C. T. Loy
K. Y. Lam
C. Shu
Department of Mechanical and Production Engineering
National University of Singapore
10 Kent Ridge Crescent
Singapore 0511

Analysis of Cylindrical Shells Using Generalized Differential Quadrature

The analysis of cylindrical shells using an improved version of the differential quadrature method is presented. The generalized differential quadrature (GDQ) method has computational advantages over the existing differential quadrature method. The GDQ method has been applied in solutions to fluid dynamics and plate problems and has shown superb accuracy, efficiency, convenience, and great potential in solving differential equations. The present article attempts to apply the method to the solutions of cylindrical shell problems. To illustrate the implementation of the GDQ method, the frequencies and fundamental frequencies for simply supported–simply supported, clamped–clamped, and clamped–simply supported boundary conditions are determined. Results obtained are validated by comparing them with those in the literature.

INTRODUCTION

The differential quadrature method developed by Bellman and Casti (1971) is an alternative discrete approach to directly solving the governing equations of engineering problems. The simplicity and ease of use of the method has gained popularity among researchers. Recently the differential quadrature method has been applied to the solutions of beam, plate, shell, and frame structure problems. Notable studies in the applications of the differential quadrature method in these areas include Bert et al. (1994), Bert and Malik (1966), Laura and Gutierrez (1993, 1994), and Striz et al. (1995).

In this article an improved version of the differential quadrature method called the generalized differential quadrature (GDQ) method developed by Shu (1991) is used to study the cylindrical shell problem. In the GDQ method the derivative of a function with respect to a space variable at a given discrete point is approximated as a weighted linear sum of all the functional values at all the discrete points. The advantages of the GDQ method include no restriction on the number of grid points used for the approximation and the weighting coefficients are determined using a simple recurrence relation instead of solving a set of linear algebraic equations as in other versions of the differential quadrature method. A more in-depth analysis of the merits of the GDQ method can be found in Du et al. (1994). The GDQ method has been applied to solutions of fluid dynamics problems by Shu and Richards (1992) and to plate problems by Shu and Du (1995a,b). In all the applications the GDQ method has shown superb accuracy, efficiency, convenience, and great potential in solving differential equations.

The objective of the present work is to extend the GDQ method to solutions of cylindrical shell problems. To illustrate the implementation, the method is used to determine the frequencies for simply supported–simply supported, clamped–

Received March 18, 1996; Accepted October 23, 1996.
Shock and Vibration, Vol. 4, No. 3, pp. 193–198 (1997)

CCC 1070-9622/97/030193-06

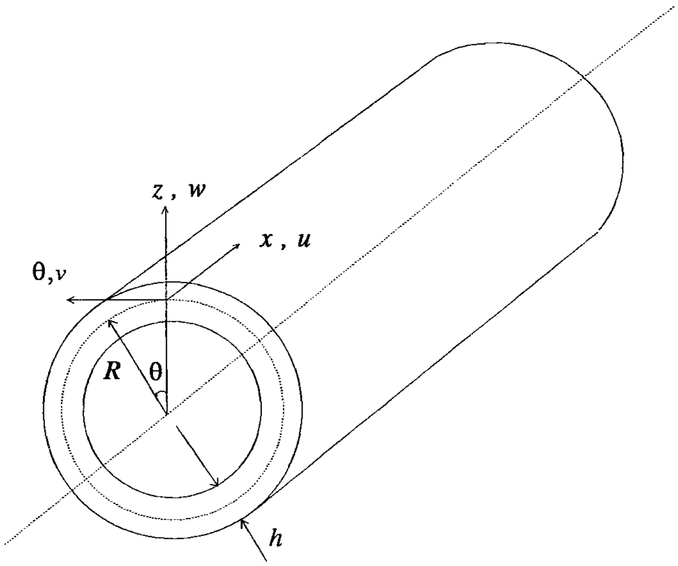

FIGURE 1 Geometry of a cylindrical shell.

clamped, and clamped–simply supported cylindrical shells.

GOVERNING EQUATIONS

Consider a cylindrical shell shown in Fig. 1: R is the radius, L is the length, h is the thickness, and (x, θ, z) is the orthogonal coordinate system fixed at the middle surface. The deformations in the x, θ, and z directions are denoted as u, v, and w, respectively.

The equations of motion for thin cylindrical shells in terms of the force N_{ij} and moment resultants M_{ij} are given as

$$N_{x,x} + \frac{1}{R}N_{x\theta,\theta} - \rho h\ddot{u} = 0, \tag{1}$$

$$N_{x\theta,x} + \frac{1}{R}N_{\theta,\theta} + \frac{1}{R}M_{x\theta,x} + \frac{1}{R^2}M_{\theta,\theta} - \rho h\ddot{v} = 0, \tag{2}$$

$$M_{x,xx} + \frac{2}{R}M_{x\theta,x\theta} + \frac{1}{R^2}M_{\theta,\theta\theta} - \frac{N_\theta}{R} - \rho h\ddot{w} = 0. \tag{3}$$

The subscripts $x, xx, \theta, x\theta$, and $\theta\theta$ denote the partial derivatives with respect to these parameters and N_{ij} and M_{ij} are given by

$$\{N_x, N_\theta, N_{x\theta}\} = \int_{-h/2}^{h/2} \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\}\, dz, \tag{4}$$

$$\{M_x, M_\theta, M_{x\theta}\} = \int_{-h/2}^{h/2} \{\sigma_x, \sigma_\theta, \sigma_{x\theta}\} z\, dz, \tag{5}$$

where σ_x and σ_θ are the normal stresses in the x and θ directions and $\sigma_{x\theta}$ is the shearing stress in the $x\theta$ plane. For thin cylindrical shells the stresses σ_x, σ_θ, and $\sigma_{x\theta}$ are related to the strains e_x, e_θ, and $e_{x\theta}$ by

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \end{Bmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{Bmatrix} e_x \\ e_\theta \\ e_{x\theta} \end{Bmatrix}, \tag{6}$$

where the reduced stiffnesses Q_{ij} for isotropic materials are defined as

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}, \qquad Q_{12} = \frac{\nu E}{1-\nu^2}, \quad Q_{66} = \frac{E}{2(1+\nu)}, \tag{7}$$

where E is the Young's modulus and ν is the Poisson's ratio. Using Love's first approximation shell theory (1927), the strain components are written as

$$e_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \qquad e_\theta = \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right) - \frac{z}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial v}{\partial \theta}\right), \qquad e_{x\theta} = \frac{\partial v}{\partial x} + \frac{1}{R}\frac{\partial u}{\partial \theta} - \frac{2z}{R}\left(\frac{\partial^2 w}{\partial x \partial \theta} - \frac{\partial v}{\partial x}\right). \tag{8}$$

Substituting Eqs. (4)–(8) into Eqs. (1)–(3), the governing equations for thin cylindrical shells can be obtained as

$$L_{11}u + L_{12}v + L_{13}w = 0, \tag{9}$$

$$L_{21}u + L_{22}v + L_{23}w = 0, \tag{10}$$

$$L_{31}u + L_{32}v + L_{33}w = 0, \tag{11}$$

where L_{ij} are the partial differential operators of x and θ. The general solutions for modal vibration can be written as

$$\begin{aligned} u(x, \theta, t) &= \phi(x)\cos(n\theta)\cos(\omega t), \\ v(x, \theta, t) &= \gamma(x)\sin(n\theta)\cos(\omega t), \\ w(x, \theta, t) &= \alpha(x)\cos(n\theta)\cos(\omega t). \end{aligned} \tag{12}$$

Substituting the displacement fields from Eq. (12) into Eqs. (9)–(11), the governing equations can be obtained as

$$\begin{aligned} J_{111}\phi + J_{112}\frac{\partial^2\phi}{\partial x^2} + J_{121}\frac{\partial\gamma}{\partial x} \\ + J_{131}\frac{\partial\alpha}{\partial x} + J_{132}\frac{\partial^3\alpha}{\partial x^3} = 0, \end{aligned} \tag{13}$$

$$J_{211}\frac{\partial\phi}{\partial x} + J_{221}\gamma + J_{222}\frac{\partial^2\gamma}{\partial x^2} + J_{231}\alpha + J_{232}\frac{\partial^2\alpha}{\partial x^2} = 0, \tag{14}$$

$$\begin{aligned} J_{311}\frac{\partial\phi}{\partial x} + J_{312}\frac{\partial^3\phi}{\partial x^3} + J_{321}\gamma + J_{322}\frac{\partial^2\gamma}{\partial x^2} \\ + J_{331}\alpha + J_{332}\frac{\partial^2\alpha}{\partial x^2} + J_{333}\frac{\partial^4\alpha}{\partial x^4} = 0, \end{aligned} \tag{15}$$

where J_{ijk} are some constant coefficients.

For cylindrical shells with simply supported–simply supported, clamped–clamped, and clamped–simply supported boundary conditions, these boundary conditions are expressed mathematically as simply supported–simply supported:

$$v = w = N_x = M_x = 0 \quad x = 0, L; \tag{16}$$

clamped–clamped:

$$u = v = w = \frac{\partial w}{\partial x} = 0, \quad x = 0, L; \tag{17}$$

clamped–simply supported:

$$\begin{aligned} u = v = w = \frac{\partial w}{\partial x} = 0 \quad x = 0; \\ v = w = N_x = M_x = 0 \quad x = L. \end{aligned} \tag{18}$$

For the solutions given in Eq. (12), these boundary conditions can be further written as simply supported–simply supported:

$$\gamma(x) = \alpha(x) = \frac{\partial\phi(x)}{\partial x} = \frac{\partial^2\alpha}{\partial x^2} = 0 \quad x = 0, L; \tag{19}$$

clamped–clamped:

$$\gamma(x) = \alpha(x) = \frac{\partial\phi(x)}{\partial x} = \frac{\partial^2\alpha}{\partial x^2} = 0 \quad x = 0, L; \tag{20}$$

clamped–simply supported:

$$\begin{aligned} \phi(x) = \gamma(x) = \alpha(x) = \frac{\partial\alpha(x)}{\partial x} = 0 \quad x = 0, \\ \gamma(x) = \alpha(x) = \frac{\partial\phi(x)}{\partial x} = \frac{\partial^2\alpha(x)}{\partial x^2} 0 \quad x = L. \end{aligned} \tag{21}$$

GDQ

The basic idea of GDQ is to approximate a derivative of a function $\psi\,(x, t)$ at the ith discrete point in a domain by a weighted linear sum of all the functional values in the domain. For the mth order derivative of $\psi\,(x, t)$, it is approximated as

$$\frac{\partial^m\psi}{\partial x^m}_{x=x_i} = \sum_{j=1}^{N} c_{ij}^{(m)}\psi(x_j, t), \tag{22}$$

where $c_{ij}^{(m)}$ are the weighting coefficients associated with the mth order derivative and N is the number of grid points used in the approximation. The weighting coefficients $c_{ij}^{(m)}$ can be easily obtained using the GDQ method; for details see Shu (1991). For the first-order derivative, the weighting coefficients are given by

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)} \quad i \neq j, \tag{23}$$

$$c_{ij}^{(1)} = -\sum_{j=1, j\neq i}^{N} c_{ij}^{(1)} \quad i = 1, 2, \ldots, N, \tag{24}$$

for $i, j = 1, 2, \ldots, N$, where

$$M^{(1)}(x_i) = \prod_{j=1, j\neq i}^{N} (x_i - x_j). \tag{25}$$

For the second- and higher order derivatives, the weighting coefficients can be computed by using a recurrence relationship as follows:

$$c_{ij}^{(m)} = m\left(c_{ii}^{(m-1)}c_{ij} - \frac{c_{ij}^{(m-1)}}{x_i - x_j}\right) \quad i \neq j, m = 2, 3, \ldots, N-1, \quad i, j = 1, 2, \ldots, N, \tag{26}$$

$$c_{ii}^{(m)} = -\sum_{j=1, j\neq i}^{N} c_{ij}^{(m)} \quad i = 1, 2, \ldots, N, \tag{27}$$

and the grid points are chosen as

$$x_i = \left(\frac{1 - \cos\left[\frac{i-1}{N-1}\right]\pi}{2}\right)L \quad i = 1, 2, \ldots, N. \tag{28}$$

APPLICATION OF GDQ METHOD TO CYLINDRICAL SHELL PROBLEM

To illustrate the implementation of the GDQ method to analyze cylindrical shell problems, the following boundary conditions are considered: simply supported–simply supported, clamped–clamped, and clamped–simply supported. To apply the GDQ method, the partial derivatives in the governing equations of Eqs. (13)–(15) and the boundary conditions of Eqs. (19)–(21) are first approximated as in Eq. (22). After spatial discretization, Eqs. (13)–(15) become

$$\begin{aligned} J_{111}\phi_j + J_{112}\sum_{j=1}^{N} c_{ij}^{(2)}\phi_j + J_{121}\sum_{j=1}^{N} c_{ij}^{(1)}\gamma_j \\ + J_{131}\sum_{j=1}^{N} c_{ij}^{(1)}\alpha_j + J_{132}\sum_{j=1}^{N} c_{ij}^{(3)}\alpha_j = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} J_{211}\sum_{j=1}^{N} c_{ij}^{(1)}\phi_j + J_{221}\sum_{j=1}^{N} c_{ij}\gamma + J_{222}\sum_{j=1}^{N} c_{ij}^{(2)}\gamma_j \\ + J_{231}\sum_{j=1}^{N} c_{ij}\alpha_j + J_{232}\sum_{j=1}^{N} c_{ij}^{(2)}\alpha_j = 0, \end{aligned} \tag{30}$$

$$\begin{aligned} J_{311}\sum_{j=1}^{N} c_{ij}^{(1)}\phi_j + J_{312}\sum_{j=1}^{N} c_{ij}^{(3)}\phi_j + J_{321}\gamma_j \\ + J_{322}\sum_{j=1}^{N} c_{ij}^{(2)}\gamma_j + J_{331}\alpha_j + J_{332}\sum_{j=1}^{N} c_{ij}^{(2)}\alpha_j \\ + J_{333}\sum_{j=1}^{N} c_{ij}^{(4)}\alpha_j = 0, \end{aligned} \tag{31}$$

and Eqs. (19)–(21) become simply supported–simply supported:

$$\begin{aligned} \gamma_1 = \alpha_1 = \sum_{j=1}^{N} c_{1j}^{(1)}\phi_j = \sum_{j=1}^{N} c_{1j}^{(2)}\alpha_j = 0, \\ \gamma_N = \alpha_N = \sum_{j=1}^{N} c_{Nj}^{(1)}\phi_j = \sum_{j=1}^{N} c_{Nj}^{(2)}\alpha_j = 0; \end{aligned} \tag{32}$$

clamped–clamped:

$$\begin{aligned} \phi_1 = \gamma_1 = \alpha_1 = \sum_{j=1}^{N} c_{1j}^{(1)}\alpha_j = 0, \\ \phi_N = \gamma_N = \alpha_N = \sum_{j=1}^{N} c_{Nj}^{(1)}\alpha_j = 0; \end{aligned} \tag{33}$$

clamped–simply supported:

$$\begin{aligned} \phi_1 = \gamma_1 = \alpha_1 = \sum_{j=1}^{N} c_{1j}^{(1)}\alpha_j = 0, \\ \gamma_N = \alpha_N = \sum_{j=1}^{N} c_{Nj}^{(1)}\phi_j = \sum_{j=1}^{N} c_{Nj}^{(2)}\alpha_j = 0. \end{aligned} \tag{34}$$

Substituting the above boundary conditions, Eqs. (32)–(34), into Eqs. (29)–(31), the resulting set of equations can be written in the form

$$\mathbf{A}\cdot\mathbf{x} = \lambda\mathbf{x}, \tag{35}$$

where $\mathbf{A}$ is a matrix, $\mathbf{x}$ is a column vector defined as

$$\mathbf{x}^T = \{\phi_2 \quad \phi_3 \cdots \phi_{N-1} \quad \gamma_2 \quad \gamma_3 \cdots \gamma_{N-1} \quad \alpha_3 \quad \alpha_4 \cdots \alpha_{N-2}\}, \tag{36}$$

and λ is a parameter defined as $\omega^2\rho h$. Solving for the eigenvalues of matrix $\mathbf{A}$ and equating to $\omega^2\rho h$, the natural frequencies ω of the cylindrical shell are obtained.

NUMERICAL RESULTS AND DISCUSSION

To examine the GDQ method for the analysis of cylindrical shells, a comparison of the results with

Table 1. Comparison of Frequency Parameter $\Omega = \omega R\sqrt{[(1-\nu^2)\rho]/E}$ for Simply Supported–Simply Supported Cylindrical Shell (m = 1, R/L = 0.05)

h/R	n	Ω: Markus (1988)[a]	Ω: Present	Ω: % Diff.
0.05	0	0.0929296	0.09295	0.0002
	1	0.0161063	0.01610	−0.04
	2	0.0392332	0.03930	0.17
	3	0.109477	0.109824	0.32
	4	0.209008	0.210284	0.61
0.002	0	0.0929296	0.09293	0.0004
	1	0.0161011	0.016101	−0.0006
	2	0.00545243	0.005453	0.01
	3	0.00503724	0.005042	0.09
	4	0.00853409	0.008534	−0.001

[a]Three-dimensional elasticity solution.

those in the literature were carried out. Table 1 shows the comparison of results for the clamped–clamped boundary condition, Table 2 shows the comparison of results for the simply supported–simply supported boundary condition, and Table 3 shows the comparison of the results for the clamped–simply supported boundary condition. In these tables m is the axial wave number and n is the circumferential wave number. In all the comparisons the present results were computed using 21 grid points approximations. As one can see from the comparisons, very good agreement with those in the literature was obtained.

As an illustration of the application of the GDQ method, the frequency parameter $\Omega = \omega R\sqrt{[(1-\nu^2)\rho]/E)}$ and the fundamental frequency parameter $\Omega_f = \omega_f R\sqrt{[(1-\nu^2)\rho]/E)}$ for the simply supported–simply supported, clamped–clamped, and clamped–simply supported boundary conditions for various circumferential wave numbers n and L/R ratios are presented in Tables 4 and 5.

Table 2. Comparison of Frequency Parameter $\Omega = \omega R\sqrt{[(1-\nu^2)\rho]/E}$ for Clamped–Clamped Cylindrical Shell (ν = 0.3, m = 1)

Case	Ω: Dym (1973)	Ω: Present
$L/R = 10$, $R/h = 500$, $n = 4$	0.01508	0.01512
$L/R = 10$, $R/h = 20$, $n = 2$	0.05784	0.05789
$L/R = 2$, $R/h = 20$, $n = 3$	0.3118	0.3119

Table 3. Comparison of Frequency Parameter $\Omega = \omega R\sqrt{[(1-\nu^2)\rho]/E}$ for Clamped–Simply Supported Cylindrical Shell (m = 1, L/R = 20; h/R = 0.002, ν = 0.3)

n	Ω: Lam and Loy (1995)	Ω: Present
1	0.024830	0.023974
2	0.008410	0.008223
3	0.005897	0.005842
4	0.008717	0.008705
5	0.013682	0.013679
6	0.019974	0.019973
7	0.027461	0.027460
8	0.036113	0.036112
9	0.045924	0.045923
10	0.056891	0.056890

Table 4. Frequency Parameter $\Omega = \omega R\sqrt{[(1-\nu^2)\rho]/E}$ for Simply Supported–Simply Supported (SS–SS), Clamped–Clamped (C–C), and Clamped–Simply Supported (C–SS) Cylindrical Shell (m = 1, L/R = 20; h/R = 0.01, ν = 0.3)

n	Ω: SS–SS	Ω: C–C	Ω: C–SS
1	0.016101	0.032885	0.023974
2	0.009382	0.013932	0.011225
3	0.022105	0.022672	0.022310
4	0.042095	0.042208	0.042139
5	0.068008	0.068046	0.068024
6	0.099730	0.099748	0.099738
7	0.137239	0.137249	0.137244
8	0.180527	0.180535	0.180531
9	0.229594	0.229599	0.229596
10	0.284435	0.284439	0.284437

Table 5. Fundamental Frequency Parameter $\Omega_f = \omega_f R\sqrt{[(1-\nu^2)\rho]/E}$ for SS–SS, C–C, and C–SS Cylindrical Shell (m = 1, h/R = 0.01 ν = 0.3)

L/R	Ω_f: SS–SS	Ω_f: C–C	Ω_f: C–SS
2	0.112275 (5)	0.153272 (6)	0.135651 (5)
5	0.044272 (3)	0.062767 (4)	0.054442 (4)
10	0.021957 (2)	0.030686 (3)	0.026776 (3)
20	0.009382 (2)	0.013932 (2)	0.011225 (2)
50	0.002648 (1)	0.005911 (1)	0.004110 (1)
100	0.000665 (1)	0.001505 (1)	0.001038 (1)

See Table 4 for abbreviations. Parameters in parentheses indicate the circumferential numbers at which the fundamental frequencies occur.

CONCLUSIONS

The article has presented the analysis of cylindrical shells using the GDQ method. Results obtained using the method have been evaluated against those available in the literature and the agreement has been found to be good. Frequency parameters and fundamental frequency parameters for the simply supported–simply supported, clamped–clamped, and clamped–simply supported boundary conditions for various circumferential wave numbers n and L/R ratios are also presented. The GDQ method can be easily extended to other mixed boundary conditions and shell structures.

REFERENCES

Bellman, R. E., and Casti, J., 1971, "Differential Quadrature and Long-Term Integration," *Journal of Math Anal. Appl.*, Vol. 34, pp. 235–238.

Bert, C. W., and Malik, M., 1996, "Free Vibration Analysis of Thin Cylindrical Shells by the Differential Quadrature Method," *ASME Journal of Pressure Vessel Technology*, Vol. 118, pp. 1–12.

Bert, C. W., Wang, X., and Striz, A. G., 1994, "Static and Free Vibrational Analysis of Beams and Plates by Differential Quadrature Method," *Acta Mechanica*, Vol. 102, pp. 11–24.

Chung, H., 1981, "Free Vibration Analysis of Circular Cylindrical Shells," *Journal of Sound and Vibration*, Vol. 74, pp. 331–350.

Du, H., Lim, M. K., and Lin, R. M., 1994, "Application of Generalized Differential Quadrature Method to Structural Problems," *International Journal for Numerical Methods in Engineering*, Vol. 37, pp. 1881–1896.

Dym, C. L., 1973, "Some New Results for the Vibrations of Circular Cylinders," *Journal of Sound and Vibration*, Vol. 29, pp. 189–205.

Lam, K. Y., and Loy, C. T., 1995, "Effects of Boundary Conditions on Frequencies Characteristics for a Multi-Layered Cylindrical Shell," *Journal of Sound and Vibration.*

Laura, P. A. A., and Gutierrez, R. H., 1993, "Analysis of Vibrating Timoshenko Beams Using the Method of Differential Quadrature," *Shock and Vibration*, Vol. 1, pp. 89–93.

Laura, P. A. A., and Gutierrez, R. H., 1994, "Analysis of Vibrating Rectangular Plates with Nonuniform Boundary Conditions by Using the Differential Quadrature Method," *Journal of Sound and Vibration*, Vol. 173, pp. 702–706.

Love, A. E. H., 1927, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed, Cambridge University Press, Oxford, U.K.

Markus, S., 1988, *The Mechanics of Vibrations of Cylindrical Shells*, Elsevier Science, New York.

Shu, C., 1991, *Generalized Differential–Integral Quadrature and Application to the Simulation of Incompressible Viscous Flows Including Parallel Computation*, PhD Thesis, University of Glasgow, Glasgow, Scotland.

Shu, C., and Du, H., 1995a, "A Generalized Approach for Implementing General Boundary Conditions in the GDQ Free Vibration Analysis of Plates," *International Journal of Solids and Structures.*

Shu, C., and Du, H., 1995b, "Implementation of Clamped and Simply Supported Boundary Conditions in the GDQ Free Vibration Analysis of Beams and Plates," *International Journal of Solids and Structures.*

Shu, C., and Richards, B. E., 1992, "Application of Generalized Differential Quadrature to Solve Two-Dimensional Incompressible Navier–Stokes Equations," *International Journal of Numerical Methods in Fluids*, Vol. 15, pp. 791–798.

Striz, A. G., Wang, X., and Bert, C. W., 1995, "Harmonic Differential Quadrature Method and Applications to Analysis of Structural Components," *Acta Mechanica*, Vol. 111, pp 85–94.