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Analysis of event-driven controllers for linear systems

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Most research in control engineering considers periodic or time-triggered control systems with equidistant sample intervals. However, practical cases abound in which it is of interest to consider event-driven control in which the sampling is event-triggered. Although there are various benefits of using event-driven control like reducing resource utilization (e.g., processor and communication load), their application in practice is hampered by the lack of a system theory for event-driven control systems. To provide a first step in developing an event-driven system theory, this paper considers an event-driven control scheme for perturbed linear systems. The event-driven control scheme triggers the control update only when the (tracking or stabilization) error is large. In this manner, the average processor and/or communication load can be reduced significantly. The analysis in this paper is aimed at the control performance in terms of practical stability (ultimate boundedness). Several examples illustrate the theory.

1. Introduction

Most research in control theory and engineering considers periodic or time-triggered control systems where continuous time signals are represented by their sampled values at a fixed sample rate. This leads to equidistant sample intervals and the analysis and synthesis problems can be coped with by the vast literature on system theory for sampled data systems. However, there are cases when it is of interest to consider event-driven control systems where the sampling is event-triggered rather than time-triggered. In an event-driven system it is the occurrence of an event rather than the passing of time, that decides when a next sample should be taken. The event-triggering mechanisms can vary and several examples are given in the following (Årzén 1999).

- Time-varying sample intervals occur in the control of internal combustion engines that are sampled against engine speed (Albertoni *et al*. 2005).
- The event-driven nature of the sampling can be intrinsic to the measurement method, for instance, when encoder sensors are used for measuring the angular position of a motor (Heemels *et al*. 1999). Other "event-driven" sensors include level sensors for measuring the height of a fluid in a tank (e.g., Förstner and Lunze (2001) and Lunze (2000)), (magnetic/optic) disk drives with "encoder-like" measurement devices (Phillips and Tomizuka 1995) and transportation systems where the longitudinal position of a vehicle is only known when certain markers are passed (de Bruin and van den Bosch 1998). Quantization of signals in which the sampling is induced by crossings of the quantization levels (Kofman and Braslavsky 2006) has a similar effect.
- Also in modern distributed control systems it is difficult to stick to the time-triggered paradigm. This is specially the case when control loops are closed over computer networks (Zhang *et al*. 2001, Cervin *et al*. 2003) or busses, e.g., field busses, local area networks, wireless networks (Ploplys *et al*. 2004, Kawka and Alleyne 2005), etc., that introduce varying communication delays.

Next to the various (natural) sources of event-triggering and their relevance in practice, there are many other

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reasons why the use of event-driven control is of interest:

- As stated in Årzén (1999), event-driven control is closer in nature to the way a human behaves as a controller. Indeed, when a human performs manual control his behaviour is event-driven rather than time-driven. It is not until the measurement signal has deviated sufficiently from the desired set point that a new control action is taken.
- Another important reason why event-driven control is of interest is resource utilization. An embedded controller is typically implemented on a real-time operating system. The available CPU (central processing unit) time is shared between the tasks such that it appears as if each task is running independently. Occupying the CPU resource for performing control calculations when nothing significant has happened in the process is clearly an unnecessary waste of resources. The same argument also applies to communication resources. Indeed, reducing the number of control updates leads directly to a reduction in the number of messages to be transmitted and thus to a lower (average) bus load. As communication busses have limited bandwidth, reducing their load is very beneficial. Moreover, lower bus loads also save energy. Especially for wireless communication via battery-powered devices, this is an important aspect as wireless communication is a severe power consumer (e.g., ElGamal *et al.* 2002, Stark *et al.* 2002).
- Also from a technology point of view, event-driven controllers are becoming more feasible, particularly for distributed real-time sensing and control. For example, for sensors in sensor networks, TinyOS is an event-driven operating system that is already used in several applications (Levis *et al.* 2004).

Although the above discussion indicates that in many situations it is logical to study and implement event-driven controllers, their application is scarce in both industry and academia, at least at the level of (servo) feedback control loops. These feedback loops are at the lowest level in the control hierarchy for high-tech systems at which fast dynamics dominate the behaviour. At the supervisory level of the control hierarchy it is more common to use discrete-event controllers that can be well described (often after abstracting away from the continuous dynamics) by discrete-event system theory; see e.g., Cassandras and Lafortune 1999. A major reason why, at the low level feedback loops, time-driven control still dominates is the difficulty involved in developing a system theory that fits these types of event-driven systems in which the continuous dynamics are profound. Traditional time-driven controllers are designed with a main focus on the performance of the controlled process. The aim of event-driven control is to create a better balance between this control performance and other system aspects (such as processor load, communication load, and system cost price).

In many situations one tries to circumvent the event-driven nature of the control system in order to still use the system theory for time-driven systems. For instance, when sensors are event-based (i.e., the measurement data arrives not equidistantly in time) often one designs asynchronous observers that provide estimates of the state variable of the plant at equidistant times. For instance, in Glad and Ljung (1984), de Bruin and van den Bosch (1998) and Krucinski *et al.* (1998) approaches based on Kalman filtering are used, while in Phillips and Tomizuka (1995) a Luenberger-type observer is applied. Since estimates of the state are now available at a constant sample rate, standard (state feedback) control analysis and design methods can be applied. Another example of neglecting the event-driven nature is to assume (or impose as stringent conditions for the software engineers) that the real-time platforms used for implementing controllers are able to guarantee deterministic sample intervals. In reality this is, however, seldom achieved. Computation and/or communication delays of networked control systems (Zhang *et al.* 2001, Ploplys *et al.* 2004, Kawka and Alleyne 2005, ten Berge *et al.* 2006) are inevitable and can degrade the performance significantly. To study the effects of computation and communication delays in control loops, in Cervin *et al.* (2003) the tools Jitterbug and Truetime are advocated. Other approaches extend the periodic sampled-data theory to incorporate the presence of latencies (delays) or jitter (variations on delays) in servo-loops in the control design. Typically, in this line of work (see, e.g., Hu and Michel (2000), Zhang *et al.* (2001), Lincoln (2002), Kao and Lincoln (2004), Balluchi *et al.* (2005) and Cloosterman *et al.* (2006)) the time variations in the 'event-triggering' can be considered as disturbances and one designs compensators that are robust to it. In Schinkel *et al.* (2002, 2003) time-varying sample times are considered. However, only a finite number of possible sample times are allowed. Then one designs controllers and observers that use feedback or observer gains that depend on the known sample time. Stability of the closed-loop is guaranteed via the existence of a common quadratic Lyapunov function. However, knowing the (future) sample time is unrealistic in various cases including event-driven control. Moreover, in the event-driven context as proposed here a common Lyapunov function does not exist as asymptotic stability cannot be achieved. As we will see, ultimate boundedness (Blanchini 1994, 1999) with small bounds (a kind of practical stability) is the most one can achieve.

There is another fundamental difference between the previously mentioned work and the current paper. We will study event-driven controllers for which we design both the control algorithm and the way the events are generated that determine when the control values are updated. This is in contrast with the approaches in Zhang *et al.* (2001), Lincoln (2002), Schinkel *et al.* (2002, 2003), Kao and Lincoln (2004), Balluchi *et al.* (2005) and Cloosterman *et al.* (2006), where the variations in the sample times are considered as externally imposed disturbances. In this paper the selection of the event-triggering mechanism suits a clear purpose: lowering the resource utilization of its implementation while maintaining a high control performance. The approach taken here is to update the control value only when the (tracking or stabilization) error is larger than a threshold and holding the control value if the error is small. Event-driven control strategies (Doff *et al.* 1962, Årzén 1999, Åström and Bernhardsson 2002, Årzén *et al.* 2003, Sandee *et al.* 2005, Johannesson *et al.* 2007) have been proposed before to make such a compromise between processor load and control performance. In Doff *et al.* (1962), Årzén (1999), Årzén *et al.* (2003), Sandee *et al.* (2005) and Johannesson *et al.* (2007), the potential of event-driven controllers has been indicated via various interesting examples, but theoretical results on performance of event-driven controllers are rare. Besides the mathematical analysis in Åström and Bernhardsson (2002) and Johannesson *et al.* (2007) for first order stochastic systems, the current paper is one of the first that provides a theoretical study of the performance of event-driven controllers (for higher-order systems).

To be more precise, this paper provides theory and insight to understand and tune a particular type of event-driven controlled linear systems. The performance of these novel control strategies is addressed in terms of ultimate boundedness (Blanchini 1994) and convergence rates. Depending on the particular event-triggering mechanism used for the control updates, properties like ultimate boundedness for the perturbed event-driven linear system can be derived either from a perturbed discrete-time linear system or from a perturbed discrete-time piecewise linear (PWL) system. Since results for ultimate boundedness are known for discrete-time linear systems (e.g., Blanchini (1994, 1999), Kolmanovsky and Gilbert (1998), Kerrigan (2000) and Rakovic *et al.* (2005), and piecewise linear systems see, e.g., Kvasnica *et al.* (2004) and Rakovic *et al.* (2004)), these results can be carried over to event-driven controlled systems. In this way we can tune the parameters of the controller to obtain satisfactory control performance on one hand and low processor/communication load on the other. Initial experimental studies of the achievable reduction in the processor load by the particular type of event-driven controllers proposed here, are very promising (Sandee *et al.* 2006).

The outline of the paper is as follows: in § 2 we present two numerical examples that show the potential of the proposed event-driven controllers for reducing resource utilization while maintaining a high control performance; after introducing some preliminaries in § 3, we present the problem formulation in § 4; in § 5 the approach is given and the main results are presented for two particular types of event-triggering mechanisms in § 6 (the non-uniform case) and § 7 (the uniform case). Section 8 shows how the intersample behaviour can be included in the analysis. In § 9 it is indicated how the main results can be exploited to compute ultimate bounds for event-driven linear systems in combination with existing theory for linear and piecewise linear systems. Moreover, we provide conditions that guarantee the existence of ultimate bounds that are bounded. Based on these results, we develop tuning rules for event-driven controllers as explained in § 10. In § 11 we present some examples that illustrate the theory and we end with the conclusions.

2. Motivating examples

To show the potential of event-driven control with respect to reduction of resource utilization we present two examples; one academic example using state feedback and one event-driven PID controller with the aim of velocity tracking for a DC motor, a situation occurring often in industrial practice.

2.1 *Scalar state feedback example*

Consider the following simple continuous-time plant

$$\dot{x}(t) = 0.5x(t) + 10u(t) + 3w(t) \tag{1}$$

with $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$ and $w(t) \in \mathbb{R}$ the state, control input and disturbance at time $t \in \mathbb{R}_+$, respectively. The additive disturbance satisfies $-10 \leq w(t) \leq 10$. This system will be controlled by a discrete-time controller

$$u_k = \begin{cases} -0.45x_k, & \text{if } |x_k| \geq e_T \\ u_{k-1}, & \text{if } |x_k| < e_T, \end{cases} \tag{2}$$

that runs at a fixed sample time of $T_s = 0.1$ time units. Hence, $x_k = x(kT_s)$ for $k = 0,1,2,\ldots$. Here, e_T denotes a parameter that determines the region $\mathcal{B} := \{x \in \mathbb{R} \mid |x| < e_T\}$ close to the origin in which the control values are not updated, while outside $\mathcal{B}$ the control values are updated in a 'normal fashion'. In this paper we will refer to this situation as uniform sampling. We will also consider the (locally) non-uniform case where reaching the boundary of $\mathcal{B}$ will be the event trigger—in addition to a fixed update rate outside $\mathcal{B}$ - for updating the control values. Figure 1 displays the ratio of the number of control updates in comparison to the case where the updates are performed each sample time (i.e., $u_k = -0.45x_k$ for all $k = 0, 1, 2, \dots$) and the maximal value of the state variable (after transients) $x_{\max} := \limsup_{t\to\infty} |x(t)|$, respectively, versus the parameter e_T. The results are based on simulations. Hence, one can reduce the number of control computations by 75% without degrading the control performance in terms of ultimate bound $x_{\max}$ drastically (e.g., take $e_T = 3$).

2.2 *Event-driven PID controller*

The most common controller in industry is still the proportional integral derivative (PID) controller. We will present an event-driven version of it. Also in Årzén (1999) an event-driven PID controller is proposed using a different event triggering mechanism. However, a formal analysis of its properties is not presented in Årzén (1999).

A standard transfer function for a continuous-time PID algorithm is

$$C(s) = K_p + K_i/s + K_d sL(s) \tag{3}$$

with $L(s)$, a low-pass filter to deal with high frequency measurement noise. The transfer function of this filter with a bandwidth ω_d is given in (4)

$$L(s) = \frac{\omega_d}{s+\omega_d}. \tag{4}$$

To use this controller in a discrete-event environment, the first step is to discretise the transfer function of the controller. This can be done by means of approximation formulas. A common choice for approximating the integral part is to use Forward Euler. To approximate the derivative part in combination with the filter $L(s)$, the Tustin approximation (Franklin *et al.* 1998) is used. The resulting transfer function of the PID controller in discrete-time is given in (5)

$$C(z) = K_p + K_i\frac{T_{s,k}}{z-1} + K_d\frac{2\omega_d}{2+\omega_d T_{s,k}}\,\frac{z-1}{z+(\omega_d T_{s,k}-2)/(\omega_d T_{s,k}+2)}. \tag{5}$$

The controller in (5) is suitable as an event-driven controller with $T_{s,k}$ a varying sample time and $\tau_k := \sum_{j=0}^{k-1} T_{s,j}$ is the time instant at which the kth control update is performed. We call τ_k, $k = 0, 1, 2, \dots$ the control update times. In conventional time-driven control $T_{s,k} = T_s$ is fixed and the control update times are equally spaced in time, but for event-driven control $T_{s,k}$ is allowed to change over time. In Årzén (1999) it is shown that adapting $T_{s,k}$ in (5) for every control update improves the control performance considerably although it requires additional control computations and thus a bit higher processor load (in comparison to the case where $T_{s,k}$ is kept constant as a kind of "average sampling period"). The event-triggering mechanism (selecting τ_k) is based on the tracking error e as follows:

$$\tau_{k+1} = \inf\{t \ge \tau_k + T_{s,\min} \mid |e(t)| \ge e_T\} \tag{6}$$

in which $e_T > 0$ is a threshold value and $T_{s,\min} > 0$ is the minimum sample time. We will refer to (6) as the locally non-uniform mechanism (cf. §2.1, where the uniform mechanism was introduced). This mechanism uses a uniform sample time of $T_{s,\min}$ for large tracking values $e(t)$, but only when $|e(t)| < e_T$ the control value (so "locally") is held longer and the sample time varies. For shortness we refer to this mechanism as non-uniform in the remainder of the paper.

The event-driven PID controller is used in simulations to control the angular velocity of a DC-motor. These results will be compared to a standard time-driven PID controller. A simplified motor model is taken with

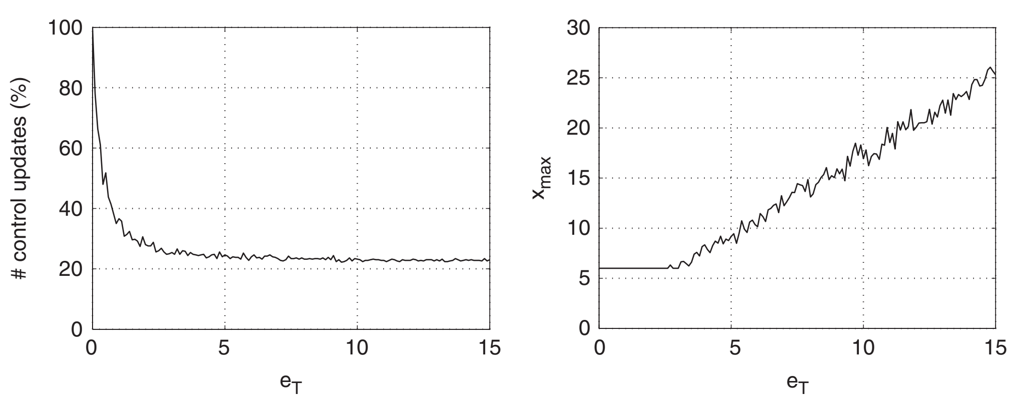

Figure 1. e_T versus the control effort and $x_{\max}$ for system (1)–(2).

input the motor voltage and output the velocity of the motor axis. The transfer function is given by

$$P(s) = \frac{A}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{7}$$

with the time constants τ_1 and τ_2 equal to 0.33 s and 0.17 s, respectively. The static gain A is 10 rad/Vs.

The gains of the continuous-time controller in (3) are determined by loop-shaping (e.g., Franklin *et al.* 2005) in $K_p = 30$ Vs/rad, $K_i = 40$ V/rad and $K_d = 2$ Vs^2/rad. In industrial practice, the sample frequency of the time-driven controller is often chosen approximately 20 times the bandwidth of the open-loop system (e.g., Franklin *et al.* (1998)). This bandwidth, defined as the zero-dB crossing of the open loop amplification, is 57 Hz in the considered example. The sample frequency is therefore chosen to be 1 kHz. To improve the performance of the controller, a feedforward term was added that feeds-forward the set-point speed multiplied by a gain of $1/A = 0.1$ Vs/rad. Furthermore, the output of the controller is saturated at +10 V and −10 V. The bandwidth of the low-pass filter f_d is chosen to be 200 Hz (and thus $\omega_d = 2\pi \cdot 200$ rad/s).

Various simulations have been carried out with the reference velocity shown in figure 2. In figure 3, simulations of the standard time-driven PID controller (5) with $T_{s,k} = T_s$ fixed and equal to 1 ms and the event-driven controller (with adaptation of $T_{s,k}$) are shown. For comparison, the parameter e_T of equation (6) is chosen such that the maximum error of the event-driven simulation approximates the maximum error obtained from the time-driven simulation. The value of $T_{s,\min}$ is chosen the same as the sample time of the time-driven controller. The values are $e_T = 5 \cdot 10^{-4}$ rad/s and $T_{s,\min} = 0.001$ s. As can be seen from figure 3, the event-driven controller does not realise a zero tracking error (in contrast with the time-driven controller). This is especially evident for the phases with (non-zero) constant deceleration (e.g., the time period from 5 s to 8 s). However, in most industrial applications there are often only requirements given for the maximum value of the error.

The third plot in figure 3 shows the number of samples that is needed for the control algorithms. This amount is equal to 10 000 for the time-driven controller as it is running on a constant sample frequency of 1 kHz for 10 seconds. The number of samples needed for the event-driven controller based on (6) is 2400, leading to a reduction of 76% in the number of control updates.

In the next simulations uniformly distributed measurement noise is added to the output of the process in the range of [−0.003, 0.003] rad/s, which is at most 3% of the maximal velocity. To obtain the best results with the event-driven controller, the value of e_T needs to be increased to make sure it will not be triggered continuously by the noise. The new value is $e_T = 7 \cdot 10^{-3}$ rad/s. The results of the simulations including measurement noise are depicted in figure 4. A slightly worse performance of the event-driven controller compared to the time-driven controller is to be accepted here, especially considering the considerable reduction of control updates to less than 3100 (69% reduction).

Of course, the reduction in control updates has to be related to its effect on resource utilization especially since the event-triggering mechanism creates some overhead as well. Depending on the ratio between the (on-line) computational complexity of the control algorithm, the overhead of the event triggering mechanisms and i/o access of the processor, the reduction of control computation indeed lowers the processor load considerably. In Sandee *et al.* (2006) the authors studied this relation both theoretically and experimentally with

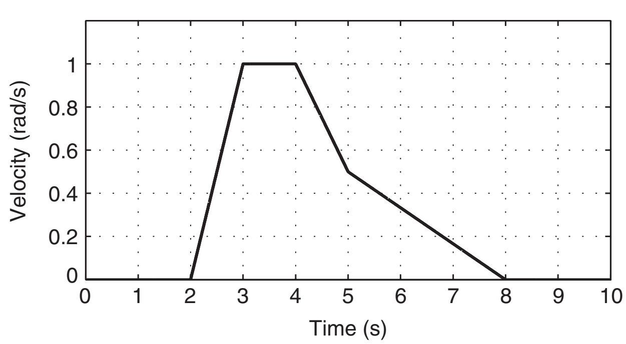

Figure 2. Reference signal.

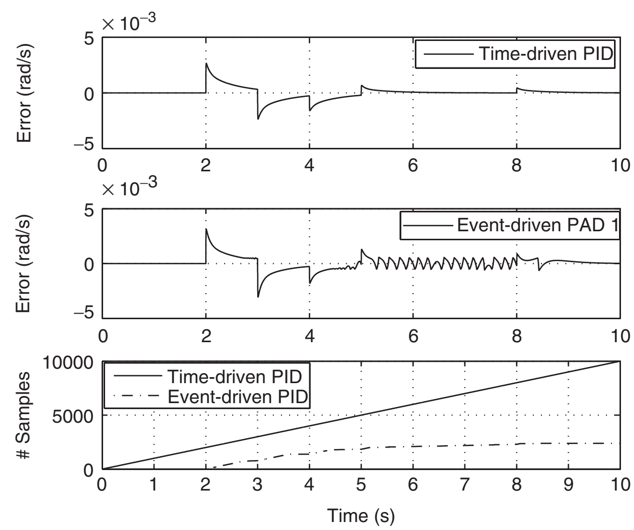

Figure 3. Simulation results of time-driven and event-driven simulation.

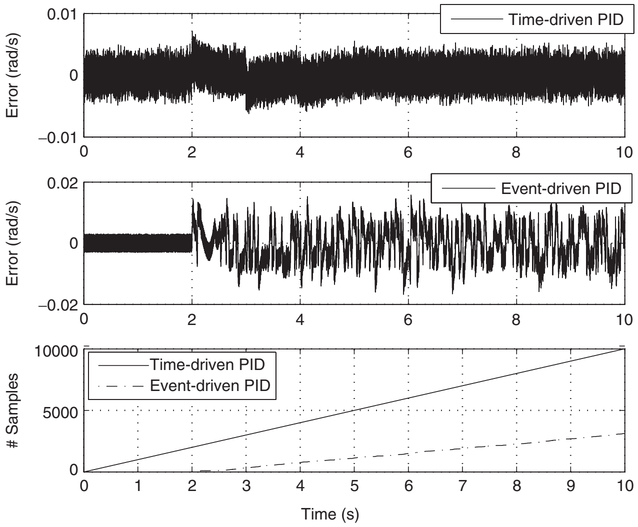

Figure 4. Simulation results with measurement noise added.

respect to processor usage. These initial results are very promising.

Both examples indicate the potential benefits of event-driven control in practice. However, as already mentioned in the introduction, a formal analysis of this type of controllers is missing in the literature, which hampers the exploitation of event-driven control. To contribute in filling this gap, we will analyse state feedback controllers using the uniform sampling of § 2.1 and the (locally) non-uniform sampling of (6). We will focus on the stabilization problem. A precise problem formulation will be given in § 1 after introducing some preliminaries next.

3. Preliminaries

For a matrix $M \in \mathbb{R}^{n\times m}$, we denote $M^T \in \mathbb{R}^{m\times n}$ as its transposed. A matrix $M \in \mathbb{R}^{n\times n}$ is called positive definite, denoted by $M > 0$, if $M = M^T$ for all $x \in \mathbb{R}^n$ with $x \neq 0$ it holds that $x^T Mx > 0$. For a set $\Omega \subseteq \mathbb{R}^n$ we denote its interior, its closure and its boundary by intΩ, clΩ and $\partial\Omega$, respectively. For two sets Ω_1 and Ω_2 of $\mathbb{R}^n$, the set difference $\Omega_1\backslash\Omega_2$ is defined as $\{x \in \Omega_1 \mid x \notin \Omega_2\}$ and the Minkowski sum as $\Omega_1 \oplus \Omega_2 := \{u+v \mid u \in \Omega_1, v \in \Omega_2\}$. The complement of $\Omega \subset \mathbb{R}^n$ is defined as $\mathbb{R}^n\backslash\Omega$ and is denoted by Ω^c.

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz, if all its eigenvalues lie in the open left half of the complex plane. The matrix is called Schur, if all its eigenvalues lie in the open unit disc. We call the matrix pair (A,B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ Hurwitz stabilizable, if there exists a matrix $F \in \mathbb{R}^{m\times n}$ such that $A + BF$ is Hurwitz. (A,B) is called Schur stabilizable, when there is an $F \in \mathbb{R}^{m\times n}$ with $A + BF$ Schur.

Consider a time-varying discrete-time system

$$x_{k+1} = f(k, x_k, w_k) \tag{8}$$

with $x_k \in \mathbb{R}^n$ the state and $w_k \in \mathcal{W}_d$ the disturbance at discrete-time $k \in \mathbb{N} := \{0, 1, 2, \ldots\}$ or a time-varying continuous-time system

$$\dot{x}(t) = f(t, x(t), w(t)) \tag{9}$$

with $x(t) \in \mathbb{R}^n$ the state variable and $w(t) \in \mathcal{W}_c$ the disturbance at time $t \in \mathbb{R}_+$. $\mathcal{W}_c$ and $\mathcal{W}_d$ denote the disturbance sets, which are assumed to be convex, compact and contain 0. We define the set $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{T} \to \mathbb{R}^p)$ as the Lebesgue space of locally integrable functions from the time interval $\mathbb{T} \subseteq \mathbb{R}$ to $\mathbb{R}^p$. Similarly, $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{T} \to \mathcal{W}_c)$ denotes all $\{w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{T} \to \mathbb{R}^p) \mid w(t) \in \mathcal{W}_c$ for almost all $t \in \mathbb{T}\}$. Analogously, for discrete-time signals we write $\mathcal{W}_d^\infty$ for the set of infinite sequences given by $\{(w_0, w_1, w_2, \ldots) \mid w_k \in \mathcal{W}_{\mathrm{d}}, k \in \mathbb{N}\}$.

Definition 1 (robust positive invariance)**:** The set $\Omega \subseteq \mathbb{R}^n$ is a robustly positively invariant (RPI) set for the discrete-time system (8) with disturbances in $\mathcal{W}_d$, if for any $x \in \Omega$, $k \in \mathbb{N}$ and any $w \in \mathcal{W}_d$ it holds that $f(k, x, w) \in \Omega$. The set $\Omega \subseteq \mathbb{R}^n$ is a robustly positively invariant (RPI) set for the continuous-time system (9) with disturbances in $\mathcal{W}_c$, if for any time $t_{ini} \in \mathbb{R}_+$, any state $x_{ini} \in \Omega$ and any disturbance signal $w \in \mathcal{L}_1^{\mathrm{loc}}([t_{ini}, \infty) \to \mathcal{W}_c)$ it holds that the corresponding state trajectory satisfies $x(t) \in \Omega$ for all $t \geq t_{ini}$. We assume implicitly the well-posedness of the system (9) in the sense that (local) existence and uniqueness of solutions to (9) given an initial condition and disturbance signal of interest is satisfied.

Definition 2 (ultimate boundedness) (Blanchini 1994)**:** We call the discrete-time difference equation (8) ultimately bounded (UB) to the set Ω with disturbances in $\mathcal{W}_d$, if for each $x_0 \in \mathbb{R}^n$ there exists a $K(x_0) > 0$ such that any state trajectory of (8) with initial condition x_0 (and any arbitrary realisation of the disturbance $w \in \mathcal{W}_d^\infty$) satisfies $x_k \in \Omega$ for all $k \geq K(x_0)$. Similarly, we call (9) ultimately bounded (UB) to the set Ω with disturbances in $\mathcal{W}_c$, if for every initial condition $x_0 \in \mathbb{R}^n$ there exists a $T(x_0) > 0$ such that any state trajectory of (9) with initial condition $x(0) = x_0$ (and any arbitrary realisation of the disturbance $w: \in \mathcal{L}_1^{\mathrm{loc}}([0, \infty) \to \mathcal{W}_c)$) satisfies $x(t) \in \Omega$ for all $t \geq T(x_0)$. We say that either the discrete-time or continuous-time system is UB for initial states in X_0, if the above properties hold for all $x_0 \in X_0$ (instead of for all $x_0 \in \mathbb{R}^n$).

4. Problem formulation

We consider the system described by

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c w(t), \tag{10}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the control input and $w(t) \in \mathcal{W}_c$ the unknown disturbance, respectively, at time $t \in \mathbb{R}_+$. The set $\mathcal{W}_c \subset \mathbb{R}^p$ is convex and compact and contains the origin. $A_c \in \mathbb{R}^{n \times n}$, $B_c \in \mathbb{R}^{n \times m}$ and $E_c \in \mathbb{R}^{n \times p}$ are constant matrices. The matrix pair (A_c, B_c) is assumed to be Hurwitz stabilizable. The control goal, which will be made more precise soon, is a 'practical stabilization problem' in the sense of controlling the state to a region close to the origin and keep it there irrespective of the presence of disturbances. Note that asymptotic stability cannot be obtained due to the possible persistence of the disturbances.

As a controller for system (10) a discrete-time state-feedback controller with gain $F \in \mathbb{R}^{m \times n}$ is considered, i.e.,

$$u_k = F x_k, \tag{11}$$

where $x_k = x(\tau_k)$, $u_k = u(\tau_k)$ using the zero-order hold $u(t) = u_k$ for all $t \in [\tau_k, \tau_{k+1})$. Hence, the system is given by

$$\dot{x}(t) = A_c x(t) + B_c u(t) + E_c w(t) \tag{12a}$$

$$u(t) = F x(\tau_k), \quad \text{for } t \in [\tau_k, \tau_{k+1}). \tag{12b}$$

The control update times τ_k are in conventional time-driven control related through $\tau_{k+1} = \tau_k + T_s$, where T_s is a fixed sample time meaning that the control value is updated every T_s time units according to (11). To reduce the number of required control calculations, we propose not to update the control value if the state $x(\tau_k)$ is contained in a set $\mathcal{B}$ close to the origin. As such, we consider a set $\mathcal{B}$ that is open bounded and contains the origin. Note that the openess of $\mathcal{B}$ is merely a technical condition to make the following exposition more compact and clear. It is not a restrictive condition. We will consider two event-triggering mechanisms, which were already discussed in § 2.

- The non-uniform mechanism as used in the example of § 2.2 is given by

$$\tau_1 = \inf\{t \geq \tau_0 \mid x(t) \notin \mathcal{B}\} \quad \text{and} \quad \tau_{k+1} = \inf\{t \geq \tau_k + T_s \mid x(t) \notin \mathcal{B}\}, \quad k > 0. \tag{13}$$

- The uniform mechanism as used in the example of § 2.1 is given by

$$\tau_{k+1} = \inf\{jT_s > \tau_k \mid j \in \mathbb{N},\ x(jT_s) \notin \mathcal{B}\}. \tag{14}$$

We will use $\tau_0 = 0$ as the first control update time (irrespective if $x(0) \in \mathcal{B}$ or not). In the non-uniform case we have a kind of "start-up behaviour" as τ_1 is defined a bit differently than τ_{k+1} for $k > 0$. In both the uniform and non-uniform case the system is controlled with a fixed sample time T_s when the state $x(t)$ is far away from $\mathcal{B}$. In the uniform case still every T_s time units it is checked, whether or not the state $x(jT_s)$ lies in $\mathcal{B}$ and the set of control update times is a subset of $\{jT_s \mid j \in \mathbb{N}\}$. The latter set can be considered the collection of control check times. The non-uniform case does not have this constant checking rate, but has locally (inside $\mathcal{B}$) a non-uniform character as new control updates are triggered by reaching the boundary of $\mathcal{B}$. It might be the case that for certain initial conditions $x(0) = x_0$ and disturbance signals $w \in \mathcal{L}_1^{\text{loc}}([0, \infty \to \mathcal{W}_c)$ there are only a finite number of control update times (i.e., $\tau_{k+1} = \infty$ for some k and thus $\tau_{k+2}, \tau_{k+3}, \ldots$ do not exist). In this case we have that the corresponding state trajectory denoted by $x_{x_0,w}$ lies inside $\mathcal{B}$ for all times $t > \tau_k + T_s$ in the non-uniform mechanism and for all control check times $jT_s > \tau_k$ in case of the uniform mechanism. Hence, for state trajectories where this phenomenon occurs we already have some ultimate boundedness properties. We introduce the notation $\mathcal{S}(x_0,w)$ in this context as the index set corresponding to all finite control update times for initial state x_0 and disturbance signal w. The notation $\mathcal{S}'(x_0,w)$ is the index set corresponding to all control update times τ_κ that are not only finite themselves, but also the next control update time $\tau_{\kappa+1}$ is finite. In the above situation with $\tau_k < \infty$ and $\tau_{k+1} = \infty$, $\mathcal{S}(x_0,w) = \{0, \ldots, k\}$ and $\mathcal{S}'(x_0,w) = \{0, \ldots, k-1\}$.

With regard to practical implementation, it has to be observed that the uniform mechanism is easier to implement, although it is more difficult to analyse, as we will see.

As already mentioned, the control objective is a "practical stabilization problem" in the sense of controlling the state towards a region Ω close to the origin and keeping it there.

Problem 1: Let a desired ultimate bound $\Omega_d \subset \mathbb{R}^n$ containing 0 in the interior be given. Construct T_s, F and $\mathcal{B}$ such that the system (12) with the control update times given by either (13) or (14) is UB to Ω_d.
For the moment, we ignore the transient behaviour of the event-driven system. The reason is that this is easily inherited from properties of the discrete-time linear system with the fixed sample time T_s (cf. (15) below). We will return to this issue later in § 10 in which it is explained how to tune the controller to get a satisfactory ultimate bound Ω_d and convergence rate towards Ω_d.

5. Approach

Problem 1 will be solved in two stages as is typical for sampled-data systems. First properties on UB to a set Ω

are obtained for the event-driven system (12) on the control update times only. Next bounds on the intersample behaviour (see § 8 below) will be derived that enlarge Ω to $\tilde{\Omega}$ such that the ultimate bound $\tilde{\Omega}$ is guaranteed for all (continuous) times t.

We first introduce the formal definitions of robust positive invariance and ultimate boundedness "on the control update times" for the system (12) together with a particular event-triggering mechanism

Definition 3: Consider the system (12) with either (13) or (14) as event-triggering mechanism.

- For this system the set $\Omega \subseteq \mathbb{R}^n$ is called robustly positively invariant (RPI) on the control update times for disturbances in $\mathcal{W}_c$, if for any initial state x_0 and $w \in \mathcal{L}_1^{\text{loc}}([0,\infty) \to \mathcal{W}_c)$ the corresponding state trajectory $x_{x_0,w}$ of the system has the property that $x_{x_0,w}(\tau_k) \in \Omega$ for some $k \in \mathcal{S}'(x_0,w)$ implies $x_{x_0,w}(\tau_{k+1}) \in \Omega$.
- This system is ultimately bounded (UB) on the control update times to the set Ω for disturbances in $\mathcal{W}_c$, if for any initial condition x_0 there exists a $K(x_0)$ such that for any $w \in \mathcal{L}_1^{\text{loc}}([0,\infty) \to \mathcal{W}_c)$ the corresponding state trajectory $x_{x_0,w}$ satisfies for all $k \in \mathcal{S}(x_0,w)$, $k \geq K(x_0)$ that $x_{x_0,w}(\tau_k) \in \Omega$.

Note that we only impose robust positive invariance or UB-related conditions on the finite control update times and not for time instants beyond. However, as noted in the previous section, for state trajectories $x_{x_0,w}$ with $\mathcal{S}(x_0,w) = \{0,\ldots,k\}$ a finite collection, it holds that $x_{x_0,w}(t) \in \mathcal{B}$ for $t > \tau_k + T_s$ in the non-uniform mechanism and $x_{x_0,w}(jT_s) \in \mathcal{B}$ for all $jT_s > \tau_k$ in case of the uniform mechanism. Hence, we already have some ultimate boundedness properties with respect to the set $\mathcal{B}$ in this situation. The situation with $\mathcal{S}(x_0,w)$ is an infinite set is the case of interest here.

In the analysis of the event-driven control scheme the discretized version of (10) and (11) using the fixed sample time T_s given by

$$x_{k+1}^d = (A + BF)x_k^d + w_k^d = A_{cl}x_k^d + w_k^d \tag{15}$$

with

$$\left.\begin{aligned} A &:= e^{A_c T_s} \\ B &:= \int_0^{T_s} e^{A_c\theta}\, \mathrm{d}\theta\, B_c \\ w_k^d &:= \int_{\tau_k}^{\tau_{k+1}} e^{A_c(\tau_{k+1}-\theta)} E_c w(\theta)\mathrm{d}\theta \\ A_{cl} &:= A + BF \end{aligned}\right\} \tag{16}$$

will play an important role. Indeed, for both the uniform and non-uniform sampling case, the system behaves away from the set $\mathcal{B}$ (at the control update times) as (15).

We use the shorthand notation $x_{x_0,w}(\tau_k) = x_k^d$ here. This system is only representing the system (12) at the control update times, when $\tau_{k+1} = \tau_k + T_s$. The bounds on $w(t)$ via $\mathcal{W}_c$ are transformed into bounds on w_k^d given by

$$\mathcal{W}_d := \left\{ \int_0^{T_s} e^{A_c(T_s-\theta)} E_c w(\theta)\mathrm{d}\theta \mid w \in \mathcal{L}_1^{loc}([0, T_s] \to \mathcal{W}_c) \right\}. \tag{17}$$

Since $\mathcal{W}_c$ is convex, compact and contains 0, also $\mathcal{W}_d$ is convex, compact and contains 0. Since (A_c, B_c) is assumed to be Hurwitz stabilizable, it follows that for almost all choices of $T_s > 0$, that (A, B) is Schur stabilizable. (see exercise 3.20 in Trentelman *et al.* (2001) for a detailed discussion and conditions on T_s that guarantee that Hurwitz stabilizability of (A_c, B_c) transfers into Schur stabilizability of its discretized version (A, B)). In general F will be chosen such that $A + BF$ is Schur. In § 9.1 we will provide conditions that guarantee the existence of a bounded set Ω to which the event-driven system will be UB to.

6. Main results for the non-uniform mechanism

The first theorem below states that ultimate bounds for the linear discrete-time system (15) can be used to find ultimate bounds for the event-driven system (12) on the control update times $\{\tau_k\}_k$ with non-uniform sampling (13).

Theorem 1: *Consider the system* (12)–(13) *with* $\mathcal{W}_c$ *a closed, convex set containing* 0 *and* $\mathcal{B}$ *an open set containing the origin. Let* $\mathcal{W}_d$ *be given by* (17).

(i) *If* Ω *is a RPI set for the* linear discrete-time system (15) *with disturbances in* $\mathcal{W}_d$ *and* $cl\mathcal{B} \subseteq \Omega$, *then* Ω *is a RPI set for the event-driven system* (12)–(13) *on the control update times for disturbances in* $\mathcal{W}_c$.

(ii) *If the* linear discrete-time system (15) *with disturbances in* $\mathcal{W}_d$ *is UB to the RPI set* Ω *and* $cl\mathcal{B} \subseteq \Omega$, then the *event-driven system* (12)–(13) *on the control update times is UB to* Ω *for disturbances in* $\mathcal{W}_c$.

Proof: (i) Let an arbitrary initial state x_0 and $w \in \mathcal{L}_1^{\text{loc}}([0,\infty) \to \mathcal{W}_c)$ be given and consider the corresponding state trajectory $x_{x_0,w}$ of the system (12)–(13) and assume that $x_{x_0,w}(\tau_k) \in \Omega$ for some $k \in \mathcal{S}'(x_0,w)$. Since τ_{k+1} is finite as well, there are two possibilities:

- $\tau_{k+1} = \tau_k + T_s$. This means that the update of the state over the interval $[\tau_k, \tau_{k+1}]$ is governed by (15) for some $w_k^d \in \mathcal{W}_d$. Since Ω is a RPI set for (15) with disturbances in $\mathcal{W}_d$, this means that $x_{x_0,w}(\tau_{k+1}) \in \Omega$ (irrespective of the realisation of the disturbance).

- $\tau_{k+1} \neq \tau_k + T_s$. Note that for $k > 0$ it holds that $\tau_{k+1} > \tau_k + T_s$. Only for $k = 0$, it may hold that $\tau_1 \leq \tau_0 + T_s$. According to (13) this implies that $x_{x_0,w}(\tau_{k+1}) \in \partial\mathcal{B} \subset \mathrm{cl}\mathcal{B}$ Since $\mathrm{cl}\mathcal{B} \subseteq \Omega$, it holds that $x_{x_0,w}(\tau_{k+1}) \in \Omega$.

This proves that Ω is RPI for (12)–(13) on the control update times for disturbances in $\mathcal{W}_c$.

(ii) Consider an initial state x_0. If $x_0 \in \Omega$, then due to the RPI property of Ω on the control update times as proven in the first part of the proof, the state trajectory $x_{x_0,w}$ stays within Ω on the control update times (irrespective of the disturbance signal $w \in \mathcal{L}_1^{\mathrm{loc}}([0,\infty) \to \mathcal{W}_c)$), i.e., $x_{x_0,w}(\tau_k) \in \Omega$ for $k \in \mathcal{S}(x_0,w)$. Hence, one can take $K(x_0) = 0$ in Definition 3. Suppose $x_0 \notin \Omega$. Since (15) is UB to Ω for disturbances in $\mathcal{W}_d$, there exists a time $K(x_0)$ such that any discrete-time trajectory x^d with initial condition $x_0^d = x_0$ of (15) satisfies $x_k^d \in \Omega$ for $k \geq K(x_0)$ (irrespective of the disturbance signal $w \in \mathcal{W}_d^\infty$). We claim that this $K(x_0)$ satisfies also Definition 3 for x_0.

Indeed, let $w \in \mathcal{L}_1^{\mathrm{loc}}([0,\infty) \to \mathcal{W}_c)$ and consider the trajectory $x_{x_0,w}$ of (12)–(13) for initial condition $x(0) = x_0$ and disturbance signal w. Let $\bar{k} \in \mathcal{S}(x_0, w)$ satisfy $\bar{k} \geq K(x_0)$. We proceed by contradiction. Assume that $x_{x_0,w}(\tau_k) \notin \Omega$ for $k = 0, \ldots, K(x_0)$. Since $x_{x_0,w}(\tau_k) \notin \Omega$ for $k = 0, \ldots, \bar{k}$, and thus $x_{x_0,w}(\tau_k) \notin \mathcal{B}$ for $k = 0, \ldots, \bar{k}$, it holds that $\tau_{k+1} = \tau_k + T_s$ and $x_{x_0,w}(\tau_{k+1})$ and $x_{x_0,w}(\tau_k)$ are related through (15) for some $w_k^d \in \mathcal{W}_d$ for all $k = 0, \ldots, \bar{k} - 1$. Hence, $x_{x_0,w}(\tau_k) = x_k^d$ for $k = 0, \ldots, \bar{k}$. However, $x_{K(x_0)}^d \in \Omega$ and $\bar{k} \geq K(x_0)$. We reached a contradiction. Hence, there is a $k \in \{0, \ldots, K(x_0)\}$, say $\tilde{k}$ such that $x_{x_0,w}(\tau_{\tilde{k}}) \in \Omega$. Since Ω is RPI for (12)–(13) on the control update times, we have $x_{x_0,w}(\tau_k) \in \Omega$ for all $k \geq \tilde{k}$ and $k \in \mathcal{S}(x_0,w)$. As $K(x_0) \geq \tilde{k}$, this completes the proof of statement (ii). □

7. Main results for the uniform mechanism

As mentioned before, the non-uniform update scheme is hard to implement in practice. Uniform sampling might be more relevant from a practical point of view. However, in contrast to non-uniform sampling the properties of the discrete-time linear system do not transfer to the event-driven system in this case. As we will see, we will need a discrete-time piecewise linear (PWL) model (see, e.g., Sontag (1981) and Heemels *et al.* (2001)) to analyse event-driven systems using uniform sampling. We will present two approaches to this problem. A first PWL model uses $(x^T(\tau_k), u^T(\tau_{k-1}))^T$ as state variable, while the second PWL model only uses $x(\tau_k)$. This implies that the first model has a higher $(n+m)$-dimensional state variable than the second (n-dimensional), but as we will see next, it only needs two linear submodels, while the second PWL model might need much more.

7.1 *A bimodal higher order piecewise linear system*

To derive results on ultimate boundedness on the control update times for the event-driven system with the uniform mechanism, we will embed the control update times $\{\tau_k \mid k \in \mathcal{S}(x_0,w)\}$ in its superset $\{jT_s \mid j \in \mathbb{N}\}$. From (12)–(14) together with the discretisation (15) it can be observed that the behaviour of the system (12) and (14) on the control check times $\{jT_s \mid j \in \mathbb{N}\}$ can be included in

$$\begin{pmatrix} x_{k+1}^d \\ u_k^d \end{pmatrix} = \begin{cases} \begin{pmatrix} A+BF & 0 \\ F & 0 \end{pmatrix}\begin{pmatrix} x_k^d \\ u_{k-1}^d \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} w_k^d, & \text{if } x_k^d \notin \mathcal{B} \\ \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}\begin{pmatrix} x_k^d \\ u_{k-1}^d \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} w_k^d, & \text{if } x_k^d \in \mathcal{B} \end{cases} \tag{18}$$

for $w_k^d \in \mathcal{W}_d$. Note that the fact $\tau_0 = 0$, i.e., at the initial time a control update $u_0 = Fx_0$ is performed, can be included by considering initial states in the set $X_0 := \{(x_0^T, u_{-1}^{d\,T})^T \mid u_{-1}^d = Fx_0\}$.

We need the following definition.

Definition 4: Let Γ be a subset of $\mathbb{R}^n \times \mathbb{R}^m$. The projection $\Pi_n(\Gamma)$ of Γ on the first n components of the vector space $\mathbb{R}^n \times \mathbb{R}^m$ is defined as $\{x \in \mathbb{R}^n \mid \exists\, u \in \mathbb{R}^m \text{ such that } (x^T, u^T)^T \in \Gamma\}$.

As $\{\tau_k \mid k \in \mathcal{S}(x_0, w)\} \subseteq \{jT_s \mid j \in \mathbb{N}\}$ for any x_0 and disturbance signal w, we will formulate a result on ultimate boundedness on the control check times, a concept that can be defined analogously as UB on the control update times as in Definition 3.

Theorem 2: *Consider the system* (12) *and* (14) *with $\mathcal{W}_c$ a closed, convex set containing* 0 *and $\mathcal{B}$ an open set containing the origin. Let $\mathcal{W}_d$ be given by* (17). *If the piecewise linear discrete-time system* (18) *with disturbances in $\mathcal{W}_d$ is UB for initial states in X_0 to the set Γ, then the event-driven system* (12) and (14) *on the control check times is UB to $\Pi_n(\Gamma)$ for disturbances in $\mathcal{W}_c$.*

Proof: As system (12), (14) coincides with (18) on the control check times and (18) is UB to Ω, we have that for any x_0 there exists a $K(x_0, u_{-1}^d)$ with $u_{-1}^d = Fx_0$ such that $(x_k^{d\,T}, u_{k-1}^{d\,T})^T \in \Gamma$ for all $k \geq K(x_0, u_{-1}^d)$. Since u_{-1}^d is a function of x_0, it holds that there is a $\tilde{K}(x_0) := K(x_0, Fx_0)$ such that $x_{x_0,w}(kT_s) \in \Pi_n(\Gamma)$ for all $k \geq \tilde{K}(x_0)$.

7.2 *A multi-modal lower order piecewise linear system*

As already mentioned, the above approach leads to a piecewise linear system with an $(m+n)$-dimensional state vector. As this might be prohibitive for numerical tools computing ultimate bounds for PWL systems (when the number of control inputs m is large), an alternative approach is presented in this section.

7.2.1 The unperturbed case. Consider (12) on the control check times, which can be described by

$$\left.\begin{aligned} x_{k+1}^d &= Ax_k^d + Bu_k^d \\ u_k^d &= \begin{cases} Fx_k^d, & \text{if } x_k^d \notin \mathcal{B} \\ u_{k-1}^d, & \text{if } x_k^d \in \mathcal{B}. \end{cases} \end{aligned}\right\} \tag{19}$$

with $x_0^d := x_0$ and $u_{-1}^d = Fx_0$. We make the following observation. When the state x_k^d is outside $\mathcal{B}$ the state update in (18) does not depend on u_{k-1}^d and hence, the information on u_{k-1}^d is not necessary. Only in case $x_k^d \in \mathcal{B}$ the previous control value has to be known as it is going to be held for at least one, but possibly multiple check times. However, we can explicitly compute this update relation depending on how many check times the control value is held. This update relation will map the state just before entering $\mathcal{B}$ (at a control update time) to the state just after leaving $\mathcal{B}$ again (at the next control update time). It will turn out that in this way a piecewise linear (PWL) model is obtained in which we abstract away from the number of discrete-time steps that the system is inside $\mathcal{B}$. Using this PWL system properties related to UB can again be translated to the original system (12) and (14) on the control update times. The advantage with respect to (18) is that we only have an n-dimensional state vector now.

To define the map h_p for the different periods of time (denoted by p) that the state stays in $\mathcal{B}$, we consider first the case $p=0$, i.e., $x_k^d \notin \mathcal{B}$ and $x_{k+1}^d \notin \mathcal{B}$ the system update matrix is given by

$$x_{k+1}^d := h_0(x_k^d) = (A+BF)x_k^d. \tag{20}$$

For $p=1$ we assume that $x_k^d \notin \mathcal{B}$, $x_{k+1}^d \in \mathcal{B}$ and then $x_{k+2}^d \notin \mathcal{B}$. The function h_1 defines the mapping from x_k^d to x_{k+2}^d in this case. This update of the state variable is given by $x_{k+i}^d = Ax_{k+i-1}^d + Bu_{k+i-1}^d$, $i=1,2$ with $u_{k+1}^d = u_k^d = Fx_k^d$ (since the control value is held). Hence,

$$x_{k+2}^d = h_1(x_k^d) := [A(A+BF)+BF]x_k^d. \tag{21}$$

Similarly, suppose we stay p steps in $\mathcal{B}$ before leaving $\mathcal{B}$ again (i.e., $x_k^d \notin \mathcal{B}$, then $x_{k+1}^d \in \mathcal{B}$, $x_{k+2}^d \in \mathcal{B}, \ldots, x_{k+p}^d \in \mathcal{B}$ and then $x_{k+p+1}^d \notin \mathcal{B}$). We obtain the function h_p that maps x_k^d to x_{k+p+1}^d as follows by using repetitively $x_{k+i}^d = Ax_{k+i-1}^d + Bu_{k+i-1}^d$, for $i=1,\ldots,p+1$. Since $u_{k+p}^d = u_{k+p-1}^d = \cdots = u_k^d = Fx_k^d$, we can express x_{k+p+1}^d as a function of x_k^d,

$$\begin{aligned} x_{k+p+1}^d = h_p(x_k^d) := \{A^{p+1} + [A^p + A^{p-1} \\ + A^{p-2} + \cdots + I]BF\}x_k^d. \end{aligned} \tag{22}$$

Now that the maps h_p are defined, the regions D_p must be determined in which the map h_p is active. As D_p is given by those $x_k^d \notin \mathcal{B}$ that satisfy $x_{k+1}^d \in \mathcal{B}$, $x_{k+2}^d \in \mathcal{B}, \ldots, x_{k+p}^d \in \mathcal{B}$ and $x_{k+p+1}^d \notin \mathcal{B}$, we have for $p=0,1,2,\ldots$

$$\begin{aligned} D_p := \{x \notin \mathcal{B} \mid h_j(x) \in \mathcal{B} \text{ for} \\ j=0,1,\ldots,p-1 \text{ and } h_p(x) \notin \mathcal{B}\}. \end{aligned} \tag{23}$$

We also define the set of states that remain inside $\mathcal{B}$ forever after entering it from outside $\mathcal{B}$

$$D_\infty := \{x \notin \mathcal{B} \mid h_j(x) \in \mathcal{B} \text{ for all } j=0,1,\ldots\}. \tag{24}$$

Note that $D_i \cap D_j = \emptyset$ if $i \neq j$.

Finally, we introduce the set $R_\mathcal{B}$ which contains all possible values of x_k^d outside $\mathcal{B}$, that can reach $\mathcal{B}$ within one discrete time-step

$$R_\mathcal{B} := \{x \notin \mathcal{B} \mid h_0(x) \in \mathcal{B}\}.$$

With these definitions, it holds that

$$\mathbb{R}^n = \mathcal{B} \cup R_\mathcal{B} \cup D_0 \quad \text{and} \quad R_\mathcal{B} = D_\infty \cup \bigcup_{i=1}^{\infty} D_i. \tag{25}$$

To obtain a finite representation of the piecewise linear system, we need the existence of a $p_{\max}$ such that

$$R_\mathcal{B} = D_\infty \cup \bigcup_{i=1}^{p_{\max}} D_i \tag{26}$$

in which $p_{\max}$ is the maximal (finite) number of discrete steps that the system (19) can stay inside $\mathcal{B}$ after entering it from outside $\mathcal{B}$.

Deriving conditions for which the existence of such a finite $p_{\max}$ is guaranteed is an open issue. One of the complications is for instance that $D_i = \emptyset$ does not necessarily imply that $D_{i+1} = \emptyset$. Also the computation of D_∞ is not straightforward. A computational approach can be obtained by increasing $p_{\max}$ until the right-hand side of (26) is equal to the left-hand side (see § 11.2 for an example). However, still analytical results proving the existence of a finite $p_{\max}$ and possibly an upper bound for it, would be very beneficial. One such condition is formulated in § 7.2.2.

The iteration parameter k related to the control check times kT_s is replaced by a new "discrete-time variable" l corresponding to the control update times τ_l after abstracting away from the motion of the system's state inside $\mathcal{B}$. Therefore, we replace $x_{k+p+1}^d = h_p(x_k^d)$ by

$x^d_{l+1} = h_p(x^d_l)$ and obtain the piecewise linear system $x^d_{l+1} = f_{\mathrm{PWL}}(x^d_l)$ with

$$x^d_{l+1} = \begin{cases} h_p(x^d_l), & \text{when } x^d_l \in D_p,\ p = 0, 1, \ldots, p_{\max} \\ 0, & \text{when } x^d_l \in D_\infty \cup \mathcal{B}. \end{cases} \tag{27}$$

Some observations on the PWL system (27) are in order.

- The dynamics of the event-driven system (12), (14) and the PWL system (27) coincide on $\mathcal{B}^c \setminus D_\infty = \bigcup_{i=0}^{p_{\max}} D_i$ for the control update times in the sense that $x_{x_0}(\tau_{l+1}) = x^d_{l+1} = f_{\mathrm{PWL}}(x^d_l) = f_{\mathrm{PWL}}(x_{x_0}(\tau_l))$ for $x^d_l = x_{x_0}(\tau_l) \in \mathcal{B}^c \setminus D_\infty$, where x_{x_0} denotes the solution of the event-driven system (12), (14) for initial condition $x(0) = x_0$ and τ_l with $l \in \mathcal{S}'(x_0)$ a control update time. Moreover, $x^d_l = x_{x_0}(\tau_l) \in D_\infty$ implies that $\tau_{l+1} = \infty$.
- On $D_\infty \cup \mathcal{B}$ the piecewise linear model was completed by adding dynamics to the system for the case when $x^d_l \in D_\infty$ and $x^d_l \in \mathcal{B}$. As will be proven below, it is not important how the dynamics are chosen exactly on these sets as long as they do not map outside $\mathcal{B}$.
- A set D_p is in general not convex. It might even not be connected. See, the example in § 11.2.

We state now the main result of this section.

Theorem 3: *Consider system* (12) *and* (14) *without disturbances* (*i.e.* $\mathcal{W}_c = \{0\}$) *and* $\mathcal{B}$ *is an open set containing the origin. Assume that there exists a* $p_{\max} < \infty$ *such that* (26) *holds. If the PWL system* (27) *is UB to the positively invariant set* Ω *and* $\mathcal{B} \subseteq \Omega$, *then the event-driven system* (12) *and* (14) *is UB to* Ω on the control check times.

Proof: The system (12) and (14) on the control check times is described by (19). Therefore, we consider solutions in terms of trajectories x^d of (19) in this proof.

If $x^d_0 \in \Omega$ then we either have that the state trajectory of (19) satisfies $x^d_k \in \Omega$ for all $k = 0, 1, 2, \ldots$ (which is in accordance with the properties of the theorem) or the state trajectory leaves Ω for some control check time. Hence, without loss of generality we can consider the case that there exists a k_0 (take the smallest) for which $x^d_{k_0} \notin \Omega$ and thus $x^d_{k_0} \in \mathcal{B}^c = D_\infty \cup \bigcup_{i=0}^{p_{\max}} D_i$ because $\mathcal{B} \subseteq \Omega$. Observe that the dynamics of (19) and (27) coincide on $\bigcup_{i=0}^{p_{\max}} D_i$ (modulo the motion inside $\mathcal{B}$, which lies in Ω and therefore is not affecting the UB property to Ω). Hence, since $x^d_{k_0} \in D_\infty \cup \bigcup_{i=0}^{p_{\max}} D_i$, the system (19) follows the dynamics of the PWL system (27) (modulo the motion inside $\mathcal{B}$) for $k \geq k_0$ until D_∞ is reached—if ever (say at $k_1 \geq k_0$ with k_1 possibly equal to ∞). If D_∞ is reached, the state x^d_k of (19) stays inside $\mathcal{B} \subseteq \Omega$ for all $k > k_1$ by definition of D_∞. Hence, on the discrete-time interval $[k_0, k_0+1, \ldots, k_1)$ the state of system (19) follows the motion of (27) and hence, the inheritance of the UB property follows. □

Remark 1: The theorem also holds for $p_{\max} = \infty$. However, the use of theorem in practice is lost due to the infinite character of the piecewise linear system.

Note that the larger p is, the more event times we are not updating the control value and thus we are not using the CPU for performing control computations. So, the larger $p_{\max}$ the more we can potentially save on processor load, but the more complex (the more regions) the resulting PWL model will be for the performance analysis. Advantageously, the computation of the ultimate bounds is performed off-line.

7.2.2 Finite PWL representations. In this section we present a sufficient condition that guarantees the existence of a finite PWL representation (27) (i.e., the existence of a finite $p_{\max}$ such that (26) holds).

Theorem 4: *Consider system* (19) *with* $\mathcal{B}$ *a set that satisfies* $0 \in \operatorname{int} \mathcal{B}$. *Assume that all the eigenvalues of the matrix* A *lie outside the closed unit circle of the complex half plane and* $A + BF$ *does not have an eigenvalue* 1 (*which is typically the case as* $A + BF$ *is chosen such that all eigenvalues are inside the open unit circle*). *Then* (26) *holds for a finite* $p_{\max}$ *and* $D_\infty = \emptyset$.

Proof: We need two algebraic results in the proof, that will be established next.

- Since A has all its eigenvalues outside the closed unit circle, A^{-1} is Schur (i.e., all eigenvalues inside the unit circle). This implies that there is a positive definite and symmetric matrix P (denoted by $P > 0$) such that $(A^{-1})^T P A^{-1} - P < 0$. Premultiplying the latter inequality by A^T and postmultiplying by A and noting that A is invertible yields $P - A^T P A < 0$. Hence, there exists a matrix $P > 0$ and a $\gamma > 1$ such that

$$A^T P A > \gamma P. \tag{28}$$

- Next we prove that the matrix $(A + BF) - (I - A)^{-1} BF$ is invertible. Suppose that $(A + BF)z = (I - A)^{-1} BFz$. This implies $A(I - A - BF)z = 0$. Since A is invertible this yields $(A + BF)z = z$. As $A + BF$ does not have an eigenvalue 1, this give $z = 0$ and hence, the invertibility of $(A + BF) - (I - A)^{-1}\, BF$ is proven.

To finish the proof, we recall that $p_{\max}$ is the maximal number of discrete steps that the system (19) can stay inside $\mathcal{B}$ after entering it from outside $\mathcal{B}$. Let x_0 be the last state outside $\mathcal{B}$ and $x_1 := (A + BF)x_0$ the first state inside $\mathcal{B}$. Inside $\mathcal{B}$ the state is governed by

$$x_{k+1} = Ax_k + BFx_0. \tag{29}$$

as the input is held at the value Fx_0. For shortness of notation, we omit superscript d here, the system (29) has an (unstable) equilibrium at $x_{eq} := (I-A)^{-1}BFx_0$. If we define $\Delta x_k := x_k - x_{eq}$, $k = 1, 2, \ldots$, then we can observe that $\Delta x_{k+1} = A\Delta x_k$. Together with (28) this yields $\Delta x_k^T P \Delta x_k > \gamma^{k-1}\Delta x_1^T P \Delta x_1$. The latter inequality indicates that x_k will move arbitrarily far away from $x_1 \in \mathcal{B}$ for sufficiently large k. Indeed,

$$\begin{aligned}(x_k - x_1)^T P(x_k - x_1) &\\ = (\Delta x_k - \Delta x_1)^T P(\Delta x_k - \Delta x_1) &= \|P^{1/2}(\Delta x_k - \Delta x_1)\|^2\\ \geq (\|P^{1/2}\Delta x_k\| - \|P^{1/2}\Delta x_1\|)^2 &> \left(\sqrt{\gamma^{k-1}} - 1\right)^2 \|P^{1/2}\Delta x_1\|^2.\end{aligned} \tag{30}$$

Note that $\Delta x_1 = [(A+BF) - (I-A)^{-1}BF]x_0$. Since $\mathcal{B}$ contains 0 in its interior, $(A+BF) - (I-A)^{-1}BF$ is invertible and P is positive definite, there exists a $\mu > 0$ such that for all $x_0 \notin B$ we have $\Delta x_1^T P \Delta x_1 \geq \mu$. Hence, using (30) we obtain that $(x_k - x_1)^T P(x_k - x_1) \geq (\sqrt{\gamma^{k-1}} - 1)^2 \mu$ for all $k = 1, 2, \ldots$ (as long as $x_k \in \mathcal{B}$). Since $\mathcal{B}$ is bounded the expression $\delta := \sup\{(x-y)^T P(x-y) \mid x \in \mathcal{B}, y \in \mathcal{B}\}$ is finite. Hence, if k is large enough to satisfy $(\sqrt{\gamma^{k-1}} - 1)^2\mu > \delta$, it follows that x_k must be outside $\mathcal{B}$ (as x_1 lies inside $\mathcal{B}$). This completes the proof.

An upper bound on $p_{\max}$ follows from the proof above.

Corollary 1: *Consider system* (19) *with $\mathcal{B}$ a set that satisfies $0 \in \operatorname{int}\mathcal{B}$. Assume that all the eigenvalues of the matrix A lie outside the closed unit circle of the complex half plane and $A + BF$ does not have an eigenvalue* 1.

- *Let $P > 0$ be a solution to $A^T PA > \gamma P$ for a $\gamma > 1$, which is known to exist.*
- *$\mu := \min_{z \notin \mathcal{B}} z^T[(A+BF) - (I-A)^{-1}BF]^T P[(A+BF) - (I-A)^{-1}BF]z > 0$.*
- *$\delta := \sup\{(x-y)^T P(x-y) \mid x \in \mathcal{B}, y \in \mathcal{B}\}$.*

Let $k_{\min}$ be the smallest integer k that satisfies $(\sqrt{\gamma^{k-1}} - 1)^2\mu > \delta$. Then $p_{\max} \leq k_{\min}$.

7.2.3 The perturbed case. In this subsection we briefly indicate how the derivation given above needs to be modified in order to include additive disturbance in the event-driven system (12) and (14). At the control check times the trajectory of the system (12) and (14) is described by the discrete-time system

$$\left.\begin{aligned} x_{k+1}^d &= Ax_k^d + Bu_k^d + w_k^d\\ u_k^d &= \begin{cases} Fx_k^d & \text{if } x_k^d \notin \mathcal{B}\\ u_{k-1}^d & \text{if } x_k^d \in \mathcal{B}\end{cases}\end{aligned}\right\} \tag{31}$$

for some realisation of the disturbance $w_k^d \in \mathcal{W}_d$, $k = 0, 1, 2, \ldots$. In this case we will also compute a PWL system, but now the mappings h_p will depend not only on the state x_k^d but also on the disturbance sequence $(w_k^d, w_{k+1}^d, \ldots, w_{k+p}^d)$. Suppose the state trajectory stays p steps in $\mathcal{B}$ before leaving $\mathcal{B}$ again (i.e., $x_k \notin \mathcal{B}$, then $x_{k+1}^d \in \mathcal{B}$, $x_{k+2}^d \in \mathcal{B}, \ldots, x_{k+p}^d \in \mathcal{B}$ and then $x_{k+p+1}^d \notin \mathcal{B}$). We obtain the function h_p that maps x_k^d to x_{k+p+1}^d similarly as in § 7.2.1

$$\begin{aligned} x_{k+p+1}^d &= h_p(x_k^d, w_{k+p}^d, \ldots, w_k^d)\\ &= Ah_{p-1}(x_k^d, w_{k+p-1}^d, \ldots, w_k^d) + BFx_k^d + w_{k+p}^d\\ &= \{A^{p+1} + [A^p + A^{p-1} + A^{p-2} + \cdots + I]BF\}x_k^d\\ &\quad + [A^p w_k^d + A^{p-1}w_{k+1}^d + \cdots + w_{k+p}^d]. \end{aligned} \tag{32}$$

One can observe that the dynamics depend on different sizes of the disturbance sequence

$$(w_k^d, w_{k+1}^d, \ldots, w_{k+p}^d) \in \underbrace{\mathcal{W}_d \times \cdots \times \mathcal{W}_d}_{p+1 \text{ times}} =: \mathcal{W}_d^{p+1}.$$

In this sense we could describe the system by using an "embedding" in the product space $\mathbb{R}^n \times l_\infty^n$, where l_∞^n denotes the space of (infinite) sequences $(w_0^d, w_1^d, w_2^d, \ldots)$ that are bounded in the sense that $\sup_{k\in\mathbb{N}} \|w_k^d\| < \infty$. Indeed, all the maps h_p can be reconsidered as having arguments in $\mathbb{R}^n \times l_\infty^n$ by defining for $p = 0, 1, 2, \ldots$

$$x_{k+p}^d = H_p(x_k^d, \mathbf{w}_k^d) = h_p(x_k^d, w_{k+p}^d, \ldots, w_k^d), \tag{33}$$

for $(x_k^d, \mathbf{w}_k^d) \in \mathbb{R}^n \times l_\infty^n$, where $\mathbf{w}_k^d = (w_k^d, \ldots, w_{k+p-1}^d, w_{k+p}^d, \ldots)$. Each map H_p is valid on regions D_p that can be determined as

$$\begin{aligned} D_p := \{(x_k^d, \mathbf{w}_k^d) \in \mathcal{B}^c \times \mathcal{W}_d^\infty \mid H_j(x_k^d, \mathbf{w}_k^d) \in \mathcal{B} \quad &\text{for}\\ j = 0, 1, \ldots, p-1 \text{ and } H_p(x_k^d, \mathbf{w}_k^d) \notin \mathcal{B}\}.& \end{aligned} \tag{34}$$

In a similar manner as for the unperturbed case, we also define the set of states and disturbance sequences that remain inside $\mathcal{B}$ forever after entering it from outside $\mathcal{B}$

$$\begin{aligned} D_\infty := \{(x_k^d, \mathbf{w}_k^d) \in \mathcal{B}^c \times \mathcal{W}_d^\infty \mid H_j(x_k^d, \mathbf{w}_k^d) \in \mathcal{B} \text{ for all}&\\ j = 0, 1, 2, \ldots\}.& \end{aligned} \tag{35}$$

Note that $D_i \cap D_j = \emptyset$ if $i \neq j$. Moreover, observe that in this case the 'switching' of the dynamics is dependent on the disturbance input as well and not solely on the state as in the unperturbed case.

Finally, we introduce the set $R_{\mathcal{B}}$ which contains all possible values $(x_k^d, \mathbf{w}_k^d)$ in $\mathcal{B}^c \times l_\infty^n$ for which $\mathcal{B}$ is reached within one discrete time-step

$$R_{\mathcal{B}} := \{(x_k^d, \mathbf{w}_k^d) \in \mathcal{B}^c \times \mathcal{W}_d^\infty \mid H_0(x_k^d, \mathbf{w}_k^d) \in \mathcal{B}\}.$$

Similarly to the unperturbed case, (25) holds. However, it does not hold in $\mathbb{R}^n$, but in the embedding space $\mathbb{R}^n \times \mathcal{W}_d^\infty$. Moreover, to obtain a finite representation of the PWL system, we need the existence of a $p_{\max}$ such that (26) holds, where $p_{\max}$ is the maximal (finite) number of discrete steps that the system (19) can stay inside $\mathcal{B}$ after entering it from outside $\mathcal{B}$ (for a particular disturbance realisation). In the perturbed case, there are two reasons for the "infinite representation" of the PWL system. First of all the number of regions can be infinite (as in the unperturbed case), but also the length of the disturbances sequence determining the update from x_k^d to x_{k+p+1}^d can be infinite. Hence, the existence of a finite $p_{\max}$ leads on one hand to a finite number of regions of the PWL system and on the other implies that the infinitely dimensional space $\mathbb{R}^n \times l_\infty^n$ can be replaced by $\mathbb{R}^n \times (\mathbb{R}^n)^{p_{\max}+1}$. Indeed, if we abstract away from the motion inside $\mathcal{B}$ and replace the iteration parameter k corresponding to the control check times by the new discrete-time variable l corresponding to the control update times, we obtain the PWL system $x_{l+1}^d = f_{\mathrm{PWL}}(x_l^d, \mathbf{w}_l^d)$ with

$$x_{l+1}^d = \begin{cases} H_p(x_l^d, \mathbf{w}_l^d), & \text{when } (x_l^d, \mathbf{w}_l^d) \in D_p,\ p = 0, 1, \ldots, p_{\max} \\ 0, & \text{when } (x_l^d, \mathbf{w}_l^d) \notin \bigcup_{p=0}^{p_{\max}} D_p \end{cases} \tag{36}$$

with $\mathbf{w}_l^d \in \mathcal{W}_d^{p_{\max}+1}$. Note that there is a slight abuse of notation in (36) as we replaced $\mathbf{w}_k^d \in \mathcal{W}_d^\infty$ by $\mathbf{w}_l^d \in \mathcal{W}_d^{p_{\max}+1}$ in both H_p and the representations of the sets D_p.

A similar result as Theorem 3 can be derived in this case as well.

8. Including intersample behaviour

The above results only provide statements on the control update or control check times. The behaviour of the system in between these control check/update times is not characterised. However, since at the control check/update times we obtain UB to a set Ω, we know that the state trajectories enter Ω in finite time. Using this observation, an ultimate bound including the intersample behaviour of (12) together with (13) or (14) can be computed from

$$\begin{aligned} x_{x_0,w}(t) - x_{x_0,w}(\tau_k) = {} & [e^{A_c(t-\tau_k)} - I]x_{x_0,w}(\tau_k) \\ & + \int_{\tau_k}^t e^{A_c(t-\theta)} B_c u(\tau_k) \mathrm{d}\theta \\ & + \int_{\tau_k}^t e^{A_c(t-\theta)} E_c w(\theta) \mathrm{d}\theta, \end{aligned} \tag{37}$$

where $t \in [\tau_k, \tau_{k+1})$.

In the non-uniform case we either have $x_{x_0,w}(t) \in \mathcal{B}$ or $t - \tau_k < T_s$ in (37). In the latter case using the boundedness of $\mathcal{W}_c$ we can easily see that

$$\begin{aligned} \|x_{x_0,w}(t) - x_{x_0,w}(\tau_k)\| \le {} & CT_s(\|x_{x_0,w}(\tau_k)\| + 1 \\ & + \|F\| \|x_{x_0,w}(\tau_k)\|) \end{aligned} \tag{38}$$

for all $T_s \in [0, T_s^{\max}]$. The constant $C = C(A_c, B_c, E_c, T_s^{\max}, \mathcal{W}_c)$ depends on the system parameters, A_c, B_c, E_c, $\mathcal{W}_c$ and $T_s^{\max}$. Hence, if the system (15) is UB to a RPI set Ω with $\mathrm{cl}\mathcal{B} \subseteq \Omega$ (as in Theorem 1), then the event-driven system (12)–(13) is UB to the set $\tilde{\Omega} := \Omega \oplus B(0, \varepsilon)$ with $\varepsilon := \sup_{x \in \Omega} CT_s(\|x\| + 1 + \|F\|\|x\|)$ and $B(0, \varepsilon) := \{x \in \mathbb{R}^n \mid \|x\| \le \varepsilon\}$.

In the uniform case a similar bound holds as in (38) with the minor modification

$$\begin{aligned} \|x_{x_0,w}(t) - x_{x_0,w}(jT_s)\| \le {} & CT_s(\|x_{x_0,w}(jT_s)\| + 1 \\ & + \|F\| \|x_{x_0,w}(\tau_k)\|), \end{aligned} \tag{39}$$

where jT_s and τ_k are the largest control check time and largest control update time smaller than t, respectively. Two situations can occur: (i) $\mathcal{S}(x_0,w)$ is a finite set or (ii) $\mathcal{S}(x_0,w)$ is an infinite set. In the latter case both $x_{x_0,w}(jT_s)$ and $x_{x_0,w}(\tau_k)$ lie ultimately in Ω, which yields a similar bound as in the uniform case.

In the former case, there is a $\bar{k}$ such that $x_{x_0,w}(kT_s) \in \mathcal{B}$ for $k \ge \bar{k}$, but it might be the case that $x_{x_0,w}(\tau_k)$ is outside Ω (it is just the last state at which a control update was performed). The only information on $x_{x_0,w}(\tau_k)$ is that it lies in $R_{\mathcal{B}}$ (in the unperturbed case) or there exists a $\mathbf{w}_k^d \in \mathcal{W}_d^\infty$ such that $(x_{x_0,w}(\tau_k), \mathbf{w}_k^d) \in R_{\mathcal{B}}$ (the perturbed case). In case that $A + BF$ is invertible, the set $R_{\mathcal{B}}$ is bounded, which gives a bound on $\|x_{x_0,w}(\tau_k)\|$. Hence, using (39) a bound on the intersample behaviour can be derived. Note that $D_\infty = \emptyset$ implies that this case is actually absent. Hence, if the conditions of Theorem 4 are fulfilled, then $\mathcal{S}(x_0)$ is infinite for all $x_0 \notin \mathcal{B}$ and this situation does not have to be considered.

Alternatively, in case there are physical reasons that the control inputs are restricted to a bounded set or if the method outlined in § 7.1 is used in which also ultimate boundedness of the u-variables is proven, a bound like $CT_s(\|x_k\| + 1)$ can be directly computed independent of the designed controller gain F.

9. Existence and computational aspects of the ultimate bounds

In this section we will provide conditions that guarantee the existence of bounded ultimate bounds for the event-driven systems at hand. Moreover, we will also present techniques that can be used to compute the ultimate bounds.

9.1 *Existence of ultimate bounds that are bounded*

As already remarked before, we assume that the pair (A_c, B_c) is Hurwitz stabilizable in (12), which implies that for almost all values of $T_s > 0$ the pair (A, B) as in (16) is Schur stabilizable as discussed in Excercise 3.20 in Trentelman *et al.* (2001). As such, we assume that $T_s > 0$ is chosen such that (A, B) is Schur stabilizable and the control gain $F \in \mathbb{R}^{m\times n}$ is such that $A+BF$ is Schur. When $\mathcal{W}_c$ is a bounded set, we also have that $\mathcal{W}_d$ is a bounded set. We will start by considering the non-uniform case and we will define the minimal robustly positive invariant (mRPI) set for the discrete-time linear system in (15).

Definition 9.1 (Minimal Robustly Positively Invariant Set $\mathcal{F}_\infty$)**:** The set $\mathcal{F}_\infty$ is the minimal robustly positively invariant set of the discrete-time linear system (15) with disturbances in $\mathcal{W}_d$, if the following statements hold.

(i) $0 \in \mathcal{F}_\infty$;
(ii) $\mathcal{F}_\infty$ is robustly positively invariant of (15) with disturbances in $\mathcal{W}_d$,
(iii) any other robustly positively invariant set $\mathcal{F}$ of (15) with disturbances in $\mathcal{W}_d$ satisfies $\mathcal{F}_\infty \subseteq \mathcal{F}$.

It is shown in Kolmanovsky and Gilbert (1998) that the mRPI set $\mathcal{F}_\infty$ of the system (15) with disturbances in $\mathcal{W}_d$ is bounded in case $A+BF$ is Schur and $\mathcal{W}_d$ is a bounded set. Moreover, it is well-known Blanchini (1994) that $\mu\mathcal{F}_\infty$ for $\mu > 1$ forms also a RPI set for the system (15) with disturbances in $\mathcal{W}_d$ and is even an ultimate bound if $A+BF$ is Schur. Selecting $\mu > 1$ large enough such that $\mathrm{cl}\mathcal{B} \subseteq \mu\mathcal{F}_\infty$ gives an ultimate bound for (15) satisfying the conditions of Theorem 6.1. Hence, using the results in Section 8 on the intersample behaviour, this reasoning shows that a *bounded* ultimate bound for the event-driven system (12)–(13) with the non-uniform mechanism exists when $A+BF$ is Schur, $\mathcal{B}$ bounded and $\mathcal{W}_c$ bounded.

The uniform case is a bit more complicated as illustrated by the following example.

Example 9.2: Consider the continuous-time system with scalar state given by

$$\dot{x}(t) = x(t) + \frac{1}{e-1}u(t), \tag{40}$$

where e denotes Euler's number being approximately 2.718. When we use a sample time $T_s = 1$ its discretized version is given by

$$x^d_{k+1} = ex^d_k + u^d_k. \tag{41}$$

The event-driven controller using the uniform mechanism is selected as

$$u^d_k = \begin{cases} -ex^d_k, & \text{when } |x_k| \ \geq \varepsilon, \\ u^d_{k-1}, & \text{when } |x_k| \ < \varepsilon. \end{cases} \tag{42}$$

Hence, $A=e$, $B=1$ and $F=-e$. Consequently, $A+BF=0$. If we now take initial condition $x^d_0 = M \geq \varepsilon$, then it is not hard to see that

$$u^d_k = \begin{cases} (-e)^{1+(k/2)}M, & \text{when } k \text{ is even} \\ (-e)^{1+(k-1)/2}M, & \text{when } k \text{ is odd;} \end{cases}$$

$$x^d_k = \begin{cases} (-e)^{k/2}M, & \text{when } k \text{ is even} \\ 0 \in \mathcal{B}, & \text{when } k \text{ is odd.} \end{cases}$$

Since M was arbitrary this implies that the event-driven system does not have a *bounded* ultimate bound and becomes even unstable although the corresponding discrete-time system $x^d_{k+1} = (A+BF)x^d_k = 0$ is asymptotically stable. Note that in case the non-uniform event triggering mechanism is used, the set $\mathrm{cl}\mathcal{B} = [-\varepsilon, \varepsilon]$ forms an ultimate bound.

The reason that the boundedness of $\mathcal{B}$ and $\mathcal{W}_d$ together with $A+BF$ being Schur is not sufficient for the existence of a *bounded* ultimate bound, is that $\mathcal{R}_\mathcal{B}$ might not be bounded. Hence, one might reach $0 \in \mathcal{B}$ from a point x^d_k arbitrarily far away from the set $\mathcal{B}$ using a large control value $u^d_k = Fx^d_k$. Since the control value will be held at the next step, $x^d_{k+1} = Fx^d_k$ is also very large again. This might lead to undesirable phenomena such as an instability or limit cycles and (thus) the absence of a *bounded* ultimate bound. However, if $A+BF$ is invertible, the set $R_\mathcal{B}$ is bounded and the following result can be derived.

Theorem 9.3: *Consider system* (12) *with a bounded disturbance set* $\mathcal{W}_c$ *and* $\mathcal{B}$ *is a bounded and open set containing the origin. Assume that* $A+BF$ *is Schur. Then the following statements hold.*

- *The event-driven system* (12) *using the non-uniform mechanism* (13) *is UB to some bounded set* Ω.
- *If in addition* $A+BF$ *is invertible, then the event-driven system* (12) *using the uniform mechanism* (14) *is UB to some bounded set* Ω.

Guaranteeing the existence of bounded ultimate bounds is certainly of interest. Additionally, it would be beneficial to have computational means to construct such ultimate bounds. This will be the topic of the next sections, after which we will discuss in § 10 how they depend on the set $\mathcal{B}$.

9.2 *Computational aspects for the non-uniform case*

In Theorem 1 it was shown that properties of robust invariance of sets and UB for the discrete-time linear system (15) carry over to the event-driven system (12)–(13) on the control update times. As such it is of importance to be able to compute RPI sets and UB for discrete-time linear systems. We already introduce the mRPI set in Definition 5, which is one useful RPI set for which computational algorithms are available. For instance, the forward algorithm of Kolmanovsky and Gilbert (1998) can be used to compute $\mathcal{F}_\infty$. If $\mathcal{W}_d$ contains the origin in its interior, then it is even known that the algorithm terminates in finite time when $A+BF$ is chosen to be Schur.

Besides the forward algorithm to find $\mathcal{F}_\infty$, there are various other ways to compute RPI sets for discrete-time linear systems (e.g., Blanchini (1994, 1999), Kolmanovsky and Gilbert (1998), Kerrigan (2000), Rakovic *et al.* (2005)). We will present here one approach based on ellipsoidal sets as in Kolmanovsky and Gilbert (1998). To use the ellipsoidal approach of Kolmanovsky and Gilbert (1998), we assume that $\mathcal{W}_d$ is included in an ellipsoid of the form $\mathcal{E}_{R^{-1}} := \{w \mid w^T R^{-1} w \le 1\}$ for a matrix $R > 0$. Techniques to find such an over-approximation are given in Boyd *et al.* (1994).

Along the lines of Kolmanovsky and Gilbert (1998) it can be shown that feasibility of

$$P - \gamma^{-1} A_{cl} P A_{cl}^T - (1-\gamma)^{-1} R > 0 \quad \text{and} \quad P > 0 \tag{43}$$

for some $\gamma \in (0,1)$ yields (using Schur complements and suitable pre- and postmultiplications) that

$$(A_{cl}x + w)^T P^{-1} (A_{cl}x + w) < \gamma x^T P^{-1} x + (1-\gamma) w^T R^{-1} w.$$

From this it is easily seen that $x^T P^{-1} x \le 1$ and $w^T R^{-1} w \le 1$ imply $(A_{cl}x + w)^T P^{-1}(A_{cl}x + w) \le 1$. This shows that $\Omega = \{x \mid x^T P^{-1} x \le 1$ is a RPI set for (15). By suitable scaling such that $\mathrm{cl}\mathcal{B} \subseteq \mu\Omega$ for $\mu > 1$ again an ultimate bound is obtained for the event-driven system (12)–(13) on the control update times.

9.3 *Computational aspects for the uniform case*

Also for PWL systems several ways to compute invariant sets are available (Kvasnica *et al.* 2004, Rakovic *et al.* 2004).

For the higher-order bimodal PWL system (18), we observe that in the first mode the x-evolution is given by $x_{k+1}^d = (A+BF)x_k^d + w_k^d$. This means that when $x_0^d \notin \mathcal{B}$ the corresponding trajectory will eventually satisfy $x_{k+1}^d \in \mu\mathcal{F}_\infty$ for any $\mu > 1$ with $\mathrm{cl}\mathcal{B} \subseteq \mu\mathcal{F}_\infty$, where $\mathcal{F}_\infty$ denotes the mRPI set containing 0 for $x_{k+1}^d = (A+BF)x_k^d + w_k^d$ and disturbances in $\mathcal{W}_d$. Hence, any state trajectory x^d for the bimodal PWL system (18) with an initial state $x_0^d \notin \mathcal{B}$ reaches the set

$$\left\{ \begin{pmatrix} (A+BF)x + w \\ Fx \end{pmatrix} \mid \text{for some } x \text{ and } w \text{ that satisfy} \right.$$
$$\left. (A+BF)x + w \in \mu\mathcal{F}_\infty \text{ and } w \in \mathcal{W}_d \right\}. \tag{44}$$

at some point. When $x_0^d \in \mathcal{B}$, then the update of system (18) gives

$$\begin{pmatrix} x_1^d \\ u_0^d \end{pmatrix} = \begin{pmatrix} (A+BF)x_0^d + w_0^d \\ Fx_0^d \end{pmatrix},$$

which also lies in the set defined in (44). This means that all state trajectories of (18) reach the set in (44). Hence, if one constructs a RPI set Γ for the system (18) containing the set in (44), then $\Pi_n(\Gamma)$ is an ultimate bound for the PWL system (18) and according to Theorem 2 also for the event-driven system (12) and (14) on the control check times.

In the above reasoning, one can actually replace the set $\mathcal{F}_\infty$ by any other RPI set for the linear system $x_{k+1}^d = (A+BF)x_k^d + w_k^d$ with disturbances in $\mathcal{W}_d$ that contains 0 (e.g., based on ellipsoidal sets as in the previous section).

In case of the lower-order PWL model we will present an approach based on ellipsoidal sets although techniques using reachability analysis Kvasnica *et al.* 2004) can be exploited as well. Actually, the example in § 11.2 uses both the ellipsoidal and the reachability approach for illustration purposes.

Theorem 6: *Consider the event-driven system* (12) *and* (14) *without disturbances* (*i.e.*, $\mathcal{W}_c = \{0\}$) and $\mathcal{B}$ *an open set containing the origin. Let* $P > 0$ *be a solution to* $A_{cl}^T P A_{cl} - P < 0$. *Take* α^* *small such that* $\alpha^* \ge \max_{p \in \{1,\dots,p_{\max}\}} \sup\{x^T P x \mid x \in h_p(D_p)\}$ *and* $\alpha^* \ge \max\{x^T P x \mid x \in \mathrm{cl}\mathcal{B}\}$, *where* $h_p(D_p)$ *denotes the image of the map* h_p *with its arguments in* D_p. *Define the set* $\Omega(\alpha^*) := \{x \in \mathbb{R}^n \mid x^T P x \le \alpha^*\}$. *Then the PWL system* (27) *is UB to the positively invariant set* $\Omega(\alpha^*)$ *and* $\mathrm{cl}\mathcal{B} \subseteq \Omega(\alpha^*)$ *and consequently the event-driven system* (12) *and* (14) *on the control check times is also UB to the set* $\Omega(\alpha^*)$.

For brevity we omitted the proof.

10. Tuning of the controller

In this section we indicate how the ultimate bound Ω depends on $\mathcal{B}$, thereby facilitating the selection of desirable ultimate bounds by tuning $\mathcal{B}$. We will present here results for the non-uniform case (with disturbances) and the uniform case without disturbances.

10.1 *Non-uniform sampling*

The following result can be inferred from Blanchini (1994).

Theorem 7: *Consider the system* (12)–(13) *with* W_c *a closed, convex set containing* 0, F *and* $T_s > 0$ *given and* $\mathcal{B}$ *an open set containing the origin. Let* W_d *be given by* (17).

- *If* Ω *is a RPI set for the discrete-time linear system* (15) *with disturbances in* $\mathcal{W}_d$ *containing* cl$\mathcal{B}$, *then for any* $\mu \geq 1$ $\mu\Omega$ *is a RPI set for* (15) *with disturbances in* $\mathcal{W}_d$ *containing* μcl$\mathcal{B}$.
- *If the discrete-time linear system* (15) *with disturbances in* $\mathcal{W}_d$ *is UB to* Ω *containing* cl$\mathcal{B}$, *then for any* $\mu \geq 1$ (15) *with disturbances in* $\mathcal{W}_d$ *is UB to* $\mu\Omega$ *containing* $\mu\mathcal{B}$.

This result shows that Ω scales "linearly" with $\mathcal{B}$ for scaling factors larger than one. Consider the minimal RPI set $\mathcal{F}_\infty$ containing 0. For small $\mathcal{B}$ this gives the ultimate bound for the event-driven system on the control update times as long as the chosen $\mathcal{B}$ lies inside $\mathcal{F}_\infty$. Strictly speaking, an ultimate bound is the set $\mu\mathcal{F}_\infty$ for any small $\mu > 1$ as $\mathcal{F}_\infty$ is only approached asymptotically by some trajectories of the discrete-time linear system (15). If $\mathcal{B}$ is taken larger and cl$\mathcal{B}$ is not contained in $\mathcal{F}_\infty$ anymore, the linear scaling effect as in Theorem 7 occurs. This effect is nicely demonstrated in the first example below.

For the tuning of the controller one typically selects the state feedback gain F such that $A + BF$ is Schur and guaranteeing suitable transient behaviour. Indeed, outside $\mathcal{B}$ the dynamics is given by the discrete-time linear system (15), which implies that the convergence towards the ultimate bound is determined by F. Selecting F such that $A + BF$ has desired eigenvalues, yields a desirable speed of convergence. If an ultimate bound Ω with cl$\mathcal{B} \subseteq \Omega$ is computed for a pre-selected $\mathcal{B}$, one tunes the size of the stabilization error $\mu\Omega$ by scaling $\mu\mathcal{B}$. However, a fundamental limitation is given by $\mathcal{F}_\infty$ as this is the error bound caused by the persistent disturbances when $\mathcal{B} = \{0\}$. One cannot go beyond this ultimate bound without changing F, although still some effect of the disturbance will remain present. However, in the unperturbed case any scaling factor holds for any $\mu > 0$ (as $\mathcal{F}_\infty = \{0\}$ in this case).

10.2 *Uniform sampling for the unperturbed case*

In this section we will consider the unperturbed case, i.e., $\mathcal{W}_c = \{0\}$.

Theorem 8: *Consider the system* (12) *and* (14) *with* $\mathcal{W}_c = \{0\}$, F *and* $T_s > 0$ *given and* $\mathcal{B}$ *an open set containing the origin. If the PWL system* (27) *corresponding to B is UB to the positively invariant set* Ω *and* $\mathcal{B} \subseteq \Omega$, *then for any* $\mu > 0$ *the PWL system* (27) *corresponding to* $\mu\mathcal{B}$ *is UB to the positively invariant set* $\mu\Omega$ *and* $\mu\mathcal{B} \subseteq \mu\Omega$.

Proof: In the proof we will indicate the dependence of f_{PWL}, h_p and D_p on the set $\mathcal{B}$ via superscripts, i.e., $f^{\mathcal{B}}_{\text{PWL}}$, $h^{\mathcal{B}}_p$ and $D^{\mathcal{B}}_p$, respectively. Let $\mu > 0$. The mappings $h^{\mathcal{B}}_p$ do not depend on $\mathcal{B}$, only on p, the number of discrete-time steps the control value is held. Hence, $h^{\mu\mathcal{B}}_p = h^{\mathcal{B}}_p$. This yields together with the linearity of the mappings that $D^{\mu\mathcal{B}}_p = \mu D^{\mathcal{B}}_p$ and $D^{\mu\mathcal{B}}_\infty = \mu D^{\mathcal{B}}_\infty$. Hence, $f^{\mu\mathcal{B}}_{\text{PWL}}(\mu x) = \mu f^{\mathcal{B}}_{\text{PWL}}(x)$. Indeed, if $x \in D^{\mathcal{B}}_p$, then $\mu x \in \mu D^{\mathcal{B}}_p = D^{\mu\mathcal{B}}_p$. As a consequence, it holds that $f^{\mu\mathcal{B}}_{\text{PWL}}(\mu x) = h^{\mu\mathcal{B}}_p(\mu x) = \mu h^{\mathcal{B}}_p(x) = \mu f^{\mathcal{B}}_{\text{PWL}}(x)$. The same reasoning can be applied to $x \in \mathcal{B}$ and $x \in D^{\mathcal{B}}_\infty$. If we denote the state trajectory $x^{d,x_0,\mathcal{B}}$ of the system (27) corresponding to $\mathcal{B}$ from initial state x_0, then we obtain the relation $x^{d,\mu x_0,\mu\mathcal{B}} = \mu x^{d,x_0,\mathcal{B}}$. From the latter relationship, the result in the theorem follows. □

This theorem gives a means, similarly to the non-uniform case, to tune the ultimate bound by suitably selecting the event-triggering mechanism parameterised by $\mathcal{B}$. Scaling $\mathcal{B}$ with a constant $\mu > 0$ leads to an ultimate error bound that is μ times the bound belonging to $\mathcal{B}$. Note that due to the absence of perturbations, we can scale $\mathcal{B}$ with any $\mu > 0$ instead of only $\mu > 1$.

Analogous results can be derived for the bimodal PWL system (18) without disturbances.

11. Examples

11.1 *Non-uniform sampling*

To illustrate the theory in case of non-uniform sampling (13) we will use the example (1) of §2 with $F = -0.45$. Note that in the introduction we used uniform sampling. In figure 5 the ratio of the number of control updates in comparison to the case where the updates are performed each sample time (i.e., $u^d_k = -0.45x^d_k$ for all x^d_k) and the maximal value of the state variable (after transients) $x_{\max} := \limsup_{t\to\infty} |x(t)|$ (the "minimal ultimate bound"), respectively, versus the parameter e_T are displayed, where $\mathcal{B} = \{x \in \mathbb{R} \mid |x| < e_T\}$.

The figure of the ultimate bounds can nicely be derived from the theory. First, we compute for the system (1), the discretised version (15) with sample time $T_s = 0.1$

$$x^d_{k+1} = 1.051x^d_k + 1.025u^d_k + w^d_k; \quad u^d_k = -0.45x^d_k \quad (45)$$

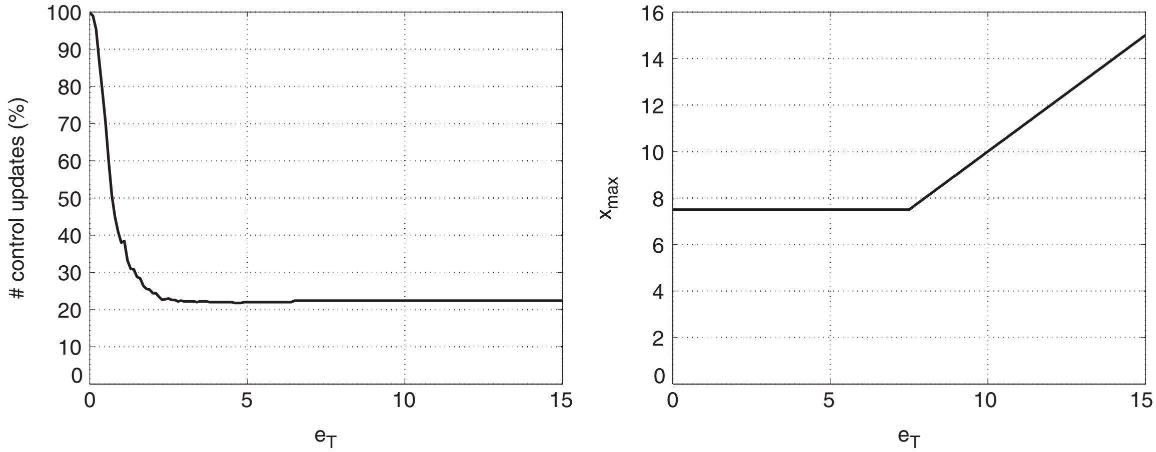

Figure 5. The control effort and $x_{\max}$ versus e_T for the example of § 11.1.

or

$$x_{k+1}^d = 0.590 x_k^d + w_k^d \tag{46}$$

with $-3.076 \le w_k^d \le 3.076$, i.e., $\mathcal{W}_c = [-10, 10]$ and $\mathcal{W}_d = [-3.076, 3.076]$. The minimal RPI set $\mathcal{F}_\infty$ for (46) containing $\{0\}$ is equal to the "ellipsoid" $[-7.50, 7.50]$. Hence, note that as long as $e_T < 7.50$ the ultimate bound of the system (12)–(13) is equal to $\mathcal{F}_\infty$ (or strictly speaking to the set $\mu\mathcal{F}_\infty$ for a small $\mu > 1$ as discussed in § 10.1). This explains the constant line in the $x_{\max}$ versus e_T plot in figure 5 up to $e_T = 7.50$. At the moment e_T becomes larger than 7.50, the condition of Theorem 1 that $\mathrm{cl}\mathcal{B} \subseteq \mathcal{F}_\infty$ does no longer hold. However, we can now use the "scaling effect" from Theorem 7. Theorem 7 implies that $(e_T/7.50)\mathcal{F}_\infty$ is RPI and the linear system (46) is UB to $(e_T/7.50)\mathcal{F}_\infty$ when $e_T > 7.50$. Since $\mathrm{cl}\mathcal{B} \subseteq (e_T/7.50)\mathcal{F}_\infty$ holds, Theorem 1 implies that $(e_T/7.50)\mathcal{F}_\infty$ is RPI for (46) and the event-driven system (12)–(13) is UB to $(e_T/7.50)\mathcal{F}_\infty$. This explains the linear part in the $x_{\max}$ versus e_T plot in figure 5. Hence, we can reduce the number of control updates with almost 80% in this set-up without reducing the control accuracy (e.g., take $e_T = 5$)!

11.2 *Uniform sampling*

To demonstrate the results of § 7.2.1 for uniform sampling, we have taken the example of an unstable system with two states ($n = 2$) given by (12) with

$$A_c = \begin{bmatrix} 1070 & 270 \\ 270 & 40 \end{bmatrix}; \quad B_c = \begin{bmatrix} 453 \\ 874 \end{bmatrix} \tag{47}$$

The controller matrix is taken to be $F = [-2.4604 - -0.2340]$. The matrices in the discrete-time version (15) are equal to

$$A = \begin{bmatrix} 3.00 & 0.50 \\ 0.50 & 1.10 \end{bmatrix} \quad B = \begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix} \tag{48}$$

for $T_s = 0.001$. Note that the the eigenvalues of $A_{cl} = A + BF$ are $0.7 \pm 0.7i$ and of A are 0.97 and 3.12. $\mathcal{B} = \{x \in \mathbb{R}^2 \mid |x_1| < e_T, |x_2| < e_T\}$ with $e_T = 6$. We computed $p_{\max}$ by continuously increasing p, calculate D_p and checking if $\bigcup_{i=1}^{p} = R_{\mathcal{B}}$ holds. This results in $p_{\max} = 3$ and thus $R_{\mathcal{B}} = \bigcup_{i=1}^{p_{\max}} D_i$. Note that this implies that $D_\infty = \emptyset$. Figure 6 displays the calculated sets $R_{\mathcal{B}}$ and D_p, $p = 1, 2, 3$ as given by equation (23).

The dynamics that are valid inside D_p, calculated with equation (22) are

$$\begin{aligned} h_0(x_l^d) &= \begin{bmatrix} 0.537 & 0.264 \\ -1.96 & 0.863 \end{bmatrix} x_l^d \\ h_1(x_l^d) &= \begin{bmatrix} -1.82 & 0.985 \\ -4.34 & 0.843 \end{bmatrix} x_l^d \\ h_2(x_l^d) &= \begin{bmatrix} -10.1 & 3.13 \\ -8.12 & 1.18 \end{bmatrix} x_l^d \\ h_3(x_l^d) &= \begin{bmatrix} -36.6 & 9.73 \\ -16.4 & 2.62 \end{bmatrix} x_l^d. \end{aligned} \tag{49}$$

As could be expected, the dynamics corresponding to h_0 is asymptotically stable. The dynamics corresponding to h_1, h_2 and h_3 are unstable.

Since we have obtained the PWL-description of the system we can apply the theory presented in § 9.3. Using the ellipsoidal approach as presented in Theorem 6 we obtain the ellipsoid Ω in figure 7, where

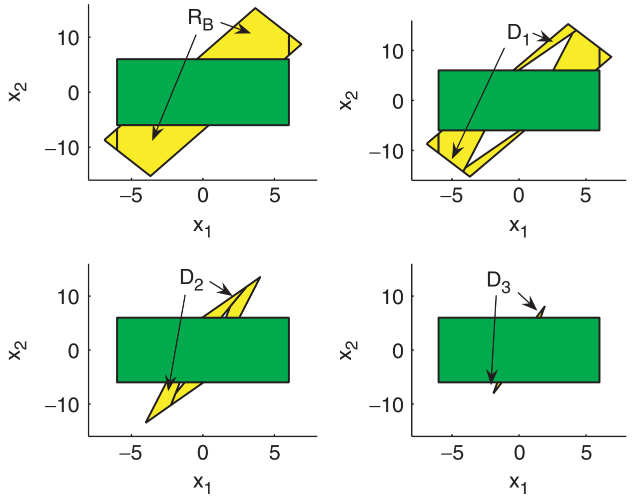

Figure 6. Sets $R_{\mathcal{B}}$, D_1, D_2, D_3 in yellow (light grey) and set $\mathcal{B}$ the green (dark grey) rectangle.

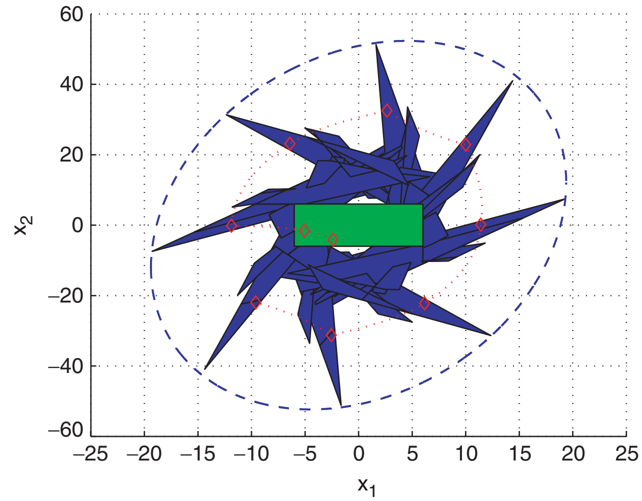

Figure 7. Ellipsoid Ω indicated by the dashed (blue) line, set $\Omega_{\min}$ in blue (dark grey) and set $\mathcal{B}$ in green (light grey).

we took the smallest value of α*. The event-driven system is UB to Ω on the control check times.We also computed the reachable set Ω_{reach} for the PWL system from points in $R_{\mathcal{B}}$. For the computation of this set a combination of tools from (Kerrigan 2000) and Kvasnica *et al*. (2004) was used. Note that Ω_{reach} is a positively invariant set for the PWL system. Since $\mathcal{B} \subseteq \Omega_{\text{reach}}$ and outside $\mathcal{B}$ the dynamics on the event times is equal to $x^d_{k+1} = A_{cl} x^d_k$, Ω_{reach} is also an ultimate bound for the event-driven system on the control check times.

Figure 7 also shows a time simulation of the continuous time system. A (red) dotted line shows the intersample behaviour in which the small (red) diamonds indicate the values at the control check times. It can be seen that the trajectory is not restricted to the depicted Ω_{reach} (in blue (dark grey)), due to the intersample behaviour. Bounds on the intersample behaviour can be obtained using the techniques as described in § 8. Note that if we would scale $\mathcal{B}$ to $\mu\mathcal{B}$ for a positive constant μ the sets Ω and Ω_{reach} would scale accordingly and hence, a desirable (arbitrarily small) stabilization error can be achieved in this case.

12. Conclusions

Although in many practical control problems it is natural and logical to use event-driven controllers, their application is still scarce in both industry and academia. A major reason why time-driven control still dominates is the absence of a system theory for event-driven systems. Due to the various benefits of event-driven control, it is worthwhile to overcome the difficulties in the analysis of this type of control.

This paper studies event-driven controllers with the purpose to reduce the required (average) processor load for the implementation of digital controllers. An introductory example already illustrated the achievable reduction of control computations (up to 80%). That this reduction of control computations indeed leads to a significantly lower processor load (in spite of the introduced overhead of the event-triggering mechanism) has been experimentally validated in Sandee *et al.* (2006). Of course, one still has to make the trade-off between this reduction in resource utilization on one hand and the control performance on the other. This paper provides the theory that gives insight in the control performance for a particular event-driven scheme. The control performance is expressed in terms of ultimate bounds and convergence rates to this bound. It is shown how these properties depend on the parameters of the control strategy. The results are based on inferring properties (like robust positive invariance, ultimate boundedness and convergence rates) for the event-driven controlled system from discrete-time linear systems (in case of non-uniform sampling) or piecewise linear systems (in case of uniform sampling). We presented computational means and tuning rules that support the design of these controllers.

This paper is one of the first that aims at deriving a formal analysis of event-driven control. Although it analyses a particular event-driven control structure, it already indicates the complexity and challenges for the analysis and synthesis of these type of control loops. Given the advantages of event-driven controllers and the various sources of event-triggering mechanisms present in industrial practice, it is fruitful to continue this line of research and developing a mature event-driven system theory. Future work will focus on the finite number of regions of the piecewise linear model, on tuning theory for the perturbed event-driven system with the uniform mechanism and on extending the current work to include reference tracking. From a broader perspective, we will consider also the analysis and synthesis of control schemes based on other event-triggering mechanisms like low resolution sensors as was initiated in Heemels *et al*. (1999).

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