Analysis of interface crack in a bi-material based on body force method

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Abstract

A method for analyzing an interface crack problem is proposed based on Body Force Method. In order to treat the so–called oscillatory stress and displacement field in the vicinity of the crack tip strictly, a new basic density function for an interface crack is introduced. In addition to the formulae for an interface crack problem, many problems of the elastic bi–material containing arbitrary cracks are solved numerically by using a personal computer and the results are demonstrated in tables and graphs.

1 Introduction

The Body Force Method (BFM)[1] is one of the indirect boundary element methods for stress analysis and has been produced a lot of highly accurate solutions which are important in practice since its original proposition in 1967 by H. Nisitani. The most characteristic point of BFM is to express the elastic boundary value problems in the form of integral equation based on the principle of superposition. Therefore, as compared with the ordinary boundary element method whose base is Somegliana's equality, BFM has the strong flexibility for introducing various types of inventions considering the physical characteristics of the target problem. For instance, in the case of analyzing a two–dimensional crack problem, the crack is replaced by the

continuously distributed pair of point forces (Body Force Doublet, BFD)[2] acting in an infinite plate along a contour to be a crack. As the magnitude of BFD at a point corresponds to the discrepancy of displacement at the point where BFD acts, crack problems are always reduced to a singular integral equation whose unknown is the density of BFD. In another words, according to the BFM, the crack problem is reduced to a boundary integral equation whose unknown is the relative displacement distribution between upper and lower crack surfaces.

In order to treat the singularity near the crack tip strictly, the unknown density of BFD is expressed by the product of the basic density function[2] for a crack and the weight function. The basic density function is a characteristic function which represents the feature of crack surface relative displacement near the crack tip, and as the basic density function we usually take the relative displacement between crack surfaces subject to constant pressure or shear stress in the problem of an infinite plate containing an isolated crack.

According to the elastic solution of crack problem, the gradient of relative displacement between crack surfaces becomes infinite at the crack tip, and therefore, we can not obtain highly accurate distributions of BFD without using the basic density function in numerical analysis. Moreover, by introducing the basic density function, we can not only calculate the singular stress field near a crack tip exactly but also obtain the value of stress intensity factor directly from the value of weight function at the crack tip.

2 Elastic field around an interface crack

Figure 1 shows an interface crack in a bi–material subject to uniform normal and shear stresses at infinity. If we use the Kolsov–Mushkelisivili's notation for complex stress functions, the solutions for material "m",($m = 1, 2$) in Fig.1 can be expressed as follows[3].

$$\Omega_m(z) = \frac{\sigma_y^\infty + \sigma_{xm}^\infty}{4} + C_m(\varepsilon)\frac{\sigma_y^\infty - i\tau_{xy}^\infty}{2\cosh(\pi\varepsilon)}\left[\frac{z - 2ic\varepsilon}{\sqrt{z^2 - c^2}}\left(\frac{z - c}{z + c}\right)^{-i\varepsilon} - 1\right] \quad (1)$$

$$\omega_m(z) = \frac{3\sigma_y^\infty - \sigma_{xm}^\infty + 4i\tau_{xy}^\infty}{4} + D_m(\varepsilon)\frac{\sigma_y^\infty - i\tau_{xy}^\infty}{2\cosh(\pi\varepsilon)}\left[\frac{z - 2ic\varepsilon}{\sqrt{z^2 - c^2}}\left(\frac{z - c}{z + c}\right)^{-i\varepsilon} - 1\right] \quad (2)$$

where c stands for the half crack length, ε denotes a bi–material constant defined as $\varepsilon = 1/2\pi \ln\left((G_2\kappa_1 + 1)/(G_1\kappa_2 + 1)\right)$ in which G_m is shear modulus, ν_m is Poisson's ratio, κ_m is equal to $(3 - \nu_m)/(1 + \nu_m)$ for plane stress and $3 - 4\nu_m$ for plane strain, $C_1(\varepsilon) = D_2(\varepsilon) = e^{-\pi\varepsilon}$, $C_2(\varepsilon) = D_1(\varepsilon) = e^{\pi\varepsilon}$, and "$m$" ($m = 1, 2$) is the number of materials.

From the condition for single valuedness of solution, the following equation between stress components at infinity is obtained.

$$\sigma_y^\infty = \frac{G_2(1 + \kappa_2)\sigma_{x1}^\infty - G_1(1 + \kappa_1)\sigma_{x2}^\infty}{G_2(1 + \kappa_2)\frac{3 - \kappa_1}{1 + \kappa_1} - G_1(1 + \kappa_1)\frac{3 - \kappa_2}{1 + \kappa_2}}$$

Substituting Eqs.(1) and (2) into the following equations, we can obtain the elastic field such as displacement (u, v), resultant force over an arc (P_x, P_y) and stress components $(\sigma_x, \sigma_y, \tau_{xy})$ in Fig.1.

$$2G(u - iv) = \kappa \overline{\int \Omega(z)dz} - \int \overline{\omega}(z)dz + (z - \overline{z})\Omega(z) \tag{3}$$

$$- P_y - iP_x = \overline{\int \Omega(z)dz} + \int \overline{\omega}(z)dz - (z - \overline{z})\Omega(z) \tag{4}$$

$$\sigma_z + \sigma_y = 4\Re[\Omega(z)] \ , \ \ \sigma_y - \sigma_x + 2i\tau_{xy} = 2[(z - \overline{z})\Omega'(z) - \Omega(z) + \overline{\omega}(z)] \tag{5}$$

Especially, the crack surface relative displacement becomes,

$$[v+iu]^+ - [v+iu]^- = \left(\frac{1 + \kappa_1}{4G_1} + \frac{1 + \kappa_2}{4G_2}\right) \sqrt{c^2 - \xi^2} \left(\frac{c - \xi}{c + \xi}\right)^{i\varepsilon} \frac{\sigma_y^\infty + i\tau_{xy}^\infty}{\cosh(\pi\varepsilon)} \tag{6}$$

in which, superscript $+, -$ denote the limiting value from the upper and lower surfaces and ξ stands for the local coordinate on an interface crack (Fig.1). Eq.(6) shows the displacement distribution with the oscillatory feature in the form of $\sqrt{c^2 - \xi^2} \times \cos\left[\varepsilon \ln\left(\frac{c-\xi}{c+\xi}\right)\right]$ or $\sqrt{c^2 - \xi^2} \times \sin\left[\varepsilon \ln\left(\frac{c-\xi}{c+\xi}\right)\right]$ along a crack, and we must express this complex feature strictly to accomplish our analysis with high accuracy.

3 Fundamentals for numerical analysis

3.1 Fundamental solutions
3.1.1 Fundamental solution for usual boundary In order to treat problems with bi–material interface efficiently, the stress field due to a point force acting in a dissimilar infinite plate was used for a fundamental solution in the present analysis (Fig.2(a)). Dunders and Hétenyi[4,5] showed the complex stress functions for Fig.2(a) as,
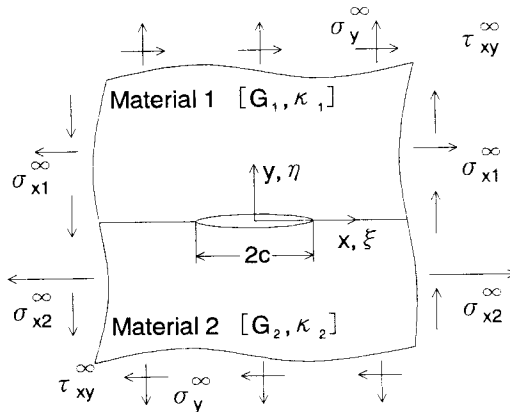

Figure 1: Infinite plate of bi–material containing an interface crack

(a) point force (fundamental solution for usual boundary)

$$\gamma_{\mathrm{I}} = \lim_{\delta \to 0}(\delta \cdot F_{\mathrm{I}})$$

$$\gamma_{\mathrm{II}} = \lim_{\delta \to 0}(\delta \cdot F_{\mathrm{II}})$$

(b) Mode I force doublet   (c) Mode II force doublet
(f.s. for Mode I crack boundary)   (f.s. for Mode II crack boundary)

Figure 2: Fundamental solutions for usual boundary and for crack boundary

$$\Omega_{lm}(z) = \frac{-1}{2\pi(1+\kappa_l)}\left[\delta_{lm}\{(1-A_l)P(z)+(1-B_l)Q(z)\}+A_lP(z)\right] \quad (7)$$

$$\omega_{lm}(z) = \frac{1}{2\pi(1+\kappa_l)}\left[\delta_{lm}\{(1-A_l)P(z)+(1-B_l)Q(z)\}+B_lQ(z)\right] \quad (8)$$

$$P(z) = \frac{F}{z-z_0} \;, \quad Q(z) = \left(\kappa_l F - \frac{z_0-\overline{z_0}}{z-z_0}\overline{F}\right)\frac{1}{z-\overline{z_0}} \qquad (l,m=1,2) \quad (9)$$

where,
$F = X + iY$: X, Y are components of point force.
l: number of material in which source point z_0 exists $(l = 1, 2)$.
m: number of material in which observing point z exists $(m = 1, 2)$.
δ_{lm}: Kronecker's delta index.

$$A_l = \frac{G_{3-l}(1+\kappa_l)}{G_l\kappa_{3-l}+G_{3-l}} \;, \quad B_l = \frac{G_{3-l}(1+\kappa_l)}{G_{3-l}\kappa_l+G_l} \qquad (l=1,2)$$

3.1.2 Fundamental solution for crack boundary The complex stress functions for Fig.2(b) and (c) are easily obtained from Eqs.(7), (8) and (9) by differentiation[2].

3.2 Basic density function for interface crack problems

If we assume the densities of distributed force doublets $\gamma_\mathrm{I}(\xi)$, $\gamma_\mathrm{II}(\xi)$ as unknowns to be determined from boundary conditions, the following complex stress functions of the material $m(m=1,2)$ are to be used for the problem shown in Fig.1.

$$\Omega_m(z) = \Omega_m^0(z) + \frac{\delta_{lm}(B_l - A_l) + A_l}{\pi(1 + \kappa_l)} \int_{-c}^{c} \left[\frac{\kappa_l - 1}{\kappa_l + 1}\gamma_\mathrm{I}(\xi) - i\gamma_\mathrm{II}(\xi) \right] \frac{d\xi}{(z - \xi)^2} \quad (10)$$

$$\omega_m(z) = \omega_m^0(z) + \frac{\delta_{lm}(A_l - B_l) + B_l}{\pi(1 + \kappa_l)} \int_{-c}^{c} \left[\frac{\kappa_l - 1}{\kappa_l + 1}\gamma_\mathrm{I}(\xi) - i\gamma_\mathrm{II}(\xi) \right] \frac{d\xi}{(z - \xi)^2} \quad (11)$$

In these expressions, the source point z_0 was replaced by a real variable ξ and the angle β was set to be zero. The functions with superscript "0" are associated with the uniform stress field at infinity.

Considering that the crack surface is to be unstressed and using Eqs.(10), (11) and (5), we can get the following integral equation with unknowns $\gamma_\mathrm{I}(\xi)$ and $\gamma_\mathrm{II}(\xi)$.

$$\sigma_y - i\tau_{xy} = \Omega_m^0(x) + \omega_m^0(x) + \frac{A_l + B_l}{\pi(1 + \kappa_l)} \int_{-c}^{c} \left[\frac{\kappa_l - 1}{\kappa_l + 1}\gamma_\mathrm{I}(\xi) - i\gamma_\mathrm{II}(\xi) \right] \frac{d\xi}{(z - \xi)^2}$$

$$= 0 \quad \text{when } (-c < x < c) \quad (12)$$

Because the gradient of unknown functions $\gamma_\mathrm{I}(\xi)$, $\gamma_\mathrm{II}(\xi)$ becomes infinite at the crack tip, we can not determine the values of $\gamma_\mathrm{I}(\xi)$, $\gamma_\mathrm{II}(\xi)$ by the numerical calculations using discretizing procedure. This is the reason why we must use the basic density function in numerical analysis by the BFD.

Considering Eq.(6) and the fact that $u^+ - u^- = \frac{1}{G}\gamma_\mathrm{II}$ and $v^+ - v^- = \frac{\kappa - 1}{G(1 + \kappa)}\gamma_\mathrm{I}$, we assume the distribution of Mode I force doublet and Mode II force doublet $\gamma_\mathrm{I}(\xi)$, $\gamma_\mathrm{II}(\xi)$ as,

$$\frac{(\kappa_1 - 1)\gamma_\mathrm{I}(\xi)}{G_1(\kappa_1 + 1)} + i\frac{\gamma_\mathrm{II}(\xi)}{G_1} = \left(\frac{1 + \kappa_1}{G_1} + \frac{1 + \kappa_2}{G_2} \right) \sqrt{c^2 - \xi^2} \left(\frac{c - \xi}{c + \xi} \right)^{i\varepsilon} \frac{W_1(\xi) + iW_2(\xi)}{\cosh \pi\varepsilon} \quad (13)$$

where $W_1(\xi)$, $W_2(\xi)$ are weight functions for Mode I force doublet and Mode II force doublet to be determined. By using Eq.(13), we can transform unknowns $\gamma_\mathrm{I}(\xi)$, $\gamma_\mathrm{II}(\xi)$ into unknowns $W_1(\xi)$, $W_2(\xi)$.

Then, the final expressions for complex stress functions correspond to Eqs.(10) and (11) become,

$$\Omega_m(z) = \Omega_m^0(z) + \frac{C_m(\varepsilon)}{2\pi} \int_{-c}^{c} \sqrt{c^2 - \xi^2} \left(\frac{c - \xi}{c + \xi} \right)^{-i\varepsilon} \frac{W_1(\xi) - iW_2(\xi)}{(z - \xi)^2} d\xi \quad (14)$$

$$\omega_m(z) = \omega_m^0(z) + \frac{D_m(\varepsilon)}{2\pi} \int_{-c}^{c} \sqrt{c^2 - \xi^2} \left(\frac{c - \xi}{c + \xi} \right)^{-i\varepsilon} \frac{W_1(\xi) - iW_2(\xi)}{(z - \xi)^2} d\xi \quad (15)$$

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Again, the notations $C_1(\varepsilon) = D_2(\varepsilon) = e^{-\pi\varepsilon}$ and $C_2(\varepsilon) = D_1(\varepsilon) = e^{\pi\varepsilon}$, and $m = 1, 2$ were used.

For the problem shown in Fig.1, the weight functions $W_1(\xi)$ and $W_2(\xi)$ become constant throughout a crack. Although the weight functions may vary with local coordinate ξ in the actual analysis, the change of weight functions along crack always much smaller than the fluctuation of $\gamma_I(\xi)$, $\gamma_{II}(\xi)$ themselves. And thus, we can obtain highly accurate solutions of crack problems even if we use a numerical procedure for determining the weight functions in Eqs. (14) and (15).

3.3 Determination of the stress intensity factor
The interface crack tip stress intensity factor may be defined as follows[6].

$$K_1 - iK_2 = (1 + e^{2\pi\varepsilon})\sqrt{2\pi} \lim_{z \to c} \sqrt{z - c} \left(\frac{z - c}{z + c}\right)^{i\varepsilon} \Omega_1(z) \qquad (16)$$

After some calculations, we obtain the following relation.

$$K_1 - iK_2 = [W_1(c) - iW_2(c)] (1 - 2i\varepsilon)\sqrt{\pi c} \qquad (17)$$

Therefore, from Eq.(17), we can obtain SIFs directory form the value of weight functions at a crack tip.

4 Examples of numerical analysis

Based on the theory presented in this paper, we introduced a versatile program for two–dimensional bi–material problems. In this section, we attempt to compare some our numerical solutions calculated by using a personal computer to those obtained by other researchers.
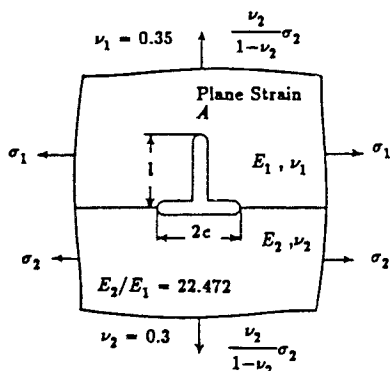
4.1 Interaction of cracks of arbitrary distribution

Fig.3: Infinite plate of bi–material with T type branched crack

Table 1: $F_{I,A} = K_{I,A}/\sigma_1\sqrt{\pi l}$ of Fig.3

c/l	Present analysis	Goree & Venezia[7]	Isida & Noguchi[8]
0.0	0.6240	0.6241	0.6240
0.005	0.6244	0.6245	0.6244
0.010	0.6251	0.6251	0.6251
0.015	0.6258	0.6258	0.6258
0.050	0.6327	0.6328	0.6327
0.075	0.6387	0.6387	0.6386
0.100	0.6447	0.6448	0.6448

4.2 Tension of interface cracked bi–material specimens
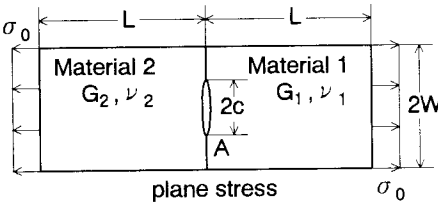
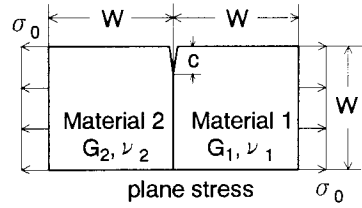plane stress

Fig.4 Center cracked specimen

plane stress

Fig.5 Edge cracked specimen

Table 2.1: $F_{1,A} = K_{1,A}/\sigma_0\sqrt{\pi c}$ of Fig.4
($L/2W = 1.0$ for present and $L/2W = \infty$ for Isida's.
$\Gamma = G_2/G_1$, $\nu_1 = \nu_2 = 0.3$, plane stress is assumed.)

Γ	c/W	0.1	0.2	0.3	0.4	0.5	0.6	0.7
1.0	Present	1.0061	1.0249	1.0583	1.1102	1.1876	1.3043	1.4893
	Isida[9]	1.0060	1.0246	1.0577	1.1094	1.1867	1.3033	1.4882
0.1	Present	0.9677	0.9865	1.0181	1.0657	1.1354	1.2388	1.4005
	Yuuki[6]	0.963	0.985	1.018	1.065	1.134	1.238	1.400
	Miyazaki[10]	0.968	0.986	1.018	1.066	1.136	1.239	1.402
0.01	Present	0.9457	0.9644	0.9947	1.0397	1.1049	1.2008	1.3500
	Yuuki[6]	0.940	0.962	0.994	1.038	1.104	1.201	1.349
	Miyazaki[10]	0.946	0.964	0.994	1.039	1.104	1.201	1.351

Table 2.2: $-F_{2,A} = -K_{2,A}/\sigma_0\sqrt{\pi c}$ of Fig.4
($\Gamma = G_2/G_1$, $\nu_1 = \nu_2 = 0.3$, plane stress is assumed.)

Γ	c/W	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0.1	Present	0.1747	0.1716	0.1713	0.1745	0.1828	0.1994	0.2306
	Yuuki[6]	0.173	0.170	0.171	0.174	0.183	0.199	0.230
	Miyazaki[10]	0.174	0.171	0.170	0.173	0.182	0.198	0.229
0.01	Present	0.2067	0.2025	0.2013	0.2039	0.2121	0.2289	0.2616
	Yuuki[6]	0.205	0.201	0.201	0.203	0.211	0.228	0.260
	Miyazaki[10]	0.206	0.201	0.200	0.203	0.210	0.228	0.260

Table 3: SIFs of Fig.5 ($\Gamma = G_2/G_1$, $\nu_1 = \nu_2 = 0.3$, plane stress)

| Γ	c/W	$F_1 = K_1/\sigma_0\sqrt{\pi c}$			$F_2 = K_2/\sigma_0\sqrt{\pi c}$		
		Present	Yuuki[6]	Miyazaki[11]	Present	Yuuki[6]	Miyazaki[11]
2.0	0.1	1.195	1.188	1.195	0.129	0.128	0.129
	0.2	1.368	1.366	1.368	0.137	0.137	0.137
	0.5	2.820	2.820	2.822	0.267	0.268	0.267
4.0	0.1	1.209	1.201	1.209	0.239	0.238	0.239
	0.2	1.368	1.387	1.368	0.250	0.254	0.250
	0.5	2.807	2.807	2.807	0.483	0.483	0.483
10.0	0.1	1.229	1.220	1.229	0.340	0.338	0.340
	0.2	1.369	1.367	1.369	0.349	0.349	0.349
	0.5	2.789	2.788	2.789	0.663	0.664	0.663
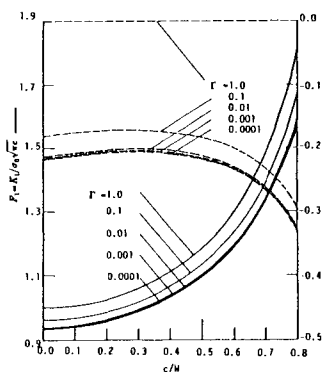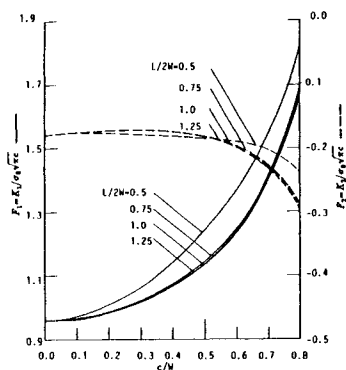
Fig.6 Effect of Γ on SIFs in Fig.4   Fig.7 Effect of $L/2W$ on SIFs in Fig.4

5 Conclusion

A method for analyzing the two–dimensional interface crack in a bi–material was investigated based on BFM. Throughout many examples for evaluating our numerical results, it was found that the present method satisfies both the conditions of the numerical accuracy and calculation efficiency.

Acknowledgement

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