n factors $F_1, F_2, \cdots, F_n$, where F_i has m_i levels for $i=1,2,\cdots,n$, there being $v = \prod_{i=1}^{n} m_i$ treatments. Kurkjian and Zelen [1963] introduced a structural property of the design which is related to the block (or column) incidence matrix N of the design. This structural property was termed Property A and was defined as follows:

A block design will be said to have Property A or will be called a PA type block design if

$$NN' = \sum_{s=0}^{n} \Big\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n} h(\delta_1, \delta_2, \cdots, \delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i} \Big\} . \tag{A}$$

where $\delta_1 = 0$ or 1 for $i=1,2,\cdots,n$, and $h(\delta_1, \delta_2, \cdots, \delta_n)$ are constants.

In this case, we obtain the following solution for equation (2.2):

$$\hat{\underline{t}} = \sum_{s=0}^{n-1} \Big\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n=s} u(\delta_1, \delta_2, \cdots, \delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i} \Big\} \underline{Q} , \tag{2.6}$$

where

$$u(\delta_1,\delta_2,\cdots,\delta_n) = \frac{1}{v}\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s}\frac{(-1)^{\Sigma\delta_i x_i}\prod_{i=1}^{n}(m_i x_i)^{1-\delta_i}}{r\theta(x_1,x_2,\cdots,x_n)}\right\} . \qquad (2.7)$$

where

$$r\theta(x_1,x_2,\cdots,x_n) = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}E_i(x_i\delta_i)\right\}$$

for $g(0,0,\cdots,0) = r - \frac{1}{k}h(0,0,\cdots,0)$, $g(\delta_1,\delta_2,\cdots,\delta_n) = -\frac{1}{k}h(\delta_1,\delta_2,\cdots,\delta_n)$ if $(\delta_1,\delta_2,\cdots,\delta_n) \neq 0$ and $\neq (1,1,\cdots,1)$, and the values of $E_i(x_i,\delta_i)$ are given by the table

		δ_i = 0	δ_i = 1
x_i	0	1	m_i
	1	1	0

(2.9)

Finally, since $\text{Cov}(\underline{Q}) = C\sigma^2$, we obtain

$$\text{Cov}(\hat{\underline{t}}) = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} u(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n} \otimes D_i^{\delta_i}\right\}\sigma^2 . \qquad (2.10)$$

Shah [1960] considered the following association scheme:

In a balanced factorial experiment with n factors $F_1,F_2,\cdots,F_n$ at $m_1,m_2,\cdots,m_n$ levels respectively, the two treatments are the $(p_1,p_2,\cdots,p_n)^{th}$ associates, where p_i=1 if the i^{th} factor occurs at the same level in both the treatments and p=0 otherwise; $\lambda_{p_1 p_2 \cdots p_n}$ will denote the number of times these treatments occur together in a block. Then we have

$$n_{p_1 p_2 \cdots p_n} = \prod_{i=1}^{n} (m_i - 1)^{1-p_i} , \tag{2.11}$$

the number of $(p_1, p_2, \cdots, p_n)^{th}$ associates.

The association scheme could be called a "binary number association scheme (BNAS)". Note: PBIB designs having BNAS are EGD/$(2^n-1)^-$ PBIB designs as defined by Hinkelmann [1964].

In the above BFE, suppose that the model (2.1) is assumed. Let $\theta_q = \theta(q_1, q_2, \cdots, q_n)$, where $q = \sum_{h=1}^{n} q_h 2^{n-h}$ for $q_i=0$ or 1, be the efficiency factor associated with the estimate of generalized interaction $F_1^{q_1} F_2^{q_2} \cdots F_n^{q_n}$ and $\lambda_p = \lambda_{p_1 p_2 \cdots p_n}$ such that $p = \sum_{h=1}^{n} p_h 2^{n-h}$ for $p_i=0$ or 1, then Shad [1958] obtained the following relation:

$$r \prod_{i=1}^{n} \otimes F(m_i) \underline{\theta} = \frac{1}{k} \underline{\lambda} . \tag{2.12}$$

where $F(m_i) = \frac{1}{m_i} \begin{bmatrix} 1 & -1 \\ 1 & m_i - 1 \end{bmatrix}$, $\underline{\theta} = (\theta_0, \theta_1, \cdots, \theta_m)'$, $\underline{\lambda} = (\lambda_0, \lambda_1, \cdots, \lambda_m)'$, $m = 2^n - 1$, and $\theta_0 = 0$, $\lambda_m = r - rk$, and conversely

$$\underline{\theta} = -\frac{1}{rk} \prod_{i=1}^{n} \otimes G(m_i) \underline{\lambda} , \tag{2.13}$$

where $G(m_i) = \begin{bmatrix} m_i - 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Kshirsager [1966] and Paik and Federer [1973b] have proved the following theorems:

Theorem 2.1 *Every PA type block design is a BFE, and conversely.*

Theorem 2.2 *Every Balanced Factorial Incomplete Block Design is a PBIB with BNAS and conversely.*

In the classical BIB design or PBIB design, no treatment appears more than once in a block. However, we may wish to apply some treatments more than once in a block. In such cases, the n-ary partially balanced block (NPBB) designs may be useful for application (see Tocher [1952] and Paik and Federer [1973b]).

Paik and Federer [1973b] defined a BNAS for an NPBB design having $v = \prod_{i=1}^{n} m_i$ treatments applied in b blocks of k plots of each. In a factorial system of n factors $F_1, F_2, \cdots, F_n$ at $m_1, m_2, \cdots, m_n$ levels respectively, the two treatments are the $(p_1, p_2, \cdots p_n)^{th}$ associates, where $p_i = 1$, if the i^{th} factor occurs at the same level in both treatments and $p_i = 0$ otherwise; $\lambda_{p_1 p_2 \cdots p_n}$ will denote the number of times these treatments occur together in the same blocks. Suppose two treatments t and t' are $(p_1, p_2, \cdots p_n)^{th}$ associates and these treatments are replicated r_{tj} and $r_{t'j}$ times in the j^{th} block respectively, then

$$\lambda_{p_1 p_2 \cdots p_n} = \begin{cases} r^* - rk \text{ if } (p_1, p_2, \cdots, p_n) = (1, 1, \cdots, 1) \\ \sum_{j=1}^{b} t_{tj} r_{t'j} \text{ otherwise} \end{cases}$$

where $r^* = \sum_j r_{tj}^2$, $r = \sum_j r_{tj}$.

If $\lambda_{p_1 p_2 \cdots p_n}$ does not depend upon a particular pair of $(p_1, p_2, \cdots, p_n)^{th}$ associates and if r is a constant for $t = 1, 2, \cdots, v$, then the above block design is an NPBB design with respect to the BNAS. Paik and Federer [1973b] concluded the following:

Theorem 2.3 Any NPBB design having BNAS is a BFE and is a PA type block design, and conversely.

Example 2.1 Consider the following block design $v = 2 \times 3$, $r = 4$, $k = 8$, and $b = 3$:

block 1,	1,	4,	2,	3,	5,	6,	1,	4
block 2,	2,	5,	1,	5,	3,	4,	6,	2
block 3,	3,	6,	6,	4,	1,	2,	5,	3

where 1 = (0,0), 2 = (0,1), 3 = (0,2), 4 = (1,0), 5 = (1,1), and 6 = (1,2).

This design is an NPBB design with $\lambda_{00} = 5$, $\lambda_{01} = 6$, $\lambda_{10} = 5$, and $\lambda_{11} = -26$; also it is a BFE and is a PA type block design. In this design, for example, treatment 1 = (0,0) is the $(1,0)^{th}$ associate with treatments 2 = (0,1) and 3 = (0,2); $(0,1)^{th}$ associate with treatment 4 = (1,0); and $(0,0)^{th}$ associate with treatments 5 = (1,1) and 6 = (1,2). Since $r_{1,1} = 2$, $r_{1,2} = 1$, $r_{1,3} = 1$ and $r_{2,1} = 1$, $r_{2,2} = 2$, $r_{2,3} = 1$, then $\lambda_{10} = 5$; also since $r_{4,1} = 2$, $r_{4,2} = 1$, $r_{4,3} = 1$, then $\lambda_{01} = 6$, and lastly, since $r_{5,1} = 1$, $r_{5,2} = 2$, $r_{5,3} = 1$, then $\lambda_{00} = 5$. Also, we obtain $r^* = 6$ and $\lambda_{11} = r^* - rk = 6 - 32 = -26$.

3. EXAMPLES OF PBB DESIGNS HAVING BNAS

A rectangular association scheme is defined for $v = m_1 m_2$ treatments as a rectangle with m_1 rows and m_2 columns, with first associates in the same row, second associates in the same column and all other pairs being third associates. In this case, if we denote $v = m_1 m_2$ treatments by (i_1, i_2), where $i_j = 0, 1, \cdots, m_j - 1$, then using the notation of BNAS,

$$n_{10} = m_2 - 1, \quad n_{01} = m_1 - 1, \quad \text{and} \quad n_{00} = (m_1 - 1)(m_2 - 1) \ ,$$

and we have λ_{10}, λ_{01}, and λ_{00} as the first, second and third associates, respectively.

From (2.13)

$$\underline{\theta} = \frac{1}{rk} \begin{bmatrix} m_1 - 1 & 1 \\ -1 & 1 \end{bmatrix} \otimes \begin{bmatrix} m_2 - 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{\lambda} \ , \qquad (3.1)$$

where $\underline{\theta} = \big(\theta(0,0), \ \theta(0,1), \ \theta(1,0), \ \theta(1,1)\big)'$ and $\underline{\lambda} = (\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11})'$, i.e.,

$$\begin{aligned} \theta(0,0) &= 0 \ , \\ \theta(0,1) &= \frac{1}{rk}\Big[(m_1 - 1)(\lambda_{00} - \lambda_{01}) + \lambda_{10} + r(k-1)\Big] \ , \\ \theta(1,0) &= \frac{1}{rk}\Big[(m_2 - 1)(\lambda_{00} - \lambda_{10}) + \lambda_{01} + r(k-1)\Big] \ , \\ \theta(1,1) &= \frac{1}{rk}\Big[\lambda_{01} + \lambda_{10} - \lambda_{00} + r(k-1)\Big] \ . \end{aligned} \qquad (3.2)$$

In a rectangular association scheme, if $m_1 = m_2$, and $\lambda_{10} = \lambda_{01}$, treatments either in the same row or in the same column, while all other pairs are second associates $\lambda_{00} \neq \lambda_{10}$, we obtain the Latin square (L_2) type association scheme.

On the other hand, if we let two treatments in the same row in a rectangular association be first, and any other two be second associates, then we obtain a so-called "group divisible" association scheme. In this case,

$$\lambda_{10},\ \lambda_{01} = \lambda_{00}, \quad \text{but} \quad \lambda_{10} \neq \lambda_{01} \ ,$$

$$n_{10} = m_2-1,\ n_{01} = m_1-1,\ n_{00} = (m_1-1)(m_2-1) \ ,$$

so, we obtain

$$\theta(0,1) = \frac{1}{rk}\left[r(k-1) + \lambda_{10}\right] = \theta(1,1)$$

$$\theta(1,0) = \frac{1}{rk}\left[r(k-1) + \lambda_{10} - m_2(\lambda_{10} - \lambda_{00})\right]$$

$$= \frac{1}{rk} m_1 m_2 \lambda_{01} \ ,$$

since $n_{10}\lambda_{10} + n_{01}\lambda_{01} + n_{00}\lambda_{00} = r - rk$.

We obtain yet another association scheme introduced by Roy (1953-4) if we denote $v = \prod_{i=1}^{n} m_i$ treatments by $(i_1, i_2, \cdots, i_n)$, where $i_j = 0, 1, \cdots, m_j-1$. Let the first associates be two treatments with all but the last subscripts equal, and generally let $(n-i)^{th}$ associates be those with the first i subscripts equal, but the $(i+1)^{th}$ ones different.

In this case, we have

$$\lambda_{11\cdots10}$$

$$\lambda_{11\cdots100} = \lambda_{11\cdots101}$$

$$\lambda_{11\cdots1000} = \lambda_{11\cdots1001} = \lambda_{11\cdots1010} = \lambda_{11\cdots1011}$$

$$\cdots\cdots\cdots$$

$$\lambda_{100\cdots0} = \lambda_{10\cdots01} = \cdots = \lambda_{101\cdots1}$$

$$\lambda_{00\cdots0} = \lambda_{00\cdots01} = \cdots = \lambda_{011\cdots1}$$

General examples of PBIB design having BNAS are partially confounded factorial experiments.

Example 3.5 — $v = 2 \times 2 \times 2$, $k = 4$, $b = 6$, $r = 3$.

	b_1	b_2	b_3	b_4	b_5	b_6
	1	5	1	3	1	2
	2	6	2	4	3	4
	3	7	5	7	5	6
Effect confounded	4	8	6	8	7	8
	A		B		C	

Treatment no. $= 4i_1 + 2i_2 + i_3 + 1$. In this case,

$$NN' = \begin{bmatrix} 3&2&2&1&2&1&1&0 \\ 2&3&1&2&1&2&0&1 \\ 2&1&3&2&1&0&2&1 \\ 1&2&2&3&0&1&1&2 \\ 2&1&1&0&3&2&2&1 \\ 1&2&0&1&2&3&1&2 \\ 1&0&2&1&2&1&3&2 \\ 0&1&1&2&1&2&2&3 \end{bmatrix} = I_2 \otimes I_2 \otimes J_2 + J_2 \otimes I_2 \otimes J_2 + J_2 \otimes J_2 \otimes I_2 \quad .$$

This design is a PA type incomplete block design and PBIB design having BNAS:

$$\lambda_{110} = 2, \quad \lambda_{101} = 2, \quad \lambda_{100} = 1, \quad \lambda_{011} = 2 \; ,$$

$$\lambda_{010} = 1, \quad \lambda_{001} = 1, \quad \lambda_{000} = 0 \quad .$$

From (2.13) or directly from (2.8)

$$\theta(001) = 8/_{12}, \quad \theta(010) = 8/_{12}, \quad \theta(011) = 1$$

$$\theta(100) = 8/_{12}, \quad \theta(101) = 1, \quad \theta(110) = 1, \quad \theta(111) = 1 \quad .$$

NOTE: Triangular and cyclic PBIB designs are not PBIB designs having BNAS.

4. ANALYSIS OF PBB DESIGNS HAVING BNAS

4.1 Intra-block analysis

The notation $u(\delta_1, \delta_2, \cdots, \delta_n)$ defined in section 2 can be expressed as follows:

$$\underline{u} = \frac{1}{rv} \left[\prod_{i=1}^{n} \otimes H(m_i) \right] \underline{\theta}^{-1} \; , \qquad (4.1)$$

where, $\underline{u} = \Big(u(0,0,\cdots,0),\ u(0,0,\cdots,1),\ \cdots,\ u(1,1,\cdots,1)\Big)'$,

$$H(m_i) = \begin{bmatrix} 0 & m_i \\ 1 & -1 \end{bmatrix}, \qquad (4.2)$$

and $\underline{\theta}^{-1} = \Big(0, \theta^{-1}(0,0,\cdots,1),\ \cdots,\ \theta^{-1}(1,1,\cdots,1)\Big)'$.

For example, if $v = m_1 m_2$

$$\begin{bmatrix} u(0,0) \\ u(0,1) \\ u(1,0) \\ u(1,1) \end{bmatrix} = \frac{1}{rv} \begin{bmatrix} 0 & 0 & 0 & m_1 m_2 \\ 0 & 0 & m_1 & -m_1 \\ 0 & m_2 & 0 & -m_2 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \theta^{-1}(0,0) \\ \theta^{-1}(1,0) \\ \theta^{-1}(1,1) \end{bmatrix} = \begin{bmatrix} \frac{1}{r}\frac{1}{\theta(1,1)} \\ \frac{1}{rm_2}\left(\frac{1}{\theta(1,0)} - \frac{1}{\theta(1,1)}\right) \\ \frac{1}{rm_1}\left(\frac{1}{\theta(0,1)} - \frac{1}{\theta(1,1)}\right) \\ \frac{1}{rm_1 m_2}\left(-\frac{1}{\theta(0,1)} - \frac{1}{\theta(1,0)} + \frac{1}{\theta(1,1)}\right) \end{bmatrix} . \quad (4.3)$$

Then,

$$\begin{aligned} \hat{\underline{t}} &= \left[u(0,0)D_1^0 \otimes D_1^0 + u(0,1)D_1^0 \otimes D_2^1 + u(1,0)D_1^1 \otimes D_2^0\right] \underline{Q} \\ &= \frac{1}{r\theta(1,1)} I_{m_1 m_2} \underline{Q} + \frac{1}{rm_2}\left(\frac{1}{\theta(1,0)} - \frac{1}{\theta(1,1)}\right) I_{m_1} \otimes J_{m_2} \underline{Q} \\ &\quad + \frac{1}{rm_1}\left(\frac{1}{\theta(0,1)} - \frac{1}{\theta(1,1)}\right) J_{m_1} \otimes I_{m_2} \underline{Q} . \end{aligned} \quad (4.4)$$

Let Q_{i_1,i_2} be the $(i_1,i_2)^{th}$ element of the vector $\underline{Q}$; then,

$$\left(I_{m_1} \otimes J_{m_2}\right) \underline{Q} = \left(\sum_j Q_{0j}, \cdots, \sum_j Q_{0j}, \sum_j Q_{1j}, \cdots, \sum_j Q_{1j}, \cdots, \sum_j Q_{m_1-1,j}, \cdots, \sum_j Q_{m_1-1,j}\right)'$$

and

$$\left(J_{m_1} \otimes I_{m_2}\right) \underline{Q} = \left(\sum_i Q_{i0}, \sum_i Q_{i1}, \cdots, \sum_i Q_{i,m_2-1}, \sum_i Q_{i0}, \cdots, \sum_i Q_{i,m_2-1}, \cdots, \sum_i Q_{i0}, \cdots, \sum_i Q_{i,m_2-1}\right)' .$$

In general, let $\underline{S} = \left(\prod_{i=1}^{n} \otimes D_i^{\delta_i}\right) \underline{Q}$ and $S(i_1,i_2,\cdots,i_n)$ be the $(i_1,i_2,\cdots,i_n)^{th}$ element of vector $\underline{S}$; then,

$$S(i_1,i_2,\cdots,i_n) = \sum_{j_1=0}^{z_1} \sum_{j_2=0}^{z_2} \cdots \sum_{j_n=0}^{z_n} Q_{j_1,j_2,\cdots,j_n} ,$$

where $z_i=m_i-1$ if $\delta_i=1$ and $z_i=0$ and $j_i=i_i$ if $\delta_i=0$.

4.2 The combined intra- and inter-block analysis

In the model (2.1), the block effects $\{\beta_j\}$ could be assumed to be independent variates with mean zero and variance σ_β^2 . Using matrix notation, model (2.1) can be expressed as follows:

$$\underline{y} = X\phi + \underline{\epsilon}$$
$$= (\underline{1},\ X_1,\ X_2)\underline{\phi} + \underline{\epsilon}\ ,$$

where $\underline{y}$ is a bk × 1 observation vector, $\underline{1}$ is a bk × 1 column vector having all elements unity, X_i is a bk × v matrix, X_2 is a bk × b matrix, $\underline{\phi}$ is a (v+b+1) × 1 parameter vector such that $\underline{\phi} = (\mu, t_1, t_2, \cdots, t_v, \beta_1, \beta_2, \cdots, \beta_b)'$, and $\underline{\epsilon}$ is a bk × 1 experimental error vector. Note that $X_1'X_1 = rI_v$, $X_1'X_2 = N$, $X_2'X_2 = kI_b$.

Let $\underline{y}^* = X_2'\underline{y}$, $X^* = (\underline{1},\ X_1)$, $\phi^* = (\mu,\ \underline{t}')'$; then, the sum of squares to be minimized is

$$w(\underline{y} - X\underline{\tilde{\phi}})'(\underline{y} - X\underline{\tilde{\phi}}) + \frac{w'}{k}(\underline{y}^* - X^*\underline{\tilde{\phi}}^*)'(\underline{y}^* - X^*\underline{\tilde{\phi}}^*)\ ,$$

where $w = \frac{1}{\sigma^2}$, $w' = \frac{1}{\sigma^2 + k\sigma_\beta^2}$, and we obtain the following normal equation:

$$w(X'\underline{y} - X'X\underline{\tilde{\phi}}) + \frac{w'}{k}(X^{*\prime}\underline{y}^* - X^{*\prime}X^*\underline{\tilde{\phi}}^*) = 0\ .$$

After some algebraic manipulation, we obtain the following reduced normal equation for $\tilde{t}$:

$$\left(rwI_v - \frac{w-w'}{k}NN' - \frac{w'r}{v}J_v\right)\underline{\tilde{t}} = w\underline{Q} + w'\left(\underline{T} - \underline{Q} - \frac{1}{v}\underline{1}_v y_{..}\right)\ , \qquad (4.6)$$

where $y_{..} = \sum_i\sum_j y_{ij}$.

Let

$$C^* = rwI_v - \frac{w-w'}{k}NN' - \frac{w'r}{v}J_v\ .$$

Then, if the design has the Property A,

$$C^* = \sum_{s=0}^{n}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g'(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}\otimes D_i^{\delta_i}\right\}\ .$$

where $g'(0,0,\cdots,0) = rw - \frac{w-w'}{k}h(0,0,\cdots,0)$

$$= rw + (w-w')g(0,0,\cdots,0)\ .$$

$g'(\delta_1,\delta_2,\cdots,\delta_n) = (w-w')g(\delta_1,\delta_2,\cdots,\delta_n)$ for $(\delta_1,\delta_2,\cdots,\delta_n) \neq 0$ and $(\delta_1,\delta_2,\cdots,\delta_n) \neq (1,1,\cdots,1)$, $g'(1,1,\cdots,1) = (w-w')g(1,1,\cdots,1) - \frac{w'r}{v}$.

If we denote the efficiency factor associated with matrix C^* by $\theta_w(x_1,x_2,\cdots,x_n)$,

$$r\theta_w(x_1,x_2,\cdots,x_n) = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g'(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n} E_i(x_i,\delta_i)\right\}$$

$$= rw' + (w-w')\sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots\delta_n=s} g(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n} E_i(x_i\delta_i)\right\} \quad (4.8)$$

$$= rw' + (w-w')r\theta(x_1,x_2,\cdots,x_n) ,$$

where $\theta(x_1,x_2,\cdots,x_n)$ is the efficiency factor associated with the matrix C.

The generalized inverse of C^* is easily obtained, i.e.,

$$C^{*+} = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} u'(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}\otimes D_i^{\delta_i}\right\} , \quad (4.9)$$

where,

$$u'(\delta_1,\delta_2,\cdots,\delta_n) = \frac{1}{rv}\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s} \frac{(-1)^{\Sigma\delta_i x_i}\prod_{i=1}^{n}(m_i x_i)^{1-\delta_i}}{w' + (w-w')\theta(x_1,x_2,\cdots,x_n)}\right\} . \quad (4.10)$$

Hence, the solution of (4.6) is

$$\tilde{\underline{t}} = C^{*+}\underline{P} , \quad (4.11)$$

where

$$\underline{P} = w\underline{Q} + w'(\underline{T} - \underline{Q} - \frac{1}{v}\underline{1}_v y_{..}) \quad . \quad (4.12)$$

Using the form of (2.6), we obtain

$$\tilde{\underline{t}} = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} u'(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}\otimes D_i^{\delta_i}\ \underline{P}\right\} . \quad (4.13)$$

If $v = m_1 m_2$, for example,

$$\tilde{\underline{t}} = u'(0,0)\ \underline{P} + u'(0,1)\left(I_{m_1}\otimes J_{m_2}\right)\underline{P} + u'(1,0)\left(J_{m_1}\otimes I_{m_2}\right)\underline{P} ,$$

where

$$u'(0,0) = \frac{1}{rw' + r(w-w')\theta(1,1)}$$

$$u'(0,1) = \frac{1}{rm_2}\left(\frac{1}{w' + (w-w')\theta(1,0)} - \frac{1}{w' + (w-w')\theta(1,1)}\right)$$

$$u'(1,0) = \frac{1}{rm_1}\left(\frac{1}{w' + (w-w')\theta(0,1)} - \frac{1}{w' + (w-w')\theta(1,1)}\right) .$$

Next, if we calculate Cov($\underset{\sim}{P}$), we obtain

$$\mathrm{Cov}(\underset{\sim}{P}) = C^{*} .$$

Then

$$\mathrm{Cov}(\underset{\sim}{\tilde{t}}) = C^{*+} . \tag{4.14}$$

Now, consider two treatment intra-block estimates $\hat{t}_i$ and $\hat{t}_j$, where $i = (i_1, i_2, \cdots, i_n)$ and $j = (j_1, j_2, \cdots, j_n)$. The variance of $\hat{t}_i - \hat{t}_j$ is given by Kurkjian and Zelen [1963] as follows:

$$\mathrm{Var}\left(\hat{t}_i - \hat{t}_j\right) = \frac{2\sigma^2}{rv} \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\cdots+x_n=s} \frac{\prod_{r=1}^{n} (m_r-1)^{x_r} + (-1)^{s+1} \prod_{r=1}^{n} (1-m_r)^{x_r p_r}}{\theta(x_1, x_2, \cdots, x_n)} \right\} \tag{4.15}$$

where

$$p_r = \begin{cases} 0 & \text{if } i_r \neq j_r \\ 1 & \text{if } i_r = j_r \end{cases} .$$

This is nothing but the variance of the difference between two treatments which are $(p_1, p_2, \cdots p_n)^{th}$ associates. For example, if $v = m_1 m_2$,

$$\mathrm{Var}\left(\hat{t}_i - \hat{t}_j\right) = \begin{cases} \dfrac{2\sigma^2}{rm_1m_2}\left(\dfrac{m_2}{\theta(0,1)} + \dfrac{m_1}{\theta(1,0)}\right) & \text{for } (p_1,p_2) = (0,0) \\ \dfrac{2\sigma^2}{rm_1m_2}\left(\dfrac{m_1}{\theta(1,0)} + \dfrac{m_1(m_2-1)}{\theta(1,1)}\right) & \text{for } (p_1,p_2) = (0,1) \\ \dfrac{2\sigma^2}{rm_1m_2}\left(\dfrac{m_2}{\theta(0,1)} + \dfrac{m_2(m_1-1)}{\theta(1,1)}\right) & \text{for } (p_1,p_2) = (1,0) \end{cases} . \tag{4.16}$$

In the case of combined intra- and inter-block analysis,

$$\mathrm{Var}\left(\tilde{t}_i - \tilde{t}_j\right) = \frac{2}{rm_1m_2} \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\cdots+x_n=s} \frac{\prod_{r=1}^{n} (m_r-1)^{x_r} + (-1)^{s+1} \prod_{i=1}^{n} (1-m_r)^{x_r p_r}}{w' + (w-w')\,\theta\,(x_1, x_2, \cdots, x_n)} \right\}, \tag{4.17}$$

where $p_r = 0$ if $i_r \neq j_r$ and 1 if $i_r = j_r$.

4.3 Illustrative example

We give below the analysis for the group divisible design with parameters $v=15$, $k=4$, $b=15$, $r=4$, $m_1=3$, $m_2=5$, $\lambda_1=0$, $\lambda_2=1$ and association scheme

1	6	11
2	7	12
3	8	13
4	9	14
5	10	15

If we denote the treatments by (i_1, i_2), where $i_j = 0,1,2,\cdots,m_j-1$ and treatment number by $5i_1 + i_2 + 1$, the above design is a PBIB design having the following BNAS:

$$\lambda_{10} = 1, \quad \lambda_{01} = 0, \quad \lambda_{00} = 1 \quad .$$

From (2.13) or (3.2)

$$\theta(0,1) = \frac{15}{16}, \quad \theta(1,0) = \frac{3}{4}, \quad \text{and } \theta(1,1) = \frac{3}{4} \ ,$$

and so from (4.3)

$$u(0,0) = \frac{1}{3}, \quad u(0,1) = 0, \quad \text{and } u(1,0) = -\frac{1}{45} \quad .$$

From the numerical example in Chapter II of Bose, Clatworthy and Shrikhande [1954], the calculation for $\hat{t}_{i_1, i_2}$ can be systematically arranged as in the Table 4.2. The entries for columns headed $S(0,0) = u(0,0)Q_{i_1, i_2}$ and $S(1,0) = u(1,0)Q_{\cdot, i_2}$ are obtained from the Table 4.1 and the last column in the Table 4.2 shows the estimates of the treatment effects t_{i_1, i_2}, obtained by adding the values of S(0,0) and S(1,0) which are shown in the corresponding rows.

TABLE 4.1

Values of Q_{i_1, i_2}

	i_1			
	0	1	2	$Q_{\cdot, i_2}$
$Q_{i_1, 0}$	0.475	0.600	0.450	1.525
$Q_{i_1, 1}$	-1.125	0.100	0.825	-0.200
$Q_{i_1, 2}$	-0.925	0.150	-0.275	-1.050
$Q_{i_1, 3}$	-0.175	0.575	-0.575	-0.175
$Q_{i_1, 4}$	0.200	-0.625	0.325	-0.100

TABLE 4.2

Intra-block Estimates of Treatment Effects

Treatment i ; (i_1, i_2)	$S(0,0)$	$S(1,0)$	$\hat{t}_{i_1, i_2}$
1 ; (0,0)	0.15833	-0.03389	0.12444
2 ; (0,1)	-0.37500	0.00444	-0.37056
3 ; (0,2)	-0.30833	0.02333	-0.28500
4 ; (0,3)	-0.05833	0.00389	-0.05444
5 ; (0,4)	0.06667	0.00222	0.06889
6 ; (1,0)	0.20000	-0.03389	0.16611
7 ; (1,1)	0.03333	0.00444	0.03777
8 ; (1,2)	0.05000	0.02333	0.07333
9 ; (1,3)	0.19166	0.00389	0.19555
10 ; (1,4)	-0.20833	0.00222	-0.20611
11 ; (2,0)	0.15000	-0.03389	0.11611
12 ; (2,1)	0.27500	0.00444	0.27944
13 ; (2,2)	-0.09167	0.02333	-0.06834
14 ; (2,3)	-0.19166	0.00389	-0.18777
15 ; (2,4)	0.10833	0.00222	0.11055

4.3.1 Intra-block analysis of variance

We make the following calculations in the usual manner:

(a) SS_t = "Treatments adjusted" sum of squares

$$= \sum_i \hat{t}_i Q_i$$

$$= (0.12444)(0.475) + (-0.37056)(-1.125) + \cdots + (0.11055)(0.325)$$

$$= 1.5641 .$$

(b) SS_b' = "Blocks unadjusted" sum of squares

$$= \sum_j B_j^2/k - (\text{Total})^2/N, \quad N = bk$$

$$= 4.9233 .$$

(c) SS = Total sum of squares corrected for the mean

$$= 9.3733$$

(d) SS_e = Error sum of squares

$$= SS - SS_t - SS_b'$$

$$= 9.3733 - 1.5641 - 4.9233$$

$$= 2.8859 \; .$$

We thus get the following intra-block analysis of variance table.

TABLE 4.3

Intra-block Analysis of Variance

Source of variation	d.f	SS	MS
Treatment (adj)	14	1.5641	$0.1117 = s_t^2$
Blocks (unadj)	14	4.9233	
Error	31	2.8859	$0.0931 = \hat{\sigma}^2$
Total	59	9.3733	

The observed F-ratio of 1.199 with 14 and 31 degrees of freedom is not significant at the 5 per cent level.

The variance of the difference between the effects of i^{th} and j^{th} treatment is given by (4.16) and is estimated by replacing σ^2 by $\hat{\sigma}^2$. Hence the estimated variance of the difference between two treatments which are $(0,0)^{th}$ associates is

$$\frac{2\hat{\sigma}^2}{rm_1 m_2}\left(\frac{m_2}{\theta(0,1)} + \frac{m_1}{\theta(1,0)} + \frac{(m_1-1)(m_2-1)-1}{\theta(1,1)}\right)$$

$$= \frac{(0.0931)(28)}{45} = 0.0579 \; .$$

Likewise the estimated variances of the difference between two treatments which are $(1,0)^{th}$ associates and $(0,1)^{th}$ associates are:

$$\frac{2\hat{\sigma}^2}{rm_1 m_2}\left(\frac{m_2}{\theta(0,1)} + \frac{m_2(m_1-1)}{\theta(1,1)}\right) = (0.0931)\,\frac{28}{45} = 0.0579$$

and

$$\frac{2\hat{\sigma}^2}{rm_1 m_2}\left(\frac{m_1}{\theta(1,0)} + \frac{m_1(m_2-1)}{\theta(1,1)}\right) = (0.0931)\,\frac{2}{3} = 0.0621 \; ,$$

respectively.

Since each treatment has eight $(0,0)^{th}$ associates, four $(1,0)^{th}$ associates and two $(0,1)^{th}$ associates, the average variance of a difference is

$$\frac{(0.0579)(8+4) + (0.0621)(2)}{14} = 0.0585 \quad .$$

4.3.2 Inter-block analysis of variance

The following calculations are to be made:

(a) SS_t' = "Treatments unadjusted (ignoring block effects)"

sum of squares

$$= \sum_i T_i^2/r - (\text{Total})^2/bk$$

$$= \tfrac{1}{4}\Big((11.100)^2 + (9.400)^2 + \cdots + (11.500)^2\Big) - (164.00)^2/64$$

$$= 3.1383 \ .$$

and is obtained in the usual manner.

(b) SS_b = "Blocks adjusted (eliminating treatment effects)"

sum of squares

$$= SS - SS_t' - SS_e$$

$$= 9.2733 - 3.1383 - 2.8859$$

$$= 3.3491 \ .$$

These sums of squares are summarized in Table 4.4.

TABLE 4.4

Auxiliary Table for Inter-block Analysis of Variance

Source of variation	d.f	SS	Expectation of the SS
Treatment (unadj)	14	SS_t'=3.1383	
Blocks (adj)	14	SS_b=3.3491	$14\sigma^2 + 45\sigma_\beta^2$
Error	31	SS_e=2.8859	$31\sigma^2$
Total	59	SS =9.3733	

(c) The estimates of σ^2 and σ_β^2 are obtained by equating SS_b and SS_e to their expectations respectively. Thus

$$SS_b = (b-1)\hat{\sigma}^2 + (bk-v)\hat{\sigma}^2 \text{ (see Appendix [a.2])}$$

$$SS_e = (bk-b-v+1)\hat{\sigma}^2$$

This gives

$$\hat{\sigma}^2 = s_e^2$$

$$\hat{\sigma}_\beta^2 = \frac{b-1}{bk-v}\left(s_b^2 - s_e^2\right)$$

where s_e^2 is the error mean square and $s_b^2 = SS_b/(b-1)$ is the "blocks adjusted" mean square. Hence

$$\hat{\sigma}^2 = 0.0931$$

$$\hat{\sigma}_\beta^2 = 0.0455$$

$$\hat{w} = \frac{1}{\hat{\sigma}^2} = 10.7411, \quad \hat{w}' = \frac{1}{\hat{\sigma}^2 + k\hat{\sigma}_\beta^2} = 3.6350$$

$$u'(0,0) = \frac{1}{r(\hat{w}' + (\hat{w}-\hat{w}')\theta(1,1))} = 0.02788$$

$$u'(0,1) = 0$$

$$u'(1,0) = \frac{1}{rm_1}\left(\frac{1}{\hat{w}' + (\hat{w}-\hat{w}')\theta(0,1)} - \frac{1}{\hat{w}' + (\hat{w}-\hat{w}')\theta(1,1)}\right)$$

$$= -0.00120 \ .$$

(d) Calculation of P_{i_1,i_2} and of the combined intra- and inter-block estimates $\tilde{t}_{i_1,i_2}$ is obtained by setting

$$P_{i_1,i_2} = \hat{w}Q_{i_1,i_2} + \hat{w}'Q'_{i_1,i_2} \ ,$$

where $Q'_{i_1,i_2} = T_{i_1,i_2} - Q_{i_1,i_2} - y_{..}/v$. The results are presented in Table 4.5.

TABLE 4.5

Calculation of P_{i_1,i_2} and $P_{\cdot,i_2}$

	i_1 = 0	1	2	Total, $P_{\cdot,i_2}$
$\hat{w}Q_{i_1,0}$	5.10202	6.44466	4.83350	
$\hat{w}'Q'_{i_1,0}$	-1.11958	0.60705	1.87930	
$P_{i_1,0}$	3.98244	7.05171	6.71280	17.74695
$\hat{w}Q_{i_1,1}$	-12.08374	1.07411	8.86141	
$\hat{w}'Q'_{i_1,1}$	-1.48308	0.97055	2.69717	
$P_{i_1,1}$	-13.56682	2.04466	11.55858	0.03642
$\hat{w}Q_{i_1,2}$	-9.93552	1.61117	-2.95380	
$\hat{w}'Q'_{i_1,2}$	-0.75608	-0.66521	-2.02833	
$P_{i_1,2}$	-10.69160	0.94596	-4.98213	-14.72777
$\hat{w}Q_{i_1,3}$	-1.87969	6.17613	-6.17613	
$\hat{w}'Q'_{i_1,3}$	3.78767	-0.75608	-2.39183	
$P_{i_1,3}$	1.90798	5.42005	-8.56796	-1.23993
$\hat{w}Q_{i_1,4}$	2.14822	-6.71319	3.49086	
$\hat{w}'Q'_{i_1,4}$	0.24355	-1.84658	0.87967	
$P_{i_1,4}$	2.39177	-8.55977	4.37053	-1.79747

Since

$$\tilde{t}_{i_1,i_2} = u'(0,0)P_{i_1,i_2} + u'(1,0)P_{\cdot,i_2} ,$$

the combined intra- and inter-block estimates $\tilde{t}_{i_1,i_2}$ of the treatment effects t_{i_1,i_2} can now be calculated and are given in Table 4.6.

TABLE 4.6

The Combined Intra- and Inter-block Estimates of the Treatment Effects

Treatment i ; (i_1, i_2)	$u^*(0,0)P_{i_1, i_2}$	$u^*(1,0)P_{\cdot, i_2}$	Total, $\tilde{t}_{i_1, i_2}$
1 ; (0,0)	0.11103	-0.02130	0.08973
2 ; (0,1)	-0.37824	-0.00004	-0.37828
3 ; (0,2)	-0.29808	0.01767	-0.28041
4 ; (0,3)	0.05319	0.00149	0.05468
5 ; (0,4)	0.06668	0.00216	0.06884
6 ; (1,0)	0.19660	-0.02130	0.17530
7 ; (1,1)	0.05701	-0.00004	0.05697
8 ; (1,2)	0.02637	0.01767	0.04404
9 ; (1,3)	0.15111	0.00149	0.15260
10 ; (1,4)	-0.23865	0.00216	-0.23649
11 ; (2,0)	0.18715	-0.02130	0.16585
12 ; (2,1)	0.32225	-0.00004	0.32221
13 ; (2,2)	-0.13890	0.01767	-0.12123
14 ; (2,3)	0.12185	0.00216	-0.23738
15 ; (2,4)	0.12185	0.00216	0.12401

For the combined intra- and inter-block analysis there is no exact test of the hypothesis of the equality of treatment means but when this hypothesis is true the quantity $\sum_i \tilde{t}_i P_i$ is approximately distributed as χ^2 with v-1 degrees of freedom.

Thus

$$\chi^2 = \sum_i \tilde{t}_i P_i = 21.0345$$

with 14 degrees of freedom and is not significant at the 5 per cent level (also, see Federer [1955], section XI-7).

The estimated variance of the difference between the estimated effects of two treatments which are $(0,0)^{th}$ associates is

$$\frac{2}{rm_1 m_2}\left(\frac{m_2}{\hat{w}' + (\hat{w}-\hat{w}')\theta(0,1)} + \frac{m_1}{\hat{w}' + (\hat{w}-\hat{w}')\theta(1,0)} + \frac{(m_1-1)(m_2-1)-1}{\hat{w}' + (\hat{w}-\hat{w}')\theta(1,1)}\right) = 0.0534 \quad .$$

Likewise the estimated variances of the difference between estimated effects of two treatments which are $(1,0)^{th}$ associates and $(0,1)^{th}$ associates are 0.0534 and 0.0558, respectively.

The average variance is therefore

$$\frac{n_{00}(0.0534) + n_{10}(0.0534) + n_{01}(0.0558)}{v-1} = 0.0537 \quad .$$

5. ANALYSIS OF PAB TYPE K-ROW BY B-COLUMN RECTANGULAR DESIGNS

5.1 Introduction

In a k-row by b-column rectangular experiment design with $v = \prod_{i=1}^{n} m_i$ treatments being replicated r times each, suppose the column incidence matrix N has Property A and the row incidence matrix $\tilde{N}$ of the experiment design has Property B as introduced by Zelen and Federer [1964], i.e.,

$$\tilde{N}\tilde{N}' = \sum_{s=0}^{n}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} \tilde{h}(\delta_1,\delta_2,\cdots,\delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i}\right\} ,$$

where the $\tilde{h}(\delta_1,\delta_2,\cdots,\delta_n)$ are constants. Then the k × b rectangular design will be said to have Property A and Property B or will be called a PAB type k × b rectangular design.

Paik and Federer [1973b] have proved the following theorems:

<u>Theorem 5.1</u> <u>If the design is an NPBB having BNAS with respect to columns, then the design is a balanced factorial rectangular experiment (BFRE) and is a PAB type rectangular design.</u>

<u>Theorem 5.2</u> <u>Every PAB type rectangular design is an NPBB rectangular design having BNAS with respect to rows and to columns, and conversely.</u>

However, a BFRE is not always a PAB type rectangular design nor an NPBB design having BNAS with respect to both rows and columns (see Kshirsager [1957]).

5.2 Analysis without recovery of the inter-row and inter-column information

If we ignore the column effects, the above k-row by b-column rectangular design could be considered a "block (row)" design having Property B. The set up assumed is

$$y_{ih} = \mu + t_i + \rho_h + e_{ih} , \tag{5.1}$$

where μ is a constant and t_i, ρ_h are the fixed effects associated respectively with the treatments and rows, while $\{e_{ih}\}$ are assumed to be independent normal variates with mean zero and variance σ_e^2 .

In this case, the reduced normal equation for estimating the treatment effect vector $\underset{\sim}{t}$ is

$$\tilde{C}\tilde{t} = \tilde{Q} , \tag{5.2}$$

where

$$\begin{aligned} \tilde{C} &= rI_v - \frac{1}{b}\tilde{N}\tilde{N}' , \\ \tilde{\underset{\sim}{Q}} &= \underset{\sim}{T} - \frac{1}{b}\tilde{N}\underset{\sim}{R} , \\ \underset{\sim}{R} &= \left(R_1, R_2, \cdots, R_k\right)' ; \end{aligned} \tag{5.3}$$

R_h = total yield of all the plots of the h^{th} row .

Since the above design has the Property B, the matrix $\tilde{C}$ has the following form:

$$\tilde{C} = \sum_{s=0}^{n}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} \tilde{g}(\delta_1,\delta_2,\cdots,\delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i}\right\} , \tag{5.4}$$

where

$$\tilde{g}(0,0,\cdots,0) = r - \tilde{h}(0,0,\cdots,0)$$

$$\tilde{g}(\delta_1,\delta_2,\cdots,\delta_n) = -\tilde{h}(\delta_1,\delta_2,\cdots,\delta_n) \text{ for } (\delta_1,\delta_2,\cdots,\delta_n) \neq 0 .$$

$$\tilde{C}^{+} = \sum_{s=1}^{n}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n} \tilde{u}(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}\otimes D_i^{\delta_i}\right\}, \tag{5.5}$$

where

$$\tilde{u}(\delta_1,\delta_2,\cdots,\delta_n) = \frac{1}{v}\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s} \frac{(-1)^{\Sigma\delta_i x_i}\prod_{i=1}^{n}(m_i x_i)^{1-\delta_i}}{r\tilde{\theta}(x_1,x_2,\cdots,x_n)}\right\},$$
$$r\tilde{\theta}(x_1,x_2,\cdots,x_n) = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} \tilde{g}(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}E_i(x_i,\delta_i)\right\}. \tag{5.6}$$

When the treatment i occurs in column j and row h, the set up could be

$$y_{ijh} = \mu + t_i + \gamma_j + \rho_h + \epsilon_{ijh}, \tag{5.7}$$

where μ is a constant, and t_i, γ_j, ρ_h are the fixed effects associated respectively with treatments, columns, and rows, while $\{\epsilon_{ijh}\}$ are assumed to be independent normal variates with mean zero and variance σ^2. (In section 5.3, the column effects $\{\gamma_j\}$ and the row effects $\{\rho_h\}$ will be assumed to be independent random variates with mean zero and variance σ_γ^2 and σ_ρ^2, respectively.)

Using matrix notation, the model (5.7) can be expressed as:

$$\begin{aligned} \underline{y} &= X\underline{\phi} + \underline{\epsilon} \\ &= (\underline{1}, X_1, X_2, X_3)\underline{\phi} + \underline{\epsilon} \end{aligned} \tag{5.8}$$

where $\underline{y}$ is a bk × 1 observation vector, $\underline{1}$ is a bk × 1 column vector with all elements unity, X_1 is a bk × v matrix, X_2 is a bk × b matrix, X_3 is a bk × k matrix, $\underline{\phi}$ is a (v+b+k+1) × 1 parameter vector such that $\underline{\phi} = (\mu, t_1, t_2, \cdots, t_v, \gamma_1, \gamma_2, \cdots, \gamma_b, \rho_1, \rho_2, \cdots, \rho_k)'$ and $\underline{\epsilon}$ is a bk × 1 experimental error vector. Note that

$$X_1'X_1 = rI_v, \quad X_1'X_2 = N, \quad X_1'X_3 = \tilde{N},$$

$$X_2'X_2 = kI_b, \quad X_2'X_3 = J_{b\times k}, \quad X_3'X_3 = bI_k, \text{ and}$$

$$X_1'\underline{y} = \underline{T}, \quad X_2'\underline{y} = \underline{B}, \quad X_3'\underline{y} = \underline{R} .$$

It is known that the reduced normal equations for estimating the treatment effect vector $\underline{t}$ may be written as:

$$(rI_v - \frac{1}{k}NN' - \frac{1}{b}\tilde{N}\tilde{N}' + \frac{r}{v}J_v)\underline{t}^* = \underline{T} - \frac{1}{k}N\underline{B} - \frac{1}{b}\tilde{N}\underline{R} + \frac{1}{v}\underline{1}_v y_{...} ,$$

where $y_{...}$ = total for all observations, or

$$\left(C + \tilde{C} - r(I_v - \frac{1}{v}J_v)\right)\underline{t}^* = \underline{Q} + \underline{\tilde{Q}} - (\underline{T} - \frac{1}{v}\underline{1}_v y_{...}) . \qquad (5.9)$$

Since we are concerned with the PAB type k × b rectangular design, this design has Property A in columns and Property B in rows. The normal equations in (5.9) can be expressed as follows:

$$\sum_{s=0}^{n}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g^*(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}\otimes D_i^{\delta_i}\right\}\underline{t}^* = \underline{Q}^* , \qquad (5.10)$$

where $g^*(0,0,\cdots,0) = g(0,0,\cdots,0) + \tilde{g}(0,0,\cdots,0) - r$, $g^*(\delta_1,\delta_2,\cdots,\delta_n) = g(\delta_1,\delta_2,\cdots,\delta_n) + \tilde{g}(\delta_1,\delta_2,\cdots,\delta_n)$ for $(\delta_1,\delta_2,\cdots,\delta_n) \neq 0$ and $\neq (1,1,\cdots,1)$ and $g^*(1,1,\cdots,1) = g(1,1,\cdots,1) + \tilde{g}(1,1,\cdots,1) + \frac{r}{v}$, and $\underline{Q}^* = \underline{Q} + \underline{\tilde{Q}} + (\underline{T} - \frac{1}{v}\underline{1}_v y_{...})$.

Let

$$r\theta^*(x_1,x_2,\cdots,x_n) = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g^*(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}E_i(x_i\delta_i)\right\}$$

$$= -r + r\theta(x_1,x_2,\cdots,x_n) + r\tilde{\theta}(x_1,x_2,\cdots,x_n) \qquad (5.11)$$

$$= r\left[\theta(x_1,x_2,\cdots,x_n) + \tilde{\theta}(x_1,x_2,\cdots,x_n) - 1\right] ;$$

then, the solution of the equation (5.10) is

$$\underline{t}^* = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} u^*(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}\otimes D_i^{\delta_i}\right\}\underline{Q}^* \qquad (5.12)$$

where

$$u^*(\delta_1, \delta_2, \cdots, \delta_n) = \frac{1}{vr} \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\cdots+x_n=s} \frac{(-1)^{\Sigma\delta_i x_i} \prod_{i=1}^{n} (m_i x_i)^{1-\delta_i}}{\theta^*(x_1, x_2, \cdots, x_n)} \right\} . \quad (5.13)$$

Example 5.1 Consider the design in Example 2.1 as a 3 × 8 rectangular arrangement.

(1). With respect to columns, $v = 2 \times 3$, $r = 4$, $k = 3$, $b = 8$, and $\lambda_{00} = 2$, $\lambda_{01} = 0$, $\lambda_{10} = 2$, and $\lambda_{11} = -8$. Using the formula (2.11)

$$\underline{\theta} = \frac{-1}{rk} \prod_{i=1}^{m} G(m_i) = \frac{-1}{12} \begin{bmatrix} 2 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ -2 & -1 & 2 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2/3 \\ 2/3 \end{bmatrix} .$$

Hence

$$r\theta(0,1) = 4, \ r\theta(1,0) = 8/3, \text{ and } r\theta(1,1) = 8/3 .$$

(2). With respect to rows, $v = 2 \times 3$, $r = 4$, $k = 8$, $b = 3$, and $\lambda_{00} = 5$, $\lambda_{01} = 6$, $\lambda_{10} = 5$, and $\lambda_{11} = -26$. In this case,

$$\tilde{\underline{\theta}} = \frac{-1}{32} \begin{bmatrix} 2 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ -2 & -1 & 2 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 5 \\ -26 \end{bmatrix} = \begin{bmatrix} 0 \\ 15/16 \\ 1 \\ 1 \end{bmatrix} .$$

Hence

$$r\tilde{\theta}(0,1) = 15/4, \ r\tilde{\theta}(1,0) = 4 \text{ and } r\tilde{\theta}(1,1) = 4 .$$

(3). From (1) and (2), the above design is a "balanced factorial 3 × 8 rectangular experiment" and is a PAB type rectangular design having the following efficiency factors:

$$\theta^*(0,1) = 15/16, \ \theta^*(1,0) = 2/3, \text{ and } \theta^*(1,1) = 2/3 .$$

Now, if we consider two treatment estimates t_i^* and t_j^*, where

$i = (i_1, i_2, \cdots, i_n)$ and $j = (j_1, j_2, \cdots, j_n)$, then the variance of $t_i^* - t_j^*$ is given by applying the formula (4.14), i.e.,

$$\mathrm{var}(t_i^* - t_j^*) = \frac{2\sigma^2}{rv} \sum_{s=1}^{n} \left\{ \sum_{x_1+\cdots+x_n=s} \frac{\prod_{r=1}^{n} (m_r-1)^{x_r} + (-1)^{s+1}(1-m_r)^{x_r p_r}}{\theta^*(x_1, x_2, \cdots, x_n)} \right\} \quad (5.14)$$

where

$$p_r = \begin{cases} 0 & \text{if } i_r \neq j_r \\ 1 & \text{if } i_r = j_r \end{cases}$$

and $\theta^*(x_1, x_2, \cdots, x_n) = \theta(x_1, x_2, \cdots, x_n) + \tilde{\theta}(x_1, x_2, \cdots, x_n) - 1$.

5.3 Analysis with recovery of inter-column and inter-row information

In the model (5.8), suppose that the column effects $\{\gamma_j\}$ and row effects $\{\rho_h\}$ are independent random variates with mean zero and variance σ_γ^2 and σ_ρ^2, respectively. Then the sum of squares to be minimized is

$$w(\underline{y}-X\tilde{\underline{\phi}})'(\underline{y}-X\tilde{\underline{\phi}}) + \frac{w_c}{k}(\underline{y}_c - X_c\tilde{\underline{\phi}}_c)'(\underline{y}_c - X_c\tilde{\underline{\phi}}_c) + \frac{w_r}{b}(\underline{y}_r - X_r\tilde{\underline{\phi}}_r)'(\underline{y}_r - X_r\underline{\phi}_r) ,$$

where

$$w = \frac{1}{\sigma^2}, \quad w_c = \frac{1}{\sigma^2 + k\sigma_\gamma^2}, \quad w_r = \frac{1}{\sigma^2 + b\sigma_\rho^2} ,$$

$$\underline{y}_c = X_2'\underline{y}, \quad \underline{y}_r = X_3'\underline{y} ,$$

$$\underline{\phi}_c = (\mu, \underline{t}', \underline{\gamma}'), \quad \underline{\phi}_r = (\mu, \underline{t}', \underline{\rho}')' ,$$

$$X_c = X_2'(\underline{1}, X_1, X_2) = (k\underline{1}_b, N', J_{b\times k}), \text{ and}$$

$$X_r = X_3'(\underline{1}, X_1, X_3) = (b\underline{1}_k, \tilde{N}', J_{k\times b}) ;$$

we obtain the following normal equation:

$$w(X'\underline{y} - X'X\tilde{\underline{\phi}}) + \frac{w_c}{k}(X_c'\underline{y} - X_c'X_c\tilde{\underline{\phi}}_c) + \frac{w_r}{k}(X_r'\underline{y} - X_r'X_r'\underline{\phi}_r) = 0 \quad .$$

After some algebraic manipulation, we obtain the following reduced normal equation for the treatment effect vector $\underline{t}$:

$$\left(C_c^* + C_r^* - rw(I_v - \frac{1}{v}J_v)\right)\underline{t}^w = \underline{P}^* , \quad (5.15)$$

where

$$
\begin{aligned}
C_c^* &= rwI_v - \frac{w-w_c}{k} NN' , \\
C_r^* &= rwI_v - \frac{w-w_r}{b} \tilde{N}\tilde{N}' , \\
\underline{P}^* &= w\underline{Q}^* + w_c \left(\frac{1}{k} N\underline{B} - \frac{1}{v}\underline{1}_v y_{\cdots}\right) \\
&\quad + w_r \left(\frac{1}{b} \tilde{N}\underline{R} - \frac{1}{v}\underline{1}_v y_{\cdots}\right) , \\
\underline{Q}^* &= \underline{Q} + \underline{\tilde{Q}} - \left(\underline{T} - \frac{1}{v}\underline{1}_v y_{\cdots}\right) .
\end{aligned}
\tag{5.16}
$$

Since we are concerned with the PAB type k X b rectangular design, normal equation (5.15) can be expressed as follows:

$$\sum_{s=0}^{n}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g_w(\delta_1,\delta_2,\cdots,\delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i}\right\} \underline{t}_w = \underline{P}^* , \tag{5.17}$$

where $g_w(0,0,\cdots,0) = g_c'(0,0,\cdots,0) + g_r'(0,0,\cdots,0) - rw = rw_c + (w-w_c)g(0,0,\cdots,0) + rw_r + (w-w_r)\tilde{g}(0,0,\cdots,0) - rw = (w-w_c)g(0,0,\cdots,0) + (w-w_r)\tilde{g}(0,0,\cdots,0) - (w-w_c-w_r)$, $g_w(\delta_1,\delta_2,\cdots,\delta_n) = g_c(\delta_1,\delta_2,\cdots,\delta_n) + g_r(\delta_1,\delta_2,\cdots,\delta_n) = (w-w_c)g(\delta_1,\delta_2,\cdots,\delta_n) + (w-w_r)\tilde{g}(\delta_1,\delta_2,\cdots,\delta_n)$ for $(\delta_1,\delta_2,\cdots,\delta_n) \neq 0$ and $\neq (1,1,\cdots,1)$, and $g_w(1,1,\cdots,1) = g_c(1,1,\cdots,1) + g_r(1,1,\cdots,1) + g_r(1,1,\cdots,1) + \frac{rw}{v} = (w-w_c)g(1,1,\cdots,1) + (w-w_r)\tilde{g}(1,1,\cdots,1) - \frac{r}{v}(w_c+w_r-w)$. The following solution for equation (5.17) is obtained:

$$\underline{t}^w = \sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} u_w(\delta_1,\delta_2,\cdots,\delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i}\right\} \underline{P}^* , \tag{5.18}$$

where

$$u_w(\delta_1,\delta_2,\cdots,\delta_n) = \frac{1}{vr}\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s} \frac{(-1)^{\Sigma\delta_i x_i}\prod_{i=1}^{n}(m_i x_i)^{1-\delta_i}}{\theta_w(x_1,x_2,\cdots,x_n)}\right\} \quad (5.19)$$

$$\theta_w(x_1,x_2,\cdots,x_n) = (w-w_c)\theta(x_1,x_2,\cdots,x_n) + (w-w_r)\tilde{\theta}(x_1,x_2,\cdots,x_n) - (w-w_c-w_r) \ .$$

In this case, if we consider two treatment estimates t_i^w and t_j^w, where $i = (i_1,i_2,\cdots,i_n)$ and $j = (j_1,j_2,\cdots,j_n)$, then the variance of $(t_i^w-t_j^w)$ is

$$\mathrm{Var}\left(t_i^w-t_j^w\right) = \frac{2}{rv}\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s} \frac{\prod_{r=1}^{n}(m_r-1)^{x_r}+(-1)^{s+1}\prod_{i=1}^{n}(1-m_r)^{x_r p_r}}{\theta_w^*(x_1,x_2,\cdots,x_n)}\right\}, \quad (5.20)$$

where

$$p_r = \begin{cases} 0 & \text{if } i_r \neq j_r \\ 1 & \text{if } i_r = j_r \end{cases} \ .$$

5.4 Illustrative example

Example 5.2 We give the analysis for a PAB type 4-row by 6-column rectangular design with $v = 2^3$. It is useful to divide the computations into a number of steps.

Step 1. Preparation of the field and treatment allocation plans. The field plan, Table 5.1, shows the row number in the first column. In the second column opposite each row number (and in the same row) are given the treatments appearing in this row, and below each treatment number is shown the corresponding observed yield.

In this design, the column incidence matrix N and row incidence matrix $\tilde{N}$ are respectively as follows:

$$N = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \text{ and } \tilde{N} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} .$$

So,

$$NN' = 4I_8 - 2I_2 \otimes I_2 \otimes J_2 - J_2 \otimes J_2 \otimes I_2 + 2J_8 ,$$

$$\tilde{N}\tilde{N}' = I_2 \otimes I_2 \otimes J_2 + 2J_8 .$$

TABLE 5.1

Field Plan

Row(h)	Columns (Blocks) (1)	(2)	(3)	(4)	(5)	(6)	Row Totals R_h	Row Average R_h/b
	Treatments and Yields							
(I)	1 19	2 12	3 17	4 18	5 18	6 17	101	16.833
(II)	4 24	3 18	1 16	7 18	8 18	2 17	111	18.500
(III)	6 20	5 17	8 22	2 16	1 19	7 21	115	19.167
(IV)	7 24	8 15	6 18	5 16	4 22	3 18	113	18.833
B_j	87	62	73	68	77	73	440	
B_j/k	21.75	15.5	18.25	17.0	19.25	18.25		18.333

Then, since $k = 4$, $b = 6$, and $r = 3$,

$$C = rI - NN'/k = 2I_8 + I_2 \otimes I_2 \otimes J_{2/2} + J_2 \otimes J_2 \otimes I_{2/4} - J_{8'/2}$$

$$\tilde{C} = rI - \tilde{N}\tilde{N}'/b = 3I_8 - I_2 \otimes I_2 \otimes J_{2/b} - J_{8/3} ;$$

$g(0,0,0) = 2$, $g(0,0,1) = 1/2$, $g(0,1,0) = 0$, $g(0,1,1) = 0$,

$g(1,0,0) = 0$, $g(1,0,1) = 0$, $g(1,1,0) = 1/4$; and

$\tilde{g}(0,0,0) = 3,\ \tilde{g}(0,0,1) = -1/6,\ \tilde{g}(0,1,0) = 0,\ \tilde{g}(0,1,1) = 0\ ,$

$\tilde{g}(1,0,0) = 0,\ \tilde{g}(1,0,1) = 0,\ \tilde{g}(1,1,0) = 0\ .$

Then, from formula (2.7), we obtain

$\theta(0,0,1) = 1,\ \theta(0,1,0) = 1,\ \theta(0,1,1) = 2/3,\ \theta(1,0,0) = 1\ ,$

$\theta(1,0,1) = 2/3,\ \theta(1,1,0) = 1,\ \theta(1,1,1) = 2/3\ ;$

$\tilde{\theta}(0,0,1) = 1,\ \tilde{\theta}(0,1,0) = 8/9,\ \tilde{\theta}(0,1,1) = 1,\ \tilde{\theta}(1,0,0) = 8/9\ ,$

$\tilde{\theta}(1,0,1) = 1,\ \tilde{\theta}(1,1,0) = 8/9,\ \tilde{\theta}(1,1,1) = 1$; and

$\theta^*(0,0,1) = 1 + 1 - 1 = 1, \quad \theta^*(0,1,0) = 1 + 8/9 - 1 = 8/9\ ,$

$\theta^*(0,1,1) = 2/3 + 1 - 1 = 2/3, \quad \theta^*(1,0,0) = 1 + 8/9 - 1 = 8/9\ ,$

$\theta^*(1,0,1) = 2/3 + 1 - 1 = 2/3, \quad \theta^*(1,1,0) = 1 + 8/9 - 1 = 8/9\ ,$

$\theta^*(1,1,1) = 2/3 + 1 - 1 = 2/3\ .$

The treatment allocation plan 1, Table 5.2, shows the treatment number in the first column. In the second column, opposite each treatment number (and in the same row) are given the columns (blocks) in which this treatment occurs, and below each block number is given the corresponding observed yield. In the next row of the second column immediately below the observed yields are shown the block averages (taken from the final column of the field plan, Table 5.1).

From Table 5.2, we obtain

$Q_{00.} = -11.00,\ Q_{01.} = 7.00,\ Q_{10.} = -4.00,\ Q_{11.} = 8.00,\ Q_{..0} = 1.00,$

$Q_{..1} = -1.00\ ,$

and from Table 5.3,

$\tilde{Q}_{00.} = -10.00,\ \tilde{Q}_{01.} = 8.667,\ \tilde{Q}_{10.} = -3.667,\ \tilde{Q}_{11.} = 5.00,\ \tilde{Q}_{..0} = 1.00,$

$\tilde{Q}_{..1} = -1.00\ .$

Also, since $Q_i^* = Q_i + \tilde{Q}_i - T_i + y_{...}/v$ we obtain

$Q^*_{000} = -4.75,\ Q^*_{001} = -5.25,\ Q^*_{010} = 1.833,\ Q^*_{011} = 6.833,\ Q^*_{100} = -0.583,$

$Q^*_{101} = -3.083,\ Q^*_{110} = 4.500,\ Q^*_{111} = 0.500,\ Q^*_{00.} = -10.000,\ Q^*_{01.} = 8.667,$

$Q^*_{10.} = -3.667,\ Q^*_{11.} = 5.000,\ Q^*_{..0} = 1.000,\ Q^*_{..1} = -1.000\ .$

TABLE 5.2

Treatment Allocation Plan 1

Treatment i $i_1, i_2, i_3)$	Block (Column) Number, Yields and Block Average			T_i and $T_i - Q_i$	Q_i
1 (0,0,0)	(1) 19 21.75	(3) 16 18.25	(5) 19 19.25	54 59.25	-5.25
2 (0,0,1)	(2) 12 15.5	(4) 16 17.0	(6) 17 18.25	45 50.75	-5.75
3 (0,1,0)	(2) 18 15.5	(3) 17 18.25	(6) 18 18.25	53 52.00	1.00
4 (0,1,1)	(1) 24 21.75	(4) 18 17.0	(5) 22 19.25	64 58.00	6.00
5 (1,0,0)	(2) 17 15.5	(4) 16 17.0	(5) 18 19.25	51 51.75	-0.75
6 (1,0,1)	(1) 20 21.75	(3) 18 18.25	(6) 17 18.25	55 58.25	-3.25
7 (1,1,0)	(1) 24 21.75	(4) 18 17.0	(6) 21 18.25	64 57.00	6.00
8 (1,1,1)	(2) 15 15.5	(3) 22 18.25	(5) 18 19.25	55 53.00	2.00

The treatment allocation plan 2, Table 5.3, shows the treatment number in the first column. In the second column, opposite each treatment number (and in the same row) are given the rows in which this treatment occurs, and below each row number is given the corresponding observed yield. In the next row of the second column immediately below the observed yields are shown the row average (taken from the final row in the field plan, Table 5.1).

TABLE 5.3

Treatment Allocation Plan 2

Treatment i (i_1, i_2, i_3)	Row Number, Yields and Row Average			T_i and $T_i - Q_i$	Q_i
1 (0,0,0)	(I) 19 16.833	(II) 16 18.500	(III) 19 19.167	54 54.5	-0.500
2 (0,0,1)	(I) 12 16.833	(II) 17 18.500	(III) 16 19.167	45 54.5	-9.500
3 (0,1,0)	(I) 17 16.833	(II) 18 18.500	(IV) 18 18.833	53 54.167	-1.167
4 (0,1,1)	(I) 18 16.833	(II) 24 18.500	(IV) 22 18.833	64 54.167	9.833
5 (1,0,0)	(I) 18 16.833	(III) 17 19.167	(IV) 16 18.833	51 54.833	-3.833
6 (1,0,1)	(I) 17 16.833	(III) 20 19.167	(IV) 18 18.833	55 54.833	0.167
7 (1,1,0)	(II) 18 18.500	(III) 21 19.167	(IV) 24 18.833	63 56.5	6.500
8 (1,1,1)	(II) 18 18.500	(III) 22 19.167	(IV) 15 18.833	55 56.5	-1.500

Step 2. Calculation of $u(\delta_1,\delta_2,\delta_3)$, $\tilde{u}(\delta_1,\delta_2,\delta_3)$, and $u^*(\delta_1,\delta_2,\delta_3)$. From the formulas (2.7) or (4.1)

$$u(0,0,0) = \frac{1}{r\theta(1,1,1)} = \frac{1}{2} = 0.5 \ ,$$

$$u(0,0,1) = \frac{1}{rm_3}\left(\frac{1}{\theta(1,1,0)} - \frac{1}{\theta(1,1,1)}\right) = -\frac{1}{12} = -0.08333 \ ,$$

$$u(0,1,0) = \frac{1}{rm_2}\left(\frac{1}{\theta(1,0,1)} - \frac{1}{\theta(1,1,1)}\right) = 0 \ ,$$

$$u(0,1,1) = \frac{1}{rm_2m_3}\left(\frac{1}{\theta(1,0,0)} - \frac{1}{\theta(1,0,1)} - \frac{1}{\theta(1,1,0)} + \frac{1}{\theta(1,1,1)}\right) = 0 \ ,$$

$$u(1,0,0) = \frac{1}{rm_1}\left(\frac{1}{\theta(0,1,1)} - \frac{1}{\theta(1,1,1)}\right) = 0 \ ,$$

$$u(1,0,1) = \frac{1}{rm_1 m_3}\left(\frac{1}{\theta(0,1,0)} - \frac{1}{\theta(0,1,1)} - \frac{1}{\theta(1,1,0)} + \frac{1}{\theta(1,1,1)}\right) = 0 \text{ , and}$$

$$u(1,1,0) = \frac{1}{rm_1 m_2}\left(\frac{1}{\theta(0,0,1)} - \frac{1}{\theta(0,1,1)} - \frac{1}{\theta(1,0,1)} + \frac{1}{\theta(1,1,1)}\right) = \frac{-1}{24}$$

$$= -0.04167 \text{ .}$$

Similarly, from (5.6) and (5.13) we have

$\tilde{u}(0,0,0) = \frac{1}{3} = 0.3333$, $\tilde{u}(0,0,1) = \frac{1}{48} = 0.02083$, $\tilde{u}(0,1,0) = 0$,

$\tilde{u}(0,1,1) = 0$, $\tilde{u}(1,0,0) = 0$, $\tilde{u}(1,0,1) = 0$, and $\tilde{u}(1,1,0) = 0$.

$u^*(0,0,0) = 0.5$, $u^*(0,0,1) = -0.06250$, $u^*(0,1,0) = 0$, $u^*(0,1,1) = 0$,

$u^*(1,0,1) = 0$, $u^*(1,1,0) = -0.04167$.

Step 3. Calculation of estimates of treatment effects.

(1). Estimation of the treatment effects eliminating column effects and ignoring row effects.

TABLE 5.4

Treatment i	$u(0,0,0)Q_{i_1 i_2 i_3}$	$u(0,0,1)Q_{i_1 i_2 \cdot}$	$u(1,1,0)Q_{\cdot\cdot i_3}$	$\hat{t}_i$
1	-2.625	0.91663	-0.04167	-1.75004
2	-2.875	0.91663	0.04167	-1.91670
3	0.500	-0.58331	-0.04167	-0.12498
4	3.000	-0.58331	0.04167	2.45836
5	-0.375	0.33332	-0.04167	-0.08335
6	-1.625	0.33332	0.04167	-1.25001
7	3.000	-0.66664	-0.04167	2.29269
8	1.000	-0.66664	0.04167	0.37503

(2). Estimation of the treatment effects ignoring column effects and eliminating row effects.

TABLE 5.5

Treatment i	$\tilde{u}(0,0,0)Q_{i_1 i_2 i_3}$	$\tilde{u}(0,0,1)Q_{i_1 i_2 \cdot}$	$\tilde{t}_i$
1	-0.16667	-0.20830	-0.37497
2	-3.16667	-0.20830	-3.37494
3	-0.38889	0.18053	-0.20836
4	3.27774	0.18053	3.45793
5	-1.27776	-0.07638	-1.35414
6	0.05556	-0.07638	-0.02082
7	2.16665	0.10415	2.27080
8	-0.50000	0.10415	-0.39585

(3). Estimation of the treatment effects eliminating column and row effects.

TABLE 5.6

Treatment i	$u^*(000)Q_{i_1 i_2 i_3}$	$u^*(001)Q_{i_1 i_2 \cdot}$	$u^*(110)Q_{\cdot \cdot i_3}$	t^*_i
1	-2.37500	0.62500	-0.04167	-1.79167
2	-2.62500	0.62500	0.04167	-1.95833
3	0.91667	-0.54167	-0.04167	0.33333
4	3.41667	0.54167	0.04167	2.91667
5	-0.29167	0.22917	-0.04167	-0.10417
6	-1.54167	0.22917	0.04167	-1.27083
7	2.25000	-0.31250	-0.04167	1.89583
8	0.25000	-0.31250	0.04167	-0.02083

Step 4. Analysis of variance

Method I.

(i). From Table 5.1, the total sum of squares (SS_T) is

$$SS_T = 19^2 + 12^2 + \cdots + 18^2 - 440^2/24 = 177.33333 \ .$$

(ii). The treatment sum of squares eliminating column and row effects (SS_{t*}) is, from step 3, (3),

$$SS_{t*} = \sum_i t_i^* Q_i^*$$

$$= (-1.79167)(-4.75) + (-1.95833)(-5.25) + \cdots$$

$$+ (-0.02083)(0.500) = 51.83331 \quad .$$

(iii). SS_c'' = "column unadjusted (ignoring treatment effects)" sum of squares

$$= \sum_j B_j^2/k - (\text{total})^2/bk$$

$$= 89.33333 \quad .$$

(iv). SS_r'' = "row unadjusted (ignoring treatment effects)" sum of squares

$$= 19.3333 \quad .$$

(v). SS_e = "error" sum of squares

$$= SS_T - SS_{t*} - SS_c'' - SS_r''$$

$$= 177.33333 - 51.83331 - 89.33333 - 19.33333$$

$$= 16.83336 \quad .$$

Table 5.7 is then obtained.

TABLE 5.7

Analysis of Variance, Form I.

Source of Variation	df	SS	MS	E(SS)
Treatment, (elim. C and R)	7	51.83331	$s_t^2 = 7.40476$	
Column, (ignor. treat.)	5	89.33333		
Row, (ignor. treat.)	3	19.33333		
Error	8	16.83336	$s_e^2 = 2.10417$	$8\sigma^2$
Total	23	177.33333		

The observed F-ratio of 3.52 with 7 and 8 degrees of freedom is significant at 5 per cent level.

Method II.

(i). From step 3, (1),

$SS_{\hat{t}}$ = "treatment sum of squares" ignoring row effect and eliminating column effects

$$= \sum_i \hat{t}_i Q_i = (-1.75004)(-5.25) + (-1.91670)(-5.75) + \cdots$$
$$+ (0.37503)(2.00) = 53.45916 .$$

(ii). SS_e' = Error sum of squares ignoring row effects

$$= SS_T - SS_{\hat{t}} - SS_c''$$
$$= 177.33333 - 53.45916 - 89.33333$$
$$= 34.54084 .$$

(iii). SS_t'' = treatment sum of squares ignoring column and row effects

$$=(1/3)(54^2 + 45^2 + \cdots + 55^2) - (440)^2/24$$
$$= 88.66667 .$$

(iv). SS_c' = column sum of squares eliminating treatment effects

$$= SS_T - SS_t - SS_e'$$
$$= 177.33333 - 88.66667 - 34.54084$$
$$= 54.58476 .$$

(v) . SS_r = row sum of squares eliminating treatment and column effects

$$= SS_T - SS_t'' - SS_c' - SS_e$$
$$= 177.33333 - 88.66667 - 54.58476 - 16.83336$$
$$= 17.70748 .$$

These results are presented in Table 5.8.

Method III.

(i). SS_t = treatment sum of squares ignoring column effects eliminating row effects

$$= \sum_i \tilde{t}_i \tilde{Q}_i$$
$$= (-0.37497)(-0.500) + (-3.37494)(-9.500) + \cdots$$
$$+ (-0.39585)(-1.500) = 86.91810 .$$

(ii). SS_e'' = error sum of squares ignoring column effects

$$= SS_T - SS_{\tilde{t}} - SS_r''$$

$$= 177.33333 - 86.91810 - 19.33333 = 71.08190 .$$

(iii). SS_r' = row sum of squares eliminating treatment effects

$$= SS_T - SS_{t*} = SS_r''$$

$$= 177.33333 - 88.66667 - 71.08190 = 17.58476 .$$

(iv). SS_c = column sum of squares eliminating treatment effects and row effects

$$= SS_T - SS_{t*} - SS_r' - SS_e$$

$$= 177.33333 - 88.66667 - 17.58476 - 16.83336 = 54.24854 .$$

We then obtain Table 5.9.

TABLE 5.8

Analysis of Variance, Form II.

Source of Variation	df	SS	Expected SS
Treat. ignor. C and R	7	88.66667	
Column elim. T ignor. R	5	54.12582	
Row elim. T. and C	3	17.70748	$3\sigma^2 + 16\sigma_\rho^2$
Error	8	16.83336	$8\sigma^2$
Total	23	177.33333	

(see Appendix [a.6])

The variance of the difference between the effects of i^{th} and j^{th} treatments is given by (5.14) and is estimated by replacing σ^2 by s_e^2 (taken from Table 5.7).

Associate	Estimation of Variance of the difference Between Two Treatments
(000)	$(19/24)s_e^2$
(001)	$(21/24)s_e^2$
(010)	$(19/24)s_e^2$
(011)	$(21/24)s_e^2$
(100)	$(19/24)s_e^2$
(101)	$(21/24)s_e^2$
(110)	$(22/24)s_e^2$

Average: $(1/7)(1/24)(19 + 21 + 19 + 21 + 19 + 21 + 22)s_e^2 = (71/84)s_e^2$.

TABLE 5.9

Analysis of Variance, Form III.

Source of Variation	df	SS	Expected SS
Treat. ignor. C and R	7	88.66667	
Column elim. T and R	5	54.58476	$5\sigma^2 + 16\sigma_\gamma^2$
Row elim. T ignor. C	3	17.58476	
Error	8	16.83336	$8\sigma^2$
Total	23	177.33333	

(see Appendix [a.7])

Step 5. Analysis with recovery of inter-column and inter-row information

(a). Estimation of σ^2, σ_γ^2, and σ_ρ^2 .

Using Tables 5.7, 5.8, and 5.9, we obtain the following equations:

$$SS_c = 5\hat{\sigma}^2 + 16\hat{\sigma}_\gamma^2 ,$$

$$SS_r = 3\hat{\sigma}^2 + 16\ \hat{\sigma}_\rho^2 , \text{ and}$$

$$SS_e = 8\hat{\sigma}^2 ,$$ (see Appendix, [a.6], [a.7]) .

Then,

$$\hat{\sigma}^2 = 2.10417 ,$$

$$\hat{\sigma}^2_{\gamma} = 0.54659 , \text{ and}$$

$$\hat{\sigma}^2_{\rho} = 0.23739 .$$

(b). Calculation of $\hat{w}$, $\hat{w}_c$, $\hat{w}_r$, and $u^*(\delta_1,\delta_2,\delta_3)$.

$$\hat{w} = 1/\hat{\sigma}^2 = 1/2.10417 = 0.47524$$

$$\hat{w}_r = 1/(\hat{\sigma}^2+b\hat{\sigma}^2_{\rho}) = 1/3.52851 = 0.28340$$

$$\hat{w}_c = 1/(\hat{\sigma}^2+k\hat{\sigma}^2_{\gamma}) = 1/4.29053 = 0.23307$$

(c). Using formula (5.19), we obtain:

$$u_w(0,0,0) = 0.84490,\ u_w(0,0,1) = -0.05528 ,$$

$$u_w(0,1,0) = 0,\ u_w(0,1,1) = 0,\ u_w(1,0,0) = 0 ,$$

$$u_w(1,0,1) = 0,\ u_w(1,1,0) = -0.03587 .$$

(d). Calculation of $\underline{P}^*$ and $\underline{t}^w$.

Using formula (5.16), we obtain:

$$P^*_{000} = -1.40854,\quad P^*_{001} = -3.62726,\quad P^*_{010} = -0.06411 ,$$

$$P^*_{011} = 3.71051,\quad P^*_{100} = -1.08193,\quad P^*_{101} = -0.75507 ,$$

$$P^*_{110} = 3.02982,\quad P^*_{111} = 0.19658 ,$$

$$P^*_{00\cdot} = -5.03580,\quad P^*_{01\cdot} = 3.64640,\quad P^*_{10\cdot} = -1.83700 ,$$

$$P^*_{11\cdot} = 3.22640 ,$$

$$P^*_{\cdot\cdot 0} = 0.47524,\quad P^*_{\cdot\cdot 1} = -0.47524 .$$

Now, we obtain the treatment effects with recovery of the inter-column and inter-row information as shown in Table 5.10.

The variance of the difference between the effects of i^{th} and j^{th} treatments, $\text{var}(t^w_i - t^w_j)$ is given by the formula (5.20), and is estimated by using the values of w, w_c, and w_r in step 5, (b).

TABLE 5.10

i	$0.84490P^*_{i_1 i_2 i_3}$	$-0.05528P^*_{i_1 i_2 \cdot}$	$0.03587P^*_{\cdot\cdot i_3}$	t^w
000	-1.19008	0.27838	-0.01705	-0.92875
001	-3.06467	0.27838	0.01705	-2.76924
010	-0.05417	-0.20157	-0.01705	-0.27279
011	3.13501	-0.20157	0.01705	2.95049
100	-0.91412	0.10155	-0.01705	-0.82962
101	-0.63796	0.10155	0.01705	-0.51936
110	2.55989	-0.17836	-0.01705	2.36448
111	0.16609	-0.17836	0.01705	0.00478

APPENDIX

I. In the model (2.1), the block effects $\{\beta_j\}$ are assumed to independent random variates with mean zero and variance σ^2_β .

Using matrix notation, model (2.1) is expressed as follows:

$$\underline{y} = \underline{1}\mu + X_1\underline{t} + X_2\underline{\beta} + \underline{\epsilon} \, , \qquad (a.1)$$

where $\underline{y}$ is a bk × 1 observation vector, $\underline{1}$ is a bk × 1 column vector having all elements unity, X_1 is a bk × v matrix, X_2 is a bk × b matrix, μ is the overall constant effect, $\underline{t}$ is a v × 1 treatment effect vector, $\underline{\beta}$ is a b × 1 random block effect vector, and $\underline{\epsilon}$ is a bk × 1 independent experimental error vector having the variance-covariance matrix $\sigma^2 I_{bk}$.

Then, the total sum of squares corrected for the mean is expressed as:

$$\underline{y}'\underline{y} - (\underline{1}'\underline{y})'(\underline{1}'\underline{y})/bk \quad .$$

Let CF = $(1'\underline{y})'1'\underline{y})/bk$. Then, the treatment sum of squares ignoring block effects is

$$SS_t' = (1/r)(X_1'\underline{y})'(X_1'\underline{y}) - CF \quad .$$

Now, since $X_1'X_1 = rI_v$, $X_2'X_2 = kI_b$, $X_1'X_2 = N$, and $E(\mu), E(\underline{t}) = \underline{t}$, $E(\underline{\beta}) = 0$, $E(\underline{\epsilon}) = 0$,

$$E(\underline{y}'\underline{y}) = E\Big((\mu\underline{1}' + \underline{t}'X_1' + \underline{\beta}'X_2' + \underline{\epsilon}')(\underline{1}\mu + X_1\underline{t} + X_2\underline{\beta} + \underline{\epsilon})\Big) \, ,$$

$$= bk\mu^2 + 2r\mu\sum_i t_i + r\sum_i t_i^2 + bk\sigma^2_\beta + bk\sigma^2 \, ,$$

and since $\underline{1}'\underline{y} = bk\mu + r\underline{1}_v'\underline{t} + k\underline{1}_b'\underline{\beta} + \underline{1}'\underline{\epsilon}$,

$$E(CF) = E\Big((1'\underline{y})'(1'\underline{y})/bk\Big) = bk\mu^2 + 2r\mu\sum_i t_i + r(\sum_i t_i)^2/v + k\sigma_\beta^2 + \sigma^2 \quad .$$

Hence,

$$E(SS_T) = E(\underline{y}'\underline{y} - CF)$$

$$= r\Big[\sum_i t_i^2 - (\sum_i t_i)^2/v\Big] + (bk-b)\sigma_\beta^2 + (bk-1)\sigma^2 \quad .$$

Next, since $X_1'\underline{y} = X_1'\underline{1}\mu + rI_r\underline{t} + N\underline{\beta} + X_1'\underline{\epsilon}$,

$$E\Big((X_1'\underline{y})'(X_1'\underline{y})/r\Big) = bk\mu^2 + 2r\mu\sum_i t_i + r\sum_i t_i^2 + v\sigma_\beta^2 + v\sigma^2 \ ,$$

so,

$$E(SS_t') = E\Big((X_1'\underline{y})'(X_1'\underline{y})/r - CF\Big)$$

$$= r\Big(\sum_i t_i^2 - (\sum_i t_i)^2/v\Big) + (v-b)\sigma_\beta^2 + (v-1)\sigma^2 \quad .$$

Then

$$E(\text{Remainder}) = E(SS_T) - E(SS_t')$$

$$= (bk-v)\sigma_\beta^2 + (bk-v)\sigma^2 \quad .$$

and

$$E(SS_e) = (bk-v-b+1)\sigma^2 \quad .$$

Therefore,

$$E(SS_b\text{, block sum of squares eliminating treatment effects})$$

$$= (SS_T - SS_t' - SS_e)$$

$$= (bk-v)\sigma_\beta^2 + (b-1)\sigma^2 \quad . \qquad (a.2)$$

II. In a k × b rectangular experiment, the set-up assumed is:

$$y_{ijh} = \mu + t_i + \gamma_j + \rho_h + \epsilon_{ijh}$$

$$i = 1,2,\cdots,v \ ;$$

$$j = 1,2,\cdots,b \ ;$$

$$h = 1,2,\cdots,k, \text{ and } bk/v = r \quad .$$

Using matrix notation

$$\underline{y} = \underline{1}\mu + X_1\underline{t} + X_2\underline{\gamma} + X_3\underline{\rho} + \underline{\epsilon} \ , \qquad (a.3)$$

where $\underline{y}$ is a bk × 1 observation vector, $\underline{1}$ is a bk × 1 column vector with all elements unity, X_1 is a bk × v matrix, X_2 is a bk × b matrix, X_3 is a bk × k matrix, μ is a constant, $\underline{t} = (t_1, \cdots, t_v)'$ is a fixed treatment effect column vector, $\underline{\gamma}$ is a b × 1 independent random column effect vector such that $E(\underline{\gamma}) = 0$, $E(\underline{\gamma}\underline{\gamma}') = \sigma^2_\gamma I_b$, $\underline{\rho}$ is a k × 1 independent row effect vector such that $E(\underline{\rho}) = 0$, $E(\underline{\rho}\underline{\rho}') = \sigma^2_\rho I_k$, and $\underline{\epsilon}$ is a bk × 1 independent random experimental error vector such that $E(\underline{\epsilon}) = 0$, $E(\underline{\epsilon}\underline{\epsilon}') = \sigma^2 I_{bk}$. Note that

$$X_1'X_1 = rI_v, \quad X_2'X_2 = kI_b, \quad X_3'X_3 = bI_k \ ,$$

$$X_1'X_2 = N, \quad X_1'X_3 = \tilde{N}, \quad X_2'X_3 = J_{bk} \ .$$

Now,

$$E(\underline{y}'\underline{y}) = bk\mu^2 + r\mu\sum_i t_i + r\mu\sum_i t_i + r\sum_i t_i^2 + bk\sigma^2_\gamma$$
$$+ bk\sigma^2_\rho + bk\sigma^2 \ ,$$

and since $\underline{1}'\underline{y} = bk\mu + r\underline{1}_v'\underline{t} + k\underline{1}_b'\underline{\gamma} + b\underline{1}_k'\underline{\rho} + \underline{1}'\underline{\epsilon}$,

$$E(CF) = E\Big((\underline{1}'\underline{y})'(\underline{1}'\underline{y})/bk\Big)$$
$$= bk\mu^2 + 2r\mu\sum_i t_i + (r/v)\Big(\sum_i t_i\Big)^2 + k\sigma^2_\gamma + b\sigma^2_\rho + \sigma^2 \ .$$

Then,

$$E(SS_T) = E(\underline{y}'\underline{y} - CF)$$
$$= r\Big(\sum_i t_i^2 - (\sum_i t_i)^2/v\Big) + (bk-k)\sigma^2_\gamma + (bk-b)\sigma^2_\rho$$
$$+ (bk-1)\sigma^2 \ .$$

Since

$$X_1'\underline{y} = r\underline{1}_v\mu + rI_v\underline{t} + N\underline{\gamma} + \tilde{N}\underline{\rho} + X_1'\underline{\epsilon} \ ,$$

$$X_2'\underline{y} = k\underline{1}_b\mu + N'\underline{t} + kI_b\underline{\gamma} + J_{bk}\underline{\rho} + X_2'\underline{\epsilon} \ , \text{ and}$$

$$X_3'\underline{y} = b\underline{1}_k\mu + \tilde{N}'\underline{t} + J_{kb}\underline{\gamma} + bI_k\underline{\rho} + X_3'\underline{\epsilon} \ ,$$

we obtain

$E(SS_t''$, ignoring column and row effects)

$$= E\Big((X_1'\underline{y})'(X_1'\underline{y})/r - CF\Big)$$

$$= r\Big(\sum_i t_i^2 - (\sum_i t_i)^2/v\Big) + (trN'N-k)\sigma_\gamma^2$$

$$+ (trN'N-b)\sigma_\rho^2 + (v-1)\sigma^2 \quad .$$

$E(SS_c''$, ignoring treatment and row effects)

$$= E\Big((X_2'\underline{y})'(X_2'\underline{y})/k - CF\Big)$$

$$= (\underline{t}'\tilde{N}\tilde{N}'\underline{t})/k - (r/v)(\sum_i t_i)^2 + (bk-b)\sigma_\rho^2 + (k-1)\sigma^2 \; ,$$

and

$$E(SS_e) = (bk-v-b-k+2)\sigma^2 \quad .$$

Next, for treatment effects ignoring row effects,

$$\underline{Q} = X_1'\underline{y} - (1/k)NX_2'\underline{y}$$

$$= \Big(rI_v - (1/k)NN'\Big)\underline{t} + \Big(X_1' - (1/k)NX_2'\Big)\underline{\epsilon}^* \; ,$$

where $\underline{\epsilon}^* = X_3\rho + \underline{\epsilon}$.

$$\underline{Q}\underline{Q}' = \Big(rI_v - (1/k)NN'\Big)\underline{t}\underline{t}'\Big(rI_v - (1/k)NN'\Big)$$

$$+ \Big(X_1' - (1/k)NX_2'\Big)\underline{\epsilon}^*\underline{\epsilon}^{*\prime}\Big(X_1 - (1/k)X_2N'\Big) \; ,$$

and since $E(\underline{\epsilon}^*\underline{\epsilon}^{*\prime}) = X_3X_3'\sigma_\rho^2 + \sigma^2 I$,

$$E(\underline{Q}\underline{Q}') = C\underline{t}\underline{t}'C' + \Big(\tilde{N}\tilde{N}' - (r^2/k)J_{vv}\Big)\sigma_\rho^2 + C\sigma^2 \; ,$$

where

$$C = rI_v - (1/k)NN' \quad .$$

$E(SS_{\hat{t}}$, treatment sum of squares eliminating column effects and ignoring row effects) $= E(\hat{\underline{t}}'\underline{Q})$

$$= E\Big((C^+\underline{Q})'\underline{Q}\Big) = E(\underline{Q}'C^+\underline{Q}) = tr\Big(C^+(\underline{Q}\underline{Q}')\Big) \quad .$$

Hence,

$$E(SS_{\hat{t}}) = tr(C^+C\underline{t}\underline{t}'C') + tr\Big(C^+(\tilde{N}\tilde{N}' - (r^2/k)J_v)\Big) + tr(C^+C\sigma^2)$$

$$= \underline{t}'C\underline{t} + tr\Big(C^+(\tilde{N}\tilde{N}' - (r^2/k)J)\Big)\sigma_\rho^2 + (v-1)\sigma^2 \quad .$$

Next,

$$E(SS_e', \text{ error sum of squares ignoring row effects})$$
$$= E(SS_T - SS_{\hat{t}} - SS_c'')$$
$$= -\mathrm{tr}\Big(C^+(\tilde{N}\tilde{N}' - (r^2/k)J_v)\Big)\sigma_\rho^2 + (bk-b)\sigma_\rho^2$$
$$+ (bk-v-b+1)\sigma^2 ,$$

and

$$E(SS_c', \text{ column sum of squares eliminating treatment effects and ignoring row effects})$$
$$= E(SS_T - SS_t'' - SS_e')$$
$$= (bk-\mathrm{tr}N'N)\sigma_\gamma^2 + \mathrm{tr}\Big(C^+(\tilde{N}\tilde{N}' - (r/k)J_v)\Big)\sigma_\rho^2$$
$$+ (b-\mathrm{tr}\tilde{N}\tilde{N}')\sigma_\rho^2 + (b-1)\sigma^2 .$$

Finally,

$$E(SS_r, \text{ eliminating treatment and column effects})$$
$$= E(SS_T - SS_t'' - SS_c' - SS_e)$$
$$= E(SS_e' - SS_e) \tag{a.4}$$
$$= \Big(bk-b-\mathrm{tr}(C^+(\tilde{N}\tilde{N}' - (r^2/k)J))\Big)\sigma_\rho^2 + (k-1)\sigma^2$$
$$= \Big(bk-b-\mathrm{tr}(C^+\tilde{N}\tilde{N}')\Big)\sigma_\rho^2 + (k-1)\sigma^2 .$$

Similarly, we obtain

$$E(SS_c, \text{ eliminating treatment and row effects})$$
$$= \Big(bk-k\ \mathrm{tr}(\tilde{C}^+NN')\Big)\sigma_\gamma^2 + (b-1)\sigma^2 . \tag{a.5}$$

From Paik and Federer [1973a], we obtain

$$\mathrm{tr}\Big(\tilde{C}^+NN'\Big) = r\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s} \frac{1-\theta(x_1,x_2,\cdots,x_n)}{\tilde{\theta}(x_1,x_2,\cdots,x_n)} \prod_{i=1}^{n}(m_i-1)^{x_i}\right\} ,$$

$$\mathrm{tr}\Big(C^+\tilde{N}\tilde{N}'\Big) = b\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s} \frac{1-\tilde{\theta}(x_1,x_2,\cdots,x_n)}{\theta(x_1,x_2,\cdots,x_n)} \prod_{i=1}^{n}(m_i-1)^{x_i}\right\} ,$$

where $r\theta(x_1,x_2,\cdots,x_n)$ and $r\tilde{\theta}(x_1,x_2,\cdots,x_n)$ are the eigenvalues of C and $\tilde{C}$, respectively, and θ and $\tilde{\theta}$ are the efficency factors defined in section 2 and section 5.

In the example 5.2,

$$tr\left(C^{+}\widetilde{NN}'\right) = 6\left(0/1 + \tfrac{1}{9}/1 + 0/\tfrac{2}{3} + \tfrac{1}{9}/1 + 0/\tfrac{2}{3} + \tfrac{1}{9}/1 + 0/\tfrac{2}{3}\right) = 0 ,$$

so

$$E\left(SS_r\right) = (24 - 6 - 2)\sigma_\rho^2 + 3\sigma^2 = 16\sigma_\rho^2 + 3\sigma^2 \qquad (a.6)$$

$$tr\left(\tilde{C}^{+}NN\right) = 4\left(0/1 + 0/\tfrac{8}{9} + \tfrac{1}{3}/1 + 0/\tfrac{8}{9} + \tfrac{1}{3}/1 + 0/\tfrac{8}{9} + \tfrac{1}{3}/1\right) = 4 ,$$

so

$$E\left(SS_c\right) = (24 - 4 - 4)\sigma_\gamma^2 + 5\sigma^2 = 16\sigma_\gamma^2 + 5\sigma^2 \quad . \qquad (a.7)$$

REFERENCES

Bose, R. C., Clatworthy, W. H. and Shrikhande, S. S. (1954) Tables of Partially Balanced Designs with Two Associate Classes. Tech. Bull. No. 107, N. C. Agri. Exp. Station.

Cochran, W. G. and Cox, G. M. (1957) Experimental Designs, Wiley, N. Y.

Federer, W. T. (1955) Experimental Design, Theory and Application, Macmillan, N. Y.

Hinkelmann, K. (1964) Extended group divisible partially balanced incomplete block designs, Ann. Math. Statist. 35, 681-695.

Kempthorne, O. (1952) The Design and Analysis of Experiments, Wiley, N. Y.

Kshirsagar, A. M. (1957) On balancing in designs in which heterogeneity is eliminated in two directions, Calcutta Statist. Ass. Bull. 7, 161-166.

Kshirsagar, A. M. (1966) Balanced factorial designs, J. R. Statist. Soc. B 28, 559-567.

Kurkjian, B. and Zelen, M. (1963) Applications of the calculus of factorial arrangements. I. Block and direct product designs, Biometrika 50, 63-73.

Paik, U. B. and Federer, W. T. (1973a) On PA-type incomplete block designs and PAB-type rectangular designs, J. R. Statist. Soc. B 35, 245-251.

Paik, U. B. and Federer, W. T. (1973b) Partially balanced designs and properties A and B, Com. Statist. 1, 331-350.

Roy, P. M. (1953-4) Hierarchical group divisible incomplete block designs with m-associate classes, Science and Culture 19, 210-211.

Shah, B. V. (1958) On balancing in factorial experiments, Ann. Math. Statist. 29, 766-779.

Shah, B. V. (1960) Balanced factorial experiments, Ann. Math. Statist. 31, 502-514.

Tocher, K. D. (1952) The design and analysis of block experiments, J. R. Statist. Soc. B 14, 45-100.

Zelen, M. and Federer, W. T. (1964) Applications of the calculus for factorial arrangements. II. Two way elimination of heterogeneity, Ann. Math. Statist. 35, 658-672.