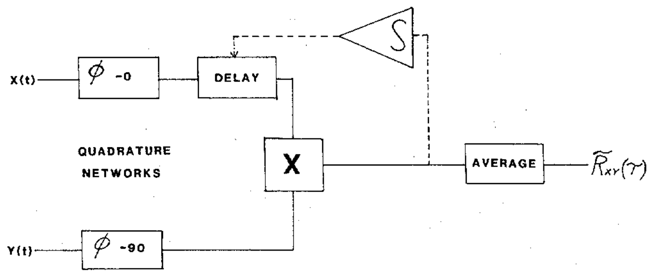

Fig. 4. Implementation of Hilbert transform correlator using analog techniques.

observed, the system will seek the maximum and stay locked to it. This assumes that the system is within the center portion of the transformed correlation function; otherwise the initial lock will not occur. Knowing the voltage to time transfer characteristic of the delay unit, the delay time may be determined. In the prototype system the delay was implemented by a bucket brigade device. This enabled the use of a frequency counter (in the period mode) to display the time delay [5].

APPLICATIONS

Aside from the more obvious applications to the arrival time of reflected signals, there are many unique applications of such a system. Processing the signals induced in the playback and record heads of a tape recorder when playing a blank tape allows measurement of speed accuracy and stability without introducing the error of a test tape. If the bandwidth of the servo loop is wide enough, wow and flutter measurements may be made by displaying the fluctuations in the delay control voltage on a meter. The servo approach may be applied to the measurement of automobile speed by using two optoelectronic sensors attached to the underside of the vehicle. The surface roughness of the road is the signal which is correlated, yielding the time required to travel the distance between sensors. This is proportional to the speed.

The system may be applied to sound localization in a similar way. A fixed 2 ms time delay is inserted in the signal received from one ear of a dummy head. A 0-4 ms variable delay is applied to the other. The differential time delay (variable minus fixed) will then be adjusted to match the interaural delay of a single sound source. A more impressive presentation may be obtained by eliminating the time delay units and placing the head on a dc motor driven turntable. The output of the Hilbert transform correlator is connected to the motor of the turntable, causing it to rotate toward the source. When there is no interaural delay (the dummy is facing the source) the correlator output is zero and the motor stops. If the source moves to either side of the head, the motor will rotate until it has eliminated the delay. Although this system suffers from complete front/back confusion, it is nonetheless entertaining. The presence of several equally intense sources will result in a weighted average localization because of the additive effects of the maxima discussed earlier.

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Analysis of Discrete Implementation of Generalized Cross Correlator

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***Abstract*–A common discrete implementation of the cross correlator uses a parabolic fit to the peak when the delay is not an integral multiple of the sampling period. This correspondence analyzes and assesses the pitfalls of this approach. It is shown that this yields a biased estimate of the time delay, with both the bias and variance of the estimate dependent on the location of the delay between samples, SNR, signal and noise bandwidths, and the prefilter or window used in the generalized correlator.**

INTRODUCTION

This correspondence examines the problems associated with digital processing and estimation of the time delay between signals received at two spatially separated sensors together with noise. Much of the investigative work in the literature centers upon the analysis of the analog processing of the time delay parameter, as opposed to the discrete-signal processing method more common in practice. Several issues affect the discrete implementation. One issue pertains to the finite observation times necessary where the received signals do not overlap exactly, causing inaccuracies in the computed spectra. Thus, attenuation of the peak as well as additional noise become prominent when the time delay is comparable to the observation time. Another point is that maximum likelihood estimation of the delay can be shown to be an application of the sin $\pi n/\pi n$ interpolating function on the discrete cross correlation. However, estimating the time delay by this method can only be approximated in practice, and can be computationally burdensome. A simple approximation which is widely used involves fitting a parabola or other polynomial in the neighborhood of the correlation peak. The parabolic fit approach is examined and shown to be a biased estimator. This limits the usefulness of the Cramér-Rao bound in interpreting digital system performance for these approximate methods.

In the next section, the estimator of time delay, a continuous parameter, for discrete signals is analyzed. Using the parabolic peak fit approach, expressions for the mean and variance of the time delay estimate are derived.

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THEORY

The signals received at the two sensors at time t_i are assumed to be of the form

$$x_1(i) = s(i) + n_1(i)$$
$$x_2(i) = as(i + D) + n_2(i), \qquad 0 \leqslant i \leqslant N - 1 \tag{1}$$

where $s(i)$ is a zero-mean stationary Gaussian process with power spectral density $G_{ss}(k)$, and $n_1(i)$, $n_2(i)$ are zero-mean stationary Gaussian processes with power spectral densities $G_{n_1 n_1}(k)$ and $G_{n_2 n_2}(k)$, $0 \leqslant k \leqslant N - 1$, uncorrelated with each other and with $s(i)$. We assume with no loss of generality that the sampling rate is 1 in the ensuing derivations.

Let $X_1(k)$ and $X_2(k)$ be the DFT of $x_1(i)$ and $x_2(i)$. Then define the generalized cross-correlation estimate as

$$\hat{R}_g(m) = \mathrm{DFT}^{-1}\,[X_1(k)\,X_2^*(k)\,W(k)] = \mathrm{DFT}^{-1}\,[\hat{G}_g(k)] \tag{2}$$

where $W(k)$ is the window used for the generalized correlator. The windows used here are described in [2], [3]. Briefly, they are

- W_U =1, no window.
- W_I Detection window for signals determined to be well-behaved, identical in form to the maximum likelihood window [2].
- W_{II} Detection window for arbitrary random signals [2].
- W_E Eckart window.
- W_S Smoothed coherence transform window [1].
- W_L Least-squares window [2].

The time delay is now expressed in terms of two components: $D = p + \delta$, where p is an integral multiple of the sampling period, and δ is the fractional part of the delay, $|\delta| \leqslant 0.5$. Estimation of the time delay is done in two steps. First, the estimate $\hat{p}$ is found by locating the maximum sample of (1). Then the estimate $\hat{\delta}$ is found by fitting a parabola to the three samples of $\hat{R}_g(m)$ about p. Fitting a parabola to these three points and solving for the peak of the parabola yields

$$\hat{\delta} = \frac{\hat{R}_g(p+1) - \hat{R}_g(p-1)}{2[-\hat{R}_g(p+1) + 2\hat{R}_g(p) - \hat{R}_g(p-1)]}. \tag{3}$$

The δ estimate can be expressed in terms of finite differences. Define two first differences and their average

$$Z_1(\delta) = \hat{R}_g(p+1) - \hat{R}_g(p);\; Z_{-1}(\delta) = \hat{R}_g(p) - \hat{R}_g(p-1);$$
$$Z(\delta) = [Z_1(\delta) + Z_{-1}(\delta)]/2 \tag{4}$$

and a second difference

$$v(\delta) = -\,[Z_1(\delta) - Z_{-1}(\delta)]. \tag{5}$$

Then (3) can be expressed as

$$\hat{\delta} = Z(\delta)/v(\delta). \tag{6}$$

The statistics of Z and v can be found via the frequency domain. Expressing the differences in the frequency domain,

$$Z(\delta) = \frac{1}{N} \sum_{k=-(N/2)+1}^{[(N/2)-1]} \hat{G}_g(k)\, e^{j(2\pi/N)kp}\, j \sin\frac{2\pi k}{N} \tag{7}$$

$$v(\delta) = \frac{1}{N} \sum_{k=-(N/2)+1}^{[(N/2)+1]} \hat{G}_g(k)\, e^{j(2\pi/N)kp} \left(2 - 2\cos\frac{2\pi k}{N}\right). \tag{8}$$

In order to make the analysis tractable we need to make assumptions similar to those in [1]. It is assumed that correct detection has occurred, and that we are looking at incremental variations of the delay. In the discrete case, this is equivalent to saying that $\hat{p}$ is correct and we are observing the statistics of δ. It can be shown that $Z(\delta)$ and $v(\delta)$ are independent. Since $\hat{G}_g(k)$ and $\hat{G}_g(l)$ are uncorrelated for $|k| \neq |l|$, and the inverse DFT and computation of finite differences are linear operations, then $Z(\delta)$ and $v(\delta)$ are nearly Gaussian and independent. In actuality, v is not Gaussian because $v \leqslant 0$ is impossible. In fact, $v > 2|Z|$; therefore v is neither Gaussian nor independent under the condition that $\hat{R}_g(p-1) < \hat{R}_g(p)$ and $\hat{R}_g(p+1) < \hat{R}_g(p)$. We shall assume $\sigma_v << E[v]$ so that the above situation is less troublesome. This will occur for relatively high SNR or optimum window functions; thus from this assumption

$$E[\hat{\delta}] \simeq E[Z(\delta)]/E[v(\delta)] \tag{9}$$

$$\sigma_{\hat{\delta}}^2 \simeq \sigma_{Z(\delta)}^2 / E^2[v(\delta)]. \tag{10}$$

It would be possible to refine (10) by expanding in a series, computing the variance, and taking the first few terms. However, the series does not converge for low SNR unless the fact that v is conditional on Z is taken into account. We show here the calculation for (10). The necessary statistics are found in a straightforward manner. Since

$$E[\hat{G}_g(k)] = aG_{ss}(k)\, W(k)\, e^{-j(2\pi/N)k\delta_0}\, e^{-j(2\pi/N)kp_0}$$

where p_0 and δ_0 are the true values of p and δ, then with $p = p_0$ and taking symmetry into account,

$$E[\hat{\delta}] \simeq \frac{E[Z]}{E[v]} = \frac{\dfrac{1}{N} \displaystyle\sum_{k=-(N/2)+1}^{(N/2)-1} aG_{ss}(k)\, W(k) \sin\frac{2\pi k\delta}{N} \sin\frac{2\pi k}{N}}{\dfrac{1}{N} \displaystyle\sum_{k=-(N/2)+1}^{(N/2)-1} aG_{ss}(k)\, W(k) \cos\frac{2\pi k\delta}{N} \left(2 - 2\cos\frac{2\pi k}{N}\right)}. \tag{11}$$

For the second moments

$$E[Z^2(\delta)] = -\frac{1}{N^2} \sum_{k,l=-(N/2)+1}^{(N/2)-1} \sum^{k=-(N/2)+1} E[\hat{G}_g(k)\, \hat{G}_g(l)] \cdot \sin\frac{2\pi k}{N} \sin\frac{2\pi l}{N}\, e^{j(2\pi kp/N)}\, e^{j(2\pi lp/N)}.$$

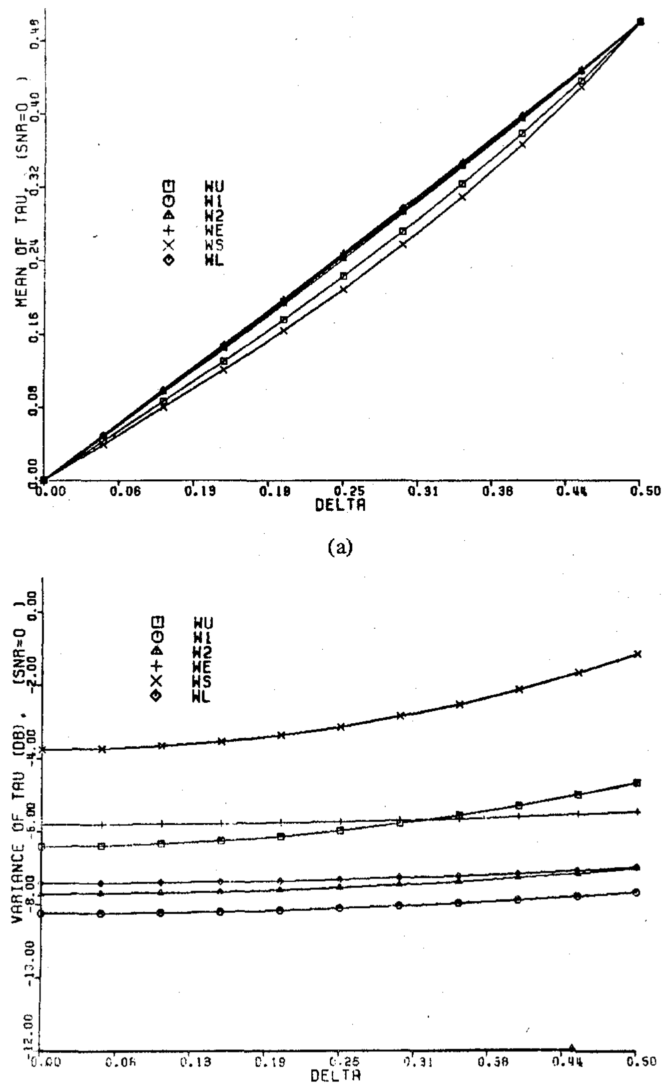
Under the Gaussian assumption for $X_1(k)$ and $X_2(k)$ and that they are uncorrelated, with $G_{ij} = E[X_i(k)\, X_j^*(k)]$, then

$$\sigma_{\hat{\delta}}^2 = \frac{\dfrac{1}{N^2} \displaystyle\sum_{k=-(N/2)+1}^{(N/2)-1} W^2(k) \sin^2\frac{2\pi k}{N}\, G_{11}(k)\, G_{22}(k) \left[1 - C_{12}(k) \cos\frac{4\pi k\delta}{N}\right]}{\left[\dfrac{1}{N} \displaystyle\sum_{k=-(N/2)+1}^{(N/2)-1} aG_{ss}(k)\, W(k) \cos\frac{2\pi k\delta}{N} \left(2 - 2\cos\frac{2\pi k}{N}\right)\right]^2} \tag{12}$$

where

$$C_{12}(k) = |G_{12}(k)|^2 / [G_{11}(k)\, G_{22}(k)].$$

Equations (11) and (12) predict the performance of the discrete generalized cross correlator for any window $W(k)$ with



(a)

(b)

Fig. 1. Theoretical mean and variance for discrete signals with parabolic peak fit, SNR = 0, white noise. (a) Mean of $\hat{\delta}$. (b) Variance of $\hat{\delta}$.

parabolic fit to locate the time delay. The values given by these equations have been computed and plotted, using the signal spectrum

$$G_{ss}(k) = \alpha^4/[\alpha^2 + (2\pi k/N)^2]^2$$

for $\alpha = 0.333$, and $G_{n_1 n_1}(k)$ and $G_{n_2 n_2}(k)$ white. Fig. 1 shows plots for SNR = 0, as a function of δ. The bias is apparent in the plot of $E[\hat{\delta}]$ in Fig. 1(a), and is worst at $\delta = \pm 0.25$. The variance of $\hat{\delta}$ in Fig. 1(b) is a function of δ, and is worst at $\delta = 0.5$, halfway between samples.

Conclusions

Parabolic fitting to the correlation peak has been shown to be a biased estimator of the time delay. The bias and variance of the estimate depend on the parameters as well as the window used. In general, we have observed that the least squares window W_L exhibits the least bias, whereas W_S and W_U exhibit the largest bias. W_I, W_{II}, and W_E lie between these extremes.

The variance of the estimate has also been observed to be a function of δ, being maximum when the true delay is halfway between sample points. The variance using W_L is generally more constant while higher than W_I or W_{II}. The windows W_S and W_U exhibit the most variability.

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A Bayesian Approach to Time Delay Estimation

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Abstract—**Estimation of time delay between two sensors or estimation of multipath delay at single sensor is carried out by calculating the *a posteriori* pdf of the time delay from a given prior distribution. When the time delay is fixed over the observation interval, the *a posteriori* pdf is obtained from a fixed set of Kalman filters, each of which is matched to a different delay. When the time delay is Markov, an expanding set of Kalman filters is required. For this case, an approximation commonly used for target tracking in multitarget environments is used to obtain an algorithm with stable memory requirements. Monte Carlo simulation results are presented to determine algorithm performance.**

I. Introduction

In this paper, a *Bayesian* approach to time delay *estimation* is developed. Although the emphasis is on estimation of *time delay* between two sensors or *multipath delay* at a single sensor, the methods developed here may be easily generalized to joint estimation of time delay and multipath delay at many sensors. For simplicity, it is assumed that the sources are stationary, so that Doppler may be ignored, and that the signal and delayed signal have the same amplitude. This is not a significant limitation, since the method is easily modified to include joint estimation of Doppler and amplitude on each propagation path. Two cases are considered for both the estimation of time delay between two sensors and estimation of multipath delay at a single sensor. In the first, the time delay is assumed constant over the observation interval. In this case, the *a posteriori* pdf of the delay is obtained from a set of *Kalman filters* matched to different time delays. In the second, the time delay is assumed Markov over the observation interval. For this case, the *a posteriori* pdf of the delay is obtained from an expanding set of Kalman filters matched to different time delay histories. Many authors [1]–[6] have considered tracking problems which lead to an expanding set of Kalman filters, and several techniques are available for obtaining stable memory approximations to the optimum algorithm. In this paper, the Singer *et al.* *N*-scan approximation

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