ANALYSIS OF CONTINUOUS H –1 LEAST-SQUARES METHODS FOR THE STEADY NAVIER-STOKES SYSTEM

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Abstract. We analyze two H^{-1}-least-squares methods for the steady Navier-Stokes system of incompressible viscous fluids. Precisely, we show the convergence of minimizing sequences for the least-squares functional toward solutions. Numerical experiments support our analysis.

Key Words. Steady Navier-Stokes system, Least-squares approach, Gradient method.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N = 2$ or $N = 3$ be a bounded connected open set whose boundary $\partial\Omega$ is Lipschitz. We denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward unit normal to Ω at any point $\mathbf{x} \in \partial\Omega$. Bold letters and symbols denote vector-valued functions and spaces; for instance $\mathbf{L}^2(\Omega)$ is the Hilbert space of the functions $\mathbf{v} = (v_1, \dots, v_N)$ with $v_i \in L^2(\Omega)$ for all i.

This work is concerned with the (numerical) approximation for the steady Navier-Stokes system

$$\begin{cases} -\nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla\pi = \mathbf{f}, \quad \nabla\cdot\mathbf{y} = 0 \quad \text{in } \Omega, \\ \mathbf{y} = \mathbf{0} \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

which describes a viscous incompressible fluid flow in the bounded domain Ω, submitted to the external force $\mathbf{f}$. Our strategy is to use a least-squares approach, much in the spirit of [2], [3], [7], but in a systematic way as in [15], having in mind some applications to control problem as described in [12, 13] for the Stokes system. From a purely analytical perspective, the following is a well-known existence theorem.

Theorem 1.1 ([17]). *For any $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, there exists at least $(\mathbf{y}, \pi) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$ solution of (1.1). Moreover, if $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ is small enough, then the couple $(\mathbf{y}, \pi)$ is unique.*

We put $L^2_0(\Omega)$ for the space of functions in $L^2(\Omega)$ with zero mean. Assuming $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, the solution of (1.1) may be investigated by considering the following functional

$$(\mathbf{y}, \pi) \to \| -\nu\Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla\pi - \mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)} + \|\nabla\cdot\mathbf{y}\|^2_{L^2(\Omega)}$$

over the space $\mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$. The minimization of this functional leads to a so-called continuous H^{-1}-least-squares type method, following the terminology of [5] and later use in [3].

Least-squares methods to solve non linear boundary value problems have been the subject of intensive developments in the last decades, as they present several advantages, notably on computational and stability viewpoints. We refer to the book [2]. The main reason of this work is to show that, under the assumption of Theorem 1.1, minimizing sequences for this so-called

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error functional do actually converge strongly to the solution of (1.1). We first consider in Section 2 the case where minimizing sequences live in $\mathbb{V} \times L^2_0(\Omega)$ with $\mathbb{V}$ defined below by (2.1), a set of divergence free fields. Then, in Section 3, we discuss the general case where the field $\mathbf{y}$ is not *a priori* assumed to be divergence free. In the two cases, we provide a sufficient condition of the convergence of the values of the error functional E in terms of the convergence of the values of its derivative. Then, in Section 4, we show that gradient methods, under hypothesis of Theorem 1.1, produce converging sequences to the unique solution of (1.1). Section 5 describes the conjugate gradient algorithm associated to the error functional E while section 6 discusses numerically the celebrated exemple of the 2D channel with a backward facing step.

2. Steady case under the div-free constraint

As indicated in the Introduction, in order to solve the boundary value problem (1.1), we use a least-squares type approach. If we insist in keeping explicitly the div-free constraint for fields, then the pressure field does not play a specific role, so that we can eliminate it from the formulation.

We consider the Hilbert space

$$\mathbb{V} := \mathbf{H}^1_{0,div}(\Omega) = \{\mathbf{y} \in \mathbf{H}^1_0(\Omega) : \nabla \cdot \mathbf{y} = 0 \text{ in } \Omega\}, \tag{2.1}$$

endowed with the norm of the gradient $\|\nabla \mathbf{y}\|_{\mathbf{L}^2(\Omega)}$ and define the functional $E : \mathbb{V} \to \mathbb{R}^+$ by putting

$$E(\mathbf{y}) = \frac{1}{2} \int_\Omega |\nabla \mathbf{v}|^2 \, d\mathbf{x} \tag{2.2}$$

where the corrector $\mathbf{v}$ is the unique minimizer in $\mathbb{V}$ for the variational problem

$$\text{Minimize in } \mathbf{v} \in \mathbb{V} : \quad \int_\Omega \left[\frac{1}{2} |\nabla \mathbf{v}|^2 + (\nu \nabla \mathbf{y} - \mathbf{y} \otimes \mathbf{y}) \cdot \nabla \mathbf{v} \right] d\mathbf{x} - \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)}.$$

Notice that the differential constrained imposed on $\mathbb{V}$, leads to the existence of a multiplier π, the pressure, such that

$$\begin{cases} -\Delta \mathbf{v} + \nabla \pi + (-\nu \Delta \mathbf{y} + div(\mathbf{y} \otimes \mathbf{y}) - \mathbf{f}) = 0, \quad \nabla \cdot \mathbf{v} = 0 \quad \text{in} \quad \Omega, \\ \mathbf{v} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.3}$$

with $div(\mathbf{y} \otimes \mathbf{y}) = (\mathbf{y}.\nabla)\mathbf{y} + \mathbf{y}\nabla \cdot \mathbf{y} = (\mathbf{y}.\nabla)\mathbf{y}$ since $\mathbf{y}$ is here divergence free.

The existence of a unique solution for this quadratic, constrained problem in $\mathbf{v}$ is standard. Optimality conditions lead directly to the weak form of (2.3).

The functional E is a so-called error functional which measures, through the corrector variable $\mathbf{v}$, the deviation of the pair $(\mathbf{y}, \pi)$ from being a solution of the underlying system (1.1). We then consider the following extremal problem

$$\inf_{\mathbf{y} \in \mathbb{V}} E(\mathbf{y}). \tag{2.4}$$

Note that the error functional E is differentiable as a functional defined on the Hilbert space $\mathbb{V}$, because the operator $\mathbf{y} \mapsto \mathbf{v}$ taking each $\mathbf{y} \in \mathbb{V}$ into its associated corrector $\mathbf{v}$, as stated above, is a differentiable operation. Indeed, $E'(\mathbf{y})$ can always be identified with an element of $\mathbb{V}$ itself. This computation is performed below in the proof of Proposition 2.4.

From Theorem 1.1, the infimum is equal to zero, and is reached by a solution of (1.1). Beyond this statement, we would like to argue why we believe it is a good idea to use a (minimization) least-squares approach to approximate the solution of (1.1) by solving (2.4).

The least-squares problem (2.2)-(2.3)-(2.4) has been actually introduced in [3], section 4.2 (and numerically discussed in [4]), in order to solve one step in time of an implicit Euler scheme, time approximation for the unsteady Navier-Stokes system. However, the analysis of the convergence of the method was not given there. In that direction, our main theorem is the following.

Theorem 2.1. *There is a positive constant C_2, such that if $\{\mathbf{y}^j\}_{j>0}$ is a sequence in*

$$\mathbb{B} := \{\mathbf{y} \in \mathbb{V} : \|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)} \leq C_2\}$$

with $E'(\mathbf{y}^j) \to 0$ as $j \to \infty$, then the whole sequence $\mathbf{y}^j$ converges strongly as $j \to \infty$ in $\mathbf{H}_0^1(\Omega)$ to the unique solution $\mathbf{y}_0$ of (1.1) guaranteed by Theorem 1.1, if $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ is small enough.

We divide the proof in two main steps.

(1) First, we use a typical a priori bound to show that leading the error functional E down to zero implies strong convergence to the unique solution of (1.1).
(2) Next, we will show that taking the derivative E' to zero actually suffices to take E to zero.

Proposition 2.2. *Assume that $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ is small enough and let $\mathbf{y}_0 \in \mathbb{V}$ be the unique solution of (1.1), mentioned in Theorem 1.1. If dimension $n \leq 4$, there is a positive constant $C \equiv C(\nu, n, \|\mathbf{f}\|_{H^{-1}(\Omega)})$ such that for every $\mathbf{y} \in \mathbb{V}$, we have*

$$\|\mathbf{y} - \mathbf{y}_0\|^2_{\mathbf{H}_0^1(\Omega)} \leq CE(\mathbf{y}).$$

This proposition very clearly establishes that as we take down the error E to zero, we get closer, in the strong norm, to the solution of the problem, and so, it justifies why a promising strategy to find good approximations of the solution of problem (1.1) is to look for global minimizers of (2.4).

The proof of this proposition basically amounts to a typical a priori estimate which is essentially the same that the proof of uniqueness in page 112 in [17]. Recall the following basic fact which will be utilized several times in what follows.

Lemma 2.3. *For all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, we have*

$$\int_\Omega (\mathbf{v} \otimes \mathbf{v} : \nabla \mathbf{u} + \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{v})\, d\mathbf{x} = \int_\Omega \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{u}\, d\mathbf{x} = 0.$$

Proof. (of Proposition 2.2) Bear in mind (2.3)

$$-\Delta \mathbf{v} + (-\nu\Delta \mathbf{y} + div(\mathbf{y} \otimes \mathbf{y}) - \mathbf{f}) + \nabla \pi = 0, \quad \text{in} \quad \Omega,$$

in addition to the boundary condition $\mathbf{v} = 0$ on $\partial\Omega$. Since $(\mathbf{y}_0, \pi_0)$ is the unique solution of problem (1.1), we can write

$$-\Delta \mathbf{v} + (-\nu\Delta \mathbf{y} + div(\mathbf{y} \otimes \mathbf{y})) - (-\nu\Delta \mathbf{y}_0 + div(\mathbf{y}_0 \otimes \mathbf{y}_0)) + \nabla(\pi - \pi_0) = 0 \quad \text{in} \quad \Omega.$$

If we put $\mathbf{Y} \equiv \mathbf{y}_0 - \mathbf{y}$, reorganizing terms it is also true that

$$\Delta \mathbf{v} - \nu\Delta \mathbf{Y} + \nabla(\pi_0 - \pi) - div(\mathbf{y} \otimes \mathbf{y} - \mathbf{y}_0 \otimes \mathbf{y}_0) = 0 \quad \text{in} \quad \Omega. \tag{2.5}$$

In a linear situation, we would immediately achieve the bound in the statement. But the presence of the non-linear (quadratic) term that is so essential to Navier-Stokes requires a bit more analysis.

Let us focus on the difference

$$Div \equiv div(\mathbf{y} \otimes \mathbf{y} - \mathbf{y}_0 \otimes \mathbf{y}_0).$$

We first put

$$-Div = div(\mathbf{y}_0 \otimes \mathbf{Y}) + div(\mathbf{Y} \otimes \mathbf{y}).$$

If we take this identity back to (2.5), multiply by $\mathbf{Y}$, and integrate by parts, taking into account boundary conditions and bearing in mind that $div\mathbf{Y} = 0$, we are led to

$$-\int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{Y}\, d\mathbf{x} + \nu \int_\Omega |\nabla \mathbf{Y}|^2\, d\mathbf{x} - \int_\Omega \mathbf{y}_0 \otimes \mathbf{Y} : \nabla \mathbf{Y}\, d\mathbf{x} - \int_\Omega \mathbf{Y} \otimes \mathbf{y} : \nabla \mathbf{Y}\, d\mathbf{x} = 0.$$

By the second part of Lemma 2.3, the last term above vanishes, while for the third term, the first part of the same lemma leads to

$$\int_\Omega \mathbf{y}_0 \otimes \mathbf{Y} : \nabla \mathbf{Y}\, d\mathbf{x} = -\int_\Omega \mathbf{Y} \otimes \mathbf{Y} : \nabla \mathbf{y}_0\, d\mathbf{x} \leq C^2(n) \|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} \|\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)},$$

where $C(n)$ is the constant of the Sobolev embedding of $H^1(\Omega)$ into $L^4(\Omega)$ provided $n \leq 4$ (see Lemmas 1.1 and 1.2 in pages 108 and 109, respectively, of [17]). By putting together all of this information, we have

$$\nu \int_\Omega |\nabla \mathbf{Y}|^2\, d\mathbf{x} = \int_\Omega \mathbf{y}_0 \otimes \mathbf{Y} : \nabla \mathbf{Y}\, d\mathbf{x} + \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{Y}\, d\mathbf{x},$$

and then

$$\nu \|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} \leq C^2(n) \|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} \|\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)} \|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}. \tag{2.6}$$

But $\mathbf{y}_0$ is precisely the solution of problem (1.1), and using it as a test function in its own system, it is again immediate to check that

$$\|\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)} \leq \frac{C}{\nu} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \tag{2.7}$$

for some $C = C(\Omega) > 0$. Altogether, we find that

$$\nu \left(1 - C\, C^2(n)\, \nu^{-2} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\right) \|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)} \leq \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)},$$

and under our hypotheses and the fact that $2E(\mathbf{y}) = \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}$, this is the statement in the proposition. □

A practical way of taking a functional to its minimum is through some (clever) use of descent directions, i.e. the use of its derivative. In doing so, the presence of local minima is always something that may dramatically spoil the whole scheme. The unique structural property that discards this possibility is the strict convexity of the functional. However, for non-linear equations like (1.1), one cannot expect this property to hold for the functional E in (2.2). Nevertheless, we insist in that for a descent strategy applied to our extremal problem (2.4), numerical procedures cannot converge except to a global minimizer leading E down to zero. In doing so, thanks to Proposition 3.4, we are establishing the strong convergence of approximations to the unique solution of (1.1).

Indeed, we would like to show that the only critical points for E correspond to solutions of (1.1). In such a case, the search for an element $\mathbf{y}$ solution of (1.1) is reduced to the minimization of E, as indicated in the preceding paragraph. Precisely, we would like to prove the following proposition, in the spirit of [16]. Our computations here follow closely those in [15].

Before proceeding to our second step for a full proof of Theorem 2.1, we stress that the error functional $E(\mathbf{y})$ is coercive in the sense

$$E(\mathbf{y}) \to \infty \quad \text{if} \quad \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} \to \infty.$$

Indeed, from (2.3), and using $\mathbf{y}$ itself as a test function, it is elementary to arrive, using

$$\int_\Omega \mathbf{y} \otimes \mathbf{y} : \nabla \mathbf{y}\, d\mathbf{x} = 0$$

according to Lemma 2.3, that

$$\nu \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} \leq C(n, \Omega) \Big(\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \Big) \tag{2.8}$$

for some constant $C(n, \Omega) > 0$, leading to the coercivity of E.

Proposition 2.4. *There is a positive constant* C_1*, such that if* $\{\mathbf{y}^j\}_{j>0}$ *is a sequence in*

$$\mathbb{B} := \{\mathbf{y} \in \mathbb{V} : \|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)} < C_1\}$$

with $E'(\mathbf{y}^j) \to 0$ *as* $j \to \infty$*, then* $E(\mathbf{y}^j) \to 0$ *as* $j \to \infty$*.*

Note that the condition on the size of $\mathbf{y}$ in this statement is coherent with our hypotheses because the norm $\|\mathbf{y}_0\|_{\mathbf{H}_0^1(\Omega)}$ of the solution $\mathbf{y}_0$ is bounded, from (2.7), by $C\nu(\nu^{-2}\|f\|_{\mathbf{H}^{-1}(\Omega)})$ which is assumed to be small.

Proof. Let us first compute the derivative of E. For $\mathbf{Y} \in \mathbb{V}$, we have

$$E'(\mathbf{y}) \cdot \mathbf{Y} = \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{V} d\mathbf{x} \tag{2.9}$$

where $\mathbf{V} \in \mathbb{V}$ solves

$$\begin{cases} -\Delta \mathbf{V} + (-\nu\Delta\mathbf{Y} + div(\mathbf{y} \otimes \mathbf{Y}) + div(\mathbf{Y} \otimes \mathbf{y})) + \nabla \Pi = 0, \quad \text{in} \quad \Omega, \\ \mathbf{V} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.10}$$

Multiplying this system by $\mathbf{v}$, and integrating by parts, we get

$$E'(\mathbf{y}) \cdot \mathbf{Y} = \int_\Omega \Big(-\nu \nabla \mathbf{v} \cdot \nabla \mathbf{Y} + (\mathbf{y} \otimes \mathbf{Y} + \mathbf{Y} \otimes \mathbf{y}) : \nabla \mathbf{v} \Big) d\mathbf{x}. \tag{2.11}$$

We plan to use two different choices for $\mathbf{Y} \in \mathbb{V}$. The first is simply $\mathbf{Y} = -\mathbf{v}$ leading to the identity

$$E'(\mathbf{y}) \cdot (-\mathbf{v}) = \int_\Omega \Big(\nu |\nabla \mathbf{v}|^2 - (\mathbf{y} \otimes \mathbf{v} + \mathbf{v} \otimes \mathbf{y}) : \nabla \mathbf{v} \Big) d\mathbf{x}. \tag{2.12}$$

For the second, we make use of the following lemma whose proof is standard after the classical Lax-Milgram lemma.

Lemma 2.5. *There is a positive constant* C_1*, such that for every* $\mathbf{y} \in \mathbb{V}$ *with* $\|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)} < C_2$*, and* $\mathbf{F} \in \mathbf{L}^2(\Omega)$*, there exists* $\mathbf{Y} \in \mathbb{V}$ *and*

$$\int_\Omega \big(\nu \nabla \mathbf{Y} - (\mathbf{Y} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{Y}) \big) : \nabla \mathbf{w} \, d\mathbf{x} - \int_\Omega \mathbf{F} \cdot \mathbf{w} \, d\mathbf{x} = 0, \tag{2.13}$$

for every $\mathbf{w} \in \mathbb{V}$*.*

Proof. For fixed $\mathbf{y} \in \mathbb{V}$, consider the bilinear form

$$a(\mathbf{Y}, \mathbf{w}) = \int_\Omega \big(\nu \nabla \mathbf{Y} - (\mathbf{Y} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{Y}) \big) : \nabla \mathbf{w} \, d\mathbf{x}.$$

It is standard to check that it is continuous. The relevant issue is to check coercivity. To this aim, we need to examine

$$a(\mathbf{Y}, \mathbf{Y}) = \int_\Omega \big(\nu \nabla \mathbf{Y} - (\mathbf{Y} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{Y}) \big) : \nabla \mathbf{Y} \, d\mathbf{x}.$$

We have already gone through this calculation above. Once again, Lemma 2.3 implies that the second term vanishes, while the third one becomes

$$\int_\Omega \mathbf{Y} \otimes \mathbf{Y} : \nabla \mathbf{y} \, d\mathbf{x}.$$

This integral, in turn, can be bounded above, as before, by

$$C^2(n) \|\mathbf{Y}\|^2_{\mathbf{H}_0^1(\Omega)} \|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)}.$$

In this way, we have

$$a(\mathbf{Y}, \mathbf{Y}) \geq (\nu - C^2(n)\|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)}) \|\mathbf{Y}\|^2_{\mathbf{H}_0^1(\Omega)}.$$

The classic Lax-Milgram lemma allows us to conclude provided $\|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)} < \nu / C^2(n)$. $\square$

If we apply this lemma for the choice $\mathbf{F} = \mathbf{v}$, and use as a test function $\mathbf{w} = \mathbf{v}$ as well, we ensure the existence of some $\mathbf{Y} \in \mathbb{V}$ with

$$\int_\Omega \big(\nu\nabla\mathbf{Y} - (\mathbf{Y}\otimes\mathbf{y} + \mathbf{y}\otimes\mathbf{Y})\big) : \nabla\mathbf{v}\, d\mathbf{x} - \int_\Omega \mathbf{v}\cdot\mathbf{v}\, d\mathbf{x} = 0.$$

But, from (2.11), we see that

$$E'(\mathbf{y})\cdot(-\mathbf{Y}) = \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)}. \tag{2.14}$$

Suppose now that $E'(\mathbf{y}) \to 0$. According to our brief discussion before the statement of the proposition we are proving, vector fields $\mathbf{y}$ remain in a bounded set of $\mathbb{V}$. We need to check that then the corresponding solutions given by Lemma 2.5 remain in a bounded set as well. This is easily accomplished by taking $\mathbf{w} = \mathbf{Y}$ in (2.13) (recall $\mathbf{F} = \mathbf{v}$)

$$\nu\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)} \le C^2(n)\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}\, \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}.$$

In checking this inequality, recall again that, by Lemma 2.3,

$$\int_\Omega \mathbf{Y}\otimes\mathbf{y} : \nabla\mathbf{Y}\, d\mathbf{x} = 0,$$

while

$$\int_\Omega \mathbf{y}\otimes\mathbf{Y} : \nabla\mathbf{Y}\, d\mathbf{x} = -\int_\Omega \mathbf{Y}\otimes\mathbf{Y} : \nabla\mathbf{y}\, d\mathbf{x}.$$

Moreover $\|\mathbf{Y}\otimes\mathbf{Y}\|_{L^2(\Omega)} \le C^2(n)\|\mathbf{Y}\|^2_{\mathbf{H}^1(\Omega)}$ provided $n \le 4$, as has already been pointed out earlier. As a matter of fact, this is the same as (2.6). In turn, from (2.3) by using $\mathbf{v}$ as a test field, we also see that

$$\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)} \le \nu\|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} + C^2(n)\|\mathbf{y}\|^2_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}.$$

Altogether, $\mathbf{Y}$ is bounded in terms of $\mathbf{y}$ (through the constant C_2) and $\mathbf{f}$, and so it is globally bounded as $E'(\mathbf{y}) \to 0$. Once we know that both fields $\mathbf{y}$ and $\mathbf{Y}$ remain in bounded sets as $E'(\mathbf{y}) \to 0$, (2.14) yields that $\mathbf{v} \to 0$ in $\mathbf{L}^2(\Omega)$. This information, taken to (2.12) and given that $E'(\mathbf{y})\cdot(-\mathbf{v}) \to 0$ (again because $\mathbf{v}$ stays in a uniform bounded set), leads to the desired result

$$2E(\mathbf{y}) = \|\mathbf{v}\|^2_{\mathbf{H}^1_0(\Omega)} \to 0.$$

□

3. Steady case without the div-free constraint

In practice, implementing the div-free constraint, as done in [3], is possibly expansive as it requires at each iteration several resolutions of the steady Stokes equation. In this section, we would like to explore to what extent a similar approach can be implemented that allows for fields without the div-free constraint. This new framework forces us to take into account the pressure field.

We define $\mathcal{A} = \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$, and define the functional $E : \mathcal{A} \to \mathbb{R}^+$

$$E(\mathbf{y}, \pi) = \frac{1}{2}\int_\Omega \left(|\nabla\mathbf{v}|^2 + |\nabla\cdot\mathbf{y}|^2\right)\, d\mathbf{x} \tag{3.1}$$

where, as above, the corrector $\mathbf{v}$ is the unique solution in $\mathbf{H}^1_0(\Omega)$ of the (elliptic) boundary value problem

$$\begin{cases} -\Delta\mathbf{v} + (-\nu\Delta\mathbf{y} + div(\mathbf{y}\otimes\mathbf{y}) + \nabla\pi - \mathbf{f}) = 0, & \text{in} \quad \Omega, \\ \mathbf{v} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

No field in this section is divergence free. If this is so, it is not true that the non-linear term $div(\mathbf{y}\otimes\mathbf{y})$ equals $(\mathbf{y}\cdot\nabla)\mathbf{y}$. However, this will be so as we take down E in (2.2) to zero.

3.1. **Some preliminaries.** Before getting into the proof of similar results as in the preceding section, it is instructive to spend some time with the following interesting discussion.

Consider the variational problem

$$\text{Minimize in } \pi \in L^2_0(\Omega): \quad \widetilde{E}(\pi) = \frac{1}{2}\int_\Omega |\nabla \mathbf{v}|^2 \, d\mathbf{x} \tag{3.3}$$

subject to

$$-\Delta \mathbf{v} + \nabla \pi - \mathbf{F} = 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \partial\Omega, \tag{3.4}$$

where $\mathbf{F}$ belongs to $\mathbf{H}^{-1}(\Omega)$. On the other hand, consider the optimization problem

$$\text{Minimize in } \mathbf{v} \in \mathbb{V}: \quad \int_\Omega \frac{1}{2}|\nabla \mathbf{v}|^2 \, d\mathbf{x} - \langle \mathbf{F}, \mathbf{v} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)}.$$

Recall that $\mathbb{V} = \mathbf{H}^1_{0,div}(\Omega)$. It is standard to check that this last problem admits a unique minimizer $\mathbf{v}_0$, which is divergence free, and is characterized by the variational identity

$$\int_\Omega \nabla \mathbf{v}_0 : \nabla \mathbf{w} \, d\mathbf{x} - \langle \mathbf{F}, \mathbf{w} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} = 0 \tag{3.5}$$

for every $\mathbf{w} \in \mathbb{V}$. Alternatively, there is unique $\pi_0 \in L^2_0(\Omega)$ such that

$$\int_\Omega (\nabla \mathbf{v}_0 : \nabla \mathbf{w} + \pi_0 \nabla \cdot \mathbf{w}) \, d\mathbf{x} - \langle \mathbf{F}, \mathbf{w} \rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega)} = 0 \tag{3.6}$$

for all $\mathbf{w} \in \mathbf{H}^1_0(\Omega)$.

Lemma 3.1. *This pair* $(\pi_0, \mathbf{v}_0)$ *is the unique solution of* (3.3)*. In particular, the minimizer* $\mathbf{v}_0$ *in* (3.3) *is divergence free. In addition, there is a constant* $C > 0$*, independent of* $\mathbf{F}$*, such that*

$$\|\pi_0\|_{L^2(\Omega)} \le C \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}.$$

Proof. Let Π be a variation of a certain π, so that $\mathbf{V}$ is the corresponding perturbation of $\mathbf{v}$. Then

$$-\Delta \mathbf{V} + \nabla \Pi = 0 \text{ in } \Omega, \quad \mathbf{V} = 0 \text{ on } \partial\Omega,$$

and, hence,

$$\widetilde{E}'(\pi) \cdot \Pi = \int_\Omega \nabla \mathbf{v} : \nabla \mathbf{V} \, d\mathbf{x} = \int_\Omega \Pi \, \nabla \cdot \mathbf{v} \, d\mathbf{x}.$$

If π is a minimizer of the problem, then this derivative must vanish for all Π, and this implies $\nabla \cdot \mathbf{v} = 0$. We therefore see that the pair $(\pi_0, \mathbf{v}_0)$ determined by (3.6) is feasible for (3.3), and π_0 is a critical point because $\mathbf{v}_0$ is divergence free. Since problem (3.3) is quadratic in π and strictly convex, we conclude that the critical pair $(\pi_0, \mathbf{v}_0)$ must be the minimizer.

Focus next on the operator taking $\mathbf{F}$ in (3.4) into the corresponding minimizer π_0. It is clear that it is a linear, continuous operator from $\mathbf{H}^{-1}(\Omega)$ to $L^2_0(\Omega)$. Note that $\mathbf{F} \equiv 0$ is taken into $\pi_0 = 0$. Thus there is a constant $C > 0$ (independent of $\mathbf{F}$) such that

$$\|\pi_0\|_{L^2(\Omega)} \le C \|\mathbf{F}\|_{\mathbf{H}^{-1}(\Omega)}.$$

□

This lemma allows us to define a certain non-linear operator $\mathbf{T} : \mathbf{H}^1_0(\Omega) \to L^2_0(\Omega)$ determined by partial minimization of our error functional $E(\mathbf{y}, \pi)$ in (2.2), in π for given $\mathbf{y}$, according to our discussion above. Since in our context $\mathbf{F} = \nu \Delta \mathbf{y} - div(\mathbf{y} \otimes \mathbf{y}) + \mathbf{f}$ in $\mathbf{H}^{-1}(\Omega)$, our discussion above amounts to having that the corresponding minimizer $\mathbf{T}(\mathbf{y})$ is such that

$$\|\mathbf{T}\mathbf{y}\|_{L^2(\Omega)} \le C\|\nu \Delta \mathbf{y} - div(\mathbf{y} \otimes \mathbf{y}) + \mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \le C(\|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{y}\|^2_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}), \tag{3.7}$$

for some positive constant $C > 0$, independent of $\mathbf{y}$.

The following fact replaces our previous Lemma 2.3. It is a version of the same lemma without the div-constraint.

Lemma 3.2. *For all* $\mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$, *we have*

$$\int_\Omega (\mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{v} + \mathbf{v} \otimes \mathbf{u} : \nabla \mathbf{u})\, d\mathbf{x} = -\int_\Omega \mathbf{v} \cdot \mathbf{u} \nabla \cdot \mathbf{u},$$

$$\int_\Omega \mathbf{u} \otimes \mathbf{v} : \nabla \mathbf{u}\, d\mathbf{x} = -\frac{1}{2} \int_\Omega |\mathbf{u}|^2\, \nabla \cdot \mathbf{v}\, d\mathbf{x}.$$

3.2. **The main result.** We are now ready to show a main result as in the previous section in this more general context.

Our strategy proceeds again in two steps. First, an estimate implying that whenever $E(\mathbf{y}, \pi) \to 0$, we will have strong convergence of $\mathbf{y}$ to the unique solution sought; secondly, the error property amounting to having $E(\mathbf{y}, \pi) \to 0$ if $E'(\mathbf{y}, \pi) \to 0$. Both steps are more involved compared to their previous counterparts, and there is an additional step to take care of the pressure. The theorem is, however, exactly the same.

Theorem 3.3. *There is a positive constant* C_3, *such that if* $\{(\mathbf{y}^j, \pi^j)\} \subset \mathcal{A}$ *is a sequence with* $\mathbf{y}^j$ *belonging to the ball*

$$\mathbb{B} := \{\mathbf{y} \in \mathbf{H}_0^1(\Omega) : \|\mathbf{y}\|_{\mathbf{H}_0^1(\Omega)} < C_3\}$$

and $E'(\mathbf{y}^j, \pi^j) \to 0$ *as* $j \to \infty$, *then the whole sequence* $\mathbf{y}^j$ *converges strongly, as* $j \to \infty$, *in* $\mathbf{H}_0^1(\Omega)$ *to the unique solution* $\mathbf{y}$ *of* (1.1) *guaranteed by Theorem 1.1, if* $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ *is small enough.*

The first step of the proof involves an upper bound of the difference $\mathbf{y} - \mathbf{y}_0$ in terms of the quantity $E(\mathbf{y}, \pi)$. Its statement makes use of the operator $\mathbf{T}$ introduced in the preceding subsection. It takes every field $\mathbf{y} \in \mathbf{H}_0^1(\Omega)$ into the corresponding optimal pressure $\mathbf{T}\mathbf{y}$ that is obtained by partial minimization of $E(\mathbf{y}, \pi)$ on π.

Proposition 3.4. *Let* $(\mathbf{y}_0, \pi_0) \in \mathcal{A}$ *be the unique solution of* (1.1), *under the appropriate hypotheses as in Theorem 1.1. If dimension* $n \le 4$ *and* $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ *is sufficiently small, there are positive constants* C_1, C_2, C_3, *depending only upon* $\nu, n, \mathbf{f}, \pi_0, \mathbf{y}_0$, *such that for every* $\mathbf{y}$ *in a fixed fall* $\mathbb{B}$ *of* $\mathbf{H}_0^1(\Omega)$, *if we put* $E = E(\mathbf{y}, \mathbf{T}\mathbf{y})^{1/2}$, *then we have*

$$\|\mathbf{y} - \mathbf{y}_0\|^2_{\mathbf{H}_0^1(\Omega)} \le \frac{E + \sqrt{E^2 + 4C_3(C_1 - C_2E)E}}{2(C_1 - C_2E)},$$

provided $C_1 - C_2E > 0$.

Note that as we take down the error $E(\mathbf{y}, \pi)$ to zero, and so $\|\nabla \cdot \mathbf{y}\|_{L^2(\Omega)} \to 0$ as well, this estimate shows the desired strong convergence because for every feasible pair $(\mathbf{y}, \pi) \in \mathcal{A}$, it is true that

$$E(\mathbf{y}, \mathbf{T}\mathbf{y}) \le E(\mathbf{y}, \pi)$$

according to the definition of the operator $\mathbf{T}$. In addition, the condition on the positivity of $C_1 - C_2E$ becomes eventually true as $E(\mathbf{y}, \pi) \searrow 0$.

The proof of this proposition consists in rechecking the various steps for the one we presented earlier, bearing in mind this time that $\nabla \cdot \mathbf{y}$ does not necessarily vanish.

Proof. We go through the same preparations as in the similar result of the previous section. If we let $\mathbf{Y} \equiv \mathbf{y}_0 - \mathbf{y}$, $\Pi \equiv \pi_0 - \pi$, where $(\mathbf{y}_0, \pi_0)$ is the unique solution of the problem that we would like to approximate, then (2.5) still holds. This system can be recast in the form

$$\Delta \mathbf{v} - \nu \Delta \mathbf{Y} - \nabla \Pi + div(\mathbf{y}_0 \otimes \mathbf{Y} + \mathbf{Y} \otimes \mathbf{y}) = 0 \quad \text{in} \quad \Omega.$$

If we multiply by $\mathbf{Y}$, and integrate by parts, bearing in mind this time that $div\mathbf{Y} = \nabla \cdot \mathbf{Y}$ does not necessarily vanish, we are led to

(3.8)

$$-\int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{Y}\, d\mathbf{x} + \nu \int_\Omega |\nabla \mathbf{Y}|^2\, d\mathbf{x} + \int_\Omega \Pi \nabla \cdot \mathbf{Y}\, d\mathbf{x} - \int_\Omega \mathbf{y}_0 \otimes \mathbf{Y} : \nabla \mathbf{Y}\, d\mathbf{x} - \int_\Omega \mathbf{Y} \otimes \mathbf{y} : \nabla \mathbf{Y}\, d\mathbf{x} = 0.$$

Recall, however, that $\mathbf{y}_0$ is divergence-free. In view of Lemma 3.2, we find that the last two terms we are interested in become, respectively,

$$\int_\Omega \mathbf{y}_0 \cdot \mathbf{Y}\,\nabla\cdot\mathbf{Y}\,d\mathbf{x} + \int_\Omega \mathbf{Y}\otimes\mathbf{Y} : \nabla\mathbf{y}_0\,d\mathbf{x}, \quad -\frac{1}{2}\int_\Omega |\mathbf{Y}|^2\,\nabla\cdot\mathbf{Y}\,d\mathbf{x}.$$

Note that $\nabla\cdot\mathbf{Y} = -\nabla\cdot\mathbf{y}$ because $\nabla\cdot\mathbf{y}_0 = 0$. The term

$$\int_\Omega \mathbf{Y}\otimes\mathbf{Y} : \nabla\mathbf{y}_0\,d\mathbf{x}$$

can be estimated, as before, by

$$C(n)^2\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)}\,\|\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)},$$

but we have to deal with three different contributions involving the divergence $\nabla\cdot\mathbf{Y}$, namely

$$\int_\Omega \left(\Pi + \mathbf{y}_0\cdot\mathbf{Y} - \frac{1}{2}|\mathbf{Y}|^2\right)\nabla\cdot\mathbf{Y}\,d\mathbf{x} = \int_\Omega \left(\Pi + \frac{1}{2}|\mathbf{y}_0|^2 - \frac{1}{2}|\mathbf{Y}-\mathbf{y}_0|^2\right)\nabla\cdot\mathbf{Y}\,d\mathbf{x}.$$

Altogether we have, from (3.8),

$$\nu\int_\Omega |\nabla\mathbf{Y}|^2\,d\mathbf{x} = -\int_\Omega \mathbf{Y}\otimes\mathbf{Y} : \nabla\mathbf{y}_0\,d\mathbf{x} + \int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{Y}\,d\mathbf{x} + \int_\Omega \left(\frac{1}{2}|\mathbf{Y}-\mathbf{y}_0|^2 - \Pi - \frac{1}{2}|\mathbf{y}_0|^2\right)\nabla\cdot\mathbf{Y}\,d\mathbf{x},$$

and then

$$\begin{aligned}\nu\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} \leq & C(n)^2\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)}\,\|\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)} \\ & + \left(\|\Pi\|_{L^2(\Omega)} + \frac{3}{2}\|\mathbf{y}_0\|^2_{L^4(\Omega)} + \|\mathbf{Y}\|^2_{L^4(\Omega)}\right)\|\nabla\cdot\mathbf{Y}\|_{L^2(\Omega)}.\end{aligned}$$

We have already used before (keep in mind (2.7)) that

$$\|\mathbf{Y}\|_{\mathbf{L}^4(\Omega)} \leq C(n)\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}, \quad \nu\|\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)} \leq C\|\mathbf{f}\|_{\mathbf{H}^{-1}_0(\Omega)}.$$

We then find that

$$\begin{aligned}\left(\nu - C\frac{C(n)^2}{\nu}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} - C(n)^2\|\nabla\cdot\mathbf{Y}\|_{L^2(\Omega)}\right)\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} \leq \|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)} \\ + \left(\|\Pi\|_{L^2(\Omega)} + C\frac{\|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)}}{\nu^2}\right)\|\nabla\cdot\mathbf{Y}\|_{L^2(\Omega)}.\end{aligned}$$

Bearing in mind that $\nabla\cdot\mathbf{Y} = -\nabla\cdot\mathbf{y}$, and taking into account that

$$\max\{\|\mathbf{v}\|^2_{\mathbf{H}^1_0(\Omega)}, \|\nabla\cdot\mathbf{Y}\|^2_{L^2(\Omega)}\} \leq 2E(\mathbf{y},\pi),$$

we can also write

$$\begin{aligned}\left(\nu - C\frac{C(n)^2}{\nu}\|\mathbf{f}\|_{\mathbf{H}^{-1}_0(\Omega)} - C(n)^2\sqrt{2}E(\mathbf{y},\pi)^{1/2}\right)\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} \leq \sqrt{2}E(\mathbf{y},\pi)^{1/2}\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)} \\ + \left(\|\Pi\|_{L^2(\Omega)} + C\frac{\|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)}}{\nu^2}\right)\sqrt{2}E(\mathbf{y},\pi)^{1/2}.\end{aligned}$$

On the other hand, if we take $\pi = \mathbf{T}\mathbf{y}$, because of (3.7), we also have

$$\begin{aligned}\|\Pi\|_{L^2(\Omega)} \leq & \|\pi_0\|_{L^2(\Omega)} + \|\mathbf{T}(\mathbf{y})\|_{L^2(\Omega)} \\ \leq & \|\pi_0\|_{L^2(\Omega)} + C(\|\mathbf{y}\|^2_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}) \\ \leq & \|\pi_0\|_{L^2(\Omega)} + C(\|\mathbf{y}\|^2_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + 1)\end{aligned}$$

provided fields $\mathbf{y}$ are taken from a fixed ball $\mathbb{B}$. In this way,

$$\|\Pi\|_{L^2(\Omega)} \leq \|\pi_0\|_{L^2(\Omega)} + C(\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{y}_0\|^2_{\mathbf{H}^1_0(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + 1)$$

and the previous inequality becomes

$$\left(\nu - C\frac{C(n)^2}{\nu}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} - \widetilde{C}(n)^2\sqrt{2}E(\mathbf{y},\pi)^{1/2}\right)\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)} \leq \sqrt{2}E(\mathbf{y},\pi)^{1/2}\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)} + \left(\widetilde{C} + C\frac{\|\mathbf{f}\|^2_{\mathbf{H}^{-1}(\Omega)}}{\nu^2}\right)\sqrt{2}E(\mathbf{y},\pi)^{1/2}, \quad \pi = \mathbf{T}\mathbf{y}.$$

Here C, $\widetilde{C}$, and $\widetilde{C}(n)$ are constants that may depend on π_0, $\mathbf{y}_0$, $\mathbf{f}$, and may eventually change from line to line. It is elementary then to have the statement in the proposition from this inequality. □

Concerning the second step, there are just minor changes in the proof.

Proposition 3.5. $\lim E(\mathbf{y},\pi) = 0$ *as* $E'(\mathbf{y},\pi) \to 0$.

Proof. Notice that the derivative involves this time a second term

$$E'(\mathbf{y},\pi)\cdot(\mathbf{Y},\Pi) = \int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{V} + (\nabla\cdot\mathbf{y})(\nabla\cdot\mathbf{Y})d\mathbf{x} \tag{3.9}$$

where again $\mathbf{V} \in \mathbf{H}^1_0(\Omega)$ solves (2.10). By using $\mathbf{v}$ as a test function in (2.10), we can also write

$$\begin{aligned} E'(\mathbf{y},\pi)\cdot(\mathbf{Y},\Pi) = &\int_\Omega \Big(-\nu\nabla\mathbf{v}\cdot\nabla\mathbf{Y} + (\mathbf{y}\otimes\mathbf{Y} + \mathbf{Y}\otimes\mathbf{y}) : \nabla\mathbf{v} + (\nabla\cdot\mathbf{v})\Pi\Big)d\mathbf{x} \\ &+ \int_\Omega (\nabla\cdot\mathbf{y})(\nabla\cdot\mathbf{Y})d\mathbf{x}. \end{aligned} \tag{3.10}$$

As before, we plan to use several appropriate choices for the direction $(\mathbf{Y},\Pi)$.

We first check that we can take $\mathbf{Y} = \mathbf{v}$ in (3.10). To be this a legitimate thing to do, one needs to check that $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ remains bounded with respect to $(\mathbf{y},\pi) \in \mathbf{H}^1_0 \times L^2(\Omega)$. To this end, by definition, the corrector $\mathbf{v}$ solves the variational formulation

$$\int_\Omega ((\nabla\mathbf{v} + \nu\nabla\mathbf{y} - \mathbf{y}\otimes\mathbf{y}) : \nabla\mathbf{w} - \pi\nabla\cdot\mathbf{w})d\mathbf{x} - <\mathbf{f},\mathbf{w}>_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} = 0$$

for every $\mathbf{w} \in \mathbf{H}^1_0(\Omega)$. Taking $\mathbf{w} = \mathbf{v}$, we get that

$$\int_\Omega |\nabla\mathbf{v}|^2 d\mathbf{x} = \int_\Omega (\mathbf{y}\otimes\mathbf{y} : \nabla\mathbf{v} - \nu\nabla\mathbf{y} : \nabla\mathbf{v} + \pi\nabla\cdot\mathbf{v})d\mathbf{x} + <\mathbf{f},\mathbf{v}>_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)},$$

so that, in view of Poincaré's inequality,

$$\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)} \leq C(\|\mathbf{y}\otimes\mathbf{y}\|_{L^2(\Omega)} + \|\mathbf{y}\|_{\mathbf{H}^1_0(\Omega)} + \|\pi\|_{L^2(\Omega)} + \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}). \tag{3.11}$$

We are therefore entitled to take $\mathbf{Y} = \mathbf{v}$ in (3.10), to get

$$\begin{aligned} E'(\mathbf{y},\pi)\cdot(\mathbf{v},\Pi) &= \int_\Omega \Big(-\nu|\nabla\mathbf{v}|^2 - (\mathbf{y}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{y}) : \nabla\mathbf{v} + (\nabla\cdot\mathbf{v})\Pi\Big)d\mathbf{x} + \int_\Omega (\nabla\cdot\mathbf{y})(\nabla\cdot\mathbf{v})d\mathbf{x} \\ &= \int_\Omega \Big(-\nu|\nabla\mathbf{v}|^2 - (\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{y} + \frac{1}{2}(\nabla\cdot\mathbf{y})|\mathbf{v}|^2\Big)d\mathbf{x} \\ &\quad + \int_\Omega (\nabla\cdot\mathbf{v})(\nabla\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{v} + \Pi)d\mathbf{x}. \end{aligned} \tag{3.12}$$

Similarly, in view of (3.11), $\Pi_s = -(\nabla\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{v}) \in L^2(\Omega)$ remains bounded too with respect to $(\mathbf{y},\pi)$, and we write

$$E'(\mathbf{y},\pi)\cdot(\mathbf{v},\Pi_s) = \int_\Omega \Big(-\nu|\nabla\mathbf{v}|^2 - (\mathbf{v}\otimes\mathbf{v}) : \nabla\mathbf{y} + \frac{1}{2}(\nabla\cdot\mathbf{y})|\mathbf{v}|^2\Big)d\mathbf{x}. \tag{3.13}$$

We now go back to Lemma 2.5. Note that (2.13) can also be recast as

$$\int_\Omega \Big((\nu\nabla\mathbf{Y} - (\mathbf{Y}\otimes\mathbf{y} + \mathbf{y}\otimes\mathbf{Y})) : \nabla\mathbf{w} + \Pi\,\nabla\cdot\mathbf{w}\Big)d\mathbf{x} - \int_\Omega \mathbf{F}\cdot\mathbf{w}d\mathbf{x} = 0$$

for some $\Pi \in L^2(\Omega)$ and for every $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$, provided fields $\mathbf{y}$ are taken from the same ball in the statement of that lemma. Lemma 2.5 can then be applied with $\mathbf{F} = \mathbf{w} = \mathbf{v}$ to find a pair $(\mathbf{Y}, \Pi)$ so that

$$\int_\Omega \Big([-\nu\nabla\mathbf{Y} + (\mathbf{y}\otimes\mathbf{Y} + \mathbf{Y}\otimes\mathbf{y})] : \nabla\mathbf{v} - \Pi\,\nabla\cdot\mathbf{v} \Big)\, d\mathbf{x} = \int_\Omega |\mathbf{v}|^2\, d\mathbf{x}.$$

But the left-hand side is exactly (see (3.10))

$$E'(\mathbf{y}, \pi)\cdot(\mathbf{Y}, \Pi)$$

which tends to zero. Conclude that $\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \to 0$. This together with (3.13) finishes the proof. □

Theorem 3.3 is a direct consequence of both Propositions 3.4 and 3.5.

4. Minimizing sequence

Theorems 2.1 and 3.3 are general convergence results that do not take into account the particular method to produce such sequence $\{\mathbf{y}^j\}$ or pairs $\{(\mathbf{y}^j, \pi^j)\}$, respectively, with

$$E'(\mathbf{y}^j) \to 0 \quad \text{or} \quad E'(\mathbf{y}^j, \pi^j) \to 0.$$

In practice, however, one would typically use a gradient method to calculate iteratively such sequences. Given that the exact solution $\mathbf{y}_0$ of the problem corresponds to an absolute minimum of the smooth functional E, for a certain small positive constant c_2, one can ensure that

$$\|\mathbf{y}^0 - \mathbf{y}_0\|_{\mathbf{H}_0^1(\Omega)} \le c_2$$

implies that the sequences computed through a gradient procedure starting from $\mathbf{y}^0$ will converge to $\mathbf{y}_0$. However, it would be interesting to have more explicit information about the size of the constant c_2 that, eventually, could be of some help to decide in practice how to select the initial guess.

We first treat the div-constrained situation. The following lemma ensures that a gradient method for the functional E in (2.2) will always converge to the solution of (1.1), provided the initial guess $\mathbf{y}^0$ is sufficiently close to the true solution $\mathbf{y}_0$ of (1.1).

Lemma 4.1. *If $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ is small enough, there is a known, specific positive constant C_4 such that if $\|\mathbf{y}^0 - \mathbf{y}_0\|_{\mathbf{H}_0^1(\Omega)} < C_4$, then a gradient method for E in* (2.2) *starting from $\mathbf{y}^0$ will always converge to $\mathbf{y}_0$.*

Proof. Our strategy is to show that the quantity

$$E'(\mathbf{y})\cdot(\mathbf{y}_0 - \mathbf{y})$$

becomes non-positive, if $\mathbf{y}$ is sufficiently close, in a precise quantitative way, to the exact solution $\mathbf{y}_0$. If this is so, then the flow of the functional E is pointing inward in those small balls centered at $\mathbf{y}_0$, and so integral curves cannot escape from those balls. Let $\mathbf{y}$ be an arbitrary field in $\mathbb{V}$, and recall formula (2.11) for the derivative of E at $\mathbf{y}$, applied to the difference $\mathbf{Y} = \mathbf{y}_0 - \mathbf{y}$

$$E'(\mathbf{y})\cdot(\mathbf{y}_0 - \mathbf{y}) = \int_\Omega \Big(-\nu\nabla\mathbf{v}\cdot\nabla(\mathbf{y}_0 - \mathbf{y}) + (\mathbf{y}\otimes(\mathbf{y}_0 - \mathbf{y}) + (\mathbf{y}_0 - \mathbf{y})\otimes\mathbf{y}) : \nabla\mathbf{v} \Big)\, d\mathbf{x},$$

where $\mathbf{v}$ is the corrector associated with $\mathbf{y}$. On the other hand, using $\mathbf{v}$ as a test function in (2.5) (which is the difference of the equations for $\mathbf{y}$ with its corrector $\mathbf{v}$ and for the exact solution $\mathbf{y}_0$), it is a matter of some careful algebra to arrive at

$$E'(\mathbf{y})\cdot(\mathbf{y}_0 - \mathbf{y}) = \int_\Omega (-|\nabla\mathbf{v}|^2 + (\mathbf{y} - \mathbf{y}_0)\otimes(\mathbf{y} - \mathbf{y}_0) : \nabla\mathbf{v})\, d\mathbf{x}. \tag{4.1}$$

We now take into account Proposition 2.2 so that

$$\|\mathbf{Y}\|^2_{\mathbf{H}_0^1(\Omega)} \le C\|\mathbf{v}\|^2_{\mathbf{H}_0^1(\Omega)},$$

for some constant C provided $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ is small enough. It is also true, as we have already used earlier several times, that

$$\|\mathbf{Y}\otimes\mathbf{Y}\|_{\mathbf{L}^2(\Omega)}\le C(n)^2\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)}.$$

By using these inequalities, (4.1) can be estimated

$$E'(\mathbf{y})\cdot(\mathbf{y}_0-\mathbf{y})\le -\|\mathbf{v}\|^2_{\mathbf{H}^1_0(\Omega)}+C\|\mathbf{v}\|^3_{\mathbf{H}^1_0(\Omega)},$$

for a certain known constant C independent of $\mathbf{y}$. If we can ensure that the size of the corrector $\mathbf{v}$ is such that

$$\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}<1/C,$$

whenever

$$\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}<C_4 \tag{4.2}$$

for some positive known constant C_4, then we would indeed have $E'(\mathbf{y})\cdot(\mathbf{y}_0-\mathbf{y})\le 0$. This sign condition is informing us that the flow of E is always pointing inwards in the ball determined by condition (4.2). If we take C_4 even smaller if necessary to guarantee that

$$\|\mathbf{y}-\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)}<C_4 \quad\text{implies}\quad \mathbf{y}\in\mathbb{B}$$

where $\mathbb{B}$ is the ball, centered at zero, in Proposition 2.4, then we would have that all integral curves starting under the condition (4.2) will converge to $\mathbf{y}_0$ since in this ball there cannot be critical points of E other than $\mathbf{y}_0$ itself, according to Theorem 2.1. It remains, hence, to quantify the continuity of E at the solution $\mathbf{y}_0$. To check this, we use $\mathbf{v}$ as a test function in (2.5) (taking into account the expression Div a few lines below that formula), to write

$$\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}\le\nu\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}+\|\mathbf{y}_0\otimes\mathbf{Y}+\mathbf{Y}\otimes\mathbf{y}\|_{\mathbf{L}^2(\Omega)}.$$

If we put

$$\mathbf{y}_0\otimes\mathbf{Y}+\mathbf{Y}\otimes\mathbf{y}=\mathbf{Y}\otimes\mathbf{Y}+2\mathbf{y}_0\otimes\mathbf{Y},$$

this last term can be estimated, once again, in the form

$$C(n)^2\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)}+C(n)^2\|\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)}\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}.$$

If we recall (2.7), we find

$$\|\mathbf{v}\|_{\mathbf{H}^1_0(\Omega)}\le\nu\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}+C(n)^2\|\mathbf{Y}\|^2_{\mathbf{H}^1_0(\Omega)}+C(n)^2\frac{C}{\nu}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}\|\mathbf{Y}\|_{\mathbf{H}^1_0(\Omega)}.$$

We clearly see that we can make the left-hand side small by making the right-hand side small in a quantified way. That was our goal. □

It is a matter of keeping track of the constants in all those inequalities used above in the proof to have an expression of the constant C_4 guaranteeing the claimed convergence. There are four quantities involved: viscosity ν, size of the source term $\|\mathbf{f}\|=\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$, Poincaré's constant C for Ω, and the constant $C(n)$ of the Sobolev compact embedding of $H^1(\Omega)$ into $L^4(\Omega)$. Computations are elementary but tedious. We find that if

$$C_4=\min\left\{\frac{\nu}{C(n)^2},\frac{\|\mathbf{f}\|}{\nu}+\frac{1}{2}\sqrt{\nu^2-4\nu+2CC(n)^2\|\mathbf{f}\|\left(1+\frac{2}{\nu}\right)+\frac{C^2C(n)^4\|\mathbf{f}\|^2}{\nu^2}}-\frac{1}{2}\nu-\frac{1}{2}\frac{CC(n)^2\|\mathbf{f}\|}{\nu}\right\},$$

then every starting field $\mathbf{y}^0$ with

$$\|\mathbf{y}^0-\mathbf{y}_0\|_{\mathbf{H}^1_0(\Omega)}<C_4$$

ensures that iterations through a gradient method for our error functional E will converge to the unique solution $\mathbf{y}_0$, provided $\nu^{-2}\|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ is small enough.

The non-divergence free situations is a bit more involved though a parallel proof would proceed along the same lines as in the case we have just explored. The main inequalities and results used in the proof above have a counterpart for the non-divergence free case. Note that (2.11) ought to be replaced by (3.9); (2.5) is still valid; Proposition 3.4 plays the role of Proposition 2.2; the continuity of E at $(\mathbf{y}_0, \pi_0)$ incorporates an additional term coming for pressure π_0. We believe it is not worth to write explicitly the constant c_2 in this case as its expression would be quite complex, mainly because the constant in Proposition 3.4 is already pretty complicated.

From a purely practical standpoint, however, checking "a posteriori" computed iterates $\mathbf{y}^j$ or pairs $(\mathbf{y}^j, \pi^j)$ will tell us whether we are getting close to the unique solution $\mathbf{y}_0$ or pair $(\mathbf{y}_0, \pi_0)$, because numbers $E(\mathbf{y}^j)$ or $E(\mathbf{y}^j, \pi^j)$ become steadily and virtually zero, or they stay bounded away from zero.

5. Conjugate gradient algorithm

The previous results hold true if we replace the cost E defined in (3.1)

$$E_\varepsilon(\mathbf{y}, \pi) := \frac{1}{2}\int_\Omega (|\nabla \mathbf{v}|^2 + |\nabla\cdot\mathbf{y} + \varepsilon\pi|^2)\, d\mathbf{x}$$

for any $\varepsilon > 0$, where $(\mathbf{y}, \pi)$ solves (3.2). The introduction of the ε term allows to fix the constant of the pressure π. The appropriate tool to produce minimizing sequence for the functional E_ε is gradient method. Among them, the Polak-Ribière version of the conjugate gradient (CG for short in the sequel) algorithm (see [8]) have shown its efficiency in the similar context analyzed in [13, 14, 12]. For the functional E_ε, the CG algorithm reads as follows :

- *Step 0: Initialization* - Given any $\eta > 0$ and any $\mathbf{z}^0 = (\mathbf{y}^0, \pi^0) \in \mathcal{A}$, compute the residual $\boldsymbol{g}^{\mathbf{0}} = (\overline{\mathbf{y}}^0, \overline{\pi}^0) \in \mathcal{A}$ solution of

$$(\boldsymbol{g}^{\mathbf{0}}, (\mathbf{Y}, \Pi))_\mathcal{A} = E'_\varepsilon(\mathbf{y}^0, \pi^0)\cdot(\mathbf{Y}, \Pi), \quad \forall(\mathbf{Y}, \Pi) \in \mathcal{A}. \tag{5.1}$$

  If $\|\boldsymbol{g}^{\mathbf{0}}\|_\mathcal{A}/\|\mathbf{z}^0\|_\mathcal{A} \le \eta$ take $\boldsymbol{z} = \boldsymbol{z}^{\mathbf{0}}$ as an approximation of a minimum of E_ε. Otherwise, set $\boldsymbol{w}^{\mathbf{0}} = \boldsymbol{g}^{\mathbf{0}}$.

  For $k \ge 0$, assuming $\mathbf{z}^k$, $\boldsymbol{g}^{\boldsymbol{k}}$, $\mathbf{w}^k$ being known with $\boldsymbol{g}^{\boldsymbol{k}}$ and $\mathbf{w}^k$ both different from zero, compute $\mathbf{z}^{k+1}, \boldsymbol{g}^{\boldsymbol{k+1}}$, and if necessary $\mathbf{w}^{k+1}$ as follows:

- *Step 1: Steepest descent* - Set $\mathbf{z}^{k+1} = \mathbf{z}^k - \lambda_k \mathbf{w}^k$ where $\lambda_k \in \mathbb{R}$ is the solution of the one-dimensional minimization problem

$$\text{minimize} \quad E_\varepsilon(\mathbf{z}^k - \lambda\mathbf{w}^k) \quad \text{over}\, \lambda \in \mathbb{R}^+. \tag{5.2}$$

  Then, compute the residual $\boldsymbol{g}^{\boldsymbol{k+1}} \in \mathcal{A}$ from the relation

$$(\boldsymbol{g}^{\boldsymbol{k+1}}, (\mathbf{Y}, \Pi))_\mathcal{A} = E'_\varepsilon(\mathbf{z}^{k+1})\cdot(\mathbf{Y}, \Pi), \quad \forall(\mathbf{Y}, \Pi) \in \mathcal{A}.$$

- *Step 2: Convergence testing and construction of the new descent direction* -
  If $\|\boldsymbol{g}^{\boldsymbol{k+1}}\|_\mathcal{A}/\|\boldsymbol{g}^{\mathbf{0}}\|_\mathcal{A} \le \eta$ take $\mathbf{z} = \mathbf{z}^{k+1}$; otherwise compute

$$\gamma_k = \frac{(\boldsymbol{g}^{\boldsymbol{k+1}}, \boldsymbol{g}^{\boldsymbol{k+1}} - \boldsymbol{g}^{\boldsymbol{k}})_\mathcal{A}}{(\boldsymbol{g}^{\boldsymbol{k}}, \boldsymbol{g}^{\boldsymbol{k}})_\mathcal{A}}, \qquad \mathbf{w}^{k+1} = \boldsymbol{g}^{\boldsymbol{k+1}} + \gamma_k \mathbf{w}^k. \tag{5.3}$$

  Then do $k = k + 1$, and return to step 1.

Two remarks are in order. First, for any $(\boldsymbol{g}_\mathbf{y}, g_\pi) \in \mathcal{A}$, the following equality (analogue to (5.1))

$$((\boldsymbol{g}_\mathbf{y}, g_\pi), (\mathbf{Y}, \Pi))_\mathcal{A} = E'_\varepsilon(\mathbf{y}, \pi)\cdot(\mathbf{Y}, \Pi), \quad \forall(\mathbf{Y}, \Pi) \in \mathcal{A}.$$

rewrites as follows

$$\int_\Omega \nabla\boldsymbol{g}_\mathbf{y}\cdot\nabla\mathbf{Y} d\mathbf{x} = \int_\Omega -\nu\nabla\mathbf{v}\cdot\nabla\mathbf{Y} + (\mathbf{y}\otimes\mathbf{Y} + \mathbf{Y}\otimes\mathbf{y})\cdot\nabla\mathbf{v} d\mathbf{x} + \int_\Omega (\nabla\cdot\mathbf{y} + \varepsilon\pi)\nabla\cdot\mathbf{Y} d\mathbf{x}, \forall\mathbf{Y} \in \mathbf{H}_0^1(\Omega) \tag{5.4}$$

for the first component $\boldsymbol{g}_{\mathbf{y}}$ of the gradient and leads to $g_\pi = \nabla\cdot\mathbf{v} + \varepsilon(\nabla\cdot\mathbf{y} + \varepsilon\pi)$ for the scalar part. Second, the scalar problem (5.2) can be solved explicitely since, for any $\mathbf{z} = (\mathbf{z}_y, z_\pi)$ and $\mathbf{w} = (\mathbf{w}_y, w_\pi) \in \mathcal{A}$, we easily get the expansion

$$\begin{aligned} E_\varepsilon(\mathbf{z} - \lambda\mathbf{w}) = E_\varepsilon(\mathbf{z}) &- \lambda\int_\Omega (\nabla\mathbf{v}\cdot\nabla\overline{\mathbf{v}} + (\nabla\cdot\mathbf{z_y} + \varepsilon z_\pi)(\nabla\cdot\mathbf{w_y} + \varepsilon w_\pi))d\mathbf{x} \\ &+ \frac{\lambda^2}{2}\int_\Omega (|\nabla\overline{\mathbf{v}}|^2 + 2\nabla\mathbf{v}\cdot\nabla\overline{\overline{\mathbf{v}}} + |\nabla\cdot\mathbf{w_y} + \varepsilon w_\pi|^2)d\mathbf{x} \\ &- \lambda^3\int_\Omega \nabla\overline{\mathbf{v}}\cdot\nabla\overline{\overline{\mathbf{v}}}d\mathbf{x} + \frac{\lambda^4}{2}\int_\Omega |\nabla\overline{\overline{\mathbf{v}}}|^2 d\mathbf{x} \end{aligned}$$

where $\mathbf{v}$, $\overline{\mathbf{v}}$ and $\overline{\overline{\mathbf{v}}}$ solves respectively

$$\begin{cases} -\Delta\mathbf{v} + (-\nu\Delta\mathbf{z_y} + div(\mathbf{z_y}\otimes\mathbf{z_y}) + \nabla z_\pi - \mathbf{f}) = 0, & \text{in} \quad \Omega, \\ \mathbf{v} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5.5}$$

$$\begin{cases} -\Delta\overline{\mathbf{v}} + (-\nu\Delta\mathbf{w_y} + div(\mathbf{z_y}\otimes\mathbf{w_y} + \mathbf{w_y}\otimes\mathbf{z_y}) + \nabla w_\pi) = 0, & \text{in} \quad \Omega, \\ \overline{\mathbf{v}} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{5.6}$$

and

$$\begin{cases} -\Delta\overline{\overline{\mathbf{v}}} + div(\mathbf{w_y}\otimes\mathbf{w_y}) = 0, & \text{in} \quad \Omega, \\ \overline{\overline{\mathbf{v}}} = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.7}$$

Consequently, each iteration of the CG algorithm requires the resolution of four Poisson problems, namely (5.4), (5.5), (5.6) and (5.7). The CG algorithm associated to the minimization over $\mathbb{V}$ of the functional E, defined in section 2, is very similar: the Poisson problems are simply replaced by Stokes problems (we refer to [3]). In both cases, the matrix (to be invert) associated to those four problems is the same and does not change from an iteration to the next one.

6. Numerical illustration: Two dimensional channel with a backward facing step

We consider the celebrated test problem of a two-dimensional channel with a backward facing step, described for instance in Section 45 of [6] (see also [10]). We use exactly the geometry and boundary conditions from this reference. The geometry is depicted Figure 1. Dirichlet conditions of the Poiseuille type are imposed on the entrant and sortant sides Γ_1 and Γ_2 of the channel: we impose $\mathbf{y} = (4(H-y)(y-h)/(H-h)^2, 0)$ on Γ_1 and $\mathbf{y} = (4(H-h)y(H-y)/H^2, 0)$ on Γ_2, with $h = 1, H = 3, l = 3$ and $L \in \{15, 30\}$ according to the value of ν. On the remaining part $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$, the fluid flow is imposed to zero. The external force $\mathbf{f}$ is zero.

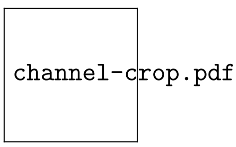

Figure 1. A two-dimensional channel with a step.

We now comment some computations performed with the FreeFem++ package developed at the University Paris 6 (see [9]). For $L = 15$ and $\nu = 1/50$, Table 2 reports the norms obtained when minimizing the functional E over $\mathbb{V}$ (section 2). Similarly, Table 3 reports the norms obtained when minimizing the functional E over $\mathcal{A}$ (section 3). In both cases, the Polak-Ribiere version of the conjugate gradient algorithm, initialized with the solution of corresponding Stokes problem. The $\mathbb{P}_1/\mathbb{P}_2$ Taylor-Hood finite element is employed. Figure 2 describes the regular

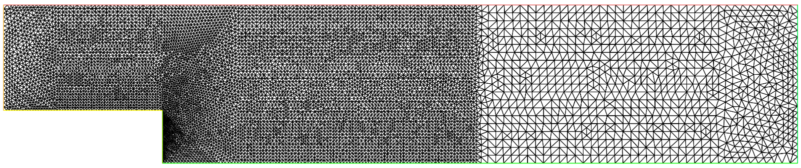

FIGURE 2. Triangular mesh of the channel - 16 425 triangles and 8 846 vertices.

triangulation used, composed of 16 425 triangles. For comparison, Table 1 reports norms for the solution of the weak variational formulation : find $\mathbf{y} \in \mathbb{V}$ solution

$$F(\mathbf{y},\mathbf{z}) := \int_\Omega \Big(\nu\nabla\mathbf{y}\cdot\nabla\mathbf{z} + (\mathbf{y}\nabla)\mathbf{y}\cdot\mathbf{z}\Big)d\mathbf{x} = 0, \quad \forall \mathbf{z}\in\mathbb{V} \tag{6.1}$$

using a Newton type method: each iteration requires to solve an Oseen equation:

$$\partial_y F(\mathbf{y}^k, z)(\mathbf{y}^{k+1}-\mathbf{y}^k) = -F(\mathbf{y}^k,\mathbf{z}), \qquad \forall \mathbf{z}\in\mathbb{V}, \quad \forall k\geq 0, \tag{6.2}$$

given an initial guess $\mathbf{y}^0 \in \mathbf{V}$ (again defined as the solution of the Stokes problem). Contrary to the CG algorithm of section 5, the matrix to be invert varies here with k. As is well-known, for large value of ν (here $\nu = 1/50$), the Newton method is very efficient : the property $\|\mathbf{y}^{k+1} - \mathbf{y}^k\|_{L^\infty(\Omega)}/\|\mathbf{y}^k\|_{L^\infty(\Omega)} \leq 10^{-8}$ is achieved in five iterates.

$\|\pi\|_{L^2(\Omega)}$	$\|y_1\|_{L^2(\Omega)}$	$\|y_2\|_{L^2(\Omega)}$	$\vert y_1\vert_{H_0^1(\Omega)}$	$\vert y_2\vert_{H_0^1(\Omega)}$	$\vert\nabla\cdot\mathbf{y}\vert_{L^2(\Omega)}$
0.1701	3.5645	0.2050	4.5370	0.3198	2.72×10^{-2}

TABLE 1. Solution of the variational equation (6.1) from the Newton algorithm (6.2); $\nu = 1/50$.

	$\|\pi\|_{L^2}$	$\|y_1\|_{L^2}$	$\|y_2\|_{L^2}$	$\vert y_1\vert_{H_0^1}$	$\vert y_2\vert_{H_0^1}$	$\vert\nabla\cdot\mathbf{y}\vert_{L^2}$	$\vert\mathbf{v}\vert_{H_0^1}$
CG - 10 iterates	0.1672	3.5281	0.2109	4.4721	0.3626	4.71×10^{-2}	8.52×10^{-3}
CG - 50 iterates	0.1701	3.5645	0.2050	4.5370	0.3198	2.65×10^{-2}	1.53×10^{-5}
BB - 50 iterates	0.1697	3.5632	0.2060	4.5334	0.3220	2.74×10^{-2}	4.07×10^{-4}

TABLE 2. Minimization of $E(\mathbf{y})$ over $\mathbb{V}$ using CG and BB algorithm; $\nu = 1/50$.

	$\|\pi\|_{L^2}$	$\|y_1\|_{L^2}$	$\|y_2\|_{L^2}$	$\vert y_1\vert_{H_0^1}$	$\vert y_2\vert_{H_0^1}$	$\vert\nabla\cdot\mathbf{y}\vert_{L^2}$	$\vert\mathbf{v}\vert_{H_0^1}$
CG - 50 iterates	0.1864	3.4878	0.2231	4.3892	0.4328	2.84×10^{-3}	1.56×10^{-2}
CG - 500 iterates	0.1747	3.5478	0.2088	4.4995	0.3367	5.81×10^{-4}	4.02×10^{-3}
CG - 750 iterates	0.1859	3.5560	0.2078	4.5171	0.3302	3.66×10^{-4}	3.06×10^{-3}
BB - 173 iterates	0.1681	3.5227	0.2120	4.4519	0.3640	1.25×10^{-3}	7.74×10^{-3}

TABLE 3. Minimization of $E(\mathbf{y},\pi)$ over $\mathcal{A}$ using CG algorithm and BB algorithm; $\nu = 1/50$.

The minimization of E over the null divergence space $\mathbb{V}$ is quite fast as well. The property $\|\mathbf{g}^k\|_{\mathbf{H}^1(\Omega)}/\|\mathbf{g}^0\|_{\mathbf{H}^1(\Omega)} \leq 10^{-3}$ ($\mathbf{g}^k$ denotes the residual at iterates k) is achieved in 39 iterates and leads to results very close to those from the resolution of (6.1), see table 2. We remind that each iterate requires the resolution of four Stokes problem. Figure 3 depicts the evolution (in log scale) of the norm $\|\mathbf{g}^k\|_{\mathbf{H}^1(\Omega)}$ of the gradient and $\sqrt{E(\mathbf{y}^k)} = |\mathbf{v}^k|_{\mathbf{H}_0^1(\Omega)}$ with respect to the iterates: the convergence to zero is sur-linear as we observe $\sqrt{E(\mathbf{y}^k)} = \mathcal{O}(e^{-0.15k})$. We also observe that the finite element approximation $\mathbb{P}_1/\mathbb{P}_1$ (which do not satisfies the Ladyzenskaia-Babushka-Brezzi condition) provides similar results in term of accuracy and convergence.

On the other hand, the minimization of the functional E over $\mathcal{A}$ is significantly slower as about 580 iterates are necessary to satisfy the property $\|\boldsymbol{g^k}\|_{\mathcal{A}}/\|\boldsymbol{g^0}\|_{\mathcal{A}} \leq 10^{-3}$. The minimization, which requires at each iterate the resolution of four Poisson problems, leads however to similar numerical results, referring to Table 3. Remark that the divergence constraint, as a part of the cost E, is paradoxically better satisfied than in the previous cases. The norm of the corrector $\mathbf{v}$, which measures how far from a solution of 1.1 the iterate $(\mathbf{y}^k,\pi^k)$ is, is much larger. Figure 4 depicts the evolution of the norm of the gradient with respect to the iterates. Among the various CG versions reported in [8], we observe that the Polak-Ribière leads to the best results in term of speed of convergence. Table 3 also displays the results obtained with the double step gradient method (named as BB method) introduced in [1] and which reads here as follows :

$$\begin{cases}(\mathbf{z}^{k+1}-\mathbf{z}^k,(\mathbf{Y},\Pi))_{\mathcal{A}} = -\alpha_k E'_\varepsilon(\mathbf{z}^k)\cdot(\mathbf{Y},\Pi), \quad \forall(\mathbf{Y},\Pi)\in\mathcal{A}, \quad k\geq 0,\\ \alpha_k = <\mathbf{z}^{k-1}-\mathbf{z}^k,\mathbf{g}^k-\mathbf{g}^{k-1}>_{\mathcal{A}}/\|\mathbf{g}^k-\mathbf{g}^{k-1}\|^2_{\mathcal{A}}, \quad \mathbf{z}^k=(\mathbf{y}^k,\pi^k).\end{cases}$$

Figure 5 displays the evolution of the relative quantity $\|\mathbf{g}^k\|_{\mathcal{A}}/\|\mathbf{g}^0\|_{\mathcal{A}}$ for both the CG and the BB algorithm. The BB one allows a speed up of the convergence: 173 iterates suffices to achieve $\|\mathbf{g}^k\|_{\mathcal{A}}/\|\mathbf{g}^0\|_{\mathcal{A}} \leq 10^{-3}$. The values of the cost $\sqrt{E(\mathbf{y}^k,\pi^k)}$ in both cases are however similar, which suggests that the functional E is flat near local minima. The main interest of the BB algorithm is on the computational viewpoint as it requires only two resolutions of Poisson problem, namely (5.4), (5.5) per iterate. The last line of Table 2 also displays the results of the BB algorithm for the minimization of E over $\mathbb{V}$ leading to similar results than CG method in term of speed of convergence.

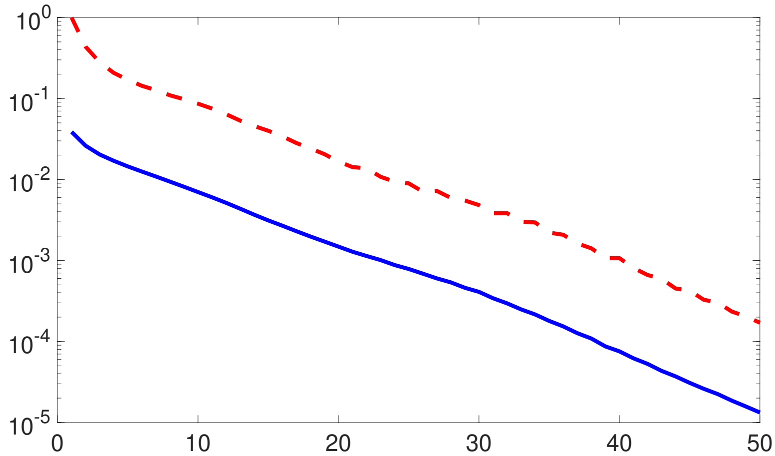

FIGURE 3. Evolution of $\sqrt{E(\mathbf{y}^k)}$ (blue full line) and the norm $\|\mathbf{g}^k\|_{\mathbf{H}^1}/\|\mathbf{g}^0\|_{\mathbf{H}^1}$ (red dashed line) w.r.t. iterate k of the CG algorithm - $\nu = 1/50$.

Remark that one may introduce a Newton type method so as to solve the following problem : find $(\mathbf{y},\pi,\mathbf{v})\in\mathcal{A}\times\mathbf{H}^1_0(\Omega)$ such that $\mathcal{F}\big((\mathbf{y},\pi,\mathbf{v}),(\mathbf{Y},\Pi,\mathbf{V})\big)=0$ for all $(\mathbf{Y},\Pi,\mathbf{V})\in\mathcal{A}\times\mathbf{H}^1_0(\Omega)$ with $\mathcal{F}:(\mathcal{A}\times\mathbf{H}^1_0(\Omega))^2\to\mathbb{R}$ defined as follows

$$\mathcal{F}\big((\mathbf{y},\pi,\mathbf{v}),(\mathbf{Y},\Pi,\mathbf{V})\big)=\bigg(E'(\mathbf{y},\pi)\cdot(\mathbf{Y},\Pi),\int_\Omega \nabla\mathbf{v}\cdot\nabla\mathbf{V}+\nu\nabla\mathbf{y}\cdot\nabla\mathbf{V}+div(\mathbf{y}\otimes\mathbf{y})\cdot\mathbf{V}+\nabla\pi\cdot\mathbf{V}\bigg).$$

If $(\mathbf{y},\pi,\mathbf{v})$ solves this problem, then from the first component of $\mathcal{F}$, $(\mathbf{y},\pi)$ is a critical point for E, while, from the second, the corrector $\mathbf{v}$ solves the boundary value problem (2.3) associated to

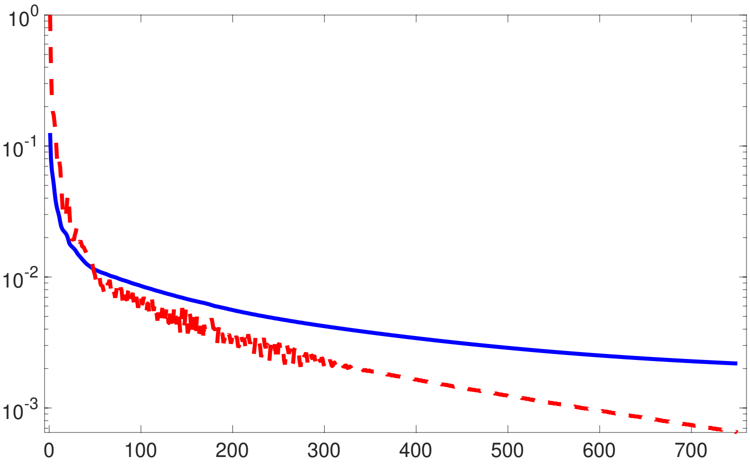

FIGURE 4. Evolution of $\sqrt{E(\mathbf{y}^k,\pi^k)}$ (blue full line) and of $\|\mathbf{g}^k\|_{\mathcal{A}}/\|\mathbf{g}^0\|_{\mathcal{A}}$ (red dashed line) w.r.t. iterate k of the CG algorithm - $\nu = 1/50$.

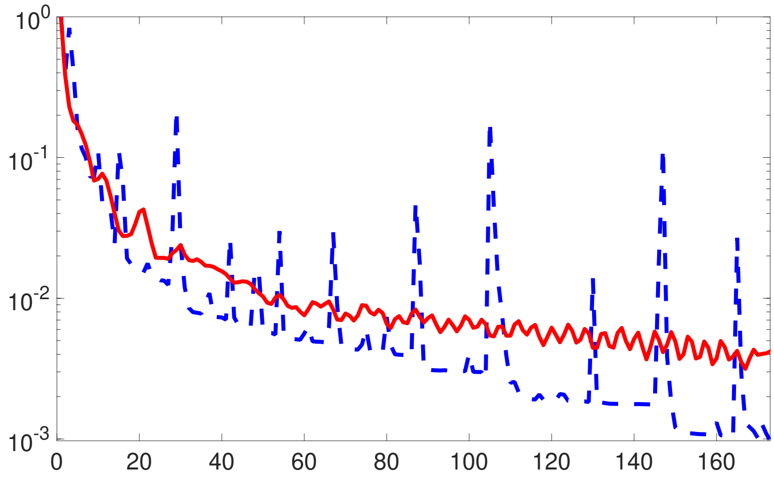

FIGURE 5. Evolution of $\|\mathbf{g}^k\|_{\mathcal{A}}/\|\mathbf{g}^0\|_{\mathcal{A}}$ for the CG method (red line) and for the BB method (blue dashed line) w.r.t. iterate k - $\nu = 1/50$.

$(\mathbf{y},\pi)$. A newton type algorithm for $\mathcal{F}$ (in the spirit of (6.2)) would be however more involved and costly than (6.2): our H^{-1} setting does not seem appropriate for Newton type method.

Figure 6 depicts the streamlines of the solution $\mathbf{y}$ from the minimization over $\mathcal{A}$ for $\nu = 1/50$ and $\nu = 1/150$. The method allows to capture the shear layer developing in the flow behind the re-entrant corner.

Let us consider now a smaller value for ν, precisely $\nu = 1/700$ with $L = 30$. The corresponding mesh is composed of 19714 triangles and 10208 vertices. For these values, we observe that the newton algorithm (6.2) - starting from the solution of the Stokes problem - does not converge. A continuation method with respect to ν is then necessary : Table 4 gives the results of the continuation method. The solution is obtained by computing 3 intermediates solution: from Stokes to $\nu = 1/50$ (5 iterates), then from $\nu = 1/50$ to $\nu = 1/141$ (16 iterates), then from $\nu = 1/141$ to $\nu = 1/400$ (14 iterates) and finally from $\nu = 1/400$ to $\nu = 1/700$ (14 iterates).

On the other hand, we observe the convergence of the least-squares method coupled with the CG algorithm, starting from the solution of the Stokes problem. Concerning the functional E of Section 2 defined over $\mathbb{V}$, Table 5 depicts the numerical values and suggests that a continuation

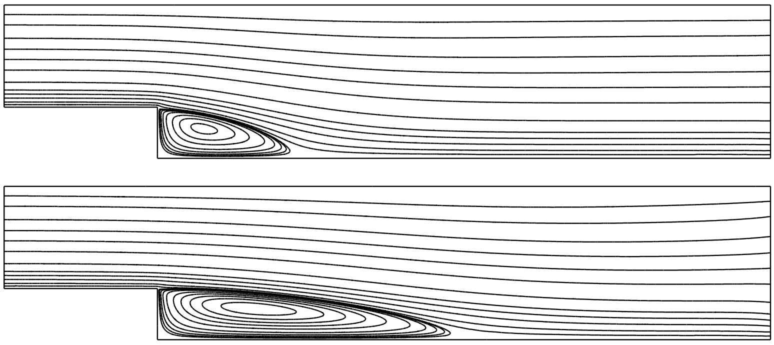

FIGURE 6. Streamlines of the steady state solution at $\nu = 1/50$ (top) and $\nu = 1/150$ (bottom); $L = 15$.

method leads to similar values but does not allow a reduction of the computational cost. The BB algorithm (from Stokes to $\nu = 1/700$) converges after 510 iterates and leads to similar values: again its allows a reduction of the computation costs (2×510 resolution of Stokes problems for BB whereas CG requires 4×357 resolution of Stokes problems). Eventually, as expected from our observations for $\nu = 1/50$, the minimization of the functional E of Section 3 defined over $\mathcal{A}$ using CG and BB algorithms requires more iterates (1020 and 439 respectively) leading to a larger computational cost but similar numerical values. However, for $\nu = 1/700$, the minimization of E using $\mathbb{P}_1/\mathbb{P}_1$ finite element approximation remains stable, contrary to the previous cases. Figure 6 depicts the streamlines of the solution $\mathbf{y}$ from the minimization of E over $\mathbb{V}$ for $\nu = 1/700$.

ν	$\sharp$ it.	$\Vert\pi\Vert_{L^2(\Omega)}$	$\Vert y_1\Vert_{L^2(\Omega)}$	$\Vert y_2\Vert_{L^2(\Omega)}$	$\vert y_1\vert_{H^1_0(\Omega)}$	$\vert y_2\vert_{H^1_0(\Omega)}$	$\vert\nabla\cdot\mathbf{y}\vert_{L^2(\Omega)}$
Stokes $\to 1/50$	5	0.7021	4.8352	0.2049	5.6947	0.3176	2.65×10^{-2}
$1/50 \to 1/141$	16	0.2163	4.9783	0.1694	6.0243	0.2622	2.12×10^{-2}
$1/41 \to 1/400$	14	0.2885	5.2441	0.1513	6.7759	1.1583	2.16×10^{-1}
$1/400 \to 1/700$	14	0.2424	5.4021	0.1641	7.3111	2.6129	7.19×10^{-1}

TABLE 4. Solution of the variational equation (6.1) from the Newton algorithm (6.2); $\nu = 1/700$.

ν	$\sharp$ it.	$\Vert\pi\Vert_{L^2}$	$\Vert y_1\Vert_{L^2}$	$\Vert y_2\Vert_{L^2}$	$\vert y_1\vert_{H^1_0}$	$\vert y_2\vert_{H^1_0}$	$\vert\nabla\cdot\mathbf{y}\vert_{L^2}$	$\vert\mathbf{v}\vert_{H^1_0}$
Stokes $\to 1/700$	357	0.2459	5.3196	0.1774	7.0156	2.4119	4.92×10^{-1}	1.06×10^{-3}
Stokes $\to 1/50$	39	0.7021	4.8353	0.2051	5.6947	0.3179	3.24×10^{-2}	9.36×10^{-5}
$1/50 \to 1/141$	114	0.2165	4.9779	0.1718	6.0240	0.2647	2.18×10^{-2}	2.06×10^{-4}
$1/141 \to 1/400$	288	0.2871	5.2956	0.1848	6.7458	1.1584	1.54×10^{-1}	6.27×10^{-4}
$1/400 \to 1/700$	220	0.2448	5.3309	0.1922	7.0538	2.3945	4.65×10^{-1}	7.91×10^{-4}

TABLE 5. Minimization of $E(\mathbf{y})$ over $\mathbb{V}$ using CG algorithm.

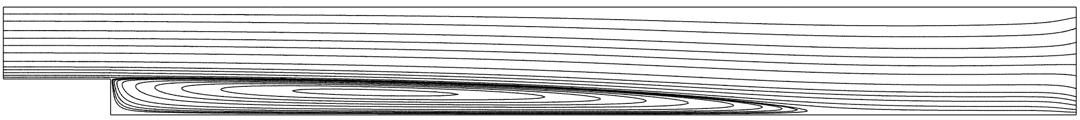

FIGURE 7. Streamlines of the steady state solution at $\nu = 1/700$; $L = 30$.

7. Conclusions and perspectives

We have analyzed two H^{-1}-least-squares methods and shown that they allow the construction of strong convergent sequences toward the solution (assumed unique) of the steady Navier-Stokes system. This study justifies in particular the least-squares approach introduced without proof in [3], which assume that the sequences are divergence free. The second least-squares approach relaxes the incompressibility constraint and incorporates an additional divergence term in the least-squares functional. As is usual in nonlinear situation, the convergence is shown if the initial guess is closed enough to the solution. Numerical experiments on a 2D channel with Poiseuille flow confirms our analysis and highlights the robustness of such methods with respect to the initial guess and also with respect to the approximation. The second least-squares functional coupled with a conjugate gradient method requires however much more iterates to achieve a satisfactory approximation. The more recent Barzilai-Borwein algorithm allows to reduce significantly the number of iterations together with the computational cost. A possible way to improve the speed of convergence is to use specific algorithms, such as the one described in [11], for so-called composite functionals, sum of two convex terms.

A natural extension of this study is the unsteady case : using ideas from [15], we may, at least in the divergence free situation of Section 2, obtain a result similar to Theorem 2.1 in the dynamic situation, and then, adapting [13, 12], examine the corresponding controllability issue.

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