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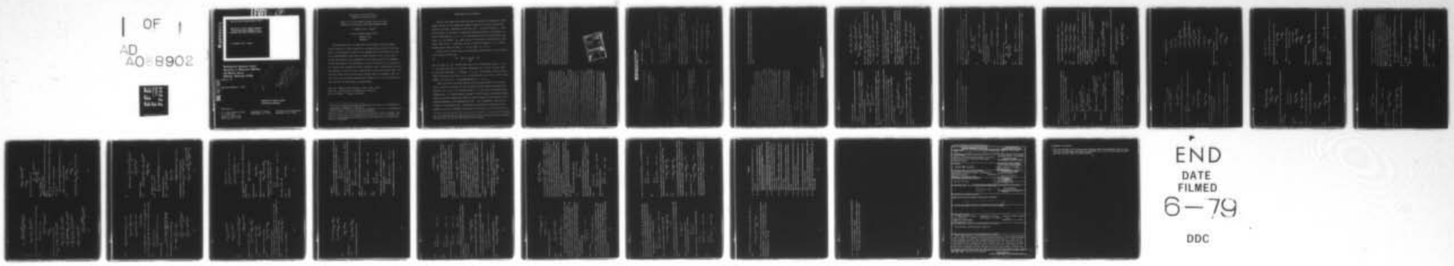

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ANALYSIS OF FINITE ELEMENT METHODS FOR SECOND ORDER BOUNDARY VALUE PROBLEMS USING MESH DEPENDENT NORMS

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ABSTRACT

This paper presents a new approach to the analysis of finite element methods based on C°-finite elements for the approximate solution of 2nd order boundary value problems in which error estimates are derived directly in terms of two mesh dependent norms that are closely related to the L_2 norm and to the 2nd order Sobolev norm, respectively, and in which there is no assumption of quasi-uniformity on the mesh family. This is in contrast to the usual analysis in which error estimates are first derived in the 1st order Sobolev norm and subsequently are derived in the L_2 norm and in the 2nd order Sobolev norm - the 2nd order Sobolev norm estimates being obtained under the assumption that the functions in the underlying approximating subspaces lie in the 2nd order Sobolev space and that the mesh family is quasi-uniform.

AMS (MOS) Subject Classifications: 65L10, 65N30, 65N15.
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SIGNIFICANCE AND EXPLANATION

Over the last twenty years there has been an extensive development of finite element methods for the approximate computer solution of problems in continuum mechanics (fracture, combustion, fluid flow, etc.). The exact solution u that we are looking for satisfies a differential equation boundary value problem. We seek an approximation u_n to u in a well-defined family of approximating functions S_n. For example, S_n might consist of continuous, piecewise linear functions, there being n subregions in each of which any member of S_n is linear. We would expect that the larger n is, the closer u_n is to u.

The key type of result used in the theoretical justification of such methods is an estimate of the form

$$(*) \qquad \|u - u_n\| \leq C \inf_{\chi \in S_n} \|u - \chi\| \quad .$$

Here the <u>norm</u> $\|u - v\|$ denotes some measure of the difference between u and v. The right hand side is a constant (independent of n) times the lower bound of the difference between u and <u>any</u> χ in S_n. This estimate reduces the problem of determining the error in u_n to a much easier one in approximation theory; namely, how closely can u be approximated by a χ in S_n.

There are several non-ideal features of the existing theory — for certain norms, estimates of the type (*) are not known (in many situations estimates of the form $\|u - u_n\| \leq C h^\alpha$ are known, where h is the size of the largest mesh subdomain and α is some positive constant), and for certain norms quasi-uniform meshes are required (meshes for which the ratio of maximum to minimum mesh size is bounded independent of the maximum mesh size). As a consequence, local mesh refinement is not accounted for. The present paper presents an approach to such methods in which (a) estimates of the type (*) are obtained, (b) the results are valid for arbitrary meshes, and (c) local mesh refinement is properly accounted for.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ANALYSIS OF FINITE ELEMENT METHODS FOR SECOND ORDER BOUNDARY VALUE PROBLEMS USING MESH DEPENDENT NORMS

I. Babuška[†] and J. Osborn[††]

1. Introduction

During the last several years there has been an extensive development of finite element methods for the approximate solution of boundary value problems for differential equations. In the most standard approach to the analysis of such methods, as they pertain to 2nd order elliptic equations, error estimates are first derived in the H^1-norm and subsequently are derived in the L_2-norm and in the H^2-norm — the H^2-norm error estimates being obtained under the assumption that the functions in the underlying approximating subspace lie in H^2 (requiring C^1-finite elements) and that the mesh family is quasi-uniform. This approach requires that the solution lies in H^1. (See Section 3 for the notation used in the paper.)

It is also known that the underlying bilinear form can be considered on $L_2 \times (H^2 \cap \overset{\circ}{H}{}^1_0)$ instead of $\overset{\circ}{H}{}^1 \times \overset{\circ}{H}{}^1$, as is usually done (see, e.g. [2, Cp. 6]). This approach leads directly to estimates in L_2 and H^2, and in a natural way handles the case in which the solution does not lie in H^1. Furthermore the error estimates are of quasi-optimal type, i.e., are estimates of the type $\|u - u_h\| \leq C \inf_{\chi \in S_h} \|u - \chi\|$, where u is the exact solution, u_h is the approximate solution, S_h is the approximating subspace, and $\|\cdot\|$ is the L_2 or H^2-norm. However, this analysis requires C^1-finite elements and quasi-uniform mesh families.

It is the purpose of this paper to extend this approach so that it covers C^0-finite elements and arbitrary mesh families. The considerations of arbitrary mesh families is important in situations in which strong mesh refinement is necessary. Toward this end two mesh dependent norms that are closely related to the L_2 and H^2-norms, respectively, are introduced. Error estimates are derived directly in terms of these norms, with no assumption of C^1-finite elements or quasi-uniform mesh families. We also treat the general situation in which the norms are based on the L_p-norm, $1 \leq p \leq \infty$, instead of just norms based on the L_2-norm. The approach is carried out in the case of a one dimensional model problem. Extensions to higher dimensional problems will be given in a forthcoming paper.

In Section 2 we review abstract results on the approximate solutions of variationally formulated boundary value problems. In Section 3 we introduce the mesh dependent norms and spaces used in the paper. In Section 4 we treat the basic bilinear form for our problem as a bilinear form on $\overset{\circ}{H}{}^1_p \times \overset{\circ}{H}{}^1_q$, with $\frac{1}{p} + \frac{1}{q} = 1$, for arbitrary $1 \leq p \leq \infty$. The results in this section are used in Section 5; they are, however, of some interest in themselves. Section 5 contains the main results of the paper — the proof of the stability condition with respect to the mesh dependent norms introduced in Section 3. Section 6 contains applications of the results in Section 5 to the finite element method for two point boundary value problems.

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2. Abstract Convergence Results

In this section we review certain results on the approximate solution of variationally formulated boundary value problems.

Let $\mathcal{H}_{1,\Delta}$ and $\mathcal{H}_{2,\Delta}$ be two reflexive Banach spaces (indexed by the parameter Δ with Δ varying over some index set) with norms $\|\cdot\|_{1,\Delta}$ and $\|\cdot\|_{2,\Delta}$, respectively, and let B_Δ be a bilinear form on $\mathcal{H}_{1,\Delta} \times \mathcal{H}_{2,\Delta}$. We suppose the following are satisfied:

(2.1) $$|B_\Delta(u,v)| \le C_1 \|u\|_{1,\Delta} \|v\|_{2,\Delta} , \quad \text{for all } u \in \mathcal{H}_{1,\Delta},\ v \in \mathcal{H}_{2,\Delta} ,$$

(2.2a) $$\inf_{\substack{u \in \mathcal{H}_{1,\Delta} \\ \|u\|_{1,\Delta}=1}} \ \sup_{\substack{v \in \mathcal{H}_{2,\Delta} \\ \|v\|_{2,\Delta}=1}} |B_\Delta(u,v)| \ge C_2 > 0 ,$$

(2.2b) $$\sup_{u \in \mathcal{H}_{1,\Delta}} |B_\Delta(u,v)| > 0, \quad \text{for each } 0 \ne v \in \mathcal{H}_{2,\Delta} ,$$

where C_1 and C_2 are constants that do not depend on Δ. As a consequence of (2.1) and (2.2) we have

<u>Theorem 1</u> [3, 1, 2]. If $f \in (\mathcal{H}_{2,\Delta})'$, then there is a unique solution u to the problem

(2.3) $$\begin{cases} u \in \mathcal{H}_{1,\Delta} \\ B_\Delta(u,v) = f(v) , \quad \text{for all } v \in \mathcal{H}_{2,\Delta} . \end{cases}$$

Moreover, u satisfies $\|u\|_{1,\Delta} \le (C_2)^{-1} \|f\|_{\mathcal{H}'_{2,\Delta}}$.

We will refer to (2.3) as an abstract, variationally formulated (boundary value) problem. In this paper we study the approximate solution of (2.3). Toward this end we suppose $S_{1,\Delta}$ and $S_{2,\Delta}$ are finite dimensional subspaces of $\mathcal{H}_{1,\Delta}$ and $\mathcal{H}_{2,\Delta}$, respectively, and we assume

(2.4a) $$\inf_{\substack{u \in S_{1,\Delta} \\ \|u\|_{1,\Delta}=1}} \ \sup_{\substack{v \in S_{2,\Delta} \\ \|v\|_{2,\Delta}=1}} |B_\Delta(u,v)| \ge C_2' > 0 ,$$

(2.4b) $$\sup_{u \in S_{1,\Delta}} |B_\Delta(u,v)| > 0, \quad \text{for each } 0 \ne v \in S_{2,\Delta} ,$$

where C_2' is a constant independent of Δ. As a consequence of (2.4) (cf. Theorem 1) there

is a unique solution u_Δ to the problem

(2.5) $$\begin{cases} u_\Delta \in S_{1,\Delta} \\ B_\Delta(u_\Delta, v) = f(v), \quad \text{for } v \in S_{2,\Delta} . \end{cases}$$

In addition, u_Δ satisfies

$$\|u_\Delta\|_{1,\Delta} \le (C_2')^{-1} \|f\|_{H'_{2,\Delta}} .$$

Since $S_{1,\Delta}$ and $S_{2,\Delta}$ are finite dimensional, (2.5) is computationally resolvable (in terms of the solution of a linear system of equations). u_Δ is the Ritz-Galerkin approximation to the exact solution u of (2.3). We note that if $\dim S_{1,\Delta} = \dim S_{2,\Delta}$, then (2.4b) follows from (2.4a). We assume $\dim S_{1,\Delta} = S_{2,\Delta}$ for the remainder of the paper.

u_Δ can also be characterized by

$$\begin{cases} u_\Delta \in S_{1,\Delta} \\ B_\Delta(u_\Delta, v) = B_\Delta(u,v), \quad \text{for all } v \in S_{2,\Delta} . \end{cases}$$

Thus we refer to u_Δ as the B_Δ - projection of u onto $S_{1,\Delta}$ with respect to $S_{2,\Delta}$, or, more briefly, as the left B_Δ - projection of u. This projection is defined for each $u \in \mathcal{H}_{1,\Delta}$. Also, for each $v \in \mathcal{H}_{2,\Delta}$ we can define the B_Δ - projection of v onto $S_{2,\Delta}$ with respect to $S_{1,\Delta}$ (right B_Δ - projection of v) by

$$\begin{cases} v_\Delta \in S_{2,\Delta} \\ B_\Delta(u, v_\Delta) = B_\Delta(u,v), \quad \text{for all } u \in S_{1,\Delta} . \end{cases}$$

We now state the fundamental estimates for the errors $u - u_\Delta$ and $v - v_\Delta$.

<u>Theorem 2</u> [1,2].

(2.6a) $$\|u - u_\Delta\|_{1,\Delta} \le (1 + C_1/C_2') \inf_{\chi \in S_{1,\Delta}} \|u - \chi\|_{1,\Delta}$$

and

(2.6b) $$\|v - v_\Delta\|_{2,\Delta} \le (1 + C_1/C_2') \inf_{\eta \in S_{2,\Delta}} \|v - \eta\|_{2,\Delta} ,$$

with C_1 and C_2' as in (2.1) and (2.4a).

These inequalities are called quasi-optimal error estimates.

In many applications of the results in this section the spaces $\mathcal{H}_{1,\Delta}$ and $\mathcal{H}_{2,\Delta}$ and the form B_Δ do not depend on Δ, i.e., $\mathcal{H}_{1,\Delta} = \mathcal{H}_1$ and $\mathcal{H}_{2,\Delta} = \mathcal{H}_2$ are fixed reflexive Banach spaces and $B_\Delta = B$ is a fixed form on $\mathcal{H}_1 \times \mathcal{H}_2$. The spaces $S_{i,\Delta}$ typically are spaces of piecewise polynomials with respect to a mesh Δ of some domain and, of course, depend on Δ. In the applications we consider, both the spaces $\mathcal{H}_{i,\Delta}$ and $S_{i,\Delta}$ will depend on Δ; the constants C_1, C_2 and C_2', however, will be independent of Δ (cf. [2, Cp. 7]). In these applications the solution u of (2.3) will lie in $\mathcal{H}_{1,\Delta}$ for all Δ. Thus the estimate (2.6a) provides a convergence estimate for $u - u_\Delta$, provided the family $\{S_{1,\Delta}\}$ satisfies an approximability assumption. For typical finite element applications, this would involve the assumption that $\inf_{\chi \in S_{1,\Delta}} \|u - \chi\|_{1,\Delta}$ tends to zero as the maximum mesh length of Δ tends to zero. *Finally, we remark that in most applications (2.4a) is the major assumption.* (2.4a) is called the stability assumption.

We will also need the general Banach space version of these results. We briefly sketch this now. Thus we let $\mathcal{H}_{i,\Delta}$, $S_{i,\Delta}$, $i = 1,2$, and B_Δ be as above, with $\mathcal{H}_{1,\Delta}$ and $\mathcal{H}_{2,\Delta}$ general Banach spaces, and suppose (2.1) and (2.4) hold. For $u \in \mathcal{H}_{1,\Delta}$ we define the left B_Δ - projection u_Δ of u by

$$\begin{cases} u_\Delta \in S_{1,\Delta} \\ B(u_\Delta, v) = B_\Delta(u,v), \quad \text{for all } v \in S_{2,\Delta} \end{cases},$$

and for $v \in \mathcal{H}_{2,\Delta}$ we define the right B_Δ - projection v_Δ of v by

$$\begin{cases} v_\Delta \in S_{2,\Delta} \\ B(u, v_\Delta) = B_\Delta(u,v), \quad \text{for all } u \in S_{2,\Delta} \end{cases}.$$

Then the estimates (2.6) are valid.

In the non-reflexive case (2.2) cannot in general hold. Note that (2.2) is most directly associated with the existence and uniqueness of solutions to (2.3) for all $f \in (\mathcal{H}_{2,\Delta})'$.

Since we will be working with a subclass of problems for which existence and uniqueness is known (from other principles), the fact that we have not assumed (2.2) will not affect our analysis.

3. Mesh dependent norms and spaces

In this section we define the mesh dependent norms, spaces, and forms that we will use in the paper.

Throughout the paper $H_p^k = H_p^k(I)$, $k = 0,1,\cdots, 1 \le p \le \infty$, will denote the kth Sobolev space on an interval I in R^1 consisting of functions with k derivatives in $L_p(I)$. On this space we have the usual norm given by

$$\|u\|_{k,p,I} = \begin{cases} \left(\sum_{j=0}^{k} \int_I |u^{(j)}|^p dx\right)^{1/p}, & 1 \le p < \infty, \\ \sum_{j=0}^{k} \operatorname{ess\,sup} |u^{(j)}|, & p = \infty. \end{cases}$$

$\overset{\circ}{H}{}_p^1(I)$ denotes the subspace of $H_p^1(I)$ of functions that vanish at the endpoints of I. In the special case $p = 2$ we use the notation $H^k = H_2^k$ and $\|u\|_{k,I} = \|u\|_{k,2,I}$. Note that $H_p^0 = L_p$.

Let $\Delta = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$, where $n = n(\Delta)$ = a positive integer, be an arbitrary mesh on the interval $I = [0,1]$ and set $h_j = x_j - x_{j-1}$ and $I_j = (x_{j-1}, x_j)$ for $j = 1,\cdots,n$, $\delta_j = (h_j + h_{j+1})/2$ for $j = 1,\cdots,n-1$, and $h = h(\Delta) = \max_j h_j$.

We now define two new spaces — $H_{p,\Delta}^2$ and $H_{p,\Delta}^0$, $1 \le p \le \infty$ — which depend on the mesh Δ. For $u \in \overset{\circ}{H}{}_p^1(I)$ let

(3.1)
$$\|u\|_{H_{p,\Delta}^0} = \begin{cases} \left(\int_0^1 |u|^p dx + \sum_{j=1}^{n-1} \delta_j |u(x_j)|^p\right)^{1/p}, & 1 \le p < \infty \\ \|u\|_{L_\infty(I)}, & p = \infty \end{cases}$$

and then define $H_{p,\Delta}^0$ to be the completion of $\overset{\circ}{H}{}_p^1(I)$ with respect to $\|\cdot\|_{H_{p,\Delta}^0}$. $H_{p,\Delta}^0$ can be identified with $L_p(I) \oplus R^{n-1}$ if $1 \le p < \infty$ and with $C(I)$ if $p = \infty$. $H_{p,\Delta}^2$ is defined by

$$H_{p,\Delta}^2 = \{u \in \overset{\circ}{H}{}_p^1(I) : u|_{I_j} \in H_p^2(I_j),\ j = 1,\cdots,n\},$$

(3.2)
$$\|u\|_{H_{p,\Delta}^2} = \begin{cases} \left(\sum_{j=1}^{n} \|u\|_{2,p,I_j}^p + \sum_{j=1}^{n-1} |Ju'(x_j)|^p \delta_j^{1-p}\right)^{1/p}, & 1 \le p < \infty \\ \max_{1 \le j \le n} \|u\|_{2,\infty,I_j} + \max_{1 \le j \le n} \delta_j^{-1} |Ju'(x_j)|, & p = \infty, \end{cases}$$

where $Ju'(x_j) = u'(x_j^+) - u'(x_j^-) = \lim_{x \searrow x_j} u'(x) - \lim_{x \nearrow x_j} u'(x)$. We note that this is well defined since $u|_{I_k} \in H_p^2(I_k)$ for each k and therefore the indicated limits exist. In the case $p = 2$ we write $H_\Delta^2 = H_{2,\Delta}^2$ and $H_\Delta^0 = H_{2,\Delta}^0$.

For $u \in H_{p,\Delta}^0$ and $v \in H_{q,\Delta}^2$, where $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we define

(3.3)
$$B_\Delta(u,v) = \sum_{j=1}^{n} \int_{I_j} u[-(av')' + cv]dx - \sum_{j=1}^{n-1} a(x_j)u(x_j)Jv'(x_j)$$

where $a \in C'(I)$, $c \in C(I)$, and $a(x) \ge a_0 > 0$, $c(x) \ge 0$ for $x \in I$. As pointed out above, if $p < \infty$, $H_{p,\Delta}^0$ can be identified with $L_p(I) \oplus R^{n-1}$. Under this identification, $\overset{\circ}{H}{}_p^1$ is considered a linear manifold in $H_{p,\Delta}^0$ through the mapping

$$\overset{\circ}{H}{}_p^1 \ni u \to (u(x), u(x_1), \ldots, u(x_{n-1})) \in L_p \oplus R^{n-1} = H_{p,\Delta}^0.$$

(More generally, $u \in L_p(I)$ with u continuous at x_j, $1 \le j \le n-1$, is considered an element in $L_p \oplus R^{n-1}$ through this mapping. Thus an element $u = (\tilde{u}, a_1, \ldots, a_{n-1}) \in L_p \oplus R^{n-1}$ is considered to be in $\overset{\circ}{H}{}_p^1$ if $\tilde{u} \in \overset{\circ}{H}{}_p^1$ and $a_j = \tilde{u}(x_j)$, $1 \le j \le n-1$. To be completely precise B_Δ should be defined by

$$B_\Delta(u,v) = \sum_{j=1}^{n} \int_{I_j} \tilde{u}[-(av')' + cv]dx - \sum_{j=1}^{n-1} a(x_j)a_j Jv'(x_j)$$

for $u = (\tilde{u}, a_1, \ldots, a_{n-1}) \in H_{p,\Delta}^0 = L_p \oplus R^{n-1}$ and $v \in H_{q,\Delta}^2$. Note that

(3.4)
$$B_\Delta(u,v) = \int_0^1 (au'v' + cuv)dx \equiv B(u,v)$$

for $u \in \overset{\circ}{H}{}_p^1(I)$ and $v \in H_{q,\Delta}^2$.

We conclude this section with the theorem corresponding to assumption (2.1).

Theorem 3. With $X_{1,\Delta} = H^0_{p,\Delta}$, $X_{2,\Delta} = H^2_{q,\Delta}$, $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and B_Δ as in (3.3), (2.1) holds with a constant C_1 that does not depend on the mesh Δ.

Proof. This result is a direct consequence of Hölders inequality.

4. Analysis of $B(u,v)$ as a bilinear form on $\mathring{H}^1_p(I) \times \mathring{H}^1_q(I)$

In this section we discuss the form $B(u,v) = \int_0^1 (au'v' + cuv)dx$ as a bilinear form on $\mathring{H}^1_p(I) \times \mathring{H}^1_q(I)$. We first state assumptions (2.1) and (2.2) for these spaces.

Theorem 4. For $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, B is a bounded bilinear form on $\mathring{H}^1_p(I) \times \mathring{H}^1_q(I)$. For $1 < p < \infty$, B satisfies

$$\inf_{\substack{u \in \mathring{H}^1_p(I) \\ \|u\|_{1,p,I}=1}} \sup_{\substack{v \in \mathring{H}^1_q(I) \\ \|v\|_{1,q,I}=1}} |B(u,v)| \ge C_2 > 0$$

and

$$\sup_{u \in \mathring{H}^1_p(I)} |B(u,v)| > 0 \quad , \quad \text{for each } 0 \ne v \in \mathring{H}^1_q(I) \ .$$

The proof of Theorem 4, which in part parallels the proof of Theorem 5 and in part is simpler than the proof of Theorem 5, will be omitted.

Next we consider (2.4) with

$$X_{1,\Delta} = \mathring{H}^1_p,\ X_{2,\Delta} = \mathring{H}^1_q,\ B_\Delta = B$$

and with

(4.1) $S_{1,\Delta} = S_{2,\Delta} = S_\Delta \equiv \{u \in C^0(I) : u\big|_{I_j} = \text{polynomial of degree } r-1,\ u(0) = u(1) = 0\}$

where $r = 2,3,\cdots$.

Theorem 5. Suppose $1 \le p \le \infty$. Then

(4.2a) $$\inf_{\substack{u \in S_\Delta \\ \|u\|_{\mathring{H}^1_p(I)}=1}} \sup_{\substack{v \in S_\Delta \\ \|v\|_{\mathring{H}^1_q(I)}=1}} |B(u,v)| \ge C_2' > 0, \quad C_2' \text{ independent of } \Delta,$$

and

(4.2b) $$\sup_{u \in S_\Delta} |B(u,v)| > 0 \quad , \quad \text{for each } 0 \ne v \in S_\Delta \ .$$

Proof. Since S_Δ is finite dimensional, (4.2b) follows from (4.2a). We also note that if (4.2) holds for $p = 1$ and $q = \infty$, then it also holds for $p = \infty$ and $q = 1$. We thus suppose $1 \le p < \infty$.

In the proof of (4.2) and in the proof of the results in Section 5 we will make use of the following preliminary results. On the space of polynomials $f(x) = \sum_{i=1}^{\ell} b_i(x-\bar{x})^i$ of degree ℓ on the interval $[\bar{x}, \bar{x}+h]$ we consider, in addition to the usual norm $\|f\|_{L_p[\bar{x},\bar{x}+h]}$, the following three norms:

$$\|f\|'_{L_p[\bar{x},\bar{x}+h]} = \begin{cases} \left(\sum_{i=0}^{\ell} |b_i|^p h^{ip+1}\right)^{1/p}, & 1 \le p < \infty \\ \max_{0\le i\le \ell} |b_i h^i|, & p = \infty, \end{cases}$$

$$\|f\|''_{L_p[\bar{x},\bar{x}+h]} = \begin{cases} \left(\sum_{i=0}^{\ell} \frac{\left|\int_{\bar{x}}^{\bar{x}+h} a(x)f(x)(x-\bar{x})^i dx\right|^p}{h^{pi+p-1}}\right)^{1/p}, & 1 \le p < \infty \\ \max_{0\le i\le \ell} \frac{\left|\int_{\bar{x}}^{\bar{x}+h} f(x)a(x)(x-\bar{x})^i dx\right|}{h^{i+1}}, & p = \infty, \end{cases}$$

and

$$\|f\|'''_{L_p[\bar{x},\bar{x}+h]} = \begin{cases} \left\{(|f(\bar{x})|^p + |f(\bar{x}+h)|^p)h + \sum_{i=0}^{\ell-2} \frac{\left|\int_{\bar{x}}^{\bar{x}+h} a(x)f(x)(x-\bar{x})^i dx\right|^p}{h^{pi+p-1}}\right\}^{1/p}, & 1 \le p < \infty \\ \max\left(|f(\bar{x})|, |f(\bar{x}+h)|, \frac{\left|\int_{\bar{x}}^{\bar{x}+h} f(x)a(x)(x-\bar{x})^i dx\right|}{h^{i+1}}, i = 0,\cdots,\ell-2\right), & p = \infty \end{cases}$$

where $a(x) \ge a_0 > 0$.

We give a brief indication of the proof that these expressions are positive definite, the other properties of a norm being obvious. The positive definiteness of $\|\cdot\|'$ is clear. Now consider $\|\cdot\|''$. Suppose $\|f\|'' = 0$. Then we easily see that $\int_{\bar{x}}^{\bar{x}+h} a f^2 dx = 0$, from which we get $f = 0$. If $\|f\|''' = 0$, then $\frac{f}{(x-\bar{x})(x-\bar{x}-h)}$ is a polynomial of degree

$\ell - 2$ and we find that $\int_{\bar{x}}^{\bar{x}+h} a(x-\bar{x})^{-1}(x-\bar{x}-h)^{-1}f^2 dx = 0$. Thus $f = 0$.

With the norms defined as above, there is a constant C such that

(4.3a) $$C^{-1}\|f\|'_{L_p[\bar{x},\bar{x}+h]} \le \|f\|_{L_p[\bar{x},\bar{x}+h]} \le C\|f\|'_{L_p[\bar{x},\bar{x}+h]},$$

(4.3b) $$C^{-1}\|f\|''_{L_p[\bar{x},\bar{x}+h]} \le \|f\|_{L_p[\bar{x},\bar{x}+h]} \le C\|f\|''_{L_p[\bar{x},\bar{x}+h]},$$

and

(4.3c) $$C^{-1}\|f\|'''_{L_p[\bar{x},\bar{x}+h]} \le \|f\|_{L_p[\bar{x},\bar{x}+h]} \le C\|f\|'''_{L_p[\bar{x},\bar{x}+h]}$$

for all polynomials f of degree ℓ, all real $\bar{x}$, and all $h > 0$. C depends on ℓ and $a(x)$ but is independent of $\bar{x}$, h and f.

For the special case $\bar{x} = 0$ and $h = 1$ these results follow from the fact that all norms on a finite dimensional space are equivalent. The general result then follows by a change of scale argument.

Now we return to the proof of (4.2). Let $u \in S_\Delta$. u' is a piecewise polynomial of degree $r - 2 \equiv s$ with zero average. We can write $u'|_{I_j} = \sum_{i=0}^{s} b_i^j (x-x_{j-1})^i$.

Choose g such that

$$g|_{I_j} = \text{polynomial of degree } s,$$

$$\int_{I_j} g(x-x_{j-1})^i dx = \left(\sum_{j=1}^{n}\sum_{i=0}^{s} |b_i^j|^p h_j^{pi+1}\right)^{\frac{2-p}{p}} |b_i^j|^{p-1} \operatorname{sgn} b_i^j h_j^{pi+1}, \quad i = 0,\cdots,s, \quad j = 1,\cdots,n.$$

It is clear that these conditions uniquely determine g (cf. the proof of (4.3b)). Then from (4.3a) we see that

(4.4) $$\int_0^1 u'g dx = \sum_{j=1}^{n}\sum_{i=0}^{s} b_i^j \int_{I_j} g(x-x_{j-1})^i dx$$

$$= \left(\sum_j\sum_i |b_i^j|^p h_j^{pi+1}\right)^{2/p} \ge C\|u'\|^2_{L_p(I)}$$

and using (4.3a) and (4.3b) we see that

$$\|g\|_{L_q(I)} \le C\left(\sum_j\sum_i \frac{|\int_{I_j} g(x-x_{j-1})^i dx|^q}{h_j^{qi+q-1}}\right)^{1/q} \tag{4.5}$$

$$\le C\left(\sum_j\sum_i |b_i^j|^p h_j^{pi+1}\right)^{1/p} \le C\|u'\|_{L_p(I)} .$$

The argument leading to (4.5) requires a slight modification if $p = 1$ and $q = \infty$.

Next let $\bar{g} = g - \dfrac{\int_0^1 \frac{g}{a}\,dx}{\int_0^1 \frac{1}{a}\,dx}$ and set $v_1(x) = \int_0^x \frac{\bar{g}}{a}\,dt$. Then from (4.4) and (4.5) we have

$$v_1(0) = v_1(1) = 0 ,$$

$$\int_0^1 au'v_1'dx = \int_0^1 u'\bar{g}dx \ge C\|u\|^2_{\overset{\circ}{H}{}^1_p(I)} \tag{4.6}$$

and

$$\|v_1\|_{\overset{\circ}{H}{}^1_q(I)} \le C\|u\|_{\overset{\circ}{H}{}^1_p(I)} . \tag{4.7}$$

v_1 is not, in general, in S_Δ. We thus approximate v_1 by $v_2 \in S_\Delta$ as follows: Let f be a piecewise polynomial of degree s defined by

$$f|_{I_j} = \text{polynomial of degree } s ,$$

$$\int_{I_j} f(x-x_{j-1})^i dx = \int_{I_j} v_1'(x-x_{j-1})^i dx,\ i = 0,\cdots,s,\ j = 1,\cdots,n .$$

Then let $v_2(x) = \int_0^x f(t)dt$. v_2 is clearly in S_Δ. Now, using the Bramble-Hilbert lemma [4] together with an argument due to Nitsche and Schatz [11] we have

$$\|v_1-v_2\|^q_{H^1_q} \le C\int_0^1 |v_1'-v_2'|^q dx \tag{4.8}$$

$$= C\int_0^1 |f - \frac{\bar{g}}{a}|^q dx$$

$$\le C\sum_j \int_{I_j} |f - \frac{\bar{g}}{a}|^q dx$$

$$\le C \sum_j h_j^{q(s+1)} \int_{I_j} |\left(\frac{\bar{g}}{a}\right)^{(s+1)}|^q dx$$

$$= C \sum_j h_j^{q(s+1)} \int_{I_j} |\sum_{k=0}^{s+1}\binom{s+1}{k}\bar{g}^{(k)}\left(\frac{1}{a}\right)^{(s+1-k)}|^q dx$$

$$= C \sum_j h_j^{q(s+1)} \int_{I_j} |\sum_{k=0}^{s}\binom{s+1}{k}\bar{g}^{(k)}\left(\frac{1}{a}\right)^{(s+1-k)}|^q dx$$

$$\le C \sum_j h_j^{q(s+1)}\sum_{k=0}^{s} \int_{I_j} |\bar{g}^{(k)}|^q dx$$

$$\le C \sum_j h_j^{q(s+1)}\sum_{k=0}^{s} h_j^{-qk} \int_{I_j} |\bar{g}|^q dx$$

$$\le C \sum_j h_j^q \int_{I_j} |v_1'|^q dx$$

$$\le C\ h^q\|v_1\|^q_{\overset{\circ}{H}{}^1_q} .$$

Combining (4.6), (4.7), and (4.8) we have

$$\int_0^1 au'v_2'dx = \int_0^1 au'v_2'dx + \int_0^1 au'(v_2'-v_1')dt \tag{4.9}$$

$$\ge (C - hC')\|u\|^2_{\overset{\circ}{H}{}^1_p(I)}$$

$$\ge \frac{C}{2}\|u\|^2_{\overset{\circ}{H}{}^1_p(I)}$$

and

$$(4.10) \qquad \|v_2\|_{\mathring{H}^1_q} \le \|v_1\|_{\mathring{H}^1_q} + \|v_2 - v_1\|_{\mathring{H}^1_q}$$

$$\le C\|u\|_{\mathring{H}^1_p(I)} + C\,h\|u\|_{\mathring{H}^1_p(I)}$$

$$\le 2C\|u\|_{\mathring{H}^1_p(I)} \quad .$$

(4.9) and (4.10) yield a proof of (4.2) for the form $\int_0^1 au'v'dx$. We now consider the complete form B.

Let w solve

$$\begin{cases} -(a\,w')' + cw = -cv_2 \\ w(0) = w(1) = 0 \quad . \end{cases}$$

w satisfies

$$(4.11) \qquad \|w\|_{H^2_q} \le C\|v_2\|_{H^0_q}$$

and

$$(4.12) \qquad \int_0^1 au'w'dx + \int_0^1 cuwdx = -\int_0^1 cuv_2dx \quad .$$

Now let $v_3 = v_2 + w$. Then from (4.9)-(4.12) we have

$$(4.13) \qquad B(u,v_3) = \int_0^1 au'v_2'dx + \int_0^1 au'w'dx + \int_0^1 cuv_2dx + \int_0^1 cuwdx$$

$$= \int_0^1 au'v_2'dx$$

$$\ge C\|u\|^2_{\mathring{H}^1_p(I)}$$

and

$$(4.14) \qquad \|v_3\|_{\mathring{H}^1_q} \le C\|u\|_{\mathring{H}^1_p(I)} \quad .$$

w, and thus v_3, is not in S_Δ. We approximate w by $\hat{w} \in S_\Delta$. Choose $\hat{w}$ so that

$$\hat{w}(x_j) = w(x_j),\ j = 1,\cdots,n \quad ,$$

$$\int_{I_j} \hat{w}(x-x_{j-1})^i dx = \int_{I_j} w(x-x_{j-1})^i dx,\ i = 0,1,\cdots,r-3,\ j = 1,\cdots,n \quad .$$

Then using the Bramble-Hilbert lemma [4] and (4.10) and (4.11) we have

$$(4.15) \qquad \|w - \hat{w}\|_{\mathring{H}^1_q} \le C\|w\|_{H^2_q}h$$

$$\le C\|u\|_{\mathring{H}^1_p}h \quad .$$

Now let $v = v_2 + \hat{w}$. Then from (4.13) and (4.15) we have

$$(4.16) \qquad B(u,v) = B(u,v_3) + B(u,v-v_3)$$

$$= B(u,v_3) + B(u,\hat{w}-w)$$

$$\ge C\|u\|^2_{\mathring{H}^1_p(I)} - C'h\|u\|^2_{\mathring{H}^1_p}$$

$$\ge \frac{C}{2}\ \|u\|^2_{\mathring{H}^1_p(I)}$$

and from (4.14) and (4.15) we have

$$(4.17) \qquad \|v\|_{\mathring{H}^1_q} \le C\|u\|_{\mathring{H}^1_p} \quad .$$

(4.16) and (4.17) yield (4.2).

We sketch briefly some applications of these results. Consider the boundary value problem

$$(4.18) \qquad \begin{cases} u \in \mathring{H}^1_p(I) \\ B(u,v) = f(v) \quad , \quad \text{for all } v \in \mathring{H}^1_q(I) \quad , \end{cases}$$

where $f \in (\mathring{H}^1_q)'$ is given and $1 < p < \infty$. Theorems 1 and 4 imply that this problem is uniquely solvable and that

$$\|u\|_{\overset{\circ}{H}{}^1_p} \le C_2^{-1}\|f\|_{(\overset{\circ}{H}{}^1_q)'} .$$

We can also consider the finite dimensional version of this problem for all $1 \le p \le \infty$:

$$\begin{cases} u_\Delta \in S_\Delta \\ B(u_\Delta, v) = f(v) \quad , \quad \text{for all } v \in S_\Delta . \end{cases}$$

From Theorem 5 we see that

(4.19) $$\|u_\Delta\|_{\overset{\circ}{H}{}^1_p} \le (C_2')^{-1}\|f\|_{(\overset{\circ}{H}{}^1_q)'} .$$

From Theorems 2 and 5 we have

(4.20) $$\|u - u_\Delta\|_{\overset{\circ}{H}{}^1_p} \le (1 + C_1/C_2') \inf_{\chi \in S_\Delta} \|u - \chi\|_{\overset{\circ}{H}{}^1_p} .$$

(4.20) with $p = \infty$ yields L_∞ estimates for the first derivatives of $u - u_\Delta$.

5. Analysis of $B_\Delta(u,v)$ as a bilinear form on $H^0_{p,\Delta} \times H^2_{q,\Delta}$

In Section 6 we apply the results of Section 2 with $\mathcal{H}_{1,\Delta} = H^0_{p,\Delta}$, $\mathcal{H}_{2,\Delta} = H^2_{q,\Delta}$, B_Δ as defined in (3.3) and $S_{1,\Delta} = S_{2,\Delta} = S_\Delta$ as defined in (4.1). It is the purpose of this section to prove assumptions (2.4a,b) with these choices. Prior to proving this result we prove (2.2a,b) - the infinite dimensional analogues of (2.4a,b) - in the case $1 < p < \infty$.

<u>Theorem 6</u>. With $\mathcal{H}_{1,\Delta} = H^0_{p,\Delta}$ and $\mathcal{H}_{2,\Delta} = H^2_{q,\Delta}$, with $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$, and B_Δ as in (3.3), (2.2a,b) hold with a constant C_2 independent of the mesh Δ.

<u>Proof</u>. Let $u \in \overset{\circ}{H}{}^1_p(I)$ be given. Choose v to be the solution of

$$\begin{cases} Lv = \|u\|^{2-p}_{L_p(I)} |u|^{p-1}\operatorname{sgn} u + \left(\sum_{i=1}^{n-1} \delta_i |u(x_i)|^p\right)^{\frac{2-p}{p}} \sum_{j=1}^{n-1} \delta_j |u(x_j)|^{p-1}(\operatorname{sgn} u(x_j)) d_{x_j} \\ v(0) = v(1) = 0 \end{cases}$$

where d_{x_j} is the Dirac distribution at x_j. v can also be characterized as the solution of

(5.1) $$\begin{cases} v \in \overset{\circ}{H}{}^1_q(I) \\ \int_0^1 (a\, v'\, \phi' + c\, v\, \phi)dx = \|u\|^{2-p}_{L_p(I)} \int_0^1 |u|^{p-1}\operatorname{sgn} u\, \phi\, dx \\ \qquad + \left(\sum_{i=1}^{n-1} \delta_i |u(x_i)|^p\right)^{\frac{2-p}{p}} \sum_{j=1}^{n-1} \delta_j |u(x_j)|^{p-1}(\operatorname{sgn} u(x_j))\, \phi(x_j) \\ \qquad\qquad \text{for all } \phi \in \overset{\circ}{H}{}^1_p(I) . \end{cases}$$

(5.1) is a boundary value problem of the type discussed at the end of Section 4 (cf. (4.18)). It is easy to see that

(5.2) $$v \in H^2_q(I_j), \text{ for each } j ,$$

(5.3) $$Lv = \|u\|^{2-p}_{L_p(I)} |u|^{p-1}\operatorname{sgn} u, \text{ on each } I_j ,$$

$$(5.4)\qquad Jv'(x_j) = -\frac{\left(\sum_{i=1}^{n-1}\delta_i|u(x_i)|^p\right)^{\frac{2-p}{p}}\delta_j|u(x_j)|^{p-1}\operatorname{sgn} u(x_j)}{a(x_j)},$$

and

$$(5.5)\qquad \|v\|_{\overset{\circ}{H}{}^1_q} \le C(\|u\|^p_{L_p(I)} + \sum_{j=1}^{n-1}\delta_j|u(x_j)|^p)^{1/p} .$$

From (5.2) we see that $v \in H^2_{q,\Delta}$ and from (3.4), (5.1) with $\phi = u$, and (3.1) we have

$$(5.6)\qquad B_\Delta(u,v) = \int_0^1 (a\,u'\,v' + c\,u\,v)dx$$

$$= (\int_0^1 |u|^p dx)^{2/p} + (\sum_{j=1}^{n-1}\delta_j|u(x_j)|^p)^{2/p}$$

$$\ge \tfrac{1}{2}\|u\|^2_{H^0_{p,\Delta}} .$$

From (3.1), (3.2), (5.3), (5.4), and (5.5) we have

$$(5.7)\qquad \|v\|_{H^2_{q,\Delta}} = (\sum_{j=1}^{n}\|v\|^q_{2,q,I_j} + \sum_{j=1}^{n-1}|Jv'(x_j)|^q\delta_j^{1-q})^{1/q}$$

$$= (\|v\|^q_{\overset{\circ}{H}{}^1_q} + \sum_{j=1}^{n}\|v''\|^q_{L_q(I_j)} + \sum_{j=1}^{n-1}|Jv'(x_j)|^q\delta_j^{1-q})^{1/q}$$

$$= (\|v\|^q_{\overset{\circ}{H}{}^1_q} + \sum_{j=1}^{n}\left\|\frac{Lv-a'v'-cv}{a}\right\|^q_{L_q(I_j)} + \sum_{j=1}^{n-1}|Jv'(x_j)|^q\delta_j^{1-q})^{1/q}$$

$$\le C(\|v\|^q_{\overset{\circ}{H}{}^1_q} + \sum_{j=1}^{n}\|Lv\|^q_{L_q(I_j)} + \sum_{j=1}^{n-1}|Jv'(x_j)|^q\delta_j^{1-q})^{1/q}$$

$$\le C\left\{\|u\|^q_{L_p(I)} + (\sum_{j=1}^{n-1}\delta_j|u(x_j)|^p)^{q/p}\right.$$

$$\left. + \left(\sum_{i=1}^{n-1}\delta_i|u(x_i)|^p\right)^{\frac{(2-p)q}{p}}\sum_{j=1}^{n-1}\delta_j|u(x_j)|^p\right\}^{1/q}$$

$$\le C\left\{\|u\|_{L_p(I)} + (\sum_{j=1}^{n-1}\delta_j|u(x_j)|^p)\right\}^{1/p}$$

$$\le C\,\|u\|_{H^0_{p,\Delta}} .$$

Using (5.6) and (5.7) we get (2.2a), i.e.,

$$\inf_{\substack{u\in H^0_{p,\Delta}\\ \|u\|_{H^0_{p,\Delta}}=1}}\ \sup_{\substack{v\in H^2_{q,\Delta}\\ \|v\|_{H^2_{q,\Delta}}=1}} |B_\Delta(u,v)| \ge C_2 > 0$$

where C_2 is independent of Δ.

The proof of (2.2b) is immediate.

__Theorem 7.__ With $S_{1,\Delta} = S_{2,\Delta} = S_\Delta$, (2.4a,b) hold with a constant C_2' which does not depend on Δ.

__Proof.__ Since S_Δ is finite dimensional, (2.4b) follows from (2.4a). We consider first the case $1 < p < \infty$.

Let $u \in S_\Delta$. Choose v to be the solution of

$$(5.8)\qquad \begin{cases} v \in S_\Delta \\ \int_0^1 (av'\phi' + cv\phi)dx = \|u\|^{2-p}_{L_p(I)}\int_0^1 |u|^{p-1}(\operatorname{sgn} u)\phi\,dx, & \text{for all } \phi \in S_\Delta . \end{cases}$$

Setting $\phi = u$ in (5.8) and using (4.3c) we get

$$(5.9)\qquad B_\Delta(u,v) = \|u\|^2_{L_p(I)}$$

$$\ge C\|u\|^2_{H^0_{p,\Delta}} , \quad C > 0 .$$

It remains to show that $\|v\|_{H^2_{q,\Delta}} \le C\|u\|_{H^0_{p,\Delta}}$. For all $\phi \in S_\Delta$ we have

$$(5.10)\qquad \|u\|^{2-p}_{L_p(I)}\int_0^1 |u|^{p-1}(\mathrm{sgn}\ u)\phi\ dx = \int_0^1 (a\ v'\phi' + c\ v\ \phi)dx$$

$$= \sum_j \int_{I_j} Lv\ \phi\ dx - \sum_{j=1}^{n-1} a(x_j)Jv'(x_j)\phi(x_j)$$

$$= \sum_j \int_{I_j} (-a\ v''\ \phi)dx + \int_0^1 (cv - a'v')\phi\ dx$$

$$- \sum_j a(x_j)Jv'(x_j)\phi(x_j)\ .$$

$-v''|_{I_j}$ = a polynomial of degree $r-3 \equiv \ell$; write $-v''|_{I_j} = \sum_{i=0}^{\ell} b_i^j (x-x_{j-1})^i$. Then $\int_{I_j} a(-v'')\phi\ dx = \sum_{i=0}^{\ell} b_i^j \int_{I_j} (x-x_{j-1})^i a\phi dx$. Now for $1 \le k \le n-1$ choose $\phi = \phi_k \in S_\Delta$ so that

$$(5.11)\qquad \begin{cases} \phi_k(x_j) = \delta_{kj},\ j = 0,\cdots,n\ , \\ \int_{I_j} (x-x_{j-1})^i \phi_k a\ dx = 0,\ i = 0,1,\cdots,\ell,\ j = 1,\cdots,n\ . \end{cases}$$

Then from (5.10) we have

$$(5.12)\qquad Jv'(x_k) = \frac{\int_0^1 (-a'v' + cv)\phi_k dx - \|u\|^{2-p}_{L_p(I)} \int_0^1 |u|^{p-1}(\mathrm{sgn}\ u)\phi_k dx}{a(x_k)}\ .$$

Now from the definition of ϕ_k in (5.11) and (4.3c) we get

$$\left|\int_0^1 (a'v' - cv)\phi_k dx\right| = \left|\int_{x_{k-1}}^{x_{k+1}} (a'v' - cv)\phi_k dx\right|$$

$$\le \left(\int_{x_{k-1}}^{x_{k+1}} |a'v' - cv|^q dx\right)^{1/q} \left(\int_{x_{k-1}}^{x_{k+1}} |\phi_k|^p dx\right)^{1/p}$$

$$\le C\left(\int_{x_{k-1}}^{x_{k+1}} |a'v' - cv|^q dx\right)^{1/q} (2\delta_k)^{1/p}$$

and

$$\left|\int_0^1 |u|^{p-1}(\mathrm{sgn}\ u)\phi_k dx\right| \le \left(\int_{x_{k-1}}^{x_{k+1}} |u|^{(p-1)q} dx\right)^{1/q} \left(\int_{x_{k-1}}^{x_{k+1}} |\phi_k|^p dx\right)^{1/p}$$

$$\le C\left(\int_{x_{k-1}}^{x_{k+1}} |u|^p dx\right)^{1/q} (2\delta_k)^{1/p}\ .$$

Thus from (5.12) we obtain

$$(5.13)\qquad \sum_j |Jv'(x_j)|^q \delta_j^{1-q} \le C\left(\int_0^1 |a'v' - cv|^q dx + \left(\int_0^1 |u|^p dx\right)^{q/p}\right)$$

$$\le C\left(\|v\|^q_{\mathring{H}^1_q} + \|u\|^q_{L_p(I)}\right)\ .$$

Next choose $\phi = \bar{\phi}$ S so that

$$(5.14)\qquad \begin{cases} \bar{\phi}(x_j) = 0,\ j = 0,\cdots,n\ , \\ \int_{I_j} (x-x_{j-1})^i a\bar{\phi}\ dx = |b_i^j|^{q-1} h_j^{qi+1} \mathrm{sgn}\ b_i^j,\ i = 0,\cdots,\ell,\ j = 1,\cdots,n\ . \end{cases}$$

Then from (5.10) and (4.2a) we get

$$(5.15)\qquad \sum_j \int_{I_j} |v''|^q dx \le C\sum_j \sum_i |b_i^j|^q h_j^{qi+1}$$

$$= C\sum_j \int_{I_j} a(-v'')\bar{\phi}\ dx$$

$$= C\|u\|^{2-p}_{L_p(I)} \int_0^1 |u|^{p-1}(\mathrm{sgn}\ u)\bar{\phi}\ dx + C\int_0^1 (a'v' - cv)\bar{\phi}\ dx\ .$$

Now from the definition of $\bar{\phi}$ in (5.14) and (4.2a,b) we have

$$(5.16)\qquad \left|\int_0^1 (a'v' - cv)\bar{\phi}\ dx\right| \le \sum_j \left(\int_{I_j} |a'v' - cv|^q dx\right)^{1/q} \left(\int_{I_j} |\bar{\phi}|^p\right)^{1/p}$$

$$\le C\left(\sum_j \int_{I_j} |a'v' - cv|^q dx\right)^{1/q} \left(\sum_j \sum_{i=0}^{\ell} \frac{\left|\int_{I_j} a\bar{\phi}(x-x_{j-1})^i dx\right|^p}{h_j^{pi+p-1}}\right)^{1/p}$$

$$\le C\|v\|_{\overset{\circ}{H}{}^1_q(I)} \Big(\sum_j \sum_i |b_i^j|^q h_j^{qi+1}\Big)^{1/p}$$

$$\le C\|v\|_{\overset{\circ}{H}{}^1_q(I)} \Big(\sum_j \int_{I_j} |v''|^q dx\Big)^{1/p}$$

and

(5.17) $$\Big|\int_0^1 |u|^{p-1}(\operatorname{sgn} u)\tilde{\phi}\, dx\Big| \le \Big(\int_0^1 |u|^p dx\Big)^{1/q} \Big(\int_0^1 |\tilde{\phi}|^p dx\Big)^{1/p}$$

$$\le C\|u\|^{p/q}_{L_p(I)} \Big(\sum_j \int_{I_j} |v''|^q dx\Big)^{1/p} .$$

Combining (5.15), (5.16), and (5.17) we get

(5.18) $$\sum_j \int_{I_j} |v''|^q dx \le C(\|v\|^q_{\overset{\circ}{H}{}^1_q(I)} + \|u\|^q_{L_p(I)}) .$$

From Theorem 5 (cf. (4.19)) and (5.8) we obtain

(5.19) $$\|v\|_{\overset{\circ}{H}{}^1_q(I)} \le C\|u\|_{L_p} .$$

Combining (5.13), (5.18), and (5.19) we have

(5.20) $$\|v\|_{H^2_{q,\Delta}} \le C\|u\|_{L_p} \le C\|u\|_{H^0_{p,\Delta}} .$$

(5.9) and (5.20) yield the proof of Theorem 7.

Let us briefly sketch the proof for the cases $p = 1$ and $p = \infty$. Let $p = 1$. Given $u \in S_\Delta$ we define v by (5.8). Then we immediately get (5.9). If ϕ_k is defined as in (5.11), a slight modification of the argument leading to (5.13) yields

(5.21) $$\max_j \delta_j^{-1}|Jv'(x_j)| \le C(\|v\|_{\overset{\circ}{H}{}^1_\infty} + \|u\|_{L_1(I)}) .$$

For k fixed let $\tilde{\phi}_k \in S_\Delta$ be defined by

$$\tilde{\phi}_k(x_j) = 0,\ j = 0,\cdots,n ,$$

$$\int_{I_j} \tilde{\phi}_k a(x-x_{j-1})^i dx = 0 \text{ for } i = 0,1,\cdots,\ell \text{ if } j \ne k ,$$

$$\int_{I_k} \tilde{\phi}_k a(x-x_{j-1})^i dx = \begin{cases} (\operatorname{sgn} b_k^{i_k}) h_k^{i_k} & \text{if } i = i_k \\ 0 & \text{if } i \ne i_k , \end{cases}$$

where $\max_{0\le i\le \ell} |b_k^i| h_k^i = |b_k^{i_k}| h_k^{i_k}$. Then, replacing $\tilde{\phi}$ by $\tilde{\phi}_k$, a slight modification of the proof of (5.18) yields

(5.22) $$\max_{I_k}|v''| \le C(\|v\|_{\overset{\circ}{H}{}^1_\infty} + \|u\|_{L_1(I)}) .$$

From Theorem 5 we have

(5.23) $$\|v\|_{\overset{\circ}{H}{}^1_\infty} \le C\|u\|_{L_1(I)} .$$

Combining (5.21), (5.22), and (5.23) we obtain

(5.24) $$\|v\|_{H^2_{\infty,\Delta}} \le C\|u\|_{L_1(I)} \le C\|u\|_{H^0_{1,\Delta}} .$$

The desired result, for $p = 1$, follows from (5.9) and (5.24).

Now let $p = \infty$. Given $u \in S_\Delta$ we here define v by

$$\begin{cases} v \in S_\Delta \\ \int_0^1 (av'\phi' + cv\phi)dx = \|u\|_{L_\infty(I)}\phi(\bar{x}) , & \text{for all } \phi \in S_\Delta , \end{cases}$$

where $\|u\|_{L_\infty(I)} = |u(\bar{x})|$. Then

(5.25) $$B_\Delta(u,v) = \|u\|^2_{L_\infty(I)} .$$

We also obtain

$$\sum_{j=1}^{n} |Jv'(x_j)| \le C(\|v\|_{\overset{\circ}{H}{}^1_1(I)} + \|u\|_{L_\infty(I)}) ,$$

$$\sum_j \int_{I_j} |v''|dx \le C(\|v\|_{\overset{\circ}{H}{}^1_1(I)} + \|u\|_{L_\infty(I)})$$

and

$$\|v\|_{\overset{\circ}{H}{}^1_1(I)} \le C\|u\|_{L_\infty(I)} .$$

from which we get

(5.26)
$$\|v\|_{H^2_{1,\Delta}} \le C\|u\|_{L_\infty(I)} = C\|u\|_{H^0_{\infty,\Delta}} .$$

The desired result now follows from (5.25) and (5.26).

6. Applications

In this section we use the mesh dependent spaces $H^0_{p,\Delta}$ and $H^2_{p,\Delta}$ to analyze the finite element method based on C^0-finite elements for the two point boundary value problem

(6.1)
$$\begin{cases} Lu \equiv -(a\,u')' + cu = f \ , & x \in I \\ u(0) = u(1) = 0 \ , \end{cases}$$

where $a \in C'(I)$, $c \in C(I)$, $a(x) \ge a_0 > 0$ and $c(x) \ge 0$ for $x \in I$, and f is given.

a) Let $p = 2$ and suppose $f \in L_2(I)$. In this case the usual variational characterization of the solution u of (6.1) is given by

(6.2)
$$\begin{cases} u \in \overset{\circ}{H}{}^1 \\ \int_0^1 (au'v' + cuv)dx = \int_0^1 f\,v\,dx \ , & \text{for all } v \in \overset{\circ}{H}{}^1 . \end{cases}$$

However, u can also be characterized in the following two ways in terms of the form B_Δ defined in (3.3):

(6.3a)
$$\begin{cases} u \in H^0_\Delta \\ B_\Delta(u,v) = \int_0^1 f\,v\,dx \ , & \text{for all } v \in H^2_\Delta \end{cases}$$

and

(6.3b)
$$\begin{cases} u \in H^2_\Delta \\ B_\Delta(v,u) = \int_0^1 f\,v\,dx \ , & \text{for all } v \in H^0_\Delta . \end{cases}$$

We note that the solution u lies in the spaces H^0_Δ and H^2_Δ for all Δ.

The usual Ritz approximation u_Δ of u is defined by

$$\begin{cases} u_\Delta \in S_\Delta \\ \int_0^1 (au_\Delta'v' + cu_\Delta v)dx = \int_0^1 f\,v\,dx \ , & \text{for all } v \in S_\Delta , \end{cases}$$

with S_Δ as defined in (4.1), and it is easily seen that u also satisfies

(6.4a) $$\begin{cases} u_\Delta \in S_\Delta \\ B_\Delta(u_\Delta, v) = \int_0^1 f\, v\, dx \quad , \quad \text{for all} \quad v \in S_\Delta \end{cases}$$

and

(6.4b) $$\begin{cases} u_\Delta \in S_\Delta \\ B_\Delta(v, u_\Delta) = \int_0^1 f\, v\, dx \quad , \quad \text{for all} \quad v \in S_\Delta . \end{cases}$$

From (6.3) and (6.4) we have

$$B_\Delta(u_\Delta, v) = B(u,v) \quad , \quad \text{for all} \quad v \in S_\Delta$$

and

$$B_\Delta(v, u_\Delta) = B(v,u) \quad , \quad \text{for all} \quad v \in S_\Delta ,$$

i.e., that u_Δ is simultaneously the left and the right B_Δ-projection of u onto S_Δ. Since assumptions (2.1), (2.2), and (2.4) hold (Theorems 3, 6, and 7) we can apply the estimates (2.6) in Theorem 2.

From (2.6a) we get

(6.5) $$\|u-u_\Delta\|_{H^0_\Delta} \le (1 + {}^{C_1}/C_2') \inf_{\chi \in S_\Delta} \|u-\chi\|_{H^0_\Delta} .$$

Thus we have obtained a quasi-optimal error estimate in the norm $\|\cdot\|_{H^0_\Delta}$.

We can estimate the right side of (6.5) in terms of the mest parameter h as follows. Let $v = \mathcal{J}u$ be the S_Δ-interpolant of u that satisfies

$$v(x_j) = u(x_j),\ j = 1,\cdots,n \ ,$$

$$\int_{I_j} v(x-x_{j-1})^i dx = \int_{I_j} u(x-x_j)^i dx,\ j = 1,\cdots,n,\ i = 0,\cdots,r-3 \ .$$

Using standard results in approximation theory we then have

(6.6) $$\|u-u_\Delta\|_{L_2(I)} \le \|u-u_\Delta\|_{H^0_\Delta} \le (1 + {}^{C_1}/C_2') \|u-\mathcal{J}u\|_{H^0_\Delta}$$

$$= (1 + {}^{C_1}/C_2') \|u-\mathcal{J}u\|_{L_2(I)}$$

$$\le Ch^k \|u\|_{k,I} \quad , \quad 0 \le k \le r \ ,$$

provided $u \in H^k(I)$. (6.6) is the standard L_2 estimate for the problem we are considering. In the usual proof of this estimate one first obtains an estimate $\|u - u_\Delta\|_{1,I}$ and than obtains an estimate for $\|u - u_\Delta\|_{0,I}$ using a duality argument due to Nitsche [9].

We have thus seen that, on the one hand, the norm $\|\cdot\|_{H^0_\Delta}$ is closely realted to the L_2 norm, and, on the other, a quasi-optimal error estimate holds with respect to $\|\cdot\|_{H^0_\Delta}$. We remark that the estimate

(6.7) $$\|u - u_\Delta\|_{0,I} \le C \inf_{\chi \in S_\Delta} \|u - \chi\|_{0,I}$$

is false, i.e., the Ritz approximation (based on C^0-finite elements) is not quasi-optimal with respect to the L_2-norm. We show this by considering the simple differential operator $Lu = -u''$ and piecewise linear approximating functions. We first observe that (6.7) implies $\|u_\Delta\|_{0,I} \le C\|u\|_{0,I}$ with C independent of u and Δ. Now consider the mesh $\Delta = \{0 < 1/2 < 1\}$ and a sequence of functions $u_j \in \mathring{H}^1(I)$ satisfying $u_j(1/2) \to \infty$ and $\|u_j\|_{0,I} \to 0$. Since for $Lu = -u''$ the Ritz approximation $u_{j,\Delta}$ is the piecewise linear interpolant of u_j, we see that $\|u_{j,\Delta}\|_{0,I} \le C\|u_j\|_{0,I}$ cannot hold with C independent of j. Finally we note that (6.7) is true for C^1-finite elements provided the mesh family is quasi-uniform (cf. [2, Cp 6]).

(6.5) is closely related to an estimate obtained by Eisenstat, Schreiber and Schultz [6]. They showed that

$$\|u - u_\Delta\|_{0,I} \le C \inf\{\|u - \chi\|_{0,I} : \chi \text{ an } S_\Delta\text{-interpolant of } u\}$$

for quasi-uniform mesh families if $r \ge 3$, and for arbitrary mesh families if $r = 2$.

Since u_Δ is also the right B_Δ-projection of u we can also apply (2.6b) to obtain

$$\|u - u_\Delta\|_{H^2_\Delta} \le (1 + {}^{C_1}/C_2') \inf_{\eta \in S_\Delta} \|u - \eta\|_{H^2_\Delta}$$

$$\leq (1 + C_{1}/C_2') \inf_{\eta\in S_\Delta \cap C^1(I)} \|u - \eta\|_{2,I}$$

$$\leq ch^{k-2}\|u\|_{k,I} \quad , \quad 2 \leq k \leq r \quad .$$

From the definition of $\|\cdot\|_{H^2_\Delta}$ we see that

(6.8a) $$\left(\sum_{j=1}^{n} \|u - u_\Delta\|^2_{0,I_j}\right)^{1/2} \leq ch^{k-2}\|u\|_{k,I}$$

and

(6.8b) $$|u_\Delta'(x_j^+) - u_\Delta'(x_j^-)| \leq ch^{k-3/2}\|u\|_{k,I} \quad , \quad j = 1,\cdots,n \quad .$$

(6.8a) provides an estimate on the L_2 norm of the 2nd derivatives of the error on each I_j. We note that the estimate $\|u - u_\Delta\|_{2,I} \lesssim ch^{k-2}\|u\|_{k,I}$ is known to hold provided $S_\Delta \subset H^2_2(I)$ (this requires C^1-finite elements) and the mesh family is quasi-uniform. (6.8b) provides an estimate on the convergence to zero of the jumps in u_Δ' at the nodes x_j.

b) We consider here a family of examples that illustrate the difference between an estimate of the type (6.5) (a quasi-optimal estimate) and one of the type (6.6) (an estimate in terms of the mesh parameter h).

Consider the family of problems

$$\begin{cases} -(au')' + cu = f \quad , \quad 0 \leq x \leq 1 \quad , \\ u(0) = u(1) = 0 \end{cases}$$

where f is such that

$$u(x) = (x+t)^{1/2} - \{t^{1/2} + x[(1+t)^{1/2} - t^{1/2}]\} \quad ,$$

where the parameter t satisfies $0 < t$. $u(x)$ has a large 2nd derivative if t is small.

Let u_Δ be the Ritz approximation to u defined by piecewise linear elements. Since these examples are all of the type considered in Subsection (a), estimates (6.5) and (6.6) are valid. We now attempt to choose the mesh Δ so as to minimize the right-hand side of (6.5). Let $\Delta = \Delta_n$ be given by $x_j^n = (j/n)^\gamma$, $j = 0,1,\cdots,n$, where $\gamma = \frac{5}{2}$. With this choice it is easily seen that

$$\inf_{\chi\in S_{\Delta_n}} \|u - \chi\|_{H^0_{\Delta_n}} \leq \|u - \mathcal{J}u\|_{L^2(I)}$$
$$\leq Cn^{-2} \quad ,$$

where $\mathcal{J}u$ is the piecewise linear interpolant of u and where C is independent of t for n sufficiently large $(n^{-5/2} \leq t)$; in fact C can be taken to be 5. Hence we have derived 2nd order estimate in the number of unknowns in the Ritz equations. If we had used (6.6) to analyze this family of examples we could only have obtained a bound of the form $Ct^{-1}n^{-2}$. Thus we see that using (6.5) we can see the effect of strong mesh refinement (i.e., refinement leading to a non quasi-uniform mesh family), whereas (6.6) does not yield this information. Note that the fact that (6.5) is valid for arbitrary (non quasi-optimal) mesh families is crucial in the analysis in this Subsection.

Finally we note that if $t = 0$ then $u \notin \overset{\circ}{H}{}^1$ and the usual analysis of finite element methods does not apply. However such a problem does fit into the theory treated in this paper and if we use a mesh family similar to that introduced above we can show that $\|u - u_{\Delta_n}\|_{0,I} \leq Cn^{-2}$.

c) Consider now the two point boundary value problem

(6.9) $$\begin{cases} Lu = -(au')' + cu = d_{\bar{x}}' \\ u(0) = u(1) = 0 \end{cases}$$

with $f = d_{\bar{x}}'$ equal to the derivative of the Dirac distribution $d_{\bar{x}}'$ at $\bar{x}$, $0 < \bar{x} < 1$. $u \notin \overset{\circ}{H}{}^1(I)$ and hence we cannot characterize u as in (6.2). Thus (6.9) and its approximate solution is not covered by the usual treatments of Ritz methods. u can, however, be characterized by

(6.10) $$\begin{cases} u \in H^0(I) = L_2(I) \\ \int_0^1 u\, Lv\, dx = -v'(\bar{x}), \qquad \text{for all} \quad v \in H^2(I) \cap \overset{\circ}{H}{}^1(I) \end{cases}$$

and also (more importantly for our purpose) by

(6.11) $$\begin{cases} u \in H^0_\Delta \\ B_\Delta(u,v) = -v'(\bar{x}) \quad , \quad \text{for all} \quad v \in H^2_\Delta \quad . \end{cases}$$

provided $\bar{x}$ is not a node of Δ. Note that $d'_{\bar{x}} \in (H^2_\Delta)'$ and thus that existence and uniqueness follow from Theorem 1. If an approximation procedure is based on (6.10) one would have to assume that the finite dimensional spaces of approximating functions lie in $H^2(I)$. However, if we base our approximation procedure on (6.11) we can use S_Δ.

The Ritz approximation u_Δ to u can now be defined by

$$\begin{cases} u_\Delta \in S_\Delta \\ B_\Delta(u_\Delta, v) = \int_0^1 (au'_\Delta v' + cu_\Delta v)dx = v'(\bar{x}) \quad , \quad \text{for all } v \in S_\Delta \ . \end{cases}$$

Thus u_Δ is the left B_Δ-projection of u and from (2.6a) we have

$$\|u - u_\Delta\|_{L_2(I)} \le \|u - u_\Delta\|_{H^0_\Delta} \le C \inf_{\chi \in S_\Delta} \|u - \chi\|_{H^0_\Delta}$$
$$\le c\sqrt{h}$$

if $r = 2$. From this estimate we get $\|u - u_\Delta\|_{L_2(I)} \le C\sqrt{h}$ for an arbitrary mesh. We also see that if we choose the mesh appropriately near the point $\bar{x}$ we obtain an estimate of the form

$$\|u - u_{\Delta_n}\|_{L_2(I)} \le cn^{-2}$$

where $n + 1$ is the number of nodes (cf. Subsection (b)).

d) We consider now estimates for general p. We again study

$$\begin{cases} Lu = f \quad , \quad x \in I \ , \\ u(0) = u(1) = 0 \end{cases}$$

where $f \in L_p(I)$. The solution u is easily seen to simultaneously satisfy

$$\begin{cases} u \in H^0_{p,\Delta} \\ B_\Delta(u,v) = \int_0^1 f\, v\, dx \quad , \quad \text{for all } v \in H^2_{q,\Delta} \end{cases}$$

and

$$\begin{cases} u \in H^2_{p,\Delta} \\ B_\Delta(v,u) = \int_0^1 f\, v\, dx \quad , \quad \text{for all } v \in H^0_{q,\Delta} \end{cases}$$

for all $1 \le p \le \infty$ and the usual Ritz approximation u_Δ is seen to simultaneously satisfy

$$\begin{cases} u_\Delta \in S_\Delta \\ B_\Delta(u_\Delta, v) = \int_0^1 f\, v\, dx \quad , \quad \text{for all } v \in S_\Delta \end{cases}$$

and

$$\begin{cases} u_\Delta \in S_\Delta \\ B_\Delta(v, u_\Delta) = \int_0^1 f\, v\, dx \quad , \quad \text{for all } v \in S_\Delta \ . \end{cases}$$

Thus u_Δ is both the left and right B_Δ- projection of u onto S_Δ for all p. Since the stability condition holds for all p (Theorem 7) we can apply Theorem 2 to obtain

(6.12) $$\|u - u_\Delta\|_{H^0_{p,\Delta}} \le C \inf_{\chi \in S_\Delta} \|u - \chi\|_{H^0_{p,\Delta}}$$

and

(6.13) $$\|u - u_\Delta\|_{H^2_{p,\Delta}} \le C \inf_{\chi \in S_\Delta} \|u - \chi\|_{H^2_{p,\Delta}} \ .$$

(6.12) and (6.13) provide quasi-optimal error estimates in the indicated norms. We emphasize that these estimates are valid for arbitrary mesh families. From these estimates we can derive estimates in terms of the mesh parameter h. For example, from (6.12) with $p = \infty$ we find

(6.14) $$\|u - u_\Delta\|_{L_\infty(I)} \le ch^k \|u\|_{H^k_\infty(I)} \quad , \quad 0 \le k \le r$$

and from (6.13) with $p = \infty$ we have

(6.15) $$\|u - u_\Delta\|_{H^2_{\infty,\Delta}} \le ch^{k-2} \|u\|_{H^k_\infty(I)} \quad , \quad 2 \le k \le r \ .$$

(6.14) was first proved by Wheeler [13]; compare also [3, 7, 10]. See Subsection (b) for remarks on the difference between estimates of type (6.12) and of type (6.14). (6.15) shows that the 2nd derivatives of $u - u_\Delta$, considered on each I_j, converge uniformly to zero.

(6.12) in the case $p = \infty$ is closely related to a recent result of Schatz [12]. He proves

$$\|u - u_h\|_{L_\infty(\Omega)} \le c(\ln h^{-1})^{\bar{r}} \inf_{\chi \in S_r^h(\Omega)} \|u - \chi\|_{L_\infty(\Omega)}$$

where Ω is a polygon in the plane, $S_r^h(\Omega)$ is the finite element space of continuous piecewise polynomials of degree $r - 1$ defined on a quasi-uniform triangulation of Ω with triangles roughly size h, $\bar{r} = \begin{cases} 1 & \text{if } r = 2 \\ 0 & \text{if } r \ge 3 \end{cases}$, u is continuous on $\bar{\Omega}$ and u_h is the usual finite element projection of u into $S_r^h(\Omega)$.

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This paper presents a new approach to the analysis of finite element methods based on C^0-finite elements for the approximate solution of 2nd order boundary value problems in which error estimates are derived directly in terms of two mesh dependent norms that are closely related to the L_2 norm and to the 2nd order Sobolev norm, respectively, and in which there is no assumption of quasi-uniformity on the mesh family. This is in contrast to the usual analysis in which error estimates are first derived in the 1st order Sobolev norm and subsequently are derived in the L_2 norm and in the 2nd order Sobolev norm - the → next page

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ABSTRACT (continued)

2nd order Sobolev norm estimates being obtained under the assumption that the functions in the underlying approximating subspaces lie in the 2nd order Sobolev space and that the mesh family is quasi-uniform.