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Research Article

Zhenhua Ma, Awais Asif, Hassen Aydi*, Sami Ullah Khan, and Muhammad Arshad

# Analysis of *F*-contractions in function weighted metric spaces with an application

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**Abstract:** In this work, we show that the existence of fixed points of $F$-contraction mappings in function weighted metric spaces can be ensured without third condition $(F3)$ imposed on Wardowski function $F:(0, \infty) \to \mathfrak{R}$. The present article investigates (common) fixed points of rational type $F$-contractions for single-valued mappings. The article employs Jleli and Samet's perspective of a new generalization of a metric space, known as a function weighted metric space. The article imposes the contractive condition locally on the closed ball, as well as, globally on the whole space. The study provides two examples in support of the results. The presented theorems reveal some important corollaries. Moreover, the findings further show the usefulness of fixed point theorems in dynamic programming, which is widely used in optimization and computer programming. Thus, the present study extends and generalizes related previous results in the literature in an empirical perspective.

**Keywords:** function weighted metric, fixed point, rational type $F$-contraction, Hardy-Rogers-type $F$-contraction

**MSC 2010:** 47H09, 47H10, 54H25

## 1 Introduction and preliminaries

Throughout this article, the notions $N$, $R$ and $R_+$ denote the set of natural numbers, real numbers and the set of positive real numbers, respectively. In recent years, various authors presented interesting generalizations of a metric space [1–7]. Among them, Jleli and Samet [8] introduced an interesting generalization of metric spaces, known as $F$-metric spaces, and proved its generality to a metric space with the help of concrete examples. They also compared the idea of $F$-metrics with $b$-metrics and $s$-relaxed metrics. A fixed point theorem of Banach contraction was established in the frame of $F$-metric spaces. For other related results, see [9]. In contrast, Wardowski [10] extended the Banach contraction principle to a more generalized form, known as $F$-contractions, and established a fixed point theorem in complete metric spaces. Klim and Wardowski [11] further discussed $F$-contractions in the frame of dynamic process and proved fixed point results of $F$-contractions for multivalued mappings. Along the same line, the class of $F$-contractions was further extended by various authors, see [12–25]. Also, numerous results on $F$-contractions have been proved, while imposing the contraction on the closed ball.

The present article relaxes the restrictions of the function $F$ by eliminating its third condition and we prove common fixed point results of both locally and globally rational type $F$-contractions in $F$-metric spaces. The article is organized in four sections. Section 1 contains a short history of the previous literature

*** Corresponding author: Hassen Aydi,** Nonlinear Analysis Research Group, Ton Duc Thang University, Ho Chi Minh City, Vietnam; Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam, e-mail: hassen.aydi@tdtu.edu.vn
**Zhenhua Ma:** Hebei University of Architecture, Zhangjiakou, 075024, P. R. China, e-mail: mazhenghua_1981@163.com
**Awais Asif:** Department of Math and Stats, International Islamic University Islamabad, Pakistan, e-mail: awais.phdma69@iiu.edu.pk
**Sami Ullah Khan:** Department of Mathematics, Gomal University, Dera Ismail Khan, 29050, KPK, Pakistan, e-mail: gomal85@gmail.com, dr.sami@gu.edu.pk
**Muhammad Arshad:** Department of Math and Stats, International Islamic University Islamabad, Pakistan, e-mail: marshadzia@iiu.edu.pk

that becomes a departure point for this article. There are some basic definitions and a lemma which help us in the sequel. In Section 2, we establish theorems ensuring the existence of (common) fixed points of single-valued $F$-contractions in $F$-metric spaces. An example is provided to explain the results. Section 3 deals with fixed point theorems of rational type $F$-contractions on closed balls in $F$-metric spaces. A related example is constructed in support of the result. Additionally, Section 4 is concerned with the application of the aforementioned results to the functional equations in dynamic programming.

**Definition 1.1.** [8] Let $\mathcal{F}$ be the collection of functions $f:(0,\infty)\to R$ such that the following conditions hold:
($F1$) $f$ is non-decreasing, i.e., $0 < a < u$ implies $f(a) \le f(u)$;
($F2$) For any sequence $(\omega_n) \subset (0,\infty)$, we have

$$\lim_{n\to\infty} \omega_n = 0 \text{ if and only if } \lim_{n\to\infty} f(\omega_n) = 0.$$

The definition of an $F$-metric space is as follows.

**Definition 1.2.** [8] Suppose $U$ is a non-empty set and $d\colon U\times U\to[0,\infty)$ is a given function. Suppose that there exists $(f,\sigma)\in\mathcal{F}\times[0,\infty)$ such that
($D1$) $(a,u)\in U\times U$, $d(a,u)=0$ if and only if $a=u$;
($D2$) for all $(a,u)\in U\times U$, $d(a,u)=d(u,a)$;
($D3$) for every $(a,u)\in U\times$ for each $n'\in N$, $n'\ge 2$ and for every $(\omega_m)_{m=1}^{n'}\subset X$ with $(\omega_1,\omega_{n'})=(a,u)$, we have

$$d(a,u)>0 \Rightarrow f(d(a,u)) \le f\left(\sum_{m=1}^{n'-1} d(\omega_m,\omega_{m+1})\right)+\sigma.$$

The function $d$ is known as an $F$-metric on $U$, and the pair $(U,d)$ is called an $F$-metric space.

It is also called a function weighted metric space. From now on, we keep this last notation.

**Example 1.1.** [8] Let $U=N$ and $d\colon U\times U\to(0,\infty)$ be defined as

$$d(a,u)=\begin{cases}(a-u)^2, & (a-u)\in[0,3]\times[0,3],\\ |a-u|, & (a-u)\notin[0,3]\times[0,3],\end{cases}$$

for all $(a,u)\in U\times U$ with $f(t)=\ln t$, $t>0$ and $\alpha=\ln 3$, then $d$ is a function weighted metric on $U$.

**Example 1.2.** [8] Let $U=N$ and $d\colon U\times U\to(0,\infty)$ be defined as

$$d(a,u)=\begin{cases}0, & a=u,\\ e^{|a-u|}, & a\ne u,\end{cases}$$

for all $(a,u)\in U\times U$. Then, $d$ is a function weighted metric on $U$.

**Definition 1.3.** [8] Let $d$ be an $\mathcal{F}$-metric space. Suppose $(a_n)$ is a sequence in $U$. Then,
(i) $(a_n)$ is $\mathcal{F}$-convergent to a point $a\in U$ if $\lim_{n\to\infty} d(a_n,a)=0$;
(ii) $(a_n)$ is $\mathcal{F}$-Cauchy if $\lim_{n\to\infty} d(a_n,a_m)=0$;
(iii) the space $(U,d)$ is $\mathcal{F}$-complete if every Cauchy sequence $(a_n)\subset U$ is convergent to a point $a\in U$.

**Definition 1.4.** [8] Let $(U,d)$ be a function weighted metric space. A subset $O$ of $U$ is said to be $F$-open if for every $a\in O$, *there is some* $r'>0$ such that $B(a,r')\subset O$, where

$$B(a,r')=\{u\in U\colon d(a,u)<r'\}.$$

We say that a subset $C$ of $U$ is $F$-closed if $U\backslash C$ is $F$-open.

**Proposition 1.1.** [8] *Let $(U, d)$ be a function weighted metric space and $V$ be a non-empty subset of $U$. Then, the following statements are equivalent:*
(i) *$V$ is $\mathcal{F}$-closed.*
(ii) *For any sequence $(a_n) \subset V$, we have*

$$\lim_{n\to\infty} d(a_n, a) = 0, a \in U$$

*implies $a \in V$.*

**Theorem 1.1.** [8] *Suppose $(U, d)$ is a complete function weighted metric space, and let $g$: $U \to U$ be a given mapping. Suppose that there exists $k \in (0,1)$ such that*

$$d(g(a), g(u)) \leq d(a, u), (a, u) \in U \times U.$$

*Then, $g$ has a unique fixed point $a^* \in U$. Moreover, for any $a_0 \in U$, the sequence $(a_n) \subset U$ defined by $a_{n+1} = g(a_n)$, $n \in N$ is convergent to $a^*$.*

**Theorem 1.2.** [26] *Suppose $U$ is a complete metric space ($d$ is the metric) and let $g$: $U \to U$ be a function such that*

$$d(g(a), g(u)) \leq \alpha d(a, u) + \beta d(a, g(a)) + \gamma d(u, g(u))$$

*for all $a, u \in U$, where $\alpha, \beta, \gamma$ are non-negative and satisfy $\alpha + \beta + \gamma < 1$. Then, $g$ has a unique fixed point. Next, let $W$: $(-\infty, 0] \to R$ be given continuous bounded initial function. Let $C$ be the space of all continuous functions from $R$ to $R$ and define the set $B(W)$ by $B(W) = \{\phi: R \to R, \phi(t) = W(t)$ if $t \leq 0, \phi(t) \to 0$ as $t \to \infty, \phi \in C\}$. Then, equipped with the supremum norm $\|\cdot\|$, $S\xi$ is a Banach space.*

**Lemma 1.1.** [27] *The Banach space $B(W)$, $\|\cdot\|$ endowed with the metric $d$ defined by*

$$d(g, h) = \|g - h\| = \max_{n\in W} |g(a), h(u)|$$

for *$h \in B(W)$, is a function weighted metric space.*

## 2 Fixed point results of globally rational type *F*-contractions

This section deals with fixed point theorems of rational type $F$-contractions in the setting of function weighted metric spaces.

**Definition 2.1.** Let $(U, d)$ be a function weighted metric space and $(F, \tau), (f, \sigma) \in \mathcal{F} \times [0, \infty)$ with $\sigma < \tau$. Let $S, T$: $U \to U$ be self-mappings. Then, the pair $(T, S)$ is called a rational type $F$-contraction if there exist $\alpha, \beta, \gamma, \delta, \lambda, \mu, \eta \in [0, \infty)$ such that $\alpha + 2\beta + 2\gamma + \delta + \mu < 1$ for all $(a, u) \in U \times U$, we have

$$d(Sa, Tu) > 0 \text{ implies } \tau + F(d(Sa, Tu)) \leq F(M_1(a, u)), \tag{1}$$

where

$$M_1(a, u) = \alpha d(a, u) + \beta(d(a, Sa) + d(u, Tu)) + \gamma(d(u, Sa) + d(a, Tu)) + \delta\frac{d(u, Tu)[1 + d(a, Sa)]}{1 + d(a, u)}$$
$$+ \lambda\frac{d(u, Sa)[1 + d(a, Tu)]}{1 + d(a, u)} + \mu\frac{d(a, u)[1 + d(a, Sa) + d(u, Sa)]}{1 + d(a, u)} + \eta d(u, Sa).$$

**Theorem 2.1.** *Let $U, d$ be a complete function weighted metric space and $S, T$: $U \to U$ be self-mappings such that $(T, S)$ is a rational type $F$-contraction. Then, $S$ and $T$ have at most one common fixed point in $U$.*

**Proof.** Suppose $s_0$ is an arbitrary point. Define a sequence $(s_n)$ by

$$Ss_{2v} = s_{2v+1} \text{ and } Ts_{2v+1} = s_{2v+2}, \quad \text{where } v = 0, 1, 2, \ldots. \tag{2}$$

Using (1) and (2) and letting $a = s_{2v}$ and $u = s_{2v+1}$, we can write

$$\begin{aligned}
\tau + F(d(s_{2v+1}, s_{2v+2})) &= \tau + F(d(Ss_{2v}, Ts_{2v+1})) \\
&\leq F[a(d(s_{2v}, s_{2v+1}) + \beta d(Ss_{2v}, s_{2v}) + d(s_{2v+1}, Ts_{2v+1})] \\
&\quad + \gamma(d(s_{2v+1}, Ss_{2v}) + d(s_{2v}, Ts_{2v+1})) + \delta\frac{d(s_{2v+1}, Ts_{2v+1})(1 + d(s_{2v}, Ss_{2v}))}{1 + d(s_{2v}, s_{2v+1})} \\
&\quad + \lambda\frac{d(s_{2v+1}, Ss_{2v+1})(1 + d(s_{2v}, Ts_{2v+1}))}{1 + d(s_{2v}, s_{2v+1})} \\
&\quad + \mu\frac{d(s_{2v}, s_{2v+1})(1 + d(s_{2v}, Ss_{2v})) + d(s_{2v+1}, Ss_{2v})}{1 + d(s_{2v}, s_{2v+1})} + \eta d(s_{2v+1}, Ss_{2v}) \\
&= F[\alpha d(s_{2v}, s_{2v+1}) + \beta(d(s_{2v}, s_{2v+1}) + d(s_{2v+1}, s_{2v+2})) \\
&\quad + \gamma d(s_{2v}, s_{2v+2}) + \delta d(s_{2v+1}, s_{2v+2}) + \mu d(s_{2v}, s_{2v+1})].
\end{aligned}$$

By $(D3)$,

$$\begin{aligned}
\tau + F(d(s_{2v+1}, s_{2v+2})) \leq F[&\alpha d(s_{2v}, s_{2v+1}) + \beta(d(s_{2v}, s_{2v+1}) + d(s_{2v+1}, s_{2v+2})) + \gamma d(s_{2v}, s_{2v+1}) \\
&+ d(s_{2v+1}, s_{2v+2}) + \delta d(s_{2v+1}, s_{2v+2}) + \mu d(s_{2v}, s_{2v+1})] + \sigma.
\end{aligned}$$

By hypothesis and $(F1)$, we can write

$$\begin{aligned}
d(s_{2v+1}, s_{2v+2}) < \alpha d(s_{2v}, s_{2v+1}) &+ \beta(d(s_{2v}, s_{2v+1}) + d(s_{2v+1}, s_{2v+2})) + \gamma d(s_{2v}, s_{2v+1}) \\
&+ d(s_{2v+1}, s_{2v+2}) + \delta d(s_{2v+1}, s_{2v+2}) + \mu d(s_{2v}, s_{2v+1}).
\end{aligned}$$

That is,

$$d(s_{2v+1}, s_{2v+2}) < \frac{\alpha + \beta + \gamma + \mu}{1 - \beta - \gamma - \delta} d(s_{2v}, s_{2v+1}) = \omega d(s_{2v}, s_{2v+1}),$$

where $\frac{\alpha + \beta + \gamma + \mu}{1 - \beta - \gamma - \delta} = \omega$.

Note that $\omega < 1$. Similarly,

$$d(s_{2v+2}, s_{2v+3}) < \frac{\alpha + \beta + \gamma + \mu}{1 - \beta - \gamma - \delta} d(s_{2v+1}, s_{2v+2}) = \omega d(s_{2v+1}, s_{2v+2}).$$

Hence,

$$d(s_n, s_{n+1}) < \omega d(s_{n-1}, s_n), \quad \text{for all } n \in N.$$

It yields that

$$d(s_n, s_{n+1}) < \omega d(s_{n-1}, s_n) < \omega^2 d(s_{n-2}, s_{n-1}) \cdots < \omega^n d(s_0, s_1), n \in N,$$

i.e.,

$$d(s_n, s_{n+1}) < \omega^n d(s_0, s_1), n \in N.$$

Using the above inequality, we can write

$$\sum_{k=n}^{m-1} d(s_k, s_{k+1}) < \omega^n[1 + \omega + \omega^2 + \cdots \omega^{m-n-1}] d(s_0, s_1) \leq \frac{\omega^n}{1 - \omega} d(s_0, s_1), m > n.$$

Since

$$\lim_{n\to\infty} \frac{\omega^n}{1 - \omega} d(s_0, s_1) = 0,$$

for any $\delta > 0$, there exists some $n' \in N$ such that

$$0 < \frac{\omega^n}{1-\omega} d(s_0, s_1) < \delta, \quad n \geq n'. \tag{3}$$

Furthermore, suppose $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ such that $(D3)$ is satisfied. Suppose $\varepsilon > 0$ is fixed. By $(F2)$ there is some $\delta > 0$ such that

$$0 < t < \delta \text{ implies } f(t) < f(\varepsilon) - \alpha. \tag{4}$$

Using (3) and (4), we write

$$f\left(\sum_{k=n}^{m-1} d(s_k, s_{k+1})\right) \leq f\left(\frac{w^n}{1-\omega} d(s_0, s_1)\right) < f(\varepsilon) - \alpha, \quad m > n \geq n'.$$

Using $(D3)$ and the above equation, we obtain

$$d(s_n, s_m) > 0, m > n > n' \text{ implies } f(d(s_n, s_m)) < f(\varepsilon),$$

which shows

$$d(s_n, s_m) < \varepsilon, m > n \geq n'.$$

Therefore, it is proved that the sequence $(s_n)$ is Cauchy in $U$. Since $(U, d)$ is complete, there exists $s^* \in U$ such that $(s_n)$ is convergent to $s^*$, i.e.,

$$\lim_{n\to\infty} d(s_n, s^*) = 0.$$

To prove that $s^*$ is a fixed point of $S$, assume that $d(Ss^*, s^*) > 0$. Then,

$$\begin{aligned}
\tau + F(d(Ss^*, s_{2v+2})) &= F(d(Ss^*, Ts_{2v+1})) \\
&\leq F[\alpha d(s^*, s_{2v+1}) + \beta(d(s^*, Ss^*) + d(s_{2v+1}, s_{2v+2})) + \gamma d(s_{2v+1}, Ss^*) + d(s^*, Ts_{2v+1}) \\
&\quad + \delta\frac{d(s_{2v+1}, s_{2v+2})(1 + d(s^*, Ss^*))}{1 + d(s^*, s_{2v+1})} + \lambda\frac{d(s_{2v+1}, Ss^*)(1 + d(s^*, Ts_{2v+1}))}{1 + d(s^*, s_{2v+1})} \\
&\quad + \mu\frac{d(s_{2v+1}, s^*)(1 + d(s^*, Ss^*)) + d(s_{2v+1}, Ss^*)}{1 + d(s^*, s_{2v+1})} + \eta d(s_{2v+1}, Ss^*)].
\end{aligned}$$

Using (2) and $(F1)$ and letting $j \to \infty$, we get

$$(1 - \beta - \gamma - \lambda - \eta) d(s^*, Ss^*) < 0.$$

It is a contradiction. Hence, $d(s^*, Ss^*) = 0$, i.e., $Ss^* = s^*$. Following the same steps, we get $Ts^* = s^*$. Hence, $Ss^* = s^* = Ts^*$.

Uniqueness: say $s^{**}$ is another common fixed point of $S$ and $T$, i.e., $d(s^*, s^{**}) > 0$. Then,

$$\begin{aligned}
\tau + F(d(s^*, s^{**})) &= F(d(Ss^*, Ts^{**})) \\
&\leq F[\alpha d(s^*, s^{**}) + \beta(d(s^*, Ss^*) + d(s^{**}, Ts^{**})) + \gamma d(s^{**}, Ss^*) + d(s^*, Ts^{**}) \\
&\quad + \delta\frac{d(s^{**}, Ts^{**})(1 + d(s^*, Ss^*))}{1 + d(s^*, s^{**})} + \lambda\frac{d(s^{**}, Ss^*)(1 + d(s^*, Ts^{**}))}{1 + d(s^*, s^{**})} \\
&\quad + \mu\frac{d(s^{**}, s^*)(1 + d(s^*, Ss^*)) + d(s^{**}, Ss^*)}{1 + d(s^*, s^{**})} + \eta d(s^{**}, Ss^*)] \\
&= F[\alpha d(s^*, s^{**}) + \gamma d(s^*, s^{**} + d(s^{**}, s^*)) + \lambda d(s^{**}, s^*) + \mu d(s^*, s^{**}) + \eta d(s^{**}, s^*)].
\end{aligned}$$

Using $(F1)$, we write

$$(1 - 2\gamma - \lambda - \mu - \eta) d(s^*, s^{**}) < 0,$$

which is a contradiction. Hence, $d(s^{**}, s^*) = 0$, i.e., $s^{**} = s^*$. □

**Example 2.1.** Suppose $U = N$, $F(a) = \ln a = f(a)$ and $S, T\colon U \to U$ and $d\colon U \times U \to R$ are defined as

$$Sa = \begin{cases} 1, & a = 1, \\ 2, & a = 2, \\ a - 2, & a > 2, \end{cases}$$

$$Ta = \begin{cases} 1, & a = 1, 2, \\ a - 1, & a > 2, \end{cases}$$

and

$$d(a, u) = \begin{cases} 0, & a = u, \\ e^{|a-u|}, & a \notin u, \end{cases}$$

for $(a, u) \in U \times U$. Observe that $f$ and $F$ satisfy $(F1) - (F2)$ and $d$ is the function weighted metric on $U$. Fix $\alpha = \beta = \gamma = \lambda = \delta = \mu = 0$ and $\eta = e^2$. Now, if $d(Sa, Tu) > 0$, then

$$\begin{aligned} F(d(Sa, Tu)) &= F(d(a-2, u-1)) \\ &= \ln(e^{|a-u-1|}) < \ln(e^2 e^{|a-u-2|}) < F(\eta d(u, Sa)) \\ &= F\Bigg[0 \cdot d(a, u) + 0 \cdot (d(a, Sa) + d(u, Tu)) + 0 \cdot (d(u, Sa) + d(a, Tu)) + 0 \cdot \frac{d(u, Tu)(1 + d(a, Sa))}{1 + d(a, u)} \\ &\quad + 0 \cdot \frac{d(u, Sa)(1 + d(a, Tu))}{1 + d(a, u)} + 0 \cdot \frac{d(a, u)(1 + d(a, Sa) + d(u, Sa))}{1 + d(a, u)} + \eta d(u, Sa)\Bigg] \end{aligned}$$

and

$$\tau \in (0, \ln(e^2 e^{|a-u-2|}) - \ln(e^{|a-u-1|})) = (0, \ln e)$$

and $\sigma$ can be chosen according to $\tau$. Therefore (1) holds. Moreover, it is clear that 1 is the unique common fixed point of $S$ and $T$. Taking $S = T$ in Theorem 2.1, the following result for single-valued mappings is obtained.

**Corollary 2.1.** *Suppose $(U, d)$ is a complete function weighted metric space and $(F, \tau), (f, \sigma) \in \mathcal{F} \times [0, \infty)$ with $\sigma < \tau$. Let $T\colon U \to U$ be a self-mapping and there exist $\alpha, \beta, \gamma, \delta, \lambda, \mu, \eta \in [0, \infty)$ with $\alpha + 2\beta + 2\gamma + \delta + \mu < 1$ such that for all $(a, U) \in U \to U$,*

$$\tau + F(d(Ta, Tu)) \le F(M_1'(a, u)),$$

*where*

$$\begin{aligned} M'_1(a, u) &= \alpha d(a, u) + \beta(d(a, Ta) + d(u, Tu)) + \gamma(d(u, Ta) + d(a, Tu)) + \delta \frac{d(u, Tu)[1 + d(a, Ta)]}{1 + d(a, u)} \\ &\quad + \lambda \frac{d(u, Ta)[1 + d(a, Tu)]}{1 + d(a, u)} + \mu \frac{d(a, u)[1 + d(a, Ta) + d(u, Ta)]}{1 + d(a, u)} + \eta d(u, Ta) \end{aligned}$$

*provided that $d(Ta, Tu) > 0$. Then, $T$ has at most one fixed point in $U$.*

*Put $\delta = \lambda = \mu = \eta = 0$ in* Theorem 2.1, *the following result for Hardy-Rogers-type F-contraction is obtained.*

**Corollary 2.2.** *Suppose $(U, d)$ is a complete function weighted metric space and $(F, \tau), (f, \sigma) \in \mathcal{F} \times [0, \infty)$ with $\sigma < \tau$. Let $S, T\colon U \to U$ be a self-mapping and there exist $\alpha, \beta, \gamma \in [0, \infty)$ with $\alpha + 2\beta + 2\gamma < 1$ such that for all $(a, U) \in U \to U$,*

$$\tau + F(d(Ta, Tu)) \le F(M(a, u)),$$

*where*

$$M(a, u) = \alpha d(a, u) + \beta(d(a, Sa) + d(u, Tu)) + \gamma(d(u, Sa) + d(a, Tu))$$

*provided that* $d(Ta, Tu) > 0$. *Then, S and T have at most one common fixed point in U. Similarly, by putting* $\alpha = \beta = 0$, $\alpha = \gamma = 0$ *and* $\gamma = 0$, *we can obtain fixed point theorems of Chatterjea-type F-contractions, Kannan-type F-contractions and Reich-type F-contractions, respectively.*

# 3 Fixed point results of locally rational type *F*-contractions

In this portion, we ensure the existence of fixed points of rational type $F$-contraction mappings on the closed ball rather than that on the whole function weighted metric space. As a consequence, Hardy and Rogers-type $F$-contraction is also established. An example is provided to illustrate the result.

**Definition 3.1.** Suppose $(U, d)$ is a complete function weighted metric space and $(F, \tau), (f, \sigma) \in \mathcal{F} \times [0, \infty)$ with $\sigma < \tau$. Let $S, T: B(a_0, r') \to U$ be a self-mapping. Then, the pair $(T, S)$ is called a rational type $F$-contraction on $B(a_0, r')$ if there exist $\alpha, \beta, \gamma, \delta, \lambda, \mu, \eta \in [0, \infty)$ with $\alpha + 2\beta + 2\gamma + \delta + \mu < 1$ such that for all $(a, u) \in B(a_0, r') \times B(a_0, r')$,

$$d(Sa, Tu) > 0 \text{ implies } \tau + F(d(Sa, Tu)) \leq F(M_1(a, u)), \tag{5}$$

where $F(M_1(a, u))$ is given in Definition 2.1.

**Theorem 3.1.** *Let* $(U, d)$ *be a complete function weighted metric space and* $S, T: U \to U$ *be self-mappings such that the pair* $(T, S)$ *is a rational type F-contraction on* $B(s_0, r')$. *Let* $r' > 0$ *so that the following conditions hold:*

(a) $B(s_0, r')$ *is* $\mathcal{F}$*-closed.*

(b) $d(s_0, s_1) \leq (1 - \omega)r'$, *for* $s_1 \in U$ *and* $\omega = \frac{\alpha + \beta + \gamma + \mu}{1 - \beta - \gamma - \delta}$.

(c) *There exists* $0 < \varepsilon < r'$ *such that* $B(s_0, r')$ $f((1 - \omega^{k+1})r') \leq f(\varepsilon) - \alpha$, *where* $k \in N$. *Then, S and T have at most one common fixed point in* $B(s_0, r')$.

**Proof.** Suppose $s_0$ is an arbitrary point. Define a sequence $(s_n)$ by

$$Ss_{2v-1} = s_{2v}, \; Ts_{2v} = s_{2v+1}, \quad \text{where } j = 0, 1, 2, \ldots.$$

We need to show that $(s_n)$ is in $B(s_0, r')$ for all $n \in N$. It is clear from definition that $\omega < 1$, therefore, by using mathematical induction and by (b), we write

$$d(s_0, s_1) \leq (1 - \omega)r' < r'.$$

Therefore $s_1 \in B(s_0, r')$. Suppose $s_2, \ldots, s_k \in B(s_0, r')$ for some $k \in N$. Now, if $s_{2v+1} \leq s_k$, then by 5, we write

$$\begin{aligned}
\tau + F(d(s_{2v}, s_{2v+1})) &= \tau + F(d(Ss_{2v-1}, Ts_{2v})) \\
&\leq F[\alpha d(s_{2v-1}, s_{2v}) + \beta(d(s_{2v-1}, Ss_{2v-1}) + d(s_{2v}, Ts_{2v})) \\
&\quad + \gamma(d(s_{2v}, Ss_{2v-1}) + d(s_{2v-1}, Ts_{2v})) + \delta\frac{d(s_{2v}, Ts_{2v})(1 + d(s_{2v-1}, Ss_{2v-1}))}{1 + d(s_{2v-1}, s_{2v})} \\
&\quad + \lambda\frac{d(s_{2v}, Ss_{2v-1})(1 + d(s_{2v-1}, Ts_{2v}))}{1 + d(s_{2v-1}, s_{2v})} \\
&\quad + \mu\frac{d(s_{2v-1}, s_{2v})(1 + d(s_{2v-1}, Ss_{2v-1})) + d(s_{2v}, Ss_{2v-1})}{1 + d(s_{2v-1}, s_{2v})} + \eta d(s_{2v}, Ss_{2v-1})] \\
&= F[\alpha d(s_{2v-1}, s_{2v}) + \beta(d(s_{2v-1}, s_{2v} + d(s_{2v}, s_{2v+1}))) \\
&\quad + \gamma d(s_{2v-1}, s_{2v+1}) + \delta d(s_{2v}, s_{2v+1}) + \mu d(s_{2v-1}, s_{2v})].
\end{aligned}$$

Using (D3), we write

$$\tau + F(d(s_{2v}, s_{2v+1})) \leq F[\alpha d(s_{2v-1}, s_{2v}) + \beta(d(s_{2v-1}, s_{2v}) + d(s_{2v}, s_{2v+1})) + \gamma d(s_{2v-1}, s_{2v}) + d(s_{2v}, s_{2v+1}) + \delta d(s_{2v}, s_{2v+1}) + \mu d(s_{2v-1}, s_{2v})] + \sigma.$$

Using $(F1)$ we write

$$d(s_{2v}, s_{2v+1}) < \frac{\alpha + \beta + \gamma + \mu}{1 - \beta - \gamma - \delta} d(s_{2v-1}, s_{2v}) = \omega d(s_{2v-1}, s_{2v}). \tag{6}$$

Similarly, if $s_{2v} \leq s_k$, then

$$d(s_{2v-1}, s_{2v}) < \frac{\alpha + \beta + \gamma + \mu}{1 - \beta - \gamma - \delta} d(s_{2v-2}, s_{2v-1}) = \omega d(s_{2v-2}, s_{2v-1}). \tag{7}$$

Therefore, from inequalities (6) and (7), we write

$$d(s_{2v}, s_{2v+1}) < \omega d(s_{2v-1}, s_{2v}) < \cdots < \omega^{2v} d(s_0, s_1) \tag{8}$$

and

$$d(s_{2v-1}, s_{2v+1}) < \omega d(s_{2v-2}, s_{2v-1}) < \cdots < \omega^{2v-1} d(s_0, s_1). \tag{9}$$

From (8) and (9), we write

$$d(s_k, s_{k+1}) < \omega^k d(s_0, s_1) \quad \text{for some } k \in N. \tag{10}$$

Now, using the above equation, we write

$$\begin{aligned} f(d(s_0, s_{k+1})) &< f\left(\sum_{i=1}^{k+1} d(s_{i-1}, s_i)\right) + \sigma \\ &= f(d(s_0, s_1) + \cdots + d(s_k, s_{k+1})) + \sigma \\ &\leq f[(1 + \omega + \omega^2 + \cdots + \omega^k)(d(s_0, s_1))] + \sigma \\ &= f\left[\frac{1 - \omega^{k+1}}{1 - \omega} d(s_0, s_1)\right] + \sigma. \end{aligned}$$

Using (b) and (c), we can write

$$f(d(s_0, s_{k+1})) \leq f((1 - \omega^{k+1})r') + \sigma \leq f(\varepsilon) < f(r').$$

Hence by $(F1)$, we argue that

$$s_{k+1} \in B(s_0, r').$$

Therefore, $s_n \in B(s_0, r')$ for all $n \in N$. Again, we have by 5,

$$\begin{aligned} \tau + F(d(s_{2v+1}, s_{2v+2})) &= \tau + F(d(Ss_{2v}, Ts_{2v+1})) \\ &\leq F[\alpha d(s_{2v}, s_{2v+1}) + \beta(d(s_{2v}, Ss_{2v}) + d(s_{2v+1}, Ts_{2v+1})) \\ &\quad + \gamma(d(s_{2v+1}, Ss_{2v}) + d(s_{2v}, Ts_{2v+1})) + \delta \frac{d(s_{2v+1}, Ts_{2v+1})(1 + d(s_{2v}, Ss_{2v}))}{1 + d(s_{2v}, s_{2v+1})} \\ &\quad + \lambda \frac{d(s_{2v+1}, Ss_{2v})(1 + d(s_{2v}, Ts_{2v+1}))}{1 + d(s_{2v}, s_{2v+1})} + \mu \frac{d(s_{2v}, s_{2v+1})(1 + d(s_{2v}, Ss_{2v})) + d(s_{2v+1}, Ss_{2v})}{1 + d(s_{2v}, s_{2v+1})} \\ &\quad + \eta d(s_{2v+1}, Ss_{2v})]. \end{aligned}$$

Following the same steps of Theorem 2.1 and using (a), we obtain that the sequence $(s_n)$ is convergent to some $s^*$ in $B(s_0, r')$. $s^*$ can be proved as a common fixed point of $S$ and $T$ in the same way as in Theorem 2.1. □

Putting $S = T$ in Theorem 3.1, the following result for single mappings is obtained.

**Corollary 3.1.** *Suppose $(f, \alpha) \in \mathcal{F} \times [0, \infty)$. Let $(U, d)$ be a complete function weighted metric space and $T\colon U \to U$ be a self-mapping and there exist $\alpha, \beta, \gamma, \delta, \lambda, \mu, \eta \in [0, \infty)$ with $\alpha + 2\beta + 2\gamma + \delta + \mu < 1$. Suppose $a_0 \in U$ and $r' > 0$ such that the following conditions are satisfied:*
(a) *$B(a_0, r')$ is $F$-closed;*
(b) *$d(a_0, a_1) \le (1 - \omega)r$, for $a_1 \in U$ and $\omega = \frac{\alpha+\beta+\gamma+\mu}{1-\beta-\gamma-\delta}$;*
(c) *There exists $0 < \varepsilon < r'$ such that $f((1 - \omega^{k+1})r') \le f(\varepsilon) - \alpha$, where $k \in N$;*
(d) *For all $(a, u) \in B(a_0, r') \times B(a_0, r')$, which $d_f(Ta, Tu) > 0$ implies $\tau + F(d(Ta, Tu)) \le F(M_1(a, u))$, where*

$$M_1(a, u) = \alpha d(a, u) + \beta(d(a, Tu) + d(a, Tu)) + \gamma(d(a, Ta) + d(a, Tu)) + \delta\frac{d(u, Tu)[1 + d(a, Ta)]}{1 + d(a, u)} + \lambda\frac{d(u, Ta)[1 + d(a, Tu)]}{1 + d(a, u)} + \mu\frac{d(a, u)[1 + d(a, Ta) + d(u, Ta)]}{1 + d(a, u)} + \eta d(u, Ta).$$

*Then, $T$ has at most one fixed point in $B(a_0, r')$.*

**Example 3.1.** Let $U \in R_0^+$ and $F(u) = \ln u = f(u)$. Define $T\colon U \to U$ by

$$Tu = \begin{cases} \frac{u}{3}, & u \in [0, 1], \\ u^2, & u \in (1, \infty) \end{cases}$$

and define $d$ as

$$d(a, u) = \begin{cases} (a - u)^2, & (a, u) \in [0, 1] \times [0, 1], \\ |a - u|, & (a, u) \notin [0, 1] \times [0, 1]. \end{cases}$$

It can be easily verified that $d$ is an $F$-metric and both $f$ and $F$ satisfy ($F1$)–($F2$). Fix $a_0 = r' = \frac{1}{2}$, then $B(a_0, r') = [0, 1]$. Clearly, $B(a_0, r')$ is $F$-closed, so condition (a) of Corollary 3.1 is satisfied. Now if $\alpha = \frac{2}{9}$ and $\beta = \gamma = \delta = \mu = \lambda = \eta = 0$, then $\omega = \alpha$ and

$$d(a_0, a_1) = d(a_0, Ta_0) = \left(\frac{1}{2} - \frac{1}{6}\right)^2 = \frac{1}{9} < \left(1 - \frac{2}{9}\right)\frac{1}{2} = (1 - \omega)r',$$

which shows that condition (b) is fulfilled. Furthermore, suppose $k = 1$, then

$$f((1 - \omega^{k+1})r') = \ln\left(\left(1 - \frac{1}{9}\right)^2\frac{1}{2}\right) = \ln\frac{77}{162} = \ln\frac{78}{162} - \ln\frac{78}{77} = f(\varepsilon) - \sigma$$

is satisfied, i.e., $\varepsilon = \frac{78}{162} \le \frac{1}{2} = r'$ and $\sigma = \ln\frac{78}{77}$. Similarly, for all values of $k \in N$, we can find some $0 < \varepsilon < r'$ and $\sigma$ such that condition (c) is fulfilled. Now, if $(a, u) \in B(a_0, r') \times B(a_0, r')$, then

$$\begin{aligned} F(d(Ta, Tu)) = \ln\left(\left(\frac{a}{3} - \frac{u}{3}\right)^2\right) < \ln\left(\frac{2}{9}(a - u)^2\right) &= \ln[\alpha d(a, u) + 0\cdot(d(a, Ta) + d(u, Tu)) + 0\cdot(d(u, Ta) + d(a, Tu)) \\ &\quad + 0\cdot\frac{d(u, Tu)(1 + d(a, Ta))}{1 + d(a, u)} + 0\cdot\frac{d(u, Ta)(1 + d(a, Tu))}{1 + d(a, u)} \\ &\quad + 0\cdot\frac{d_f(a, u)(1 + d(a, Ta) + d(u, Ta))}{1 + d(a, u)} + 0\cdot d(u, Ta)], \end{aligned}$$

where

$$\tau \in \left(\ln\frac{78}{77},\ \ln\frac{\frac{2}{9}(a - u)^2}{\left(\frac{a}{3} - \frac{u}{3}\right)^2}\right) = \left(\ln\frac{78}{77},\ \ln\frac{\frac{2}{9}(a - u)^2}{\frac{1}{9}(a - u)^2}\right) = \left(\ln\frac{78}{77},\ \ln 2\right) = (\sigma,\ \ln 2).$$

Therefore, for all $(a, u) \in B(a_0, r') \times B(a_0, r')$, condition (d) is also satisfied. On the other side, if $(a, u) \notin B(a_0, r') \times B(a_0, r')$, i.e., $a = 2$ and $u = 3$, then

$$F(d(Ta, Tu)) = \ln|2^2 - 3^2| > \ln\left(\frac{2}{9}|2 - 3|\right) = F(\alpha d(a, u)) = F(M_1(a, u)).$$

Hence, condition (d) holds only for $B(a_0, r')$ and not on $U \times U$. Moreover, $0 \in B(a_0, r')$ is the fixed point of $T$.

Fixed point result of Hardy-Rogers-type $F$-contraction can be obtained by putting $\delta = \mu = \lambda = \eta = 0$ in Theorem 3.1.

**Corollary 3.2.** *Let $(U, d)$ be a complete function weighted metric space and $S, T\colon U \to U$ be a self-mapping such that $(T, S)$ is a rational type F-contraction on $B(a_0, r)$. Suppose $a_0 \in B$ and $r' > 0$ such that the following conditions are satisfied:*
a) *$B(a_0, r')$ is F-closed;*
b) *$d(a_0, a_1) \le (1 - \omega)r'$, for $a_1 \in U$ and $\omega = \frac{\alpha+\beta+\gamma}{1-\beta-\gamma}$;*
c) *There exists $0 < \varepsilon < r'$ such that $f((1 - \omega^{k+1})r') \le f(\varepsilon) - \alpha$, where $k \in N$;*
d) *For all $(a, u) \in B(a_0, r') \times B(a_0, r')$,*

$$\tau + F(d(Ta, Tu)) \le F[\alpha d(a, u) + \beta(d(a, Sa) + d(u, Tu)) + \gamma(d(u, Sa) + d(a, Tu))].$$

*Then, S and T have at most one common fixed point in $B(a_0, r')$.*

**Proof.** Taking $\delta = \lambda = \mu = \eta = 0$ in proof of Theorem 3.1, we get the required proof. □

# 4 Application in dynamic programing

In this part, we use our results to assure the existence of a unique common solution of functional equations in dynamic programming. The problems of dynamic programming comprise two main parts. One is a state space, which is a collection of parameters representing different states including transitional states, initial states and action states. The other one is a decision space, that is, the series of decisions taken to solve the problems. This setting formulates the problems of a computer programming and a mathematical optimization. Particularly, the problems of dynamic programming are transformed into the problems of functional equations:

$$p(a) = \max_{b \in B}\{G(a, b) + g_1(a, b, p(\eta(a, b)))\}, \quad \text{for } a \in A \tag{11}$$

and

$$q(a) = \max_{b \in B}\{G(a, b) + g_2(a, b, p(\eta(a, b)))\}, \quad \text{for } a \in A, \tag{12}$$

where $U$ and $V$ are Banach spaces such that $A \subseteq U$, $B \subseteq U$ and

$$\eta\colon\ A \times B \to A \quad G\colon\ A \times B \to R \quad g_1, g_2\colon\ A \times B \times R \to R.$$

Assume that the decision spaces and state spaces are $A$ and $B$, respectively. Our aim is to assure that equations (11) and (12) have at most one common solution. Suppose $W(A)$ represents the collection of all real-valued bounded mappings on $A$. Suppose that $h$ is an arbitrary element of $W(A)$. Define $\|h\| = \max_{a \in A}|h(a)|$. Then, $(W(A), \|\cdot\|)$ is a Banach space along with the metric $d_f$ is given by

$$d(h, k) = \max_{a \in A}|h(a) - k(a)|. \tag{13}$$

Assume the following conditions hold:

(C1) $G, g_1, g_2$ are bounded;

(C2) For $a \in A$ and $h \in W(A)$, define $S, T: W(A) \to W(A)$ by

$$\mathrm{Sh}(a) = \max_{b\in B}\{G(a,b) + g_1(a,b,h(\eta(a,b)))\}, \quad \text{for } a \in A, \tag{14}$$

$$\mathrm{Th}(a) = \max_{b\in B}\{G(a,b) + g_2(a,b,h(\eta(a,b)))\}, \quad \text{for } a \in A. \tag{15}$$

Clearly, if the functions $G, g_1$ and $g_2$ are bounded, then $S$ and $T$ are well-defined.

(C3) For $\sigma < \tau: R_+ \to R_+$, $(a, b) \in A \times B$, $h, k \in W(A)$ and $t \in A$, we have

$$|g_1(a,b,h(t)) - g_2(a,b,k(t))| \le e^{-\tau} M_1(h,k), \tag{16}$$

where

$$M_1(h,k) = \alpha d(h,k) + \beta(d(h,Sh) + d(k,Tk)) + \gamma(d(k,Sh) + d(h,Tk)) + \delta\frac{d(k,Tk)[1+d(h,Sh)]}{1+d(h,k)} + \lambda\frac{d(k,Sh)[1+d(h,Tk)]}{1+d(h,k)} + \mu\frac{d(h,k)[1+d(h,Sh)+d(k,Sh)]}{1+d(h,k)} + \eta d(k,Sh)$$

for $\alpha, \beta, \gamma, \delta, \lambda, \mu, \eta \in [0, \infty)$ such that $\alpha + 2\beta + 2\gamma + \delta + \mu < 1$, where $\min\{d(Sh, Tk), M_1(h,k)\} > 0$. Now, we prove the following result.

**Theorem 4.1.** *Suppose conditions* (C1)–(C3) *hold, then equations* (11) *and* (12) *have at most one common bounded solution.*

**Proof.** By Lemma 1.1, we have $(W(A), d)$ is a complete function weighted metric space, where $d$ is given by (13). By (C1), $S$ and $T$ are self-mappings on $W(A)$. Suppose $\lambda$ is an arbitrary positive number and $h_1, h_2 \in W(A)$. Take $a \in A$ and $b_1, b_2 \in B$ such that

$$\mathrm{Sh}_j < G(a,b_j) + g_1(a,b_j,h_j(\eta(a,b_j))) + \lambda, \tag{17}$$

$$\mathrm{Th}_j < G(a,b_j) + g_2(a,b_j,h_j(\eta(a,b_j))) + \lambda, \tag{18}$$

$$\mathrm{Sh}_1 \ge G(a,b_2) + g_1(a,b_2,h_1(\eta(a,b_2))) \tag{19}$$

and

$$\mathrm{Th}_2 \ge G(a,b_1) + g_2(a,b_1,h_2(\eta(a,b_1))). \tag{20}$$

Then, using (17) and (20), we get

$$\begin{aligned}\mathrm{Sh}_1(a) - \mathrm{Th}_2(a) &< g_1(a,b_1,h_1(\eta(a,b_1))) - g_2(a,b_1,h_2(\eta(a,b_1))) + \lambda\\ &\le |g_1(a,b_1,h_1(\eta(a,b_1))) - g_2(a,b_1,h_2(\eta(a,b_1)))| + \lambda\\ &\le e^{-\tau}M_1(h_1(a),h_2(a)) + \lambda.\end{aligned}$$

Similarly, by (18) and (19), we get

$$\mathrm{Th}_2(a) - \mathrm{Sh}_1(a) < e^{-\tau}M_1(h_1(a),h_2(a)) + \lambda.$$

Combining the above two inequalities, we get

$$|\mathrm{Sh}_1(a) - \mathrm{Th}_2(a)| < e^{-\tau}M_1(h_1(a),h_2(a)) + \lambda$$

for all $\lambda > 0$. Hence,

$$d(\mathrm{Sh}_1(a), \mathrm{Th}_2(a)) \le e^{-\tau}M_1(h_1(a),h_2(a)),$$

that is,

$$d(\mathrm{Sh}_1, \mathrm{Th}_2) \le e^{-\tau}M_1(h_1,h_2)$$

for each $a \in A$. Using logarithms, we have

$$\tau + \ln(d(\mathrm{Sh}_1, \mathrm{Th}_2)) \leq \ln(e^{-\tau} M_1(h_1, h_2)).$$

Now, it is clear that, for the mapping $F\colon R_+ \to R$ defined as $F(a) = \ln a \in \mathcal{F}$, all the conditions of Theorem 3.1 are fulfilled, so by applying Theorem 3.1, $S$ and $T$ have a unique common and bounded solution of equations (11) and (12). ☐

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