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ANALYSIS OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS.(U)

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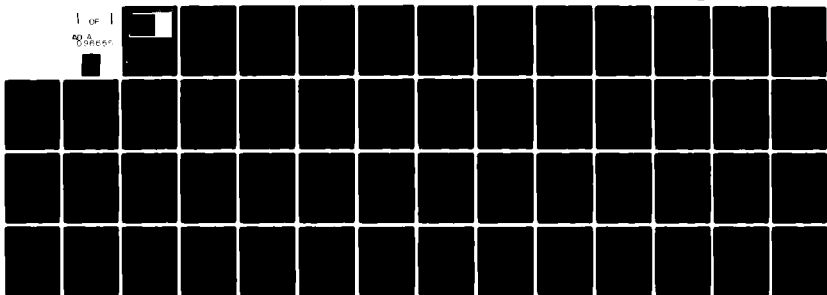
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ON INFINITE INTERVALS

Peter A. Markowich

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ABSTRACT

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In this paper boundary value problems on infinite intervals are treated. There is a theory for problems of this kind which requires the fundamental matrix of the system of differential equations to have certain decay properties near infinity. The aim of this paper is to establish a theory which holds under weaker and more realistic assumptions. The analysis for linear problems is done by determining the fundamental matrix of the system of differential equations asymptotically. For inhomogeneous problems a suitable particular solution having a 'nice' asymptotic behaviour is chosen and so global existence and uniqueness theorems are established in the linear case. The asymptotic behaviour of this solution follows immediately. Non-linear problems are treated by using perturbation techniques meaning linearization near infinity and by using the methods for the linear case. Moreover, some practical problems from fluid dynamics and thermodynamics are dealt with and they illustrate the power of the asymptotic methods used.

AMS(MOS) Subject Classification: 34B15, 34C05, 34C11, 34D05, 34B05.

Key Words: Nonlinear boundary value problems, singular points, boundedness, asymptotical properties, asymptotic expansions.

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SIGNIFICANCE AND EXPLANATION

Boundary value problems on infinite intervals occur in many areas of physics, for example in fluid mechanics when similarity solutions of the Navier-Stokes equations describing the flow over an infinite medium are sought. Problems which are dealt with in this paper have the following form. We look for a solution of a system of differential equations over an infinite interval which is continuous at infinity and which fulfills certain boundary conditions at a finite point and at infinity. The following questions arise immediately: which condition on the differential equation and on the boundary conditions assure the existence and the uniqueness of a solution and how does this solution depend on the data. There is a well known theory for problems of this kind, but it can only be applied to problems where the fundamental matrix of the system of differential equations has certain convergence properties near infinity. However this assumption is not fulfilled for many practical problems. This paper answers the above named question under very weak assumptions on the problems. Moreover some fluid dynamical problems illustrating the power of the theory developed are dealt with.

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ANALYSIS OF BOUNDARY VALUE PROBLEMS
ON INFINITE INTERVALS

PETER A. MARKOWICH

1. Introduction.

This paper is concerned with the analysis of boundary value problems on infinite intervals posed in the following way

$$(1.1) \quad y' = t^\alpha f(t, y), \quad 1 < t < \infty, \quad \alpha \in \mathbb{N}_0$$

$$(1.2) \quad y \in C([1, \infty]): \Leftrightarrow y \in C([1, \infty)) \text{ and } \lim_{t \rightarrow \infty} y(t) = y(\infty) \text{ exists}$$

$$(1.3) \quad b(y(1), y(\infty)) = 0$$

where y is an n -vector, f and b are nonlinear mappings.

Equation (1.1) has a singularity of the second kind at $t = \infty$ of rank $\alpha + 1$, because we assume that f is continuous in $(\infty, y(\infty))$. The goal is to establish existence and uniqueness theorems for very general f 's and b 's, to gain information on the behaviour of y for large t and - in the linear inhomogeneous case - to investigate the dependence of y on the boundary data and the inhomogeneity.

Problems of this kind frequently occur in fluid dynamics when stationary similarity solutions of the Navier-Stokes equations for certain flow-constellations are sought (see for example McLeod (1969); Markowich (1980); Lentini and Keller (1980), Cohen, Fokas and Lagerstrom (1978)).

For application in other areas of physics see for example Lentini (1978).

Much analytical work has been done on singular boundary value problems of the second kind. F. de Hoog and Richard Weiss (1980a,b) investigated the case where $\frac{\partial f}{\partial y}(\infty, y(\infty))$ has

no eigenvalue on the imaginary axis by linearizing f around $y^{(\infty)}$ and evaluating at $t = \infty$ so getting the constant coefficient problem

$$(1.5) \quad z' = t^\alpha f_y(\infty, y^{(\infty)})z$$

and by employing perturbation techniques which are based on estimates derived for a certain particular solution of linear inhomogenous problems. They could establish uniqueness and existence theorems. M. Lentini and H. B. Keller (1980) extended this approach neglecting the assumption on the eigenvalues but they required that the projection of $f_y(t, y^{(\infty)})$ onto the direct sum of invariant subspaces of $f_y(\infty, y^{(\infty)})$ which correspond to an imaginary eigenvalue, converge, at least as $t^{-(\alpha+1)r-\epsilon}$, where r is the largest dimension of these subspaces and $\epsilon > 0$. It turns out that this assumption is crucial and the perturbation approach breaks down if $f_y(\infty, y^{(\infty)})$ has eigenvalues with real part zero and if the convergence requirement is neglected. However many practical problems do not fulfill this convergence-requirement in the presence of imaginary eigenvalues (see for example Cohen, Fokas and Lagerstrom (1978) and M. Lentini and V. Pereyra (1977) and therefore a more general approach is necessary.

In this paper asymptotic series are used in order to determine asymptotically fundamental solution matrices of linear systems of the form:

$$(1.6) \quad y' = t^\alpha A(t)y \quad t > \delta.$$

The basic assumption is that A is analytical in $[\delta, \infty]$ for some $\delta > 1$, so that

$$(1.7) \quad A(t) = \sum_{i=0}^{\infty} A_i t^{-i} \quad \text{where} \quad A_i = \frac{1}{i!} \lim_{x \rightarrow 0^+} \frac{d^i}{dx^i} A\left(\frac{1}{x}\right).$$

Then a fundamental matrix $\Phi(t)$ of (1.6) can be calculated as an asymptotic (formal) logarithmic series from the coefficients A_i by a recursive algorithm (see Coddington and Levinson (1955) and Wasow (1965)).

Assumption (1.7) can be weakened so that only a finite but large enough number of these derivatives exist. Equation (1.1) is treated by linearization around $y^{(\infty)}$ obtaining the variable coefficient problem

$$(1.8) \quad z' = t^\alpha f_y(t, y^{(\infty)})z$$

and again by employing similar perturbation techniques. The advantages of the formal-series approach is twofold. Firstly no restrictions (except (1.7)) have to be made

concerning the convergence behaviour of $f_y(t, y^{(\infty)})$, secondly the asymptotic behaviour of the (basic) solutions is obtained directly. The asymptotic behaviour is crucial for the determination of appropriate numerical procedures for problems (1.1), (1.2), (1.3). (see Lentini and Keller (1980), Markowich (1980) and de Hoog and Weiss (1980b)).

This paper is organized in the following way, in chapter 1 some remarks are made on linear inhomogenous constant coefficient problems (see Lentini and Keller (1980)), in Chapter 2 the case where $A^{(\infty)}$ has distinct eigenvalues is treated, in Chapter 4 we admit a general Jordan canonical form of $A^{(\infty)}$ and in Chapter 5 we get to nonlinear problems of the form (1.1), (1.2), (1.3). Chapter 6 is concerned with practical examples which illustrate the power of the used asymptotic methods.

2. Linear problems with constant coefficients.

We consider problems of the form

$$(2.1) \quad y' - t^\alpha Ay = t^\alpha f(t) \quad 1 < t < \infty, \quad \alpha \in \mathbb{R}, \quad \alpha > -1$$

$$(2.2) \quad y \in C([1, \infty))$$

$$(2.3) \quad B_1 y(1) + B_\infty y(\infty) = \hat{\beta}$$

where the $n \times n$ -Matrix $A \neq 0$.

At first we transform A to its Jordan canonical form J

$$(2.4) \quad A = E J E^{-1}$$

and substitute

$$(2.4a) \quad u = E^{-1} y.$$

So we get the new problem

$$(2.5) \quad u' - t^\alpha J(t)u = t^\alpha E^{-1} f(t)$$

$$(2.6) \quad u \in C([1, \infty))$$

$$(2.7) \quad B_1 E u(1) + B_\infty E u(\infty) = \hat{\beta}.$$

Without loss of generality we can assume that J has the block diagonal form

$$(2.8) \quad J = \text{diag}(J^+, J^0, J^-)$$

where the real part of the eigenvalues of J^+ are positive. The real parts of the eigenvalues of J^0 are equal to zero and the real parts of the eigenvalues of J^- are negative. This structure can always be obtained by reordering the columns of E . Let the dimensions of these three matrices be r_+ , r_0 resp. r_- .

The diagonal projections D_+ , D_0 , D_- are obtained by taking the main-diagonal of J and by replacing every eigenvalue with positive, zero resp. negative real part by 1 and all others by zero so that

$$(2.9) \quad I = D_+ + D_0 + D_-$$

holds.

Furthermore let \hat{D}_0 be the projection onto the direct sum of eigenspaces of J associated with zero eigenvalues, which is obtained by replacing by zero every (diagonal) element of D_0 which is not associated with the first column of a Jordan block of J belonging to a zero eigenvalue.

Let the number of nonzero columns of \bar{D}_0 which equals the geometrical multiplicity of the eigenvalue zero, be \bar{r}_0 . The general solution of the homogenous problem (2.5), (2.6) is:

$$(2.10) \quad u_h(t) = \phi(t)(\bar{D}_0 + D_-)\xi = \exp\left(J \frac{t^{\alpha+1}}{\alpha+1}\right) (\bar{D}_0 + D_-)\xi, \quad \xi \in C^n.$$

In order to solve the inhomogenous problem (2.5) we look for a particular solution

$$u_p \in C([1, \infty)).$$

F. de Hoog and R. Weiss (1980a,b) and M. Lentini and H. B. Keller (1980) suggested the following choice:

$$(2.11) \quad \begin{aligned} u_p(t) = (Hf)(t) = & \phi(t) \int_{\infty}^t D_+ \phi^{-1}(s) E^{-1} f(s) s \, ds + \\ & + \phi(t) \int_{\infty}^t D_0 \phi^{-1}(s) E^{-1} f(s) s^{\alpha} \, ds + \\ & + \phi(t) \int_{\delta}^t D_- \phi^{-1}(s) E^{-1} f(s) s^{\alpha} \, ds \end{aligned}$$

with $\delta \in [1, \infty)$.

We denote the three terms on the right hand side of (2.11) by

$u_{p_+} = H_+ f$, $u_{p_0} = H_0 f$ resp. $u_{p_-} = H_- f$. F. de Hoog and Richard Weiss (1980a) showed that u_{p_+} and u_{p_-} are in $C([1, \infty))$ if $D_- E^{-1} f$ and $D_+ E^{-1} f$ are in $C([1, \infty))$ and that $J(D_+ + D_-)u_p(\infty) = -(D_+ + D_-)E^{-1} f(\infty)$ holds.

M. Lentini and H. B. Keller (1980) showed that

$$\|u_{p_0}(t)\| = O(t^{-\epsilon}) \quad \text{if} \quad \|D_0 E^{-1} f\| = O(t^{-(\alpha+1)r-\epsilon})$$

where $\epsilon > 0$ and r is the maximal dimension of the invariant subspaces of J associated with eigenvalues on the imaginary axis. Therefore the operator H operates on the space of all functions f , which fulfill:

$$(2.12) \quad f \in C([1, \infty)) \quad \text{and} \quad D_0 E^{-1} f(t) = F_0(t) t^{-(\alpha+1)r-\epsilon}$$

with $F_0 \in C_b([1, \infty))$, where $C_b([1, \infty))$ is the space of functions which are continuous on $[1, \infty)$ and bounded as $t \rightarrow \infty$.

Inserting the general solution of (2.5), (2.6) into (2.7) we get

$$(2.13) \quad \begin{aligned} & [(B_1 E + B_\infty E) \bar{D}_0 + B_1 \exp(\frac{J}{\alpha+1}) D_-] \xi = \\ & = \hat{\beta} - [B_1 E u_p(1) + B_\infty E u_p(\infty)]. \end{aligned}$$

Therefore we conclude

Theorem 2.1. The problem (2.1), (2.2), (2.3) has a unique solution y for

all f which fulfill (2.12) and $\hat{\beta} \in \mathbb{R}^{\bar{r}_0 + r_-}$, if and only if

$$(2.14) \quad \text{rank}[(B_1 E + B_\infty E) \bar{D}_0 + B_1 \exp(\frac{J}{\alpha+1}) D_-] = \bar{r}_0 + r_-$$

where B_1 and B_∞ are $(\bar{r}_0 + r_-) \times n$ matrices.

In this case y depends continuously (in the norm $\|y\|_{[1,\infty]}$) :=
 $:= \max_{t \in [1,\infty]} \|y(t)\|$) on the data $\hat{\beta}$, $(D_+ + D_-)E^{-1}f$ and F_0 .

The 1st statement follows directly from estimates given in the papers cited above. We see that (2.2) is an additional boundary condition at $t = \infty$ of the rank $r_+ + (r_0 - \bar{r}_0)$.

Now we investigate the decay properties of u_p in dependence of the decay properties of f :

Theorem 2.2. If f fulfills (2.12) then the following estimates hold for $t > \delta$:

$$(2.15) \quad \|(H_+ f)(t)\| < \text{const.} \|D_+ E^{-1} f\|_{[t,\infty]}$$

$$(2.16) \quad \|(H_0 f)(t)\| < \text{const.} t^{-\varepsilon} \max_{s > t} \|s^{(\alpha+1)r+\varepsilon} D_0 E^{-1} f(s)\|$$

with $\varepsilon > 0$.

If $\gamma > 0$ then

$$(2.17) \quad \|(H_- f)(t)\| < \text{const.} t^{-\gamma} \max_{\delta \leq s \leq t} \|s^\gamma D_- E^{-1} f(s)\|$$

all constants are independent of f and δ .

The first two estimates have been proved by M. Lentini and H. B. Keller (1980).

The third estimate follows from

$$\|u_{P_-}(t)\| < c \cdot \max_{i=1(1)k} \left(e^{-\frac{\lambda}{\alpha+1}t^{\alpha+1}} \int_{\delta}^t e^{\frac{\lambda}{\alpha+1}s^{\alpha+1}} (t^{\alpha+1} - s^{\alpha+1})^{i-1} s^{\alpha-\gamma} ds \right) \max_{\delta < s < t} \|s^{\gamma} D_{-} E^{-1}(s)\|$$

where $-\lambda$ is the largest real part of eigenvalues of J with $\lambda > 0$ and k is the maximal dimension of the associated Jordan-blocks.

By applying Taylor's formula to

$$g_i(t) = \frac{\int_{\delta}^t e^{\frac{\lambda}{\alpha+1}s^{\alpha+1}} (t^{\alpha+1} - s^{\alpha+1})^{i-1} s^{\alpha-\gamma} ds}{e^{\frac{\lambda}{\alpha+1}t^{\alpha+1}} t^{-\gamma}}$$

it is easy to conclude that $\lim_{t \rightarrow \infty} g_i(t) = \text{const.}$ In particular Theorem 2.2 tells us that inhomogenities which converge to zero as a power function produce a particular solution which converges as a power function whose exponent is increased by $(\alpha + 1)r$.

Now we want to investigate exponentially decreasing inhomogenities.

If $\int_{\delta}^{\infty} D_{-} \phi^{-1}(s) E^{-1} f(s) s^{\alpha} ds$ exists we can substitute H_{-} by \tilde{H}_{-} which is defined by

$$(2.18) \quad (\tilde{H}_{-} f)(t) = \phi(t) \int_{\infty}^t D_{-} \phi^{-1}(s) E^{-1} f(s) s^{\alpha} ds.$$

Now we prove:

Theorem 2.3. Let J^{-} consist of Jordan blocks belonging to the same eigenvalue $-\lambda$ and let k be the dimension of the largest of these blocks.

Furthermore let $D_{-} E^{-1} f(t) = t^{\beta} \exp(-\frac{\omega}{\alpha+1} t^{\alpha+1}) F_{-}(t)$ with $F_{-} \in C_D([1, \infty))$, $\beta \in \mathbb{R}$ and $\omega > 0$. Then for $t > \delta$:

$$(2.19) \quad \|(H_{-} f)(t)\| < \text{const.} \cdot t^{\beta} \exp(-\frac{\omega}{\alpha+1} t^{\alpha+1}) \|F_{-}\|_{[\delta, t]}$$

if $\text{Re} \lambda - \omega > 0$.

$$(2.20) \quad \|(H_{-} f)(t)\| < \text{const.} \exp(-\frac{\omega}{\alpha+1} t^{\alpha+1}) t^{(\alpha+1)k+\beta} \ln t \cdot \|F_{-}\|_{[\delta, t]}$$

if $\text{Re} \lambda - \omega = 0$ and $\beta > -k(\alpha+1)$ The factor $\ln t$ only appears

if $\beta = -k(\alpha+1)$.

$$(2.21) \quad \|\tilde{H}_{-} f(t)\| < \text{const.} \exp(-\frac{\omega}{\alpha+1} t^{\alpha+1}) \cdot t^{k(\alpha+1)+\beta} \|F_{-}\|_{[t, \infty]}$$

if $\text{Re} \lambda - \omega = 0$ and $\beta < -k(\alpha+1)$.

$$(2.22) \quad \|\tilde{H}_- f(t)\| \leq \text{const.} \exp\left(-\frac{\omega}{\alpha+1} t^{\alpha+1}\right) t^{\beta} \|F_-\|_{[t, \infty]}$$

if $\text{Re} \lambda - \omega < 0$.

All constants are independent of f and δ .

The proofs work analogously to Theorem 2.2. Theorem 2.3 implies that exponentially decaying inhomogenities produce inhomogenities which converge with the same exponential factor, only the algebraic factor may change and a logarithmic factor may appear. If \tilde{H}_- exists then it cuts off the terms of the particular solution which are already included in $\phi(t)D_-\xi$.

Assume now that J^- consists of more than one Jordan block with different eigenvalues and that $D_- E^{-1} f(t)$ has the form as in Theorem 3. Then H_- and \tilde{H}_- may be used in order to gain a particular solution which decays as fast as possible according to the different cases of Theorem 2.3. Doing this D_- has to be split up into the projections onto the direct sums of the invariant subspaces associated with different eigenvalues with negative real part and H_- resp \tilde{H}_- have to be applied to the resulting subsystems. We will call the resulting operator \bar{H} . Its composition depends on the decay properties of f and on J^- .

3. Linear Variable Coefficient Problems - Distinct Eigenvalues.

Now we analyze

$$(3.1) \quad y' - t^\alpha A(t)y = t^\alpha f(t), \quad \alpha \in \mathbb{N}_0$$

$$(3.2) \quad y \in C([1, \infty))$$

$$(3.3) \quad B_1 y(1) + B_\infty y(\infty) = \hat{\beta}$$

The $n \times n$ matrix $A(t)$ fulfills:

$$(3.4) \quad A \in C([1, \infty)), \quad A(\infty) \neq 0$$

$$(3.5) \quad A \text{ is analytic on } [\delta, \infty) \text{ for some } \delta > 1$$

so that

$$(3.6a) \quad A(t) = \sum_{i=0}^{\infty} A_i t^{-i} \text{ for } t \text{ sufficiently large where}$$

$$(3.6b) \quad A_i = \frac{1}{i!} \lim_{x \rightarrow 0^+} \frac{d^i}{dx^i} A\left(\frac{1}{x}\right).$$

The basic assumption of this chapter is that the eigenvalues of $A_0 = A(\infty)$ are distinct.

Let J_0 be the Jordan canonical form of A_0 obtained by the transformation

$$(3.7) \quad A_0 = E J_0 E^{-1}$$

and let the J_i 's be defined by

$$(3.8) \quad A_i = E J_i E^{-1}.$$

The matrices J_i are the coefficients of the series

$$(3.9) \quad J(t) = E^{-1} A(t) E = \sum_{i=0}^{\infty} J_i t^{-i} \text{ for } t \rightarrow \infty.$$

We set

$$(3.10) \quad J_0 = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \lambda_i \neq \lambda_j \text{ for } i \neq j$$

and assume that:

$$(3.11) \quad \left. \begin{array}{l} \text{Re} \lambda_i > 0 \quad \text{for } 1 \leq i \leq r_+ \\ \text{Re} \lambda_{r_+ + j} = 0 \quad \text{for } 1 \leq j \leq r_0 \\ \text{Re} \lambda_{r_+ + r_0 + k} < 0 \quad \text{for } 1 \leq k \leq r_- \end{array} \right\} r_+ + r_0 + r_- + n.$$

Again we substitute

$$(3.12) \quad u = E^{-1} y$$

and get the problem

$$(3.13) \quad u' = t^\alpha J(t)u = t^{\alpha-1} E^{-1} f(t), \quad 1 < t < \infty$$

$$(3.14) \quad u \in C([1, \infty)).$$

For the following we need the definition of an asymptotic series. A function $P(t)$ is said to be represented asymptotically by a formal series:

$$(3.15) \quad P(t) \sim \sum_{i=0}^{\infty} P_i t^{-i}, \quad t \rightarrow \infty$$

if

$$(3.16) \quad t^m [P(t) - \sum_{i=0}^m P_i t^{-i}] \rightarrow 0 \text{ for } t \rightarrow \infty \text{ and } m > 0.$$

Therefore

$$(3.17) \quad P(t) = \sum_{i=0}^m P_i t^{-i} + O(t^{-m-1}) \text{ for } t \rightarrow \infty \text{ and } m > 0$$

holds. To get more information on asymptotic series see Wasow (1965).

Coddington and Levinson (1955) and Wasow (1965) prove an asymptotic representation of the fundamental matrix of the homogenous system (3.13) which we state in

Theorem 3.1. Under the given assumptions on $J(t)$ the homogenous system (3.13) possesses a fundamental solution of the form:

$$\phi(t) = P(t) t^D e^{Q(t)}.$$

Here $D = \text{diag}(d_1, \dots, d_n)$, $Q(t) = \text{diag}(q_1(t), \dots, q_n(t))$ where the $q_i(t)$ are polynomials in t of degree $\alpha + 1$, so that

$$Q(t) = J_0 \frac{t^{\alpha+1}}{\alpha+1} + Q_1 \frac{t^\alpha}{\alpha} + \dots + Q_\alpha t$$

with diagonal matrices Q_i . $P(t)$ admits the asymptotic expansion:

$$P(t) \sim \sum_{i=0}^{\infty} P_i t^{-i} \quad \text{for } t \rightarrow \infty$$

and $P_0 = P(\infty) = I$.

The unknown coefficients P_i , Q_i and D can be calculated by algebraic operations from the J_i 's. We get an algorithm for the calculation of these coefficients by inserting the asymptotic series for ϕ and J into the differential equation, by formal differentiation of ϕ and by comparing the coefficients of the corresponding powers of t (see Wasow (1965)).

For an arbitrary matrix B let $\text{diag}(B)$ denote the matrix which has only zero off-diagonal entries and the same main diagonal as B .

In the case $\alpha = 0$ we find $D = \text{diag}(J_1)$. For $\alpha > 0$ we pursue in the following way:

1) Set $Q_1 := \text{diag}(J_1)$ and let \tilde{P}_1 be the solution matrix of the equation:

$$\tilde{P}_1 J_0 - J_0 \tilde{P}_1 = J_1 - Q_1$$

with $\text{diag}(\tilde{P}_1) = 0$. This equation is uniquely soluble because the diagonal elements of J_0 are distinct.

2) \tilde{P}_k and Q_k are determined recursively. Set

$$Q_k = \text{diag} \left(\sum_{l=1}^{k-1} (J_l \tilde{P}_{k-l} - \tilde{P}_{k-l} Q_l) + J_k \right) \quad \text{for } 2 \leq k \leq \alpha$$

and let \tilde{P}_k with $\text{diag}(\tilde{P}_k) = 0$ be the solution matrix of:

$$\tilde{P}_k J_0 - J_0 \tilde{P}_k = \sum_{l=1}^{k-1} (J_l \tilde{P}_{k-l} - \tilde{P}_{k-l} Q_l) + (J_k - Q_k).$$

3) Set $D := \text{diag} \left(\sum_{l=1}^{\alpha} (J_l \tilde{P}_{\alpha+1-l} - \tilde{P}_{\alpha+1-l} Q_l) + J_{\alpha+1} \right)$.

The matrices $J_0, \dots, J_{\alpha+1}$ determine Q_0, \dots, Q_{α} and D . In the presence of eigenvalues of $A(\infty)$ with a real part zero it is therefore not sufficient to know $A(\infty) = A_0$ in order to determine whether the fundamental solutions which correspond to these eigenvalues are in $C([1, \infty])$ or not. More (at most $\alpha+2$) coefficients of the series expansion of $\lambda(t)$ have to be known. Moreover an eigenvalue with real part zero can produce a basic solution which is exponentially increasing or decreasing, algebraically increasing or decreasing, constant or undampenedly oscillating as t approaches infinity. So the i -th basic solution is in $C([1, \infty])$ if and only if:

$$(3.18) \quad \lim_{t \rightarrow \infty} e^{\int_1^t q_i(\tau) d\tau} \text{ exists.}$$

Lentini and Keller (1980) treated the case $\alpha = 0$ and $D_0 J_1 = 0$ where D_0 is again the diagonal projection onto the direct sum of invariant subspaces associated with eigenvalues of J_0 with real part zero (D_-, D_+ are also defined as in Chapter 2). Under this assumptions a zero eigenvalue of J_0 generates a solution which has a power-series expansion in t^{-1} without an exponential and algebraic factor. Therefore it is sufficient to investigate the system

$$(3.19) \quad v' - t^\alpha J_0 v = 0$$

and to apply a perturbation approach.

Let \tilde{r}_0 be the number of solutions generated by imaginary eigenvalues of J_0 fulfilling the condition (3.18) and let \tilde{D}_0 be the projection onto the direct sum of invariant subspaces associated with these eigenvalues. Then the general solution of the homogenous problem (3.13), (3.14) is

$$(3.20) \quad u_h = \phi(t)(\tilde{D}_0 + D_-)\xi, \quad \xi \in \mathbb{C}^n.$$

Now we have to determine a particular solution $u_p \in C([1, \infty))$.

In order to do this we denote by D_{0i} for $1 < i < r_0$ the diagonal projection onto the eigenspace associated with the i -th imaginary eigenvalue of J_0 . D_{0i} has only one nonzero entry in the $(r_+ + i)$ -th column of the main diagonal.

Now we impose conditions on f :

$$(a) \quad f \in C([1, \infty)),$$

$$(b) \quad D_{0i} P^{-1}(t) E^{-1} f(t) = t^{-k_i} F_{0i}(t) \in C([1, \infty))$$

(3.21)

$$\text{if } |\operatorname{Re} q_{r_++i}(t)| = 0(t^{\alpha+1-k_i}) \text{ and } \operatorname{Re} q_{r_++i} \neq 0$$

$$\text{for } i = 1(1)r_0, \quad 1 < k_i < \alpha$$

$$(c) \quad D_{0i} P^{-1}(t) E^{-1} f(t) = t^{-\alpha-1-\epsilon_i} F_{0i}(t),$$

$$F_{0i} \in C_b([1, \infty)), \quad \epsilon_i > 0 \text{ if } \operatorname{Re} q_{r_++i} \equiv 0.$$

Therefore we require at most that

$$(3.22) \quad \|f(t)\| = O(t^{-\alpha-1-\epsilon}), \quad \epsilon > 0.$$

Now we define:

$$(3.23) \quad (a) \quad u_p = u_{p_+} + u_{p_0} + u_{p_-}, \quad (b) \quad u_{p_0} = \sum_{j=1}^{r_0} u_{p_{0j}}$$

$$(3.24) \quad (a) \quad u_p = Hf, \quad u_{p_+} = H_+f, \quad u_{p_0} = H_0f, \quad u_{p_-} = H_-f,$$

$$(b) \quad u_{p_{0i}} = H_{0i}f$$

and

$$(3.25) \quad u_{p_+}(t) = \phi(t) \int_{\infty}^t D_+ \phi^{-1}(s) s^{\alpha-1} f(s) ds$$

$$(3.26) \quad u_{p_-}(t) = \phi(t) \int_{\delta}^t D_- \phi^{-1}(s) s^{\alpha-1} f(s) ds$$

$$(3.27) \quad u_{p_{0i}}(t) = \begin{cases} \phi(t) \int_{\infty}^t D_{0i} \phi^{-1}(s) s^{\alpha-1} f(s) ds, & \text{if (A) holds} \\ \phi(t) \int_{\delta}^t D_{0i} \phi^{-1}(s) s^{\alpha-1} f(s) ds, & \text{if (B) holds} \end{cases}$$

where:

$$(A) \quad \operatorname{Re} q_{r+i}(t) \rightarrow \infty \quad \text{or} \quad (\operatorname{Re} q_{r+i} \equiv 0 \quad \text{and} \quad \epsilon_i > -\operatorname{Re} d_{r+i})$$

$$(B) \quad \operatorname{Re} q_{r+i}(t) \rightarrow -\infty \quad \text{or} \quad (\operatorname{Re} q_{r+i} \equiv 0 \quad \text{and} \quad \epsilon_i < -\operatorname{Re} d_{r+i}).$$

Then $u_p^{(\infty)}$ exists and fulfills the linear equation:

$$J_0(D_+ + D_-)u_p^{(\infty)} = -(D_+ + D_-)E^{-1}f^{(\infty)}$$

We prove that u_{p_+} is continuous in $t = \infty$, the proofs for u_{p_0} and u_{p_-} are similar.

Using Theorem 3.1 and (3.28) we get

$$u_{p_+}(t) = P(t) t^D e^{Q(t)} \int_{\infty}^t D_+ e^{-Q(s)} s^{-D+\alpha I} P^{-1}(s) E^{-1} f(s) ds.$$

Looking at the i -th component of $P^{-1}(t)u_{p_+}(t)$ we find,

$$(P^{-1}(t)u_{p_+}(t))_i = \int_{\infty}^t e^{q_i(t) - q_i(s)} \left(\frac{t}{s}\right)^{d_i} s^{\alpha} (P^{-1}(s)E^{-1}f(s))_i ds$$

Applying del'Hospital's limit theorem to

$$g_i(t) = \frac{\int_{-\infty}^t e^{-q_i(s)} s^{\alpha-d_i} (P^{-1}(s)E^{-1}f(s))_i ds}{e^{-q_i(t)} t^{-d_i}}$$

we get because of the convergence of the integral in the numerator

$$(u_{P_+}(\infty))_i = \lim_{t \rightarrow \infty} g_i(t) = \lim_{t \rightarrow \infty} \frac{(P^{-1}(t)E^{-1}f(t))_i}{-q_i'(t)t^{-\alpha} - d_i t^{-\alpha-1}} = \frac{(E^{-1}f(\infty))_i}{-\lambda_i}.$$

We have used that $P(t) = I + O(t^{-1})$ and $P^{-1}(t) = I + O(t^{-1})$ are continuous in $t = \infty$.

As in paragraph 2 we substitute $u_h + u_p$ into the boundary condition:

$$(B_1 E \phi(1)(\tilde{D}_0 + D_-) + B_\infty E \tilde{D}_{00}) \xi = \hat{\beta} - B_1 E u_p(1) - B_\infty E u_p(\infty).$$

There \tilde{D}_{00} is the projection onto the direct sum of invariant subspaces associated with those imaginary eigenvalues of J_0 , which produce basic solutions which do not vanish at infinity.

Theorem 3.2. The problem (3.1), (3.2), (3.3) under the assumptions (3.4), (3.5), (3.10)

has a unique solution y for all f fulfilling (3.21) and $\hat{\beta} \in R^{\tilde{r}_0 + r_-}$ if and only if

$$\text{rank}[B_1 E \phi(1)(\tilde{D}_0 + D_-) + B_\infty E \tilde{D}_{00}] = \tilde{r}_0 + r_-$$

where B_1 and B_∞ are $(\tilde{r}_0 + r_-) \times n$ matrices. This solution depends continuously on $\hat{\beta}$, $(D_+ + D_-)E^{-1}P^{-1}f$ and F_0 where F_0 fulfills $D_{0i}F_0 = F_{0i}$.

For the proof of the continuity statement we need

Theorem 3.3. If f fulfills (3.21) then the following estimates hold for $t > \delta$:

$$(3.28) \quad \|(H_+ f)(t)\| < \text{const.} \|D_+ P^{-1} E^{-1} f\|_{[t, \infty]}.$$

If $\gamma > 0$ then:

$$(3.29) \quad \|(H_- f)(t)\| < \text{const.} t^{-\gamma} \max_{\delta < s < t} \|s^\gamma D_- P^{-1} E^{-1} f(s)\|.$$

For $i = 1(1)r_0$:

If $\text{Re } q_{r_+ + i} \rightarrow +\infty$ then

$$(3.30) \quad \|(H_{0i} f)(t)\| < \text{const.} \|F_i\|_{[t, \infty]}.$$

If $\operatorname{Re} \sigma_{r_+ + i} = 0$ and $\epsilon_1 > -\operatorname{Re} d_{r_+ + i}$ then

$$(3.31) \quad |(H_{01} f)(t)| < \text{const. } t^{-\epsilon_1} |F_1|_{[t, \infty]}.$$

If $\operatorname{Re} \sigma_{r_+ + i} \rightarrow -\infty$ and $\gamma > 0$ then

$$(3.32) \quad |(H_{01} f)(t)| < \text{const. } t^{-\gamma} \max_{\delta < s < t} |s^\gamma F_1(s)|.$$

If $\operatorname{Re} \sigma_{r_+ + i} \equiv 0$ and $\epsilon_1 < -\operatorname{Re} d_{r_+ + i}$ then

$$(3.33) \quad |(H_{01} f)(t)| < \text{const. } t^{-\epsilon_1} |F_1|_{[\delta, t]}$$

If $\operatorname{Re} \sigma_{r_+ + i} \equiv 0$ and $\epsilon_1 = -\operatorname{Re} d_{r_+ + i}$ then

$$(3.34) \quad |(H_{01} f)(t)| < \text{const. } t^{-\epsilon_1} \ln t |F_1|_{[\delta, t]}.$$

All constants are independent of δ and f .

We prove estimate (3.30) and the others are proven quite analogously.

From (3.12) and (3.27) we derive

$$|H_{01} f)(t)| < \text{const. } \frac{\int_0^\infty \exp(-\operatorname{Re} \sigma_{r_+ + i}(s)) s^{-\operatorname{Re} d_{r_+ + i} + \alpha - k_1} ds}{\exp(-\operatorname{Re} \sigma_{r_+ + i}(t)) t^{-\operatorname{Re} d_{r_+ + i}}} |F_1|_{[t, \infty]}$$

Applying del'Hospital's rule and using $|\operatorname{Re} \sigma_{r_+ + i}(t)| = O(t^{\alpha+1-k_1})$ the estimate (3.30) follows. Herewith Theorem (3.2) is proven completely. An inhomogeneity f fulfilling (3.22) produces a particular solution which decays at least as fast as $t^{-\epsilon} \ln t$.

Now we again investigate exponentially decaying inhomogenities.

We define \tilde{H}_- as in Chapter 2, \tilde{H}_{01} as (3.27a), \hat{H}_{01} as (3.27b) no matter whether (A) or (B) holds. Of course the definitions make sense only for inhomogenities which make the appearing integrals exist.

We assume that

$$(3.35) \quad f(t) = e^{p(t)} t^\beta F(t), \quad F \in C_b([1, \infty)), \quad \beta \in \mathbb{R}$$

where $p(t)$ is a real polynomial of maximal degree $\alpha + 1$ and $p(t) \rightarrow -\infty$ for $t \rightarrow \infty$. We split up the operators H_- , \tilde{H}_- into H_{-j} , \tilde{H}_{-j} by substituting the projection D_- by D_{-j} (for $j = 1(1)r_-$) which is the projection onto the invariant subspace generated by the j -th eigenvalue of J_0 with real part smaller than zero.

Theorem 3.4. Let (3.35) hold. Then the following estimates can be derived for $t > \delta$:

$$\| (H_{-j} f)(t) \| \leq \text{const. } e^{p(t)} t^{\beta + \alpha + 1 - \ell_i} \|F\|_{[\delta, t]}$$

if $p(t) = \text{Re}(q_{r_+ + r_0 + j}(t)) \rightarrow +\infty$ and ℓ_i is the degree of the polynomial $p(t) = \text{Re}(q_{r_+ + r_0 + j}(t))$

$$(3.37) \quad \| (H_{-j} f)(t) \| \leq \text{const. } e^{p(t)} t^{\alpha + \beta + 1} \|F\|_{[\delta, t]}$$

if $p(t) = \text{Re}(q_{r_+ + r_0 + j}(t))$ and $\beta > \text{Re}(d_{r_+ + r_0 + j}) - \alpha - 1$.

$$(3.38) \quad \| (H_{-j} f)(t) \| \leq \text{const. } e^{p(t)} t^{\alpha + \beta + 1} \ln t \|F\|_{[\delta, t]}$$

if $p(t) = \text{Re}(q_{r_+ + r_0 + j}(t))$ and $\beta = \text{Re}(d_{r_+ + r_0 + j}) - \alpha - 1$

$$(3.39) \quad \| (\tilde{H}_{-j} f)(t) \| \leq \text{const. } e^{p(t)} t^{\alpha + \beta + 1} \|F\|_{[t, \infty]}$$

if $p(t) = \text{Re}(q_{r_+ + r_0 + j}(t))$ and $\beta < \text{Re}(d_{r_+ + r_0 + j}) - \alpha - 1$.

$$(3.40) \quad \| (\tilde{H}_{-j} f)(t) \| \leq \text{const. } e^{p(t)} t^{\beta + \alpha + 1 - \ell_i} \|F\|_{[t, \infty]}$$

if $p(t) = \text{Re}(q_{r_+ + r_0 + j}(t)) \rightarrow -\infty$.

The estimates (3.37), (3.38), (3.39) are also valid if H_{-j} resp. \tilde{H}_{-j} and the index $r_+ + r_0 + j$ are substituted by \hat{H}_{-j} resp. \tilde{H}_{0i} and the index $r_+ + i$ for $i = 1(1)r_0$. The proof of Theorem 3.4 is quite similar to that of Theorem 3.3.

As in the constant coefficient case exponentially decaying inhomogeneities produce particular solutions which converge with the same exponential factor, only the algebraic factor may change and $\ln t$ as a factor may appear. If the inhomogeneity f contains $(\ln t)^s$ with $s \in \mathbb{N}$ as a factor then all derived estimates are valid if their right hand sides are multiplied by $(\ln t)^s$. This follows by direct estimation of the appearing integrals and by using the Theorems 3.3 and 3.4. The operator which produces the best

order of convergence according to Theorems 3.3 and 3.4 and which is composed of the operators $H_+, \tilde{H}_{01}, \hat{H}_{01}, H_{-1}, \tilde{H}_{-1}$ is again denoted by \bar{H} . The composition of \bar{H} depends on the decay properties of f and on $J_0, J_1, \dots, J_{\alpha+1}$.

We want to weaken the condition (3.6) which is a very strong assumption on $A(t)$. The basic idea for this is that the determination of the matrices Q_0, \dots, Q_α, D only requires the knowledge of $A_0, \dots, A_{\alpha+1}$ while the matrices $A_{\alpha+2}, \dots$ do not influence the exponential and algebraic factors of the fundamental matrix, they only influence the power series factor $P(t)$.

We assume that

$$(3.41) \quad A\left(\frac{1}{t}\right) \in C^{\alpha+2}\left(\left[0, \frac{1}{\delta}\right]\right), \quad \delta > 1$$

$$A \in C([1, \infty))$$

and therefore we can write

$$(3.42) \quad A(t) = A_0 + t^{-1}A_1 + \dots + t^{-\alpha-1}A_{\alpha+1} + \tilde{A}(t), \tilde{A}(t) = a(t)t^{-\alpha-2-\beta}$$

where $a \in C_b([1, \infty))$ and $\beta > 0$.

The equation

$$y' = t^\alpha A(t)y + f(t), \quad t > 1, \quad y \in C([1, \infty))$$

where f fulfills (3.22) can be rewritten as

$$(3.43) \quad v' = t^\alpha \left(\sum_{i=0}^{\alpha+1} A_i t^{-i} \right) y + (\tilde{A}(t)y + f(t)), \quad y \in C([1, \infty)).$$

The homogenous problem (3.43) with $f \equiv 0$ has the general solution

$$(3.44) \quad y_h = E \phi(\tilde{D}_0 + D_-) \xi + E \bar{H}_1 \tilde{A} y_h, \quad \xi \in C^n$$

where E transforms A_0 into its Jordan form and $E\phi$ is the fundamental matrix of the unperturbed problem

$$\tilde{y}_h = t^\alpha \left(\sum_{i=0}^{\alpha+1} A_i t^{-i} \right) \tilde{y}_h.$$

Now let

$$(3.45) \quad \phi(t)(\tilde{D}_0 + D_-) = p(t)t^{\alpha} e^{a(t)} = p(t)\sigma_h(t), \quad p \in C([1, \infty))$$

\bar{H}_1 is composed so that inhomogenities which decay as $t^{-\alpha-2-\beta}\sigma_h(t)$ produce fast decaying particular solutions regarding Theorems 3.3 and 3.4.

From (3.44) we get the equation:

$$(3.46) \quad (I - E\bar{H}_1\tilde{A})y_h = E\phi(\tilde{D}_0 + D_-)\xi$$

for which we take as basic Banach-space:

$$(3.47) \quad (A_{\sigma_h, \delta} = \{u|u(t) = U(t)\sigma_h(t), U \in C_b([\delta, \infty)), \|u\|_{\sigma_h, \delta} = \|U\|_{[\delta, \infty)}\}.$$

If $\sigma_h(t) \equiv 1$ then we set $A_{\sigma_h, \delta} = C([\delta, \infty))$.

We get the estimate

$$(3.48) \quad \|\bar{H}_1\tilde{A}\|_{\sigma_h, \delta} = \max_{\|y_h\|_{\sigma_h, \delta} < 1} \|\bar{H}_1\tilde{A}y_h\|_{\sigma_h, \delta} < \text{const.} \cdot \delta^{-1-\beta} \ln \delta < \frac{1}{2|\bar{E}|}$$

if δ is sufficiently large. Therefore $(I - E\bar{H}_1\tilde{A})^{-1}$ exists as an operator on $A_{\sigma_h, \delta}$ and for $\xi \in C^n$:

$$(3.49) \quad y_h = (I - E\bar{H}_1\tilde{A})^{-1}E\phi(\tilde{D}_0 + D_-)\xi = \psi^0(\tilde{D}_0 + D_-)\xi \in A_{\sigma_h, \delta}.$$

As particular solution y_p of (3.43) we set,

$$(3.50) \quad y_p = E\bar{H}_3\tilde{A}y_p + E\bar{H}_2f.$$

The inhomogeneity f fulfills

$$(3.51) \quad \|f(t)\| = O(t^{\bar{d}} e^{\bar{q}(t)}) \quad \text{and} \quad \sigma_p(t) = t^{\alpha+1+\bar{d}} \ln t e^{\bar{q}(t)} + o$$

and \bar{H}_2 is composed so that $(\bar{H}_2f)(t)$ decays as fast as possible with regard to

Theorems 3.3 and 3.4. Then

$$(3.52) \quad \|(\bar{H}_2f)(t)\| = O(\sigma_p(t))$$

\bar{H}_3 is composed to make particular solutions belonging to inhomogeneities which decay as $t^{-\alpha-2-\beta} \sigma_p(t)$ decrease as fast as possible. As basic space we now take $A_{\sigma_p, \delta}$ and

conclude the invertibility of $(I - E\bar{H}_3\tilde{A})$ on $A_{\sigma_p, \delta}$

with δ sufficiently large and get

$$(3.53) \quad y_p = \psi(f) = (I - E\bar{H}_3\tilde{A})^{-1}E\bar{H}_2f \in A_{\sigma_p, \delta}$$

Obviously

$$(3.54) \quad y(t) = y_h(t) + y_p(t) \in C([\delta, \infty))$$

holds. By substituting H , which is defined by (3.24), for \bar{H}_1 in (3.44) it is easily shown that the solution manifold y_h (with the parameters $(\tilde{D}_0 + D_-)\xi$) is unique in $C([\delta, \infty))$, because $\Lambda_{\sigma_h, \delta} \in C([\delta, \infty))$ and because the solution space is $\tilde{r}_0 + r_-$ dimensional. Therefore y defined in (3.54) is unique in $C([\delta, \infty))$ (as manifold).

In order to get the solution in $C([1, \infty))$ we solve the 'regular' problem:

$$(3.55) \quad y' = t^\alpha \Lambda(t)y + t^\alpha f(t) \quad 1 < t < \delta$$

$$(3.56) \quad y(\delta) = y_h(\delta) + y_p(\delta).$$

From (3.44) and (3.49) we get, using the expansion

$$(I - G)^{-1} = \sum_{i=0}^{\infty} G^i \quad \text{for } |G| < 1$$

the following estimates which hold for $t > \delta$:

$$(3.57) \quad \|\psi_-^0(t) - E\phi(t)(\tilde{D}_0 + D_-)\| < \text{const. } t^{-1-\beta} \ln t \cdot \sigma_h(t)$$

and

$$(3.58) \quad \|(\psi(f))(t) - E(\bar{H}_2 f)(t)\| < \text{const. } t^{-1-\beta} \ln t \cdot \sigma_p(t).$$

Theorem 3.2 remains valid if $E\phi(1)(\tilde{D}_0 + D_-)$ is substituted by $\psi_-^0(1)$ where $\psi_-^0(t)$ has been continued to $[1, \infty)$.

4. Linear Variable Coefficient Problems - General case.

In this paragraph problems of the form (3.1), (3.2), (3.3) fulfilling (3.4), (3.5), (3.6) but not (3.10) are investigated. So we assume a general Jordan-structure of J_0 in the block diagonal form

$$(4.1) \quad J_0 = \text{diag}(J_0^+, J_0^0, J_0^-), \quad \dim(J_0^+) = r_+, \quad \dim(J_0^0) = r_0, \quad \dim(J_0^-) = r_-$$

where J_0^+, J_0^0 resp. J_0^- contain the Jordan blocks which are associated with eigenvalues with real part greater, equal resp. smaller than zero. As in the case of distinct eigenvalues of J_0 we determine an asymptotic expression for the fundamental matrix. This expression is given by Wasow (1965):

Theorem 4.1. Under the given assumption on $J(t)$ there is a fundamental matrix which has the form

$$\phi(t) = P(t)t^D e^{Q(t)}$$

where $Q(t)$ is a diagonal matrix:

$$Q(t) = \text{diag}(J_0) \frac{t^{\alpha+1}}{\alpha+1} + Q_1 \frac{t^{\alpha+1-\frac{1}{p}}}{p(\alpha+1)-1} + Q_2 \frac{t^{\alpha+1-\frac{2}{p}}}{p(\alpha+1)-2} + \dots$$

$$\dots + Q_{p(\alpha+1)-2} \frac{t^{\frac{2}{p}}}{2} + Q_{p(\alpha+1)-1} \frac{1}{t^{\frac{1}{p}}}$$

with some $p \in \mathbb{N}$, D is a constant matrix in Jordan canonical form and

$$P(t) = P_1(t)P_2(t)$$

where

$$P_1(t) \sim I + \sum_{i=1}^{\infty} P_{1i} t^{-i}, \quad T \rightarrow \infty$$

and

$$P_2(t) \sim \sum_{i=0}^{\infty} P_{2i} t^{-\frac{i}{p}}, \quad t \rightarrow \infty$$

The (diagonal) elements of $Q(t)$ which correspond to a particular Jordan block of D are equal. Therefore t^D and $e^{Q(t)}$ commute. Moreover the block structure of D is a subdivision of that blocking of J_0 which is obtained by gathering all Jordan blocks of J_0 belonging to the same eigenvalue. Also $P_2(t)$ has a block structure which is identical to the above mentioned blocking of J_0 .

Contrary to the case of distinct eigenvalues $A(t)$ is now a polynomial in $t^{\frac{1}{p}}$, where p is some positive integer and $P(t)$ has an expansion in descending powers of

$t^{\frac{1}{p}}$, D is no longer strictly diagonal, it is in Jordan canonical form and maybe the most important difference is, that $P(\infty) = P_{20}$ does no longer have to be regular which implies that $P(t)^{-1}$ may grow undoundedly as $t \rightarrow \infty$.

However, the proof of this asymptotic expansion for $\phi(t)$ given by Wasow (1965) is constructive and therefore contains an algorithm for the calculation of P , D and Q . We assume that J_0 has the different eigenvalues $\lambda_1, \dots, \lambda_k$ and the block diagonal form

$$(4.2) \quad J_0 = \text{diag}(J_0^1, \dots, J_0^k), \quad \dim(J_0^i) = r_i$$

J_0^i has the only eigenvalue λ_i . Then the following algorithm results:

1) Substitute $u = T_1(t)u_1$ and determine

$$T_1(t) \sim \sum_{j=1}^{\infty} T_1^j t^{-j}$$

so that the resulting system

$$u_1' = t^\alpha B_1(t)u_1, \quad B_1(t) \sim J_0 + \sum_{j=1}^{\infty} B_1^j t^{-j}$$

split up into two separate ones. The first of them, which is r_1 -dimensional, has the form

$$u_{(1)}' = t^\alpha J_{(1)}(t)u_{(1)} \quad \text{with} \quad J_{(1)}(t) \sim J_0^1 + \sum_{j=1}^{\infty} J_{(1)}^j t^{-j}$$

where $u_{(1)}$ is the vector consisting of the first r_1 components of u_1 . The algorithm for the recursive determination of T_1^j and B_1^j is given by Wasow (1965), paragraph 11.

The remaining $(n-r_1)$ -dimensional system is treated in the same way. A r_2 -dimensional system with J_0^2 as leading matrix is split off. Finally we get k separate systems

$$(4.3) \quad u_{(i)}' = t^\alpha J_{(i)}(t)u_{(i)} \quad \text{with} \quad J_{(i)}(t) \sim J_0^i + \sum_{j=1}^{\infty} J_{(i)}^j t^{-j}.$$

The transformation

$$(4.4) \quad u = P_1(t) \begin{bmatrix} u_{(1)} \\ \vdots \\ u_{(k)} \end{bmatrix}$$

where

$$(4.12) \quad (a) \quad h_i = (\alpha + 1 - g_i) p_i^{-1}, \quad (b) \quad C_{(i)}(x_i) \sim \sum_{j=0}^{\infty} C_{(i)}^j x_i^{-j}$$

4a) If $C_{(i)}^0$ has only 0 as eigenvalue then we have reduced the rank of the system or it splits up into separate subsystems of lower order. Applying more shearing transformations to (4.11) we end up with a system whose leading matrix has either more than one different eigenvalue or has rank equal to 0. In the second case we have a system with a singularity of the first kind for which the fundamental matrix $\phi_{(i)}(t)$ has the form

$$(4.13) \quad \phi_{(i)}(t) = P_{(i)}(t) t^{D_{(i)}}, \quad P_{(i)}(t) = \sum_{j=0}^{\infty} P_{(i)}^j t^{-j}$$

and $D_{(i)}$ is a constant matrix in Jordan canonical form (see Wasow (1965)).

4b) Now we assume that $C_{(i)}^0$ has at least two different eigenvalues. Then we transform $C_{(i)}^0$ to its Jordan canonical form $\tilde{C}_{(i)}^0$

$$(4.14) \quad C_{(i)}^0 = E_{(i)} \tilde{C}_{(i)}^0 E_{(i)}^{-1}$$

and substitute

$$(4.15) \quad w_{(i)}(x_i) = E_{(i)} z_{(i)}(x_i)$$

getting a system whose leading matrix is $\tilde{C}_{(i)}^0$, that means in Jordan form

$$(4.16) \quad z'_{(i)}(x_i) = x_i^{h_i} \tilde{C}_{(i)}(x_i) z_{(i)}(x_i).$$

5) We apply the transformation explained in 1) to the system (4.16) in order to get separate subsystems of lower order whose leading matrices have the only eigenvalue $\mu_{(i)j}$. By the means of 2) we normalize these systems so that their leading matrices have only the eigenvalue 0. These transformations split off the factors

$$(4.17) \quad \exp\left(\mu_{(i)j} \frac{x_i^{h_i+1}}{h_i+1}\right) \quad \text{for } i = 1(1)k.$$

Resubstituting in (4.10) and using (4.12)(a) we notice that the argument in (4.17) is of the order $t^{\alpha+1-g_i}$ that means of order lower than $t^{\alpha+1}$ which is the order of the argument of the first exponential factor because if $g_i = 0$ the system remained unchanged.

Applying another set of shearing transformations 3) we arrive at 4a) or 4b).

6) So a finite chain of all the in 1) -5) described transformations result in a set of one dimensional systems and systems with a singularity of the first kind. Setting $p = \bar{p}^{-m}$ where \bar{p} is the smallest common multiple of all the p 's used in the sequence of shearing transformations, and m is the number of these transformations which split the system into a set of systems described above, we get the formula for the fundamental matrix $\phi(t)$ given in Theorem 4.1 by taking into account (4.13). Moreover we get:

$$(4.18) \quad P_2(t) = \left(\prod_{i=1}^m S_i(t) E_i P_{2i}(t) \right) P_4(t), \quad P_{2i}(t) = I + \sum_{j=1}^{\infty} P_{2i}^j t^{-j}$$

and the $S_i(t)$ are composed of submatrices $S_{(ij)}(t)$ defined in (4.9). The E_i 's are regular and $P_4(t)$ is derived by solving the systems with singularities of the first kind using (4.13).

We define:

$$(4.19) \quad P_3(t) = \prod_{i=1}^m S_i(t) E_i P_{2i}(t).$$

An estimate of $P_3^{-1}(t)$ can be obtained in the following way.

Let D_i be the projection onto the direct sum of eigenspaces associated with the eigenvalue λ_i of J_0 . Then

$$(4.20) \quad \|P_3^{-1}(t) D_i\| \leq \text{const. } t^{[(r_i-1)(g_{i1} + \frac{g_{i2}}{p_{i1}} + \frac{g_{i3}}{p_{i1}p_{i2}} + \dots + \frac{g_{im}}{p_{i1} \dots p_{im-1}})]}$$

holds. The sum in the exponent of t is derived by estimating $S_i^{-1}(t)$ and by taking into account the block structures of the E_i^{-1} and $P_{2i}^{-1}(t)$. p_{ij} and g_{ij} are as in (4.10) and represent that sequence of shearings starting off from the i -th r_i -dimensional subsystem and giving the largest exponent in (4.20).

For this sequence we can calculate the ranks of the corresponding sequence of subsystems as in (4.12)

$$\begin{aligned}
 h_{i0} + 1 &= \alpha + 1 \\
 h_{i1} + 1 &= p_{i1}(h_{i0} + 1 - g_{i1}) \\
 h_{i2} + 1 &= p_{i2}(h_{i1} + 1 - g_{i2}) \\
 &\vdots \\
 h_{i,m} + 1 &= p_{im}(h_{i,m-1} + 1 - g_{im}) .
 \end{aligned}
 \tag{4.21}$$

By assumption $h_{i,m} + 1 > 0$ holds and we get recursively:

$$\alpha + 1 > g_{i1} + \frac{g_{i2}}{p_{i1}} + \dots + \frac{g_{im}}{p_{i1} \dots p_{i,m-1}}
 \tag{4.22}$$

and therefore the estimate

$$\|P_3^{-1}(t)D_1\| < \text{const. } t^{(r_i-1)(\alpha+1)}$$

holds.

The basic solutions φ_i with $\phi(t) = (\varphi_1(t), \dots, \varphi_{m_i}(t))$ fulfill

$$\|\varphi_i(t)\| < p_i(t) e^{\frac{q_i(t)}{t} d_i} t^{j_i} \ln t, \quad p_i \in C([1, \infty)).
 \tag{4.24}$$

Eigenvalues of J_0 with positive real part produce exponentially increasing, eigenvalues with negative real part produce exponentially decreasing basic solutions. Imaginary eigenvalues of J_0 can produce exponentially and algebraically increasing and decreasing constant and oscillating and logarithmically increasing solutions. The asymptotic behaviour of a particular basic solution φ_i can be determined by the knowledge of $Q(t)$, D and p_0, \dots, p_{m_i} where m_i is sufficiently large. Therefore the solution of the homogenous problem (3.16), (3.17) (under the assumptions of Chapter 4) is:

$$u_h(t) = \phi(t)(\tilde{D}_0 + D_-)\xi, \quad \xi \in C^n
 \tag{4.25}$$

where \tilde{D}_0 and D_- are defined as in Chapter 3.

Now we construct a particular solution u_p of the problem

$$(4.26) \quad u_p' = t^\alpha J(t)u_p + t^\alpha E^{-1}f(t) \quad 1 < t < \infty, f \in C([1, \infty))$$

$$(4.27) \quad u_p \in C([1, \infty)).$$

We substitute

$$(4.28) \quad u_p(t) = P_1(t)v_p(t) \quad \text{and} \quad v_p(t) = \begin{bmatrix} v_{P_+}(t) \\ v_{P_0}(t) \\ v_{P_-}(t) \end{bmatrix}$$

and define

$$(4.29) \quad u_{P_+}(t) = P_1(t) \begin{bmatrix} v_{P_+}(t) \\ 0 \\ 0 \end{bmatrix}, \quad u_{P_0}(t) = P_1(t) \begin{bmatrix} 0 \\ v_{P_0}(t) \\ 0 \end{bmatrix}$$

$$u_{P_-}(t) = P_1(t) \begin{bmatrix} 0 \\ 0 \\ v_{P_-}(t) \end{bmatrix}$$

so that the definitions (3.23a) and (3.24a) hold. We get the separated problems

$$\begin{bmatrix} v_{P_+} \\ v_{P_0} \\ v_{P_-} \end{bmatrix}' = t^\alpha (\text{diag}(J_0^+ + J^+(t), J_0^0 + J^0(t), J_0^- + J^-(t))) \begin{bmatrix} v_{P_+} \\ v_{P_0} \\ v_{P_-} \end{bmatrix} + t^\alpha P_1^{-1}(t)E^{-1}f(t)$$

where $J^+(t)$, $J^0(t)$, $J^-(t)$ have an asymptotic power series expansion in

t^{-1} without a constant term. We define the block components v_{P_+}, v_{P_-} as in de Hoog and

Weiss (1980a,b) as solutions of the operator equations:

$$(4.30)(a) \quad v_{P_+} = \hat{H}_+ J^+ v_{P_+} + \hat{H}_+ (P_1^{-1} E^{-1} f)_+$$

$$(4.30)(b) \quad v_{P_-} = \hat{H}_- J^- v_{P_-} + \hat{H}_- (P_1^{-1} E^{-1} f)_-$$

where $(P_1^{-1}E^{-1}f)_+$ resp. $(P_1^{-1}E^{-1}f)_-$ are the first r_+ resp. the last r_- components of $P_1E^{-1}f$ and \hat{H}_+, \hat{H}_- are the operators defined similar to (2.11).

$$(4.30)(c) \quad (\hat{H}_+g)(t) = \int_{\infty}^t \exp\left(\frac{J_0^+}{\alpha+1} (t^{\alpha+1} - s^{\alpha+1})\right) s^\alpha g(s) ds$$

$$(4.30)(d) \quad (\hat{H}_-g)(t) = \int_{\delta}^t \exp\left(\frac{J_0^-}{\alpha+1} (t^{\alpha+1} - s^{\alpha+1})\right) s^\alpha g(s) ds$$

for $\varepsilon \in C([1, \infty))$.

From (4.30) we derive

$$(4.31)(a) \quad v_{P_+} = (I - \hat{H}_+J^+)^{-1} \hat{H}_+(P_1^{-1}E^{-1}f)_+ \in C([\delta, \infty))$$

$$(4.31)(b) \quad v_{P_-} = (I - \hat{H}_-J^-)^{-1} \hat{H}_-(P_1^{-1}E^{-1}f)_- \in C([\delta, \infty))$$

with δ sufficiently large. The proof of the invertibility of $(I - \hat{H}_+, -J^+, -)$ is given in de Hoog and Weiss (1980a,b). Again we get:

$$(4.32) \quad (a) \quad v_{P_+}(\infty) = (J_0^+)^{-1}(E^{-1}f(\infty))_+, \quad (b) \quad v_{P_-}(\infty) = (J_0^-)^{-1}(E^{-1}f(\infty))_-$$

because $J^+(\infty) = 0$ and $J^-(\infty) = 0$.

Now we assume that for some $\varepsilon > 0$:

$$(4.33) \quad D_0P_1^{-1}(t)E^{-1}f(t) = F_0(t)t^{-\bar{r}(\alpha+1)-\varepsilon}, \quad F_0 \in C_b([1, \infty))$$

holds, where \bar{r} is the largest algebraic multiplicity of the eigenvalues of J_0 with real part zero.

So \bar{r} is defined differently to r in Chapter 2. However, more sophisticated assumptions on f could be used (similar to (3.21)) but we will use (4.33) for simplicity.

The system

$$(4.34) \quad v'_{P_0} = t^\alpha (J_0^0 + J^0(t)) v_{P_0}$$

is composed of separate systems, each of them associated with one imaginary eigenvalue of J_0 and \bar{r} is the maximal dimension of these subsystems.

We take one of these (inhomogeneous) subsystems

$$(4.35) \quad v'_{P_0(i)} = t^\alpha J_{(i)}(t) v_{P_0(i)} + t^\alpha (P_1^{-1}(t)E^{-1}f(t))_{0(i)}$$

where $\hat{f}_{(i)}(t) := (P^{-1}(t)E^{-1}f(t))_{0(i)}$ consists of the corresponding components of the inhomogeneity $P^{-1}(t)E^{-1}f(t)$. The leading matrix J_0^1 of $J_{(i)}(t)$ has the only eigenvalue λ_i with $\text{Re}\lambda_i = 0$.

Now we apply the transformations

$$(4.36) \quad v_{P_0(i)} = \exp\left(\frac{\lambda_i t^{\alpha+1}}{\alpha+1}\right) \cdot S_{(i)}(t) w_{P_0(i)}, \quad x_i = c(i)t^{\frac{1}{P_i}}$$

$$(4.37) \quad w_{P_0(i)}(x_i) = E_{(i)} z_{P_0(i)}(x_i)$$

as defined in (4.6), (4.9), (4.10) and (4.15) and get

$$(4.38) \quad z'_{P_0(i)}(x_i) = x_i^{h_i} \tilde{C}_{(i)}(x_i) z_{P_0(i)}(x_i) + x_i^{h_i} \hat{g}_{(i)}(x_i)$$

where

$$(4.39) \quad \hat{g}_{(i)}(x_i) = \left(\frac{x_i}{c(i)}\right)^{P_i g_i - 1} E_{(i)}^{-1} S_{(i)}^{-1} \left(\left(\frac{x_i}{c(i)}\right)^{P_i}\right) \exp\left(-\frac{\lambda_i}{\alpha+1} \left(\frac{x_i}{c(i)}\right)^{P_i(\alpha+1)}\right) \cdot \hat{f}_{(i)}\left(\left(\frac{x_i}{c(i)}\right)^{P_i}\right).$$

From (4.23), (4.33) we derive:

$$(4.40) \quad \|\hat{g}_{(i)}(x_i)\| < \text{const.} \cdot x_i^{-P_i \varepsilon - \bar{r}_{P_i}(\alpha+1-g_i)}$$

If the leading matrix of the system (4.38) has at least one eigenvalue different from zero then the separating transformation

$$z_{P_0(i)} = P_{2(i)}(x_i) z_{P_0(i)}^1, \quad P_{2(i)}^{(\infty)} = I$$

can be applied. All resulting subsystems, whose leading matrices have eigenvalues with real part different from zero can now be solved by the means of (4.30), (4.31) because the new inhomogeneity has the form

$$(4.41) \quad \hat{g}_{(i)}^1(x_i) = P_{2(i)}^{-1}(x_i) \hat{g}_{(i)}(x_i) \in C([1, \infty)).$$

For all other subsystems this sequence of substitutions is repeated as long as we arrive at systems whose leading matrices have eigenvalues with real part different from zero or one-dimensional systems or we arrive at systems with a singularity of rank zero. In the second and third cases only eigenvalues with real part zero have been split off therefore the inhomogeneities do not contain exponentially increasing or decreasing factors (see (4.39))

and therefore every inhomogeneity g_0 which occurs in this sequence of transformations fulfills after resubstitution to t as independent variable:

$$(4.42) \quad \|g_0(t)\| < \text{const.} \cdot |P_3^{-1}(t)D_i| t^{\alpha+1} |D_0 P_1^{-1}(t)E^{-1}f(t)| < \text{const.} \cdot t^{-\epsilon}$$

because (4.22) holds.

A particular solution for one-dimensional systems can be found according to Chapter 3, so we just have to treat systems with a singularity of rank zero.

We want to solve

$$(4.43) \quad z'_p = \left(\frac{1}{x} B + \frac{1}{x} \tilde{B}(x)\right) z_p + \frac{1}{x} g_0(x), \quad \tilde{B}(x) = \bar{B}(x) \frac{1}{x}$$

$$(4.44) \quad z_p(x) = z_p(x) x^{-\epsilon} (\ln x)^j, \quad z_p \in C_p([1, \infty)), \quad j \in N_0$$

where B is a constant matrix in Jordan form, $\bar{B} \in C([1, \infty))$ and g_0 fulfills (4.42).

Let

$$(4.45) \quad B = \text{diag}(B_1, \dots, B_q), \quad B_i = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & b_i & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}, \quad i = 1(1)q$$

and let D^i be the projection onto the invariant subspace associated with b_i .

Then we define

$$(4.46) \quad z_p = \tilde{G} z_p + G g_0, \quad G = G_1 + \dots + G_q$$

and

$$(4.47)(a) \quad (G_i g_0)(x) = \begin{cases} x^B \int_{\delta}^x D^i s^{-B-I} g_0(s) ds, & b_i < -\epsilon \\ x^B \int_x^{\infty} D^i s^{-B-I} g_0(s) ds, & b_i > -\epsilon. \end{cases}$$

It is easily checked that for $x > \delta$

$$(4.48) \quad \|(G g_0)(x)\| < \text{const.} \cdot (\ln x)^j \max_{s \in [x, \infty]} \|g_0(s)\|, \quad x^{-\epsilon} \cdot \max_{s \in [\delta, x]} \|s^{\epsilon} g_0(s)\|$$

holds where $j = \max(\dim(B_i))$. Therefore we get from (4.46)

$$(4.49) \quad z_p = (I - \tilde{G}B)^{-1} G g_0$$

z_p fulfills (4.44) because of a contraction argument similar to that applied in (3.47),

(3.48), (3.49). If $g_0(x) = O(x^{-\epsilon} (\ln x)^i)$ then the right hand side of (4.48) has an additional factor $(\ln x)^i$.

After having performed all necessary resubstitutions we get a solution $v_{P_0}(t)$ fulfilling $v_{P_0}(\infty) = 0$. Defining $\tilde{r}_0, \tilde{D}_{00}$ in the same way as we did in Chapter 3 we get

Theorem 4.2: The problem (3.1), (3.2), (3.3) under the assumption (3.4), (3.5), (4.1) has a unique solution y for all f fulfilling (3.21)(a), (4.33) and for all $\hat{\beta} \in R^{\tilde{r}_0 + r_-}$ if and only if

$$\text{rank}[B_1 E \phi(1)(\tilde{D}_0 + D_-) + B_\infty E \tilde{D}_{00}] = \tilde{r}_0 + r_-$$

where B_1 and B_∞ are $(\tilde{r}_0 + r_-) \times n$ matrices. This solution y depends continuously on $\hat{\beta}$, $(D_+ + D_-)E^{-1}P_1^{-1}f$ and F_0 which is defined in (4.33).

The proof is complete if the continuity statement is proven.

Theorem 4.3. If $f \in C([1, \infty))$ and (4.33) holds then the following estimates hold for

$t > \delta$:

$$(4.50) \quad \| (H_+ f)(t) \| < \text{const.} \| D_+ P_1^{-1} E^{-1} f \|_{[t, \infty)}$$

$$(4.51) \quad \| (H_- f)(t) \| < \text{const.} t^{-\gamma} \max_{s \in [\delta, t]} \| s^\gamma D_- P_1^{-1}(s) E^{-1} f(s) \| \text{ for } \gamma > 0.$$

$$(4.52) \quad \| (H_0 f)(t) \| < \text{const.} (lnt)^{j_0} \max_{s \in [\delta, t]} t^{-\epsilon} \max_{s \in [\delta, t]} \| s^{\epsilon + (\alpha+1)\tilde{r}_-} D_0 P_1^{-1}(s) E^{-1} f(s) \|,$$

$$\max_{s > t} \| s^{(\alpha+1)\tilde{r}_-} D_0 P_1^{-1}(s) E^{-1} f(s) \|$$

where $j_0 = \max_{i_k} (\dim(D_{i_k}))$ and D_{i_k} are the Jordan blocks of the matrix D in Theorem 4.1 for which the corresponding polynomials $q_{i_k}(t)$ fulfill $\text{Re} q_{i_k}(t) \equiv 0$. All constants are independent of δ and f .

Proof. From (4.31) we conclude

$$(4.53)(a) \quad v_{P_+}(t) = \sum_{i=0}^{\infty} ((\hat{H}_+ J^+)^i \hat{H}_+ (P_1^{-1} E^{-1} f)_+)(t)$$

$$(4.53)(b) \quad v_{P_-}(t) = \sum_{i=0}^{\infty} ((\hat{H}_- J^-)^i \hat{H}_- (P_1^{-1} E^{-1} f)_-)(t)$$

for δ sufficiently large.

\hat{H}_+ resp \hat{H}_- fulfill the estimates (2.15) resp. (2.17) and therefore we get:

$$(4.54) \text{ (a)} \quad \|v_{P_+}(t)\| < \text{const.} \max_{s \in [t, \infty]} \|D_{P_+}^{-1}(s)E^{-1}f(s)\| \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

$$(4.54) \text{ (b)} \quad \|v_{P_-}(t)\| < \text{const.} t^{-\gamma} \max_{s \in [\delta, t]} \|s^{\gamma} D_{P_-}^{-1}(s)E^{-1}f(s)\| \cdot \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i$$

if δ is so large that $\|J^+\|_{[\delta, \infty]} < 1/2$ and $\|J^-\|_{[\delta, \infty]} < 1/2$. The estimates (4.50), (4.51) follow by using (4.29) and (4.52) follows from the derivation of v_{P_0} and from (4.48), (4.49).

Theorem 4.4 implies that an inhomogeneity f fulfilling

$$(4.55) \quad f(t) = t^{-(\alpha+1)\bar{\Gamma}-\varepsilon} (\ell n t)^{\bar{k}} F(t), \quad F \in C_b((1, \infty))$$

produces a particular solution Hf for which the estimate

$$(4.56) \quad \| (Hf)(t) \| < \text{const.} t^{-\varepsilon} (\ell n t)^{j_0 + \bar{k}} \|F\|_{[\delta, \infty]}$$

holds.

Now we assume inhomogenities of the form

$$(4.57) \quad f(t) = e^{p(t)} t^{\beta} (\ell n t)^{\bar{k}} F(t), \quad F \in C_b((1, \infty))$$

and $e^{p(t)} t^{\beta} F(t)$ fulfills (4.33) and $p(t)$ is a polynomial in $t^{\frac{1}{p}}$ with leading term

$\frac{t^{\alpha+1}}{\alpha+1}$. It is possible to construct a particular solution $\bar{H}f = H_+f + \bar{H}_0f + \bar{H}_-f$ fulfilling

$$(4.58) \text{ (a)} \quad \|(\bar{H}_0 f)(t)\| < \text{const.} e^{p(t)} t^{\beta + \bar{\Gamma}(\alpha+1)} (\ell n t)^{j_0 + \bar{k}} \|F\|_{[\delta, \infty]}$$

$$(4.58) \text{ (b)} \quad \|(\bar{H}_- f)(t)\| < \text{const.} e^{p(t)} t^{\beta + \bar{k}(\alpha+1)} (\ell n t)^{j_- + \bar{k}} \|F\|_{[\delta, \infty]}$$

where \bar{k} is the maximal algebraic multiplicity of the eigenvalues of J_0 with negative real part and j_- is the maximal dimension of Jordan blocks in DD_- . To outline the proof we assume the system

$$(4.59) \quad v' = t^\alpha (J_- + \tilde{J}_-(t)t^{-1})v + t^\alpha g(t), \quad \tilde{J}_-(\infty) = 0$$

where \tilde{J}_- has an asymptotic series expansion and J_- has only one eigenvalue λ with $\text{Re} \lambda < 0$ and $g(t)$ fulfills (4.57). If $-\text{Re} \lambda + \text{Re} p_0 > 0$ we determine a particular solution v_{p_-} by applying (4.30)(b). If $-\text{Re} \lambda + \text{Re} p_0 < 0$ we use the operator \tilde{H}_- instead of H_- in (4.30) where \tilde{H}_- is defined by (4.30)(a) with $\delta = \infty$. In both cases the estimate $\|v_{p_-}(t)\| < \text{const.} \cdot e^{p(t)} t^\beta (\ln t)^k \|F_-\|_{[\delta, \infty]}$ follows by a contraction argument and by (2.19), (2.22). However, if $\text{Re} \lambda = \text{Re} p_0$ we perform the substitution of the form (4.36), (4.37), (4.40) as long as we arrive at systems whose eigenvalues have real parts different from the real parts of the corresponding coefficients of $p(t)$. Then we apply H_- resp. \tilde{H}_- . If $\text{Re} p(t) \equiv \text{Re} q_j(t)$ where q_j is a polynomial in the exponential factor of the fundamental matrix of the homogenous system (4.59) this sequence of transformation either leads to one dimensional systems or to a system with a singularity of the first kind (rank zero). The first has been treated in Chapter 3 and the second is solved by (4.46), (4.47). Resubstitution and application to all separate subsystems corresponding to eigenvalues with negative real part yields the estimate (4.58)(b). The estimate (4.58)(a) is gained in a similar way.

Now we assume that the matrix $A(t)$ of the system (3.1) fulfills

$$(4.60) \quad A\left(\frac{1}{\delta}\right) \in C^{(\alpha+1)\bar{l}+1}\left(\left[0, \frac{1}{\delta}\right]\right), \quad \delta > 1$$

$$A \in C([1, \infty))$$

instead of (3.5), where $\bar{l} = \max(\bar{r}, \bar{k})$ is defined for the Jordan form of $A(\infty)$. Therefore,

$$(4.61) \quad A(t) = A_0 + t^{-1}A_1 + \dots + t^{-(\alpha+1)\bar{l}} A_{(\alpha+1)\bar{l}} + \tilde{A}(t), \quad \tilde{A}(t) = a(t)t^{-(\alpha+1)\bar{l}-1-\beta}$$

where $a \in C_b([1, \infty))$, $\beta > 0$.

The system (3.1) can now be written as:

$$(4.62) \quad y' = t^\alpha (A_0 + \dots + t^{-(\alpha+1)\bar{l}} A_{(\alpha+1)\bar{l}})y + t^\alpha (\tilde{A}(t)y(t) + f(t)).$$

By regarding (4.62) as a perturbation of

$$(4.63) \quad \tilde{y}' = t^\alpha \left(\sum_{i=0}^{(\alpha+1)\bar{l}} t^{-i} A_i \right) \tilde{y} + t^\alpha f$$

and proceeding similarly to Chapter 3 we find that

$$(4.64) \quad \|\psi_-^0(t) - E\phi(t)(\tilde{D}_0 + D_-)\| < \text{const.} \cdot t^{-1-\beta} (\ln t)^{2\max(j_0, j_-)} \sigma_h(t)$$

where $E\phi(t)$ is the fundamental matrix of (4.63) $\sigma_h(t)$ is defined as in (3.45) and $\psi_-^0 \in C([\delta, \infty))$ is the matrix fulfilling $y_h(t) = \psi_-^0(t)(\tilde{D}_0 + D_-)\xi$, where y_h is the general $C([\delta, \infty))$ -solution of (4.62) with $f \equiv 0$. Similarly a particular solution $\psi(f)$ of the inhomogenous problem (4.62) is obtained fulfilling

$$(4.65) \quad \|(\psi(f))(t) - E(\bar{H}f)(t)\| < \text{const. } t^{-1-\beta} (\ln t)^{2\max(j_0, j_-)} \sigma_p(t)$$

where $E\bar{H}f$ is the particular solution of (4.3) which decays as fast as possible according to Theorem 4.3 and (4.58). σ_p is defined as:

$$(4.66) \quad \|(\bar{H}f)(t)\| = O(\sigma_p(t)).$$

Theorem 3.2 remains valid if $E\phi(1)(\tilde{D}_0 + D_-)$ is substituted by $\psi_-^0(1)$ where ψ_-^0 has been continued to $[1, \infty)$.

If only an existence theorem for $\psi_-^0(t)$ is desired without taking care of the decay properties then $\bar{\lambda} = \bar{r}$ and $\beta > -1$ in (4.61) can be chosen. In this case the right hand sides of (4.64), (4.65) do not contain eventually occurring exponentially decaying factors in $\sigma_h(t)$ resp. $\sigma_p(t)$.

Moreover it is important to consider problems where the matrix $\tilde{A}(t)$ defined in (4.61) decays exponentially:

$$(4.67) \quad \tilde{A}(t) = a(t)t^\gamma e^{q(t)}, \quad a \in C_b([1, \infty)), \quad q(t) \rightarrow -\infty$$

Using the same methods as in the case of algebraic decay we get:

$$(4.68) \quad \|\psi_-^0(t) - E\phi(t)(\tilde{D}_0 + D_-)\| < \text{const. } t^{\gamma+(\alpha+1)\bar{\lambda}} (\ln t)^{2\max(j_0, j_-)} \sigma_h(t)e^{q(t)}$$

$$(4.69) \quad \|(\psi_p(f))(t) - E(\bar{H}f)(t)\| < \text{const. } t^{\gamma+(\bar{\alpha}+1)\bar{\lambda}} (\ln t)^{2\max(j_0, j_-)} \sigma_p(t)e^{q(t)}.$$

Only the construction of the 'particular' solutions has to be changed in order to get these estimates.

Finally the author conjectures that it is possible to change \bar{r} to r defined in Chapter 2 so that all statements made should hold with r instead of \bar{r} . Maybe someone else is successful in improving the derived estimates.

5. Nonlinear Problems.

We consider problems of the form

$$(5.1) \quad y' = t^\alpha f(t, y), \quad 1 < t < \infty, \quad \alpha \in \mathbb{N}_0$$

$$(5.2) \quad y \in C([1, \infty])$$

$$(5.3) \quad b(y(1), y(\infty)) = 0.$$

From (5.2) and (5.1) we conclude that

$$(5.4) \quad f(\infty, y(\infty)) = 0.$$

Now we define for $a \in \mathbb{R}^n$, $x, \bar{t} \in \mathbb{R}$:

$$(5.5) \quad S_x(a) = \{y \in \mathbb{R}^n \mid |y - a| < x\}$$

$$(5.6) \quad C_x(\bar{t}, a) = \{(\bar{t}, y) \in \mathbb{R}^{n+1} \mid \bar{t} > \bar{t}, y \in S_x(a)\}$$

and assume that

$$(5.7) \quad f, f_y \in C_{\text{lip}}(C_x(1, y(\infty)))$$

for some $x > 0$ sufficiently large.

(5.4) is a (nonlinear) system of equations from which $y(\infty)$ can be calculated as a solution manifold $y(\infty) = y_\infty(\mu)$, $y \in S \subset \mathbb{R}^{n_1}$, $n_1 < n$, if the Problems (5.1), (5.2), (5.3) admits a solution. The dimension of this manifold $-n_1-$ can be calculated a priori if we require for a solution point y_∞ that

$$(5.8) \quad \text{rank}(f_y(\infty, y_\infty)) = n - n_1$$

$$(5.9) \quad \text{rank}(f_y(\infty, y)) < n - n_1 \quad \text{for } y \in \overset{\circ}{S}_x(y_\infty), \quad x > 0.$$

Then there is a $x_1 > 0$ and a n_1 -dimensional manifold $y_\infty(\mu)$ which fulfills the equation

$$(5.10) \quad f(\infty, y_\infty(\mu)) \equiv 0 \quad \text{for } \mu \in S \subset \mathbb{R}^{n_1}$$

$$(5.11) \quad y_\infty(\mu) \in \overset{\circ}{S}_{x_1}(y_\infty).$$

From (5.8) we conclude that n_1 equals the geometrical multiplicity of the eigenvalue zero of the matrix $f_y(\infty, y_\infty)$.

However as practical examples point out, the assumption that $f_y(\infty, y)$ does not decrease its rank in y_∞ is too strong and therefore we regard n_1 as a priori unknown but obviously the solution of the equation (5.1) determines n_1 for a practical problem.

Now we define

$$(5.12) \quad A(t, \mu) = f_y(t, y_\infty(\mu))$$

and require that A fulfills (3.4), (3.5) so that

$$(5.13) \quad A(t, \mu) = \sum_{i=0}^{\infty} A_i(\mu) t^{-i} \quad \text{for } t > \delta.$$

We transform $A_0(\mu)$ to its Jordan canonical form $J_0(\mu)$

$$(5.14) \quad A_0(\mu) = E(\mu) J_0(\mu) E^{-1}(\mu)$$

and introduce z as new dependent variable

$$(5.15) \quad E(\mu) z = y - y_\infty(\mu)$$

getting

$$(5.16) \quad z' = t^\alpha J(t, \mu) z + t^\alpha g(z, t, \mu)$$

$$(5.17) \quad z(\infty) = 0.$$

Here

$$(5.18) \quad J(t, \mu) = E^{-1}(\mu) A(t, \mu) E(\mu)$$

and

$$(5.19) \quad g(z, t, \mu) = E^{-1}(\mu) f(t, E(\mu) z + y_\infty(\mu)) - J(t, \mu) z$$

hold. The perturbation g fulfills the estimates:

$$(5.20) \quad \|g(z, t, \mu)\| \leq C_1(\mu) (\|f(t, y_\infty(\mu))\| + \|z\|^2)$$

$$(5.21) \quad \|g(z_1, t, \mu) - g(z_2, t, \mu)\| \leq C_2(\mu) (\|z_1\| + \|z_2\|) \|z_1 - z_2\|$$

where $C_i(\mu)$ depend also on the Lipschitz-constant of f_y on $C_x(1, Y_\infty(\mu))$.

We restrict μ to subsets $\tilde{S} \subset S$ which are defined as follows.

1) The projections onto the direct sums of invariant subspaces of $J_0(\mu)$, which belong to eigenvalues with positive, zero and negative real part are constant for $\mu \in \tilde{S}$.

Moreover the projections onto the invariant subspaces of $J_0(\mu)$ are constant for $\mu \in \tilde{S}$, therefore r_+, r_0, r_-, \bar{r} are defined for $J_0(\mu)$ as in the last chapters and are independent of $\mu \in \tilde{S}$.

2) $y_\infty(\mu)$, $E(\mu)$, $E^{-1}(\mu)$ are continuous for $\mu \in \tilde{S}$.

Let $\phi(t, \mu)$ be the fundamental matrix of the (homogenous) Problem (5.16).

3) The same columns $\varphi_i(t, \mu)$ of the matrix $\phi(t, \mu)$ fulfill

$$\|\varphi_i(t, \mu)\| \leq C_i(\mu) t^{-(\alpha+1)\bar{r}-\varepsilon_1(\mu)} (\ln t)^{j_i}, \quad \varepsilon_1(\mu) > 0$$

for all $\mu \in \tilde{S}$. Therefore there is a projection matrix \hat{D}_0 independent of μ in \tilde{S} so that

$$(5.23) \quad \|\phi(t, \mu)(\hat{D}_0 + D_-)\| \leq C(\mu) t^{-(\alpha+1)\bar{r}-\varepsilon_1(\mu)} (\ln t)^{j_0} \quad \text{for } \mu \in \tilde{S}$$

holds. \hat{r}_0 be the number of 1's in the main diagonal of \hat{D}_0 .

We require f to fulfill

$$(5.24) \quad \|f(t, y_\infty(\mu))\| \leq C(\mu) t^{-2(\alpha+1)\bar{r}-\varepsilon_2(\mu)}, \quad \varepsilon_2(\mu) > 0 \quad \text{for } \mu \in \tilde{S}.$$

For $\mu \in \tilde{S}$, $\xi \in C^n$ fixed we set

$$(5.25) \quad (\psi(z, \mu))(t) = \phi(t, \mu)(\hat{D}_0 + D_-)\xi + (Hg(z, \cdot, \mu))(t)$$

where H is as in (4.56) (with $E = I$). Regarding $\psi(\cdot, \mu)$ as an operator on the Banach space:

$$(5.26) \quad (A_{\varepsilon, \delta} = \{z | z(t) = Z(t) t^{-(\alpha+1)\bar{r}-\varepsilon} (\ln t)^{j_0}, Z \in C_b([\delta, \infty))\}, \|z\|_\varepsilon = \|Z\|_{[\delta, \infty)}),$$

$$\delta > 1, \quad 0 < \varepsilon = \min(\varepsilon_1, \varepsilon_2).$$

Every fixed point z of $\psi(\cdot, \mu)$ establishes a solution for all $\xi \in C^n$. At first ψ maps $A_{\varepsilon, \delta}$ on $A_{\varepsilon, \delta}$ for δ sufficiently large because of (5.20), (5.23), (5.24), (4.55), (4.56).

Now we take a sufficiently large sphere $S_\varepsilon(\rho)$ in $A_{\varepsilon, \delta}$ with center

$\phi(\cdot, \mu)(\hat{D}_0 + D_-)\xi$ and radius ρ and prove the contraction property of ψ on $S_\varepsilon(\delta)$

$$(5.27) \quad \begin{aligned} \|\psi(z_1, \mu) - \psi(z_2, \mu)\|_\varepsilon &= \|Hg(z_1, \cdot, \mu) - Hg(z_2, \cdot, \mu)\|_\varepsilon < \\ &< \text{const}(\mu) \rho \cdot \delta^{-\varepsilon} (\ln \delta)^{2j_0} \|z_1 - z_2\|_\varepsilon \end{aligned}$$

for $z_1, z_2 \in S_\varepsilon(\rho)$ because of (5.21), (4.55) and (4.56).

Moreover if $z \in S_\varepsilon(\rho)$ then

$$(5.28) \quad \begin{aligned} \|\psi(z, \mu) - \phi(\cdot, \mu)(\hat{D}_0 + D_-)\xi\|_\varepsilon &= \|Hg(z, \cdot, \mu)\|_\varepsilon \\ &< (\text{const}(\mu) + \rho)^2 \cdot \delta^{-\varepsilon} (\ln \delta)^{2j_0} \end{aligned}$$

Therefore $\psi(z, \mu) \in S_\epsilon(\rho)$ if δ is sufficiently large and from (5.27), (5.28) we conclude that $\psi(\cdot, \mu)$ has a unique fixed point $z \in S_\epsilon(\rho) \cap A_{\epsilon, \delta}$ for δ sufficiently large. The construction of H implies that $(\hat{D}_0 + D_-)P(\delta, \mu)^{-1}z(\delta) = \delta^{D(\mu)}e^{Q(\delta, \mu)}(\hat{D}_0 + D_-)\xi$. Therefore, for fixed $\mu \in \tilde{S}$, we have constructed a $\hat{r}_0 + r_-$ -dimensional solution manifold in $A_{\epsilon, \delta}$, for δ sufficiently large but fixed whenever $(\hat{D}_0 + D_-)\xi$ varies

in a compact set $K \subset C^{\hat{r}_0 + r_-}$. In order to get more information on the asymptotic behavior of the solution we now treat the important case:

$$(5.29) \quad f(t, y_\infty(\mu)) \equiv 0 \quad \text{for } t > \delta, \quad \mu \in \tilde{S}$$

and

$$(5.30) \quad \|\phi(t, \mu)(\hat{D}_0 + D_-)\| = p(t, \mu)e^{q(t, \mu)}t^{\beta(\mu)}(\ln t)^{j_0}, \quad p \in C([1, \infty))$$

where

$$(5.31) \quad q(t, \mu) \rightarrow -\infty \quad \text{for } t \rightarrow \infty \quad \text{and } \mu \in \tilde{S}$$

holds.

We define:

$$(5.32) \quad \sigma(t, \mu) = e^{q(t, \mu)}t^{\beta(\mu)}(\ln t)^{j_0}$$

and set

$$(5.33) \quad (\bar{\psi}(z, \mu))(t) = \phi(t, \mu)(\hat{D}_0 + D_-)\zeta + (\bar{H}g(z, \cdot, \mu))(t), \quad \zeta \in C^n.$$

\bar{H} is constructed according to Chapter 4 (with $E = I$) so that inhomogeneities f which decay as $\sigma^2(t, \mu)$ produce a particular solution $\bar{H}f$ which decays as $t^{(\alpha+1)\bar{k}}\sigma^2(t, \mu)(\ln t)^{\max(j_0, j_-)}$. We regard $\bar{\psi}$ as an operator on the Banach space

$$(5.34) \quad (A_{\sigma, \delta} = \{u | u = \sigma(t, \mu)U, U \in C_b([\delta, \infty))\}, \|u\|_\sigma = \|U\|_{[\delta, \infty)}).$$

The contraction mapping theorem, employed as in the case of algebraic decay, assures the existence of a unique fixed point z in $A_{\sigma, \delta}$.

From (5.33) we conclude

$$(5.35) \quad \|z(t) - \phi(t, \mu)(\hat{D}_0 + D_-)\zeta\| \leq C(\mu)t^{(\alpha+1)\bar{k}}\sigma^2(t, \mu)(\ln t)^j$$

where $j = \max(j_0, j_-)$.

It is easy to check that $\psi(\cdot, \mu)$ is also a contraction in a sphere around $\phi(\cdot, \mu)(\hat{D}_0 + D_-)\xi$ in $A_{\sigma, \delta}$. The uniqueness of the fixed point assures that

$$(5.36) \quad \bar{H}g(z, \cdot, \mu) \in A_{\sigma, \delta}$$

for every fixed point z of $\psi(\cdot, \mu)$ in $A_{\epsilon, \delta}$. Therefore

$$(5.37) \quad (\bar{\psi}(z, \mu))(t) = \phi(t, \mu)(\hat{D}_0 + D_-)\xi + ((H - \bar{H})g(z, \cdot, \mu))(t) + (\bar{H}g(z, \cdot, \mu))(t)$$

holds. Because Hg and $\bar{H}g$ are particular solutions for fixed $z \in A_{\sigma, \delta}$ we get

$$(5.38) \quad ((H - \bar{H})g(z, \cdot, \mu))(t) = \phi(t, \mu)(\hat{D}_0 + D_-)\gamma(z), \quad \gamma(z) \in C^n.$$

Choosing

$$(5.39) \quad \zeta = \xi + \gamma(z)$$

assures that every fixed point of $\psi(z, \mu)$ is also a fixed point of $\bar{\psi}(z, \mu)$ (with ζ different to ξ) and vice versa.

In general our perturbation approach does not give us all $C([1, \infty))$ solutions of the Problem (5.1), (5.2), (5.3), it only gives us all solutions, which decay at least as fast as $t^{-(\alpha+1)\bar{r}-\epsilon}$ where $\epsilon > 0$. This is illustrated by the example:

$$(5.40) \quad y' = y^2, \quad 1 < t < \infty$$

$$(5.41) \quad y \in C([1, \infty))$$

which has the solution manifold $y = -\frac{1}{t+c}$, $c > -1$ and $y \equiv 0$. Our approach gives

$$(5.42) \quad y_\infty = 0, \quad \frac{\partial f}{\partial y}(y_\infty) = 0, \quad \bar{r} = 1$$

and therefore the fixed point equation

$$(5.43) \quad y = Hy^2$$

results. In $A_{\epsilon, \delta}$ the only solution of (5.43) is $y \equiv 0$, which is the only solution of (5.40) decaying faster than t^{-1} .

If $f_y(t, y_\infty(\mu))$ is not analytical at $t = \infty$ but if it fulfills a relation of the form (4.61) then $\psi_-^0(t, \mu)(D_- + \hat{D}_0)$ resp. $\psi(g(z, \cdot, \mu))$ defined in Chapter 3.4 have to be used instead of $E(\mu)\phi(t, \mu)(\hat{D}_0 + D_-)$ resp. $E(\mu)\bar{H}g(z, \cdot, \mu)$. The results do not change.

The following theorem follows immediately:

Theorem 5.1. Let f fulfill a uniform Lipschitz condition in y on $[1, \infty)$. Then the problem (5.1), (5.2), (5.3) where (5.13) holds asymptotically has a solution

$y \equiv y(\cdot, (\hat{D}_0 + D_-)\xi, \mu) \equiv E(\mu)z(\cdot, (\hat{D}_0 + D_-)\xi, \mu) + y_\infty(\mu)$ for every root $((\hat{D}_0 + D_-)\xi, \mu)$ of the equation

$$b(E(\mu)z(1, (\hat{D}_0 + D_-)\xi, \mu) + y_\infty(\mu), y_\infty(\mu)) = 0$$

where $(\xi, \mu) \in C^n \times \tilde{S}$ and $b: R^n + R^{\hat{r}_0 + r_- + n_1}$. Here $z(t, (\hat{D}_0 + D_-)\xi, \mu)$ denotes the continued fixed points of $\psi(\cdot, \mu)$ with $\xi \in C^n$. On the other hand, if the boundary value problem (5.1), (5.2), (5.3) has a solution y , so that $y - y^{(\infty)} \in A_{\epsilon, 1}$ for some $\epsilon > 0$, then there is a $\mu \in R^{n_1}$ and a $(\hat{D}_0 + D_-)\xi$, $\xi \in C^n$ so that $z := E^{-1}(\mu)(y - y_\infty)$ has the asymptotic expansion (5.35).

6. Case Studies.

In this chapter problems from fluid dynamics resp. thermo dynamics described by boundary value problems on infinite intervals are analyzed. These problems are represented by nonautonomous system of nonlinear differential equations. An autonomous problem, namely von Karman's swirling flow problem has been investigated by Markowich (1980b). The following equation represents a model for viscous flow past a solid at low Reynolds number:

$$(6.1) \quad u'' + \frac{k}{x}u' + \alpha uu' + \beta(u')^2 = 0, \quad \delta < x < \infty, \quad \alpha > 0$$

$$(6.2) \quad u(\delta) = 0, \quad \delta > 0, \quad u(\infty) = U > 0.$$

The parameters and variables are described by Cohen, Fokers and Lagerstrom (1978). The transformation

$$(6.3) \quad y_1 = u, \quad y_2 = u', \quad Y = (y_1, y_2)^T$$

takes (6.1), (6.2) into

$$(6.4) \quad y' = \begin{bmatrix} y_2 \\ -\frac{k}{x}y_2 - \alpha y_1 y_2 - \beta y_2^2 \end{bmatrix} = f(x, y)$$

$$(6.5) \quad (a) \quad [1, 0]y(\delta) = 0, \quad (b) \quad [1, 0]y(\infty) = U$$

$$(6.6) \quad y \in C([\delta, \infty)).$$

We get $y_\infty = y_\infty(U) = \begin{pmatrix} U \\ 0 \end{pmatrix}$ and calculate

$$(6.7) \quad f_y(x, y_\infty(U)) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\alpha U \end{bmatrix}}_{A_0(U)} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -k \end{bmatrix}}_{A_1}$$

$A_0(U)$ has the distinct eigenvalues 0 and $-\alpha U$ and we get

$$(6.8) \quad J_0(U) = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha U \end{bmatrix}, \quad E(U) = \begin{bmatrix} 1 & -\frac{1}{\alpha U} \\ 0 & 1 \end{bmatrix}.$$

The transformation (5.15) with $\mu = U$ results in the system

$$(6.9) \quad z' = \left(\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\alpha U \end{bmatrix}}_{J_0(U)} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & -\frac{k}{\alpha U} \\ 0 & -k \end{bmatrix}}_{J_1(U)} \right) z + g(z, U)$$

$$(6.10) \quad z(\infty) = 0.$$

The homogenous problem $z'_h = (J_0(U) + \frac{1}{x} J_1(U))z_h$ has a fundamental matrix $\phi(x, U)$ of the form

$$(6.11) \quad \phi(x, U) = P(x, U) \begin{bmatrix} 1 & 0 \\ 0 & e^{-\alpha U x} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x^{-k} \end{bmatrix}$$

where

$$(6.12) \quad P(x, U) = I + O(x^{-1})$$

holds. This follows from Theorem 3.1 and from the algorithm for the calculation of the coefficients. Moreover

$$(6.13) \quad \hat{D}_0 = 0, \quad D_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

holds and the fixed point equation

$$(6.14) \quad z(t) = P(t, U) \begin{bmatrix} 0 \\ e^{-\alpha U x} x^{-k} \xi \end{bmatrix} + (\tilde{H}g(z, U))(t), \quad \xi \in \mathbb{R}$$

results because we are only interested in real solutions.

$g(z, U)$ fulfills

$$(6.15) \quad \|g(z, U)\| \leq \text{const}(U) \|z\|^2$$

because $f(x, y_\infty(U)) = 0$ holds.

Chapter 5 assures the existence of solutions $z(\cdot, \xi, U) \in A_{\epsilon, \delta} = \{u | u(x) = x^{-1-\epsilon} \ln x U(x), U \in C_b([1, \infty))\}$ and from (5.35) we conclude:

$$(6.16) \quad \|z(x, \xi, U) - P(x, U) \begin{bmatrix} 0 \\ e^{-\alpha U x} x^{-k} \xi \end{bmatrix}\| \leq C(U) x^{1-2k} e^{-2\alpha U x} (\ln x)^2.$$

Using Theorem 3.4 this estimate can be improved, so that its right hand side is of the order $x^{-2k} e^{-2\alpha U x}$. Resubstituting we get

$$(6.17) \quad u(x, \xi, U) = U - \frac{1}{\alpha U} (1 + O(x^{-1})) e^{-\alpha U x - k \xi} + O(e^{-2\alpha U x - 2k}), \quad x \rightarrow \infty.$$

Since $U \in \mathbb{R}^+$ is given (6.5)(a) has to be used in order to determine $\xi \in \mathbb{R}$ and the problem is well posed concerning the number of conditions as $x = \delta$ and $x = \infty$.

The second problem is a similarity equation for a combined forced and free convection flow over a horizontal plate (see Schneider (1979)):

$$(6.18) \quad \left. \begin{array}{l} (a) \quad 2f'' + ff'' + kxg = 0 \\ (b) \quad 2g' + fg = 0 \end{array} \right\} \quad 0 < x < \infty$$

$$(6.19) \quad (a) \quad f(0) = f'(0) = 0, \quad g(0) = 1, \quad (b) \quad f'(\infty) = 1, \quad g(\infty) = 0$$

The variables and the parameter k are explained in Schneider (1979). For simplicity we have set the Prandtl number to 1. Because of (6.19)(b) we substitute

$$(6.20) \quad f(x) = x + h(x)$$

and get the new problems:

$$(6.21) \quad \left. \begin{array}{l} (a) \quad 2h'' + (x + h)h'' + kxg = 0 \\ (b) \quad 2g' + (x + h)g = 0 \end{array} \right\} \quad 0 < x < \infty$$

$$(6.22) \quad (a) \quad h(0) = 0, \quad h'(0) = -1, \quad g(0) = 1, \quad (b) \quad h'(\infty) = 0, \quad g(\infty) = 0.$$

Substituting

$$(6.23) \quad y_1 = h, \quad y_2 = h', \quad y_3 = h'', \quad y_4 = g, \quad y = (y_1, y_2, y_3, y_4)^T$$

we get the system

$$(6.24) \quad y' = x \begin{bmatrix} \frac{y_2}{x} \\ \frac{y_3}{x} \\ -\frac{1}{2} \left(1 + \frac{y_1}{x}\right) y_3 - \frac{k}{2} y_4 \\ -\frac{1}{2} \left(1 + \frac{y_1}{x}\right) y_4 \end{bmatrix} = x f(x, y) \quad , \quad 0 < x < \infty$$

$$(6.25)(a) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} y(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad , \quad (b) \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} y(\infty) = \begin{bmatrix} n \\ 0 \end{bmatrix}$$

We only admit solutions fulfilling $h(\infty) \in \mathbb{R}$ therefore we require that

$$(6.26) \quad y \in C([0, \infty))$$

holds. From (6.23), (6.24) we conclude

$$(6.27) \quad y_\infty = y_\infty(h_\infty) = (h_\infty, 0, 0, 0) \quad , \quad h_\infty = h(\infty) \in \mathbb{R} .$$

We calculate:

$$(6.28) \quad f_y(x, y_\infty(h_\infty)) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{k}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}}_{A_0} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{h_\infty}{2} & 0 \\ 0 & 0 & 0 & -\frac{h_\infty}{2} \end{bmatrix}}_{A_1(h_\infty)}$$

and

$$(6.29) \quad E \equiv E(h_\infty) = \text{diag}(1, 1, 1, -\frac{2}{k}) .$$

The substitution $E(h_\infty)z = y - y_\infty(h_\infty)$ gives the system

$$(6.30) \quad z' = x \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & -1/2 \end{bmatrix} + \frac{1}{x} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{h_\infty}{2} & 0 \\ 0 & 0 & 0 & -\frac{h_\infty}{2} \end{bmatrix} z + xg(z, h_\infty)$$

J_0
 $J_1(h_\infty) = A_1(h_\infty)$

J_0 has the eigenvalues 0 with algebraic and geometric multiplicity 2 and $-1/2$ with algebraic multiplicity 2 and geometric multiplicity 1. Because $f(x, y_\infty(h_\infty)) = 0$ holds we get

$$(6.31)(a) \quad |g(z, h_\infty)| < C(h_\infty) |z|^2.$$

We have to set up the fundamental matrix $\phi(x, h_\infty)$ of the system

$$(6.31)(b) \quad z'_h = x(J_0 + \frac{1}{x} J_1(h_\infty)) z_h, \quad z_h = (z_h^1, z_h^2, z_h^3, z_h^4).$$

Because of the simple structure of $J_0 + \frac{1}{x} J_1$ we do not have to apply the algorithm of Chapter 4, we can proceed in the following way.

The last equation of (6.31) is

$$(6.32) \quad z_h^{4'} = x(-1/2 - \frac{h_\infty}{2} \frac{1}{x}) z_h^4.$$

It can be integrated at once giving

$$(6.33) \quad z_h^4 = e^{-1/4 x^2 - \frac{h_\infty}{2} x} C, \quad C \in \mathbb{R}.$$

Setting $c = 1$ we have to find a particular solution of

$$(6.34) \quad z_h^{3'} = x(-1/2 - \frac{h_\infty}{2} \frac{1}{x}) z_h^3 + x z_h^4.$$

We take

$$(6.35) \quad z_h^3 = e^{-1/4 x^2 - \frac{h_\infty}{2} \frac{1}{x}} \frac{x^2}{2} (1 - \frac{1}{x^2}).$$

Integrating

$$(6.36) \quad z_h^{2'} = z_h^3$$

we find

$$(6.37) \quad z_h^2 = e^{-\frac{1}{4}x^2 - \frac{h_\infty}{2}x} \times O(1).$$

Analogously we integrate

$$(6.38) \quad z_h^1 = z_h^2$$

and get

$$(6.39) \quad z_h^1 = e^{-\frac{1}{4}x^2 - \frac{h_\infty}{2}x} O(1).$$

The fourth column of $\phi(x, h_\infty)$ can be chosen as $(z_h^1, z_h^2, z_h^3, z_h^4)^T$.

In order to get the third column we set $C = 0$ in (6.33) and proceed as we did.

Finally we get the following fundamental matrix

$$(6.40) \quad \phi(x, h_\infty) = \begin{bmatrix} 1 & x & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} O(x^{-2}) & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} O(1) \\ 0 & 1 & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} O(x^{-1}) & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} O(1) \\ 0 & 0 & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} x^2 \left(1 - \frac{1}{x^2}\right) \\ 0 & 0 & 0 & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} \end{bmatrix}$$

which we can write as in Theorem 4.1

$$(6.41) \quad \phi(x, h_\infty) = P(x, h_\infty) x^D e^{Q(x, h_\infty)}$$

where

$$(6.42) \quad P(x, h_\infty) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{x} O(1)$$

$$(6.43) \quad D = \text{diag}(1, x, 1, x^2)$$

$$(6.44) \quad Q(x, h_\infty) = \text{diag}\left(1, 1, e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x}, e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x}\right)$$

holds.

Similarly to the first example we conclude that there are solutions $z(x, \xi_1, \xi_2, h_\infty)$ in $A_{\varepsilon, \delta}$, which is now the space of all functions in $C([\delta, \infty])$ which decay at least as $x^{-4-\varepsilon} \ln x$, $\varepsilon > 0$ because $\bar{r} = 2$ and $\alpha = 1$ holds. Moreover

$$(6.45) \quad z(x, \xi_1, \xi_2, h_\infty) = P(x, h_\infty) \begin{bmatrix} 0 \\ 0 \\ x e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} \xi_1 \\ x^2 e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} \xi_2 \end{bmatrix} + O(x^4 e^{-\frac{x^2}{2} - h_\infty x})$$

with $h_\infty, \xi_1, \xi_2 \in \mathbb{R}$ results because the exponential factor $e^{-\frac{x^2}{2} - h_\infty x}$ does not appear in the fundamental matrix.

So we get the asymptotic expansions

$$(6.46) \quad f(x) = x + h_\infty + O(1) e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} + O(x^4 e^{-\frac{x^2}{2} - h_\infty x})$$

$$(6.47) \quad g(x) = -\frac{2}{k} e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} \xi_2 + O(x^4 e^{-\frac{x^2}{2} - h_\infty x})$$

where the $O(1)$ in (6.46) depends linearly on ξ_1 and ξ_2 .

The constants $(h_\infty, \xi_1, \xi_2) \in \mathbb{R}^3$ have to be determined from the three initial conditions (6.25)(a).

The third problem to be analyzed is the well known Falkner-Skan equation (see for example Lentini and Pereyra (1977)):

$$(6.48) \quad f''' + ff'' + (1 - f'^2) = 0$$

$$(6.49) \quad f'(\infty) = 1.$$

We do not pose any initial condition because we look for a solution manifold.

Because of (6.48) we substitute

$$(6.50) \quad f(x) = x + g(x), \quad y_1 = g, \quad y_2 = g', \quad y_3 = g''$$

$$(6.51) \quad y = (y_1, y_2, y_3)^T$$

and get the system

$$(6.52) \quad y' = x \begin{bmatrix} \frac{y_2}{x} \\ \frac{y_3}{x} \\ -y_3 - \frac{y_1 y_3}{x} + \frac{2y_2}{x} + \frac{y_2^2}{x} \end{bmatrix} = x f(x, y), \quad x > \delta.$$

Moreover we require

$$(6.53) \quad y \in C([\delta, \infty))$$

so that

$$(6.54) \quad y(\infty) = (g_\infty, 0, 0)^T, \quad g_\infty := g(\infty)$$

holds. We calculate

$$(6.55) \quad f_y(x, y_\infty(g_\infty)) = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{J_0} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -g_\infty \end{bmatrix}}_{J_1(g_\infty)}.$$

The linearized system is therefore

$$(6.56) \quad z'_h = x(J_0 + \frac{1}{x} J_1(g_\infty))z_h$$

Because J_0 has the eigenvalue 0 with algebraic multiplicity 2 we apply the theory developed in Chapter 4.

At first we split up the system (6.56) by the transformation

$$(6.57) \quad z_h = P(x, g_\infty) u \quad u = (u_1, u_2, u_3)^T, \quad P(x, g_\infty) \sim I + \sum_{i=1}^{\infty} P_i(g_\infty) x^{-i}$$

to get subsystems whose leading matrices have the only eigenvalue 0 resp. -1. (6.57)

gives a system of the form:

$$(6.58) \quad u' = x B(x, g_\infty) u, \quad B(x, g_\infty) \sim J_0 + \sum_{i=1}^{\infty} B_i(g_\infty) x^{-i}.$$

From Wasow (1965) we conclude that

$$(6.59) \quad P_i = \begin{bmatrix} 0 & 0 & P_{i1} \\ 0 & 0 & P_{i2} \\ P_{i3} & P_{i3} & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} b_{i1} & b_{i2} & 0 \\ b_{i3} & b_{i4} & 0 \\ 0 & 0 & b_{i5} \end{bmatrix}$$

holds and that the recursion

$$(6.60) \quad J_0 P_i - P_i J_0 = \sum_{s=0}^{i-1} (P_s B_{i-s} - J_{i-s} P_s) - (i-2) P_{i-2}, \quad i > 0$$

with the last term absent for $i < 2$ and $J_k = 0$ for $k > 1$ holds.

From the investigation of the perturbed system we know that only the coefficients B_0, B_1, B_2, B_3 and B_4 influence the asymptotic behavior of the fundamental matrix because $(\alpha + 1)\bar{r} = 4$ for our example. (6.59) and (6.60) give

$$(6.61) \quad B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -g_\infty \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2g_\infty & 0 \\ 0 & 0 & -2g_\infty \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2+2g_\infty^2 & 0 \\ 0 & 0 & b_{45} \end{bmatrix}.$$

We do not have to know b_{45} explicitly because it does not influence the behavior of the solution of

$$(6.62) \quad u_3' = x \left(-1 - \frac{g_\infty}{x} - \frac{2}{x^2} + \frac{2g_\infty}{x^3} + \dots \right) u_3.$$

From Chapter 3 we get

$$(6.63) \quad u_3 = p_3(x, g_\infty) e^{-\frac{x^2}{2} - g_\infty x} x^{-2}, \quad p_3(x, g_\infty) \sim 1 + p_{31}(g_\infty)x^{-1} + p_{32}(g_\infty)x^{-2} + \dots$$

Moreover we get:

$$(6.64) \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{x} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \frac{1}{x^2} \begin{bmatrix} 0 & 0 \\ 0 & -2g_\infty \end{bmatrix} + \frac{1}{x^3} \begin{bmatrix} 0 & 0 \\ 0 & -2+2g_\infty^2 \end{bmatrix} + \dots \right) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

because the leading term which comes from B_0 vanishes. Therefore the coefficient of $\frac{1}{x^3}$ does not influence the behavior of u_1, u_2 . It is sufficient to solve

$$(6.65) \quad \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}' = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{x} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \frac{1}{x^2} \begin{bmatrix} 0 & 0 \\ 0 & -2g_\infty \end{bmatrix} \right) \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}.$$

We get directly by integration

$$(6.66) \quad \tilde{u}_2 = x^2 \exp\left(\frac{2g_\infty}{x}\right) \quad \text{and} \quad \tilde{u}_1 = x^3 O(1).$$

Finally we conclude

$$(6.67) \quad f = x + g_\infty + p(x, g_\infty) e^{-\frac{x^2}{2} - g_\infty x} \xi + O(e^{-x^2 - 2g_\infty x}), \quad p(x, g_\infty) \sim 1 + \sum_{i=1}^{\infty} p_i(g_\infty) x^{-i}$$

with $\xi \in \mathbb{R}$ because we look for real solutions and because there is no column of the

fundamental matrix of (6.56) which has the factor $e^{-x^2 - 2g_\infty x}$.

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established in the linear case. The asymptotic behaviour of this solution follows immediately. Nonlinear problems are treated by using perturbation techniques meaning linearization near infinity and by using the methods for the linear case. Moreover, some practical problems from fluid dynamics and thermodynamics are dealt with and they illustrate the power of the asymptotic methods used.

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